A combinatorial identity and the finite dual of infinite dihedral group algebra

Gongxiang Liu (A joint work with Fan Ge)

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Motivations

A combinatorial identity

3 The results

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Preparation and aim

- In this talk, k an algebraically closed field of characteristic zero.
- All spaces and algebras are over \mathbf{k} .
- Aim: Determine the finite dual H° of a prime regular Hopf algebra H.

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A Larson-Radford's result

 It is well-known that Larson-Radford (J. Algebra, 1988) proved the following result:

Theorem

Let H be a finite dimensional Hopf algebra, then H is semisimple if and only if H^{*} is semisimple.

• A natural question is: How about the infinite dimensional case?

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• Naively, the infinite dimensional analogue seems to be:

- A Hopf algebra *H* has finite global dimension if and only if *H** has finite global dimension?
- But *H** has no dual Hopf algebra structure in general.
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A Takeuchi's definition

• Takeuchi defined a quantum group as follows.

Definition

A quantum group *G* is defined to be a triple

$$G = (A, U, \langle , \rangle)$$

where A and U are Hopf algebras, and \langle , \rangle is a Hopf pairing on $U \times A$.

- A natural question is: Is a prime regular Hopf algebra a quantum group in the Takeuchi's sense?
- In this talk, we will determine $(\mathbb{k}\mathbb{D}_{\infty})^{\circ}$.

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Description of the identity

- To describe the identity, we need some notions at first.
- Let *m* and *n* are positive integers, define

$$U := U(m, n) = \{1, 2, ..., m + n\}.$$

• For a set $X = \{x_1, ..., x_m\}$ of nonnegative integers whose elements are listed in increasing order, we denote by V_X the Vandermonde determinant of *X*. That is,

$$V_X = \prod_{1 \le i < j \le m} (x_j - x_i).$$

• $G(m+1) := (m-1)! \cdots 1!$ is the Barnes G-function.

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Theorem Let $t \in \left[\frac{m(m+1)}{2}, \frac{m(m+1)}{2} + mn\right]$ be an integer, and let $t^* = t - \frac{m(m+1)}{2}$. We have $\sum_{\substack{X = \{x_1, \dots, x_m\} \subset U\\ \sum x_i = t}} V_X V_Y = G(m+1)G(n+1)\binom{mn}{t^*}.$

Here the sum is over all subsets X of U whose elements' sum is t, and Y = U - X.

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Description of the identity

This identity is related with the following known function:

$$\gamma_m(c) = rac{1}{m! \, G(m+1)^2} \int_{[0,1]^m} \delta(s_1 + s_2 + \dots + s_m - c) \ \prod_{i < j} (s_i - s_j)^2 \, ds_1 \dots ds_m.$$

which is used in some aspects of number theory.

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• which is used in some aspects of number theory.

Sketch of the proof

- Step 1: Explain the number $\frac{V_X}{G(m+1)}$ as the number of some semi-standard Young tableaus (SSYTs).
- Step 2: Explain the number $\frac{V_Y}{G(n+1)}$ as the number of some semi-standard Young tableaus with transpose shape with respect to SSYTs in step 1.
- Step 3: The number $\sum \frac{V_X V_Y}{G(m+1)G(n+1)}$ equals to the number of some pairs of SSYTs (P, Q).
- Step 4: Apply the Robinson-Schensted-Knuth correspondence.

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A corollary

• This observation has the following direct consequence.

Corollary

Let x be an indeterminate and A be the $2m \times 2m$ matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ x & x & \cdots & x & x^{-1} & x^{-1} & \cdots & x^{-1} \\ x^2 & 2x^2 & \cdots & 2^{m-1}x^2 & x^{-2} & 2x^{-2} & \cdots & 2^{m-1}x^{-2} \\ x^3 & 3x^3 & \cdots & 3^{m-1}x^3 & x^{-3} & 3x^{-3} & \cdots & 3^{m-1}x^{-3} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ x^M & Mx^M & \cdots & M^{m-1}x^M & x^{-M} & Mx^{-M} & \cdots & M^{m-1}x^{-M} \end{pmatrix}$$

where M = 2m - 1. Then the determinant of A is

$$|A| = G(m+1)^2 \cdot (x^{-1}-x)^{m^2}.$$

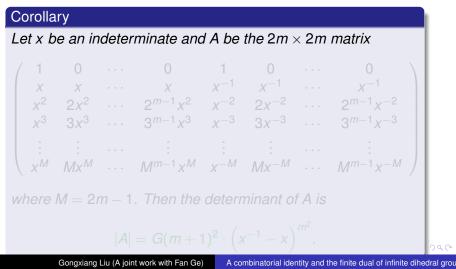
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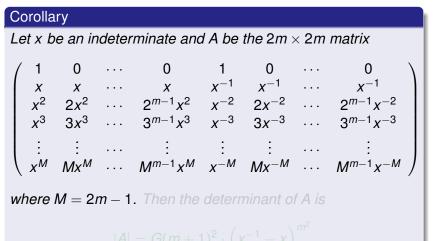
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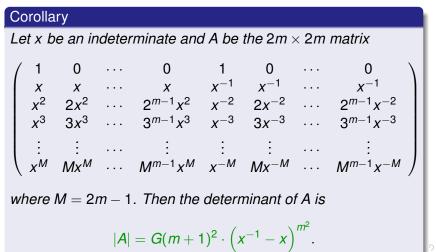
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Let H be a Hopf algebra, the finite dual H° of H is defined by

 $H^{\circ} := \{ f \in H^* | f(I) = 0, \text{ some ideal } I \text{ s.t. } \dim(H/I) < \infty \}.$

• A basic fact: H° is a Hopf algebra.

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Some known examples

Example

Let
$$H = \Bbbk[x]$$
, $\Delta(x) = 1 \otimes x + x \otimes 1$. Then we have
 $H^{\circ} \cong \Bbbk[x] \otimes kG$

where G = (k, +).

Example

Let $H = \Bbbk[x, x^{-1}], \ \Delta(x) = x \otimes x$. Then we have

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Some known examples

 There is a common point in above examples, that is, H is commutative. Therefore H° is cocommutative and thus one can apply Milnor-Moore's Theorem.

Example

Consider the quantum group $U_q(sl_n)$. Then we have

 $U_q(sl_n)^{\circ} \cong \mathcal{O}_q(SL_n) \# k\mathbb{Z}_2^{n-1}.$

This is proved by Takeuchi in 1992.

• The key point of above example is that the category of finite-dimensional representations of *U*_q(*sl*_n) is semisimple.

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Our situation

 By definition, the infinite dihedral group D_∞ is generated by two elements g and x satisfying

$$x^2 = 1$$
, $xgx = g^{-1}$.

 Note that kD∞ is not not commutative and thus (kD∞)° is not cocommutative. Also, the category of finite-dimensional representations of kD∞ is not semisimple.

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The Hopf algebra $k\mathbb{D}_{\infty^{\circ}}$

 As an algebra, kD_{∞°} is generated by F, φ_λ, ψ_λ for λ ∈ k[×] = k \ {0} and subjects to the following relations

$$\begin{split} F\phi_{\lambda} &= \phi_{\lambda}F, \ F\psi_{\lambda} = \psi_{\lambda}F, \ \phi_{1} = 1, \\ \phi_{\lambda}\psi_{\lambda'} &= \psi_{\lambda'}\phi_{\lambda} = \psi_{\lambda\lambda'}, \ \phi_{\lambda}\phi_{\lambda'} = \phi_{\lambda\lambda'}, \ \psi_{\lambda}\psi_{\lambda'} = \phi_{\lambda\lambda'} \end{split}$$

for all $\lambda, \lambda' \in \mathbb{k}^{\times}$.

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The Hopf algebra $\mathbb{kD}_{\infty^{\circ}}$

The comultiplication, counit and the antipode are given by

$$\begin{split} \Delta(F) &= F \otimes 1 + \psi_1 \otimes F, \\ \Delta(\phi_{\lambda}) &= \frac{1}{2}(\phi_{\lambda} + \psi_{\lambda}) \otimes \phi_{\lambda} + \frac{1}{2}(\phi_{\lambda} - \psi_{\lambda}) \otimes \phi_{\lambda^{-1}}, \\ \Delta(\psi_{\lambda}) &= \frac{1}{2}(\phi_{\lambda} + \psi_{\lambda}) \otimes \psi_{\lambda} - \frac{1}{2}(\phi_{\lambda} - \psi_{\lambda}) \otimes \psi_{\lambda^{-1}}, \\ \varepsilon(F) &= 0, \ \varepsilon(\phi_{\lambda}) &= \varepsilon(\psi_{\lambda}) = 1, \\ S(F) &= -\psi_1 F, \ S(\phi_{\lambda}) &= \frac{1}{2}(\phi_{\lambda^{-1}} + \psi_{\lambda^{-1}}) + \frac{1}{2}(\phi_{\lambda} - \psi_{\lambda}), \\ S(\psi_{\lambda}) &= \frac{1}{2}(\phi_{\lambda^{-1}} + \psi_{\lambda^{-1}}) - \frac{1}{2}(\phi_{\lambda} - \psi_{\lambda}) \end{split}$$

for $\lambda \in \mathbb{k}^{\times}$.

Main result

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With operations defined above, $\mathbb{kD}_{\infty^{\circ}}$ is a Hopf algebra.

Theorem

As Hopf algebras, we have

 $(\Bbbk \mathbb{D}_{\infty})^{\circ} \cong \Bbbk \mathbb{D}_{\infty^{\circ}}.$

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Sketch of the proof

Clearly, {gⁱx^j | i ∈ ℤ, j = 0, 1} is a basis of kD_∞. Denote its dual basis by f_{i,j}.

• Construct:

$$\begin{split} E &:= \sum_{i \in \mathbb{Z}} i(f_{i,0} + f_{i,1}), \\ \Phi_{\lambda} &:= \sum_{i \in \mathbb{Z}} \lambda^i (f_{i,0} + f_{i,1}), \\ \Psi_{\lambda} &:= \sum_{i \in \mathbb{Z}} \lambda^i (f_{i,0} - f_{i,1}) \end{split}$$

for $\lambda \in \mathbb{k}^{\times}$.

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Sketch of the proof

- Key: As an algebra, (kD_∞)° is generated by E, Φ_λ and Ψ_λ.
 To prove this, we need The Identity we proved before.
- Define a map

 $\Theta \colon \Bbbk \mathbb{D}_{\infty^{\circ}} \to (\Bbbk \mathbb{D}_{\infty})^{\circ}, \ F \mapsto E, \ \phi_{\lambda} \mapsto \Phi_{\lambda}, \ \psi_{\lambda} \mapsto \Psi_{\lambda}, \ (\lambda \in \Bbbk^{\times})$

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Remarks

• The connected component (Montgomery-Radford's sense) containing 1 is the Hopf subalgebra generated by E, Ψ_1 which can be described as follows

$$\begin{split} E\Psi_1 &= \Psi_1 E, \ \Psi_1^2 = 1, \\ \Delta(E) &= E \otimes 1 + \Psi_1 \otimes E, \ \Delta(\Psi_1) = \Psi_1 \otimes \Psi_1. \end{split}$$

 This verifies the infinite-dimensional case of the theorem of Larson-Radford. In our subsequent computations, we will find that the infinite-dimensional analogue of Larson-Radford's theorem is not always true.



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Thanks for your attention!

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