The structure of connected (graded) Hopf algebras

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Sept.19, 2020; BNU

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Overview

1. Background & the statement of the main result

2. Lyndon words & the proof of the main result

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  - ⋆ *GK-dimension*: a measure of the rate of growth of the algebra in terms of any generating set.
  - ⋆ Facts:
    - ◇ finitely generated commutative implies GKdim = Krull-dim;
    - ◇ for any $2 \leq r \in \mathbb{R}$, $\exists$ finitely generated algebras with GKdim $= r$;
    - ◇ for a connected graded noetherian algebra with finite gldim, then GKdim $< \infty$;
    - ◇ $\cdots\cdots$

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  ⋆ **Conjecture**: GKdim of any Hopf algebra is in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$.

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- The *coradical* of a coalgebra $= \sum$ simple subcoalgebras. The coradical filtration of a coalgebra is very useful as a vehicle for inductive arguments.
- The notion of coradically graded Hopf algebra is very natural: let H be a pointed Hopf algebra with coradical filtration $\{H_n\}_{n\geq 0}$. Then the associated graded space $\mathrm{gr}(H) = \bigoplus_{n\geq 0} H_n/H_{n-1}$ is a coradically graded Hopf algebra, moreover, $\mathrm{GKdim}(\mathrm{gr}(H)) = \mathrm{GKdim}(H)$.

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- The structure of $\mathrm{gr}(H)$ is relatively easier: it is commutative and has a nice decomposition $\mathrm{gr}(H) \cong R\#kG$, where R is a certain graded subalgebra of $\mathrm{gr}(H)$.

Basic definitions

A Hopf algebra is called

- *pointed* if every simple subcoalgebra is one-dimensional;
- *connected* if its coradical is of dimension one;
- *connected graded* if it is equipped with a (an $\mathbb{N}$-)grading which is compatible with the algebra structure and the coalgebra structure, and of one-dimensional 0^{th} component.

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An easy observation

For a Hopf algebra, we have

$$\text{connected graded} \Longrightarrow \text{connected.}$$

Connected Hopf algebras of finite GKdim

Theorem (Cartier-Gabriel-Konstant)

The assignment $\mathfrak{g} \mapsto U(\mathfrak{g})$ and $H \mapsto P(H)$ define mutually inverse equivalences between the category of Lie algebras and the category of cocommutative connected Hopf algebras. Moreover,

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Connected Hopf algebras of finite GK dimension can be viewed as

- ⋆ generalizations of enveloping algebras of finite dimensional Lie algebras.

Connected Hopf algebras of finite GKdim

Theorem (Basic facts of group schemes + Lazard's Theorem)

Let H be an commutative Hopf algebra. TFAE:

1. *H is connected and affine.*
2. *$H \cong k[x_1, \cdots, x_d]$ as algebras for some integer $d \geq 0$.*
3. *H is the coordinate ring of a connected unipotent algebraic group U.*

In this case,

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Connected Hopf algebras of finite GK dimension can be viewed as

⋆ NC counterpart of connected unipotent algebraic groups.

Connected Hopf algebras of finite GKdim

Theorem (Zhuang, 2012)

Let H be a connected Hopf algebra. Let $\mathrm{gr}(H)$ be the associated graded Hopf algebra w.r.t. the coradical filtration. Then $\mathrm{gr}(H)$ is commutative and TFAE:

1. $\mathrm{GKdim}\, H < \infty$;
2. $\mathrm{GKdim}\, \mathrm{gr}(H) < \infty$;
3. $\mathrm{gr}(H)$ *is affine;*
4. $\mathrm{gr}(H) \cong k[x_1, \cdots, x_n]$ *as algebras for some integer* $n \geq 0$.

In this case, $\mathrm{GKdim}\, H = \mathrm{GKdim}\, \mathrm{gr}(H) = n$.

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Connected Hopf algebras of finite GK dimension can be viewed as

⋆ deformations of polynomial algebras in finitely many variables.

Aims

- Detecting properties and structures of connected Hopf algebras.
- Classifying connected Hopf algebras of low GK dimensions.
- Relating connected Hopf algebras to general Hopf algebras.

Ring-theoretic and homological properties

The above Zhuang's result is the keystone theorem for many recent studies on connected Hopf algebras. An easy consequence is:

Theorem (Zhuang, 2012)

Let H be a connected Hopf algebra with $\mathrm{GKdim}\, H = n < \infty$.

1. *H is a noetherian domain of Krull dimension $\leq n$;*
2. *H is Artin-Schelter regular of global dimension n;*
3. *H is Auslander-regular, Cohen-Macaulay and skew n-Calabi-Yau.*

On the antipode

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Theorem (Brown-Gilmartin-Zhang, 2017)

If a Hopf algebra has finite GK dimension and is connected graded as an algebra, then its antipode is of order 2.

Consequently, a connected graded Hopf algebra of finite GK dimension has antipode of order 2.

On the classification problem

Theorem (Zhuang, 2012; Wang-Zhang-Zhuang, 2013)

Assume k is alg. closed. Let H be a connected Hopf algebra.

1. *If* $\mathrm{GKdim}\, H = 0$ *then* $H = k$;
2. *If* $\mathrm{GKdim}\, H = 1$ *then* $H \cong k[x]$ *with x primitive;*
3. *If* $\mathrm{GKdim}\, H = 2$ *then H is isomorphic to the enveloping algebra of one of the two Lie algebras of dimension 2;*
4. *If* $\mathrm{GKdim}\, H = 3$, *then H is either isomorphic to enveloping algebras of Lie algebras of dimension 3, or H is a member of one of two explicitly defined families;*
5. *If* $\mathrm{GKdim}\, H = 4$, *then H is either isomorphic to enveloping algebras of Lie algebras of dimension 4, or H is a member of one of 12 explicitly defined families.*

On the underlying algebra structure

Theorem (Wang-Zhang-Zhuang, 2013)

Connected Hopf algebras of GK dimension ≤ 4 are isomorphic as algebras to enveloping algebras of Lie algebras.

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Let H be a connected Hopf algebra of finite GK dimension d. If $\dim P(H) \geq d - 1$, then H is isomorphic as algebras to an enveloping algebra of some Lie algebra of dimension d.

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Theorem (Brown-Gilmartin-Zhang, 2017)

There is a connected graded Hopf algebra of GK dimension 5 *which is not isomorphic as algebras to the enveloping algebra of any Lie algebra.*

Brown-Gilmartin-Zhang's example

Generators: a, b, c, z, w of degrees 1, 2, 1, 3, 3 respectively;

Relations:

$$[b, a] = 0, \quad [c, a] = -b, \quad [c, b] = 0,$$
$$[z, a] = [z, b] = [z, c] = 0,$$
$$[w, a] = [w, b] = [w, c] = 0, \quad [w, z] = -\tfrac{1}{3} b^3;$$

Comultiplication:

$$a \mapsto 1 \otimes a + a \otimes 1, \quad b \mapsto 1 \otimes b + b \otimes 1, \quad c \mapsto 1 \otimes c + c \otimes 1,$$
$$z \mapsto 1 \otimes z + z \otimes 1 + \underline{a \otimes b - b \otimes a},$$
$$w \mapsto 1 \otimes w + w \otimes 1 + \underline{c \otimes b - b \otimes c};$$

Counit:

$$a, b, c, z, w \mapsto 0.$$

Iterated Hopf Ore extension

An effective way to construct "nontrivial" connected Hopf algebras of high GK dimension is iterated Hopf Ore extension (IHOE).

Definition (Brown-O'Hagan-Zhang-Zhuang, 2013)

A Hopf algebra H is called an IHOE if there is a chain of Hopf subalgebras

$$k = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n = H$$

such that $H_i = H_{i-1}[x_i; \sigma_i, \delta_i]$, an Ore extension of H_{i-1}, for $1 \leq i \leq n$.

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Theorem (Brown-O'Hagan-Zhang-Zhuang, 2013)

Let H be an IHOE with a defining series as above. Then H is a connected Hopf algebra of GK dimension n.

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- Commutative connected Hopf algebras of finite GK dimension are IHOE.
- Cocommutative connected graded Hopf algebras of finite GK dimension are IHOE.
- Connected Hopf algebras of GK dimension ≤ 4 are IHOE.
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Question

Is every connected graded Hopf algebra of finite GK dimension an IHOE?

The main result

Theorem (Zhou-Shen-L., *Adv. Math.*, 2020)

Let H be a connected graded Hopf algebra of finite GK dimension d. Then there exists a finite sequence $z_1, \cdots, z_d$ of homogeneous elements of H of positive degrees such that

$$\Delta_H(z_r) \in 1 \otimes z_r + z_r \otimes 1 + H^{<r} \otimes H^{<r},$$

where $H^{<r}$ is the subalgebra of H generated by $z_1, \cdots, z_{r-1}$ and

$$H = k[z_1][z_2; \mathrm{id}, \delta_2] \cdots [z_d; \mathrm{id}, \delta_d].$$

In particular, H is an IHOE.

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$$\text{connected graded as an algebra} \Longrightarrow \text{IHOE?}$$

The idea of our argument is originated from the following work:

- V. K. Kharchenko, *A quantum analogue of Poincaré-Birkhoff-Witt theorem*, *Alg. Log.*, vol 38, (1999) 259-276.

He construct a set of PBW generators for *primitively generated* connected graded braided Hopf algebras with braiding of diagonal type, in terms of Lyndon words and braided bracketing on words.

Warning!

Connected graded Hopf algebras are not necessarily primitively generated!

Lyndon words

Let $(X, \leq)$ be a well ordered set of variables.

For words $u, v \in \langle X \rangle$,

$$u <_{\text{lex}} v \iff \begin{cases} v \text{ is a proper prefix of } u, \quad \text{or} \\ u = rxs,\ v = ryt,\ \text{with } x, y \in X;\ x < y. \end{cases}$$

Definition

A word u on X is called *Lyndon* if $u \neq 1$ and $u >_{\text{lex}} wv$ for every factorization $u = vw$ with $v, w \neq 1$.

Lyndon words of length ≤ 4 in $X = \{x_1 < x_2\}$ are:

$$x_1,\ x_2,\ x_2x_1,\ x_2x_1^2,\ x_2^2x_1,\ x_2x_1^3,\ x_2^2x_1^2,\ x_2^3x_1.$$

Useful characterizations of Lyndon words

Let u be a word of length ≥ 2. TFAE:

1. u is a Lyndon word.
2. $u >_{\text{lex}} w$ for every factorization $u = vw$ with $v, w \neq 1$.
3. $v >_{\text{lex}} w$ for every factorization $u = vw$ with $v, w \neq 1$.
4. $u = vw$ with v, w both Lyndon and $v >_{\text{lex}} w$.
5. u_L is Lyndon and $u_L >_{\text{lex}} u_R$, where u_R is the proper Lyndon suffix of maximal length of u and $u_L := uu_R^{-1}$.

The pair $\mathrm{Sh}(u) := (u_L, u_R)$ is called the *Shirshov factorization* of u.

E.g. $\mathrm{Sh}(x_2^3x_1) = (x_2, x_2^2x_1)$, $\mathrm{Sh}(x_2^2x_1x_2x_1) = (x_2^2x_1, x_2x_1)$.

Some facts on Lyndon words

- Every nonempty word u can be written uniquely as a nondecreasing product $u = u_1 u_2 \cdots u_r$ of Lyndon words.
- Let $u = u_1 u_2 \cdots u_r$ be a nondecreasing product of Lyndon words. If v is a Lyndon factor of u then v is a factor of some u_i.
- Let u, v be Lyndon words such that $u >_{\mathrm{lex}} v$. Then $\mathrm{Sh}(uv) = (u, v)$ if and only if either $u \in X$ or $|u| \geq 2$ and $u_R \leq_{\mathrm{lex}} v$.
- Let w_1, w_2, w_3 be words with w_2 nonempty. If $w_1 w_2$ and $w_2 w_3$ are both Lyndon words, then $w_1 w_2 w_3$ is also a Lyndon word.
- Let $u_1 >_{\mathrm{lex}} u_2 >_{\mathrm{lex}} u'$ be nonempty words. If $u_1 u_2$ and u' are Lyndon words, then $u_1 u_2 u' >_{\mathrm{lex}} u_1 u' >_{\mathrm{lex}} u'$ and $u_1 u_2 u' >_{\mathrm{lex}} u_2 u' >_{\mathrm{lex}} u'$.

Fix a grading on $k\langle X\rangle$ by a map $X \to \mathbb{Z}_{>0}$.

Let I be an ideal of $k\langle X\rangle$. A word on X is called *I-reducible* if it is the leading word of some polynomial in I. We let

$$\mathcal{O}_I := \{\ I\text{-reducible words with proper factors } I\text{-irreducible}\ \},$$
$$\mathcal{N}_I := \{\ I\text{-irreducible Lyndon words}\ \},$$
$$\mathcal{B}_I := \{\ u_1 \cdots u_n \mid u_1 \leq_{\mathrm{lex}} \cdots \leq_{\mathrm{lex}} u_n \in \mathcal{N}_I\ \}.$$

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Easy observations

- The set of I-irreducible words is included in $\mathcal{B}_I$, and they are equal if and only if $\mathcal{O}_I$ consists of Lyndon words.
- The quotient algebra $k\langle X\rangle/I$ is generated by $\{\ u + I \mid u \in \mathcal{N}_I\ \}$.

An ideal I of $k\langle X\rangle$ is called *Lyndon* if $\mathcal{O}_I$ consists of Lyndon words.

E.g. The ideal of $k\langle d,u\rangle$ generated by

$$d^2u - \alpha dud - \beta ud^2 - \gamma d, \quad du^2 - \alpha udu - \beta ud^2 - \gamma u.$$

An observation

Let I be a $\mathbb{Z}^2$-homogeneous ideal of $k\langle x_1, x_2\rangle$ with x_1 and x_2 of degrees $(1,0)$ and $(0,1)$ respectively. If $k\langle x_1, x_2\rangle/I$ is an AS-regular domain of global dimension ≤ 5, then I is a Lyndon ideal.

Theorem (Zhou-L., 2014)

Let I be a Lyndon ideal of $k\langle X\rangle$ and let $A = k\langle X\rangle/I$. Then

$$\mathrm{GKdim}\, A = \#(\mathcal{N}_I).$$

Assume further I is homogeneous and $\mathcal{N}_I$ is finite. Then

$$H_A(t) = \prod_{u\in\mathcal{N}_I}(1 - t^{\deg(u)})^{-1},$$

A is homologically smooth in the graded sense,

$$\mathrm{pdim}_{A^e}(A) = \mathrm{gldim}\, A = \#(N_I),$$

and

$$\underline{\mathrm{Ext}}_A^d({}_Ak, {}_Ak) = \underline{\mathrm{Ext}}_A^d(k_A, k_A) = k(l),$$

where $d := \#(\mathcal{N}_I)$ and $l := \sum_{u\in\mathcal{N}_I} \deg(u)$.

Standard bracketing of words

The *standard bracketing* of words $[-] : \langle X \rangle \to k\langle X \rangle$ is defined as follows.

- first set $[1] = 1$ and $[x] := x$ for $x \in X$;
- then for words u of length ≥ 2, inductively set

$$[u] = \begin{cases} [[u_L], [u_R]], & u \text{ is Lyndon;} \\ \\ [u_L][u_R], & u \text{ is not Lyndon.} \end{cases}$$

Note that u is the leading word of $[u]$.

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Note that u is the leading word of $[u]$. Consequently, one has:

An easy observation

The quotient algebra $k\langle X \rangle / I$ is generated by $\{ \, [u] + I \mid u \in \mathcal{N}_I \, \}$.

A well-known fact

Let $\Delta_s : k\langle X\rangle \to k\langle X\rangle \otimes k\langle X\rangle$ be the algebra homomorphism given by

$$x \mapsto 1 \otimes x + x \otimes 1.$$

Theorem

For an ideal I of $k\langle X\rangle$, TFAE:

1. $\Delta_s(I) \subseteq k\langle X\rangle \otimes I + I \otimes k\langle X\rangle$.
2. *I is generated by Lie polynomials (i.e. linear combinations of standard bracketing of Lyndon words).*

In this case, $k\langle X\rangle/I \cong U(\mathrm{Lie}(X)/\mathrm{Lie}(X) \cap I)$.

Triangular comultiplication on $k\langle X\rangle$

Definition

A triangular comultiplication on $k\langle X\rangle$ is an algebra homomorphism $\Delta : k\langle X\rangle \to k\langle X\rangle \otimes k\langle X\rangle$ such that for each letter $x \in X$,

$$\Delta(x) \;=\; 1 \otimes x + x \otimes 1 + f_x + g_x$$

with $f_x \in \sum_{\substack{i,j\,>\,0 \\ i+j=\deg(x)}} k\langle X\rangle_i^{<x} \otimes k\langle X\rangle_j^{<x}$ and $g_x \in \big(k\langle X\rangle \otimes k\langle X\rangle\big)_{<\deg(x)}$.

Notation: $k\langle X\rangle^{<w}$, the subalgebra of $k\langle X\rangle$ generated by

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Warning! Such Δ is no longer coassociative in general!

The first technical observation

Let Δ be a triangular comultiplication on $k\langle X\rangle$. Then

$$\Delta([u]^n) \;=\; \sum_{p=0}^{n} \binom{n}{p} [u]^p \otimes [u]^{n-p} + f_{u,n} + g_{u,n}$$

with

$$f_{u,n} \in \sum_{\substack{r,s \;\geq\; 0 \\ r+s \;<\; n}} \; \sum_{\substack{i,j \;>\; 0 \\ i+j=(n-r-s)\deg(u)}} k\langle X\rangle_i^{<u} \cdot [u]^r \otimes k\langle X\rangle_j^{<u} \cdot [u]^s$$

and

$$g_{u,n} \in \Big(k\langle X\rangle \otimes k\langle X\rangle\Big)_{<\deg(u^n)}$$

for every Lyndon word u and every positive integer n.

The second technical observation

Assume that there exists a triangular comultiplication Δ on $k\langle X\rangle$ such that $\Delta(I) \subseteq k\langle X\rangle \otimes I + I \otimes k\langle X\rangle$.

1. For every I-reducible Lyndon words v,

$$[v] \in k\langle X|I\rangle^{<v}_{\deg(v)} + k\langle X|I\rangle_{<\deg(v)} + I.$$

2. For every pair of I-irreducible Lyndon words $u >_{\mathrm{lex}} v$,

$$[u][v] - [v][u] \in k\langle X|I\rangle^{\leq uv}_{\deg(uv)} + k\langle X|I\rangle_{<\deg(uv)} + I.$$

3. The set $\{ [w] + I \mid w \in \mathcal{B}_I \}$ form a basis of $k\langle X\rangle/I$. Consequently, $\mathcal{O}_I$ consists of Lyndon words.

Notation: $k\langle X|I\rangle^{<w}$ (resp. $k\langle X|I\rangle^{\leq w}$), the subalgebra of $k\langle X\rangle$ generated

$$\{ [u] \mid u \in \mathcal{N}_I, u <_{\mathrm{lex}} w \} \quad (\text{resp.} \{ [u] \mid u \in \mathcal{N}_I, u \leq_{\mathrm{lex}} w \}).$$

Sketch

Now let H be a connected graded Hopf algebra.

- First fix an arbitrary set X of homogeneous generators of positive degree for H.
- Then fix an arbitrary homomorphism $\Delta : k\langle X\rangle \to k\langle X\rangle \otimes k\langle X\rangle$ of graded algebras that lifts Δ_H.

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- First fix an arbitrary set X of homogeneous generators of positive degree for H.
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An easy observation

One may choose a well order on X so that the chosen lifting Δ is a triangular comultiplication.

The Structure Theorem

Theorem (Zhou-Shen-L., *Adv. Math.*, 2020)

Let H be a connected graded Hopf algebra. Then there exists an indexed family $\{z_\gamma\}_{\gamma\in\Gamma}$ of homogeneous elements of H of positive degrees and a total order $\leq$ on Γ satisfying the following conditions:

1. *for every index $\gamma \in \Gamma$,*

$$\Delta_H(z_\gamma) \in 1 \otimes z_\gamma + z_\gamma \otimes 1 + H^{<\gamma} \otimes H^{<\gamma},$$

   where $H^{<\gamma}$ is the subalgebra of H generated by $\{\ z_\delta \mid \delta < \gamma\ \}$;

2. *for every pair of indices $\gamma, \delta \in \Gamma$ with $\delta < \gamma$,*

$$[z_\gamma, z_\delta] \in H^{<\gamma};$$

3. $\{\ z_{\gamma_1} \cdots z_{\gamma_n} \mid \gamma_1 \leq \cdots \leq \gamma_n \in \Gamma\ \}$ *is a basis of H.*

Moreover, $\mathrm{GKdim}\, H = \#(\Gamma)$.

Remarks

- The total order on the index set Γ may be not compatible with the degrees of the indexed generators.
- The totally ordered set $(\Gamma, \leq)$ may be weird when Γ is infinite. It may contain no totally ordered subset isomorphic to

$$(\mathbb{N}, \leq) = \{0 < 1 < 2 < 3 < \cdots\}.$$

E.g. $(\Gamma, \leq) = \{0 < \cdots < 3 < 2 < 1\}$.

Theorem (Zhou-Shen-L., *Adv. Math.*, 2020)

Let H be a commutative Hopf algebra. Assume that H is either connected as a coalgebra or connected graded as an algebra. Then H is isomorphic as an algebra to the polynomial algebra in some family of variables.

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Many Hopf algebras arose from combinatoric (e.g. quasi-symmetric functions) are <u>NON-affine</u> commutative connected Hopf algebras.

Some corollaries

Corollary 1

Let H be a connected Hopf algebra. Then for every algebra A which is a domain, $H \otimes A$ is a domain. In particular, $H^{\otimes n}$ are domains for $n \geq 1$.

Corollary 2

Let H be a connected Hopf algebra. Then

$$\operatorname{GKdim} H = \operatorname{GKdim} \operatorname{gr}(H) \in \mathbb{N} \cup \{\infty\}.$$

Corollary 3 (Zhuang's keystone theorem, 2012)

A connected Hopf algebra H has finite GK dimension d iff $\operatorname{gr}(H)$ is a polynomial algebra in d variables.

Thanks!