A combinatorial identity and the finite dual of infinite dihedral group algebra

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Outline

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Preparation and aim

- In this talk, $\Bbbk$ an algebraically closed field of characteristic zero.
- All spaces and algebras are over $\Bbbk$.
- Aim: Determine the finite dual H° of a prime regular Hopf algebra H.

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- It is well-known that Larson-Radford (J. Algebra, 1988) proved the following result:

Theorem

Let H be a finite dimensional Hopf algebra, then H is semisimple if and only if H^* is semisimple.

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- A Hopf algebra H has finite global dimension if and only if H^* has finite global dimension?
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A Takeuchi's definition

- Takeuchi defined a quantum group as follows.

Definition

A quantum group G is defined to be a triple

$$G = (A, U, \langle\ ,\ \rangle)$$

where A and U are Hopf algebras, and $\langle\ ,\ \rangle$ is a Hopf pairing on $U \times A$.

- A natural question is: Is a prime regular Hopf algebra a quantum group in the Takeuchi's sense?
- In this talk, we will determine $(\Bbbk\mathbb{D}_\infty)^\circ$.

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1. Motivations
2. A combinatorial identity
3. The results

Description of the identity

- To describe the identity, we need some notions at first.
- Let m and n are positive integers, define

$$U := U(m,n) = \{1, 2, ..., m+n\}.$$

- For a set $X = \{x_1, ..., x_m\}$ of nonnegative integers whose elements are listed in increasing order, we denote by V_X the Vandermonde determinant of X. That is,

$$V_X = \prod_{1 \le i < j \le m} (x_j - x_i).$$

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Theorem

Let $t \in \left[\frac{m(m+1)}{2}, \frac{m(m+1)}{2} + mn\right]$ *be an integer, and let* $t^* = t - \frac{m(m+1)}{2}$. *We have*

$$\sum_{\substack{X=\{x_1,\ldots,x_m\}\subset U \\ \sum x_i = t}} V_X V_Y = G(m+1)G(n+1)\binom{mn}{t^*}.$$

Here the sum is over all subsets X *of* U *whose elements' sum is* t*, and* $Y = U - X$.

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Here the sum is over all subsets X of U whose elements' sum is t, and $Y = U - X$.

Description of the identity

- This identity is related with the following known function:

$$\gamma_m(c) = \frac{1}{m!G(m+1)^2} \int_{[0,1]^m} \delta(s_1 + s_2 + \cdots + s_m - c) \prod_{i<j} (s_i - s_j)^2 \, ds_1 \cdots ds_m.$$

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Sketch of the proof

- Step 1: Explain the number $\frac{V_X}{G(m+1)}$ as the number of some semi-standard Young tableaus (SSYTs).
- Step 2: Explain the number $\frac{V_Y}{G(n+1)}$ as the number of some semi-standard Young tableaus with transpose shape with respect to SSYTs in step 1.
- Step 3: The number $\sum \frac{V_X V_Y}{G(m+1)G(n+1)}$ equals to the number of some pairs of SSYTs (P, Q).
- Step 4: Apply the Robinson-Schensted-Knuth correspondence.

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A corollary

- This observation has the following direct consequence.

Corollary

Let x be an indeterminate and A be the $2m \times 2m$ matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ x & x & \cdots & x & x^{-1} & x^{-1} & \cdots & x^{-1} \\ x^2 & 2x^2 & \cdots & 2^{m-1}x^2 & x^{-2} & 2x^{-2} & \cdots & 2^{m-1}x^{-2} \\ x^3 & 3x^3 & \cdots & 3^{m-1}x^3 & x^{-3} & 3x^{-3} & \cdots & 3^{m-1}x^{-3} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ x^M & Mx^M & \cdots & M^{m-1}x^M & x^{-M} & Mx^{-M} & \cdots & M^{m-1}x^{-M} \end{pmatrix}$$

where $M = 2m - 1$. Then the determinant of A is

$$|A| = G(m+1)^2 \cdot \left(x^{-1} - x\right)^{m^2}.$$

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Finite Dual

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$$H^\circ := \{f \in H^* | f(I) = 0, \text{some ideal } I \text{ s.t. } \dim(H/I) < \infty\}.$$

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Some known examples

Example

Let $H = \mathbb{k}[x], \ \ \Delta(x) = 1 \otimes x + x \otimes 1$. Then we have

$$H^{\circ} \cong \mathbb{k}[x] \otimes kG$$

where $G = (\mathbb{k}, +)$.

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- There is a common point in above examples, that is, H is commutative. Therefore H° is cocommutative and thus one can apply Milnor-Moore's Theorem.

Example

Consider the quantum group $U_q(sl_n)$. Then we have

$$U_q(sl_n)^{\circ} \cong \mathcal{O}_q(SL_n)\#k\mathbb{Z}_2^{n-1}.$$

This is proved by Takeuchi in 1992.

- The key point of above example is that the category of finite-dimensional representations of $U_q(sl_n)$ is semisimple.

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Our situation

- **By definition,** the infinite dihedral group $\mathbb{D}_\infty$ is generated by two elements g and x satisfying

$$x^2 = 1, \quad xgx = g^{-1}.$$

- Note that $\Bbbk\mathbb{D}_\infty$ is not not commutative and thus $(\Bbbk\mathbb{D}_\infty)^\circ$ is not cocommutative. Also, the category of finite-dimensional representations of $\Bbbk\mathbb{D}_\infty$ is not semisimple.

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The Hopf algebra $k\mathbb{D}_{\infty^\circ}$

- As an algebra, $\Bbbk\mathbb{D}_{\infty^\circ}$ is generated by $F, \phi_\lambda, \psi_\lambda$ for $\lambda \in \Bbbk^\times = \Bbbk \setminus \{0\}$ and subjects to the following relations
- $$F\phi_\lambda = \phi_\lambda F, \quad F\psi_\lambda = \psi_\lambda F, \quad \phi_1 = 1,$$
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for all $\lambda, \lambda' \in \mathbb{k}^\times$.

The Hopf algebra $\Bbbk\mathbb{D}_{\infty}^{\circ}$

The comultiplication, counit and the antipode are given by

$$\Delta(F) = F \otimes 1 + \psi_1 \otimes F,$$

$$\Delta(\phi_\lambda) = \frac{1}{2}(\phi_\lambda + \psi_\lambda) \otimes \phi_\lambda + \frac{1}{2}(\phi_\lambda - \psi_\lambda) \otimes \phi_{\lambda^{-1}},$$

$$\Delta(\psi_\lambda) = \frac{1}{2}(\phi_\lambda + \psi_\lambda) \otimes \psi_\lambda - \frac{1}{2}(\phi_\lambda - \psi_\lambda) \otimes \psi_{\lambda^{-1}},$$

$$\varepsilon(F) = 0, \;\; \varepsilon(\phi_\lambda) = \varepsilon(\psi_\lambda) = 1,$$

$$S(F) = -\psi_1 F, \;\; S(\phi_\lambda) = \frac{1}{2}(\phi_{\lambda^{-1}} + \psi_{\lambda^{-1}}) + \frac{1}{2}(\phi_\lambda - \psi_\lambda),$$

$$S(\psi_\lambda) = \frac{1}{2}(\phi_{\lambda^{-1}} + \psi_{\lambda^{-1}}) - \frac{1}{2}(\phi_\lambda - \psi_\lambda)$$

for $\lambda \in \Bbbk^{\times}$.

Main result

Lemma

With operations defined above, $\Bbbk\mathbb{D}_{\infty^\circ}$ is a Hopf algebra.

Theorem

As Hopf algebras, we have

$$(\Bbbk\mathbb{D}_\infty)^\circ \cong \Bbbk\mathbb{D}_{\infty^\circ}.$$

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Main result

Lemma

With operations defined above, $\Bbbk\mathbb{D}_{\infty^\circ}$ is a Hopf algebra.

Theorem

As Hopf algebras, we have

$$(\Bbbk\mathbb{D}_\infty)^\circ \cong \Bbbk\mathbb{D}_{\infty^\circ}.$$

Sketch of the proof

- Clearly, $\{g^i x^j | i \in \mathbb{Z}, j = 0, 1\}$ is a basis of $k\mathbb{D}_\infty$. Denote its dual basis by $f_{i,j}$.
- Construct:

$$E := \sum_{i \in \mathbb{Z}} i(f_{i,0} + f_{i,1}),$$

$$\Phi_\lambda := \sum_{i \in \mathbb{Z}} \lambda^i (f_{i,0} + f_{i,1}),$$

$$\Psi_\lambda := \sum_{i \in \mathbb{Z}} \lambda^i (f_{i,0} - f_{i,1})$$

for $\lambda \in \mathbb{k}^\times$.

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Sketch of the proof

- Key: As an algebra, $(k\mathbb{D}_\infty)^\circ$ is generated by E, Φ_λ and Ψ_λ. To prove this, we need The Identity we proved before.
- Define a map

$$\Theta: \Bbbk\mathbb{D}_{\infty^\circ} \to (\Bbbk\mathbb{D}_\infty)^\circ, \quad F \mapsto E, \ \phi_\lambda \mapsto \Phi_\lambda, \ \psi_\lambda \mapsto \Psi_\lambda, \quad (\lambda \in \Bbbk^\times)$$

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Remarks

- The connected component (Montgomery-Radford's sense) containing 1 is the Hopf subalgebra generated by E, Ψ_1 which can be described as follows

$$E\Psi_1 = \Psi_1 E, \quad \Psi_1^2 = 1,$$
$$\Delta(E) = E \otimes 1 + \Psi_1 \otimes E, \quad \Delta(\Psi_1) = \Psi_1 \otimes \Psi_1.$$

- This verifies the infinite-dimensional case of the theorem of Larson-Radford. In our subsequent computations, we will find that the infinite-dimensional analogue of Larson-Radford's theorem is not always true.

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- Thanks for your attention!