Trigonometric Lie algebras, affine Lie algebras and vertex algebras

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Outline

1. Basics: Affine vertex algebras

2. Main Results: Trigonometric Lie algebras and Γ-vertex algebras

Vertex Algebra

- U: a vector space,

$$U[[z, z^{-1}]] = \{\sum_{n\in\mathbb{Z}} u_n z^n | u_n \in U\}.$$

- Non-associative algebra: A vector space A equipped with a bilinear operation, which is equivalent to a linear map from A to End$A$ (through left multiplication).

Vertex Algebra

Vertex algebra: A vector space V equipped with infinitely many bilinear operations parametrized by integers, which is equivalent to infinitely many linear maps:

$$V \to \operatorname{End} V$$

$$v \mapsto v_n.$$

For each v, the infinitely many associated left multiplications are written in terms of generating function as:

$$Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \in (\operatorname{End} V)[[z, z^{-1}]],$$

that is, A vector space V equipped with a linear map:

$$Y(\cdot, z) : V \longrightarrow (\mathrm{End}\, V)[[z, z^{-1}]]$$

$$v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}, \quad v_n \in \mathrm{End}\, V,$$

(1) for $u, v \in V$: $u_n v = 0$ if n sufficiently large;

(2) vacuum vector $\mathbf{1}$: $Y(\mathbf{1}, z) = id$;

(3) weak commutativity: for any $u, v \in V$, there exists $n \geq 0$ such that

$$(z_1 - z_2)^n Y(u, z_1) Y(v, z_2) = (z_1 - z_2)^n Y(v, z_2) Y(u, z_1);$$

(4) weak associativity: for any $u, v, w \in V$, there exists $l \geq 0$ such that

$$(z_0 + z_2)^l Y(u, z_0 + z_2) Y(v, z_2) w = (z_0 + z_2)^l Y(Y(u, z_0) v, z_2) w.$$

Remark. $V = (V, Y, \mathbf{1})$: VA.

Vertex Algebra

Example. Commutative associative algebra A with identity $\mathbf{1}$:
$Y(a,z)b = ab$ for $a, b \in A$.

Example. Affine VA, Virasoro VA, lattice VA

Vertex operator algebra: A vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$, $\dim V_n < \infty$, equipped with a vertex algebra structure $(V, Y, \mathbf{1})$,

+the Virasoro vector $\omega \in V_2$:

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2},$$

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12}\delta_{m+n,0}c,$$

where $c \in \mathbb{C}$ is called the central charge. Moreover,

$$Y(L(-1)v, z) = \frac{d}{dz}Y(v, z), \quad v \in V,$$

$L(0) = n$ on V_n.

Remark. $V = (V, Y, \mathbf{1}, \omega)$: VOA.

Vertex Operator Algebra

- V: VA
- Module: A vector space M equipped with a linear map:

$$Y_M(\cdot, z) : V \longrightarrow (\mathrm{End}M)[[z, z^{-1}]]$$
$$v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Z}} v_n^M z^{-n-1}, \quad v_n^M \in \mathrm{End}M$$

  + some axioms.

- Remark. Simple VA: V is an irreducible as a V-module.

- $\mathfrak{g}$: finite dimensional simple Lie algebra
- the affine Kac-Moody Lie algebra:

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where c is central and

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\langle a, b\rangle \delta_{m+n,0} c$$

for $a, b \in \mathfrak{g}$, $m, n \in \mathbb{Z}$.

Affine VA

- Set $\hat{\mathfrak{g}}_{\geq 0} = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}c$, a Lie subalgebra of $\hat{\mathfrak{g}}$. Let $k \in \mathbb{C}$.
- Denote by $\mathbb{C}_k$ the 1-dimensional $\hat{\mathfrak{g}}_{\geq 0}$-module with $\mathfrak{g}[t]$ acting trivially and with c acting as scalar k. Form the induced $\hat{\mathfrak{g}}$-module

$$V_{\hat{g}}(k,0) = U(\hat{g}) \otimes_{U(\hat{\mathfrak{g}}_{\geq 0})} \mathbb{C}_k.$$

- $V_{\hat{g}}(k,0)$ has a VA structure:

$$V = V_{\hat{g}}(k,0),$$

$$Y(a,x) = a(x) = \sum_{n\in\mathbb{Z}} (a \otimes t^n) x^{-n-1}, \ a \in \mathfrak{g},$$

$$\mathbf{1} = 1 \otimes 1.$$

- $k \neq -h^{\vee}$, $V_{\hat{g}}(k,0)$ is a VOA.

Theorem 1.1 (Li, 1996)

For any $\ell \in \mathbb{C}$, let W be any restricted $\hat{\mathfrak{g}}$-module W of level ℓ. Then there exists a $V_{\hat{\mathfrak{g}}}(\ell, 0)$-module structure on W, which is uniquely determined by

$$Y_W(a, x) = a(x) \quad \text{for } a \in \mathfrak{g}.$$

On the other hand, let (W, Y_W) be a $V_{\hat{\mathfrak{g}}}(\ell, 0)$-module. Then W affords a restricted $\hat{\mathfrak{g}}$-module of level ℓ with

$$a(x) = Y_W(a, x) \quad \text{for } a \in \mathfrak{g}.$$

[Li] H.-S. Li, Local systems of vertex operators, vertex subalgebras and modules, *J. Pure Appl. Algebra* **109** (1996), 143-195.

- $L_{\hat{\mathfrak{g}}}(\ell, 0)$: the simple quotient of the vertex algebra $V_{\hat{\mathfrak{g}}}(\ell, 0)$

Theorem 1.2 (Dong-Li-Mason, 1997)

Let ℓ be a positive integer and let W be a restricted $\hat{\mathfrak{g}}$-module. Then W is a $L_{\hat{\mathfrak{g}}}(\ell, 0)$-module if and only if W is an integrable $\hat{\mathfrak{g}}$-module of level ℓ.

[DLM] C. Dong, H.-S. Li and G. Mason, Regularity of rational vertex operator algebras, *Adv. Math.* **132** (1997), 148-166.

Vertex algebra from local system

- W : a vector space, $\mathcal{E}(W)$ =Hom$(W, W((x)))$
- A subset(subspace) U of $\mathcal{E}(W)$ is said to be local if for any $a(x), b(x) \in U$, there exists a nonnegative integer k such that

$$(x_1 - x_2)^k a(x_1)b(x_2) = (x_1 - x_2)^k b(x_2)a(x_1).$$

- It was proved by Li that any local subset U_W of $\mathcal{E}(W)$ canonically generates a vertex algebra $\langle U_W \rangle$ with W a module, where for $a(x), b(x) \in \mathcal{E}(W)$,

$$Y_{\mathcal{E}}(a(x), x_0)b(x) = \sum_{n \in \mathbb{Z}} a(x)_n b(x) x_0^{-n-1}$$

$$= x_0^{-k}((x_1 - x)^k a(x_1)b(x)) \mid_{x_1 = x + x_0}$$

- Main difficulty: How to identify $\langle U_W \rangle$.

Affine vertex algebras

For affine Kac-Moody algebra case:

- Let $a(x) = \sum_{n\in\mathbb{Z}} (a \otimes t^n) x^{-n-1}, \quad a \in \mathfrak{g}$
- The Lie relation of affine Lie algebra can be written as

$$[a(x_1), b(x_2)] = [a, b](x_2) x_1^{-1} \delta\left(\frac{x_2}{x_1}\right) + \langle a, b\rangle \frac{\partial}{\partial x_2} x_1^{-1} \delta\left(\frac{x_2}{x_1}\right) \mathbf{c}$$

  where $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$.
- Locality for affine Lie algebras: for two generating functions $a(x), b(x)$, we have

$$(x_1 - x_2)^2 a(x_1) b(x_2) = (x_1 - x_2)^2 b(x_2) a(x_1).$$

- Fact: $\langle U_W \rangle = V_{\hat{\mathfrak{g}}}(\ell, 0)$.

Trigonometric Lie algebras

- Trigonometric Lie algebras, of types A, B, C, and D, are a family of infinite-dimensional Lie algebras.

[FFZ] D. Fairlie, P. Fletcher, C. Zachos, Trigonometric structure constants for new infinite-dimensional algebras, *Phys. Lett.* **B 218** (1989) 203-206.

[F] E. G. Floratos, Spin wedge and vertex operator representations of trigonometric algebras and their central extensions, *Phys. Lett.* **B 232** (1989) 467-474.

- The rank 2 trigonometric (sine) Lie algebra $\widehat{A_\hbar}$ (of type A) with a real parameter $\hbar$ is the Lie algebra with generators $A_{\alpha,m}$ for $(\alpha, m) \in \mathbb{Z}^2$ and $\mathbf{c}$, a central element, subject to the relation

$$[A_{\alpha,m}, A_{\beta,n}] = 2i \sin \hbar(m\beta - n\alpha)A_{\alpha+\beta,m+n} + m\delta_{\alpha+\beta,0}\delta_{m+n,0}\mathbf{c}$$

for $\alpha, \beta, m, n \in \mathbb{Z}$.

Trigonometric Lie algebras

- Quantum torus A_q is generated by two invertible elements U_1, U_2, subjects to the relation ($q = e^{i\hbar}$):

$$U_2 U_1 = q^2 U_1 U_2.$$

- A_q and $A_\hbar$ (with $c = 0$) are related by $A_{m,n} = q^{mn} U_1^m U_2^n$ for $m, n \in \mathbb{Z}$.
- The quantum torus plays crucial rule in the classification of nullity two extended affine Lie algebras.

[ABP] B. Allison, S. Berman, A. Pianzola, Multiloop algebras, iterated loop algebras and extended affine Lie algebras of nullity 2, *J. Eur. Math. Soc.* 16 (2014), 327-385.

[G-KL] M. Golenishcheva-Kutuzova and D. Lebedev, Vertex operator representations of some quantum tori Lie algebras, *Commun. Math. Phys.*, **148**(1992) 403-416.

[KR] V. G. Kac and A. Radul, Quasifinite highest weight modules over the Lie algebra of differential operators on the circle, *Commun. Math. Phys.* **157** (1993) 429-457.

Trigonometric Lie algebras of B, C, D types

- Define a second order automorphism τ_B of $\widehat{A_\hbar}$ by

$$\tau_B(\mathbf{c}) = \mathbf{c}, \quad \tau_B(A_{\alpha,m}) = -(-1)^m A_{-\alpha,m} \quad \text{for } \alpha, m \in \mathbb{Z}. \tag{2.1}$$

- $\widehat{B_\hbar} = (\widehat{A_\hbar})^{\tau_B}$: the τ_B-fixed points subalgebra of $\widehat{A_\hbar}$.
- Define a second order automorphism τ_C of $\widehat{A_\hbar}$ by

$$\tau_C(\mathbf{c}) = \mathbf{c}, \quad \tau_C(A_{\alpha,m}) = -(-1)^m q^{2\alpha} A_{-\alpha,m} \quad \text{for } \alpha, m \in \mathbb{Z}. \tag{2.2}$$

- $\widehat{C_\hbar} = (\widehat{A_\hbar})^{\tau_C}$: the τ_C-fixed points subalgebra of $\widehat{A_\hbar}$.
- Define a second order automorphism τ_D of $\widehat{A_\hbar}$ by

$$\tau_D(\mathbf{c}) = \mathbf{c}, \quad \tau_D(A_{\alpha,m}) = -q^{2\alpha} A_{-\alpha,m} \quad \text{for } \alpha, m \in \mathbb{Z}. \tag{2.3}$$

- $\widehat{D_\hbar} = (\widehat{A_\hbar})^{\tau_D}$: the τ_D-fixed points subalgebra of $\widehat{A_\hbar}$.

Problems

Problems:

- Can we associate the trigonometric Lie algebras with the vertex algebras?
- How to establish the equivalence between the module categories of the trigonometric Lie algebras and their corresponding vertex algebras?

Main Results: Trigonometric Lie algebras and vertex algebras

- Recall the type A trigonometric Lie algebra $\widehat{A_\hbar}$, with $q = e^{i\hbar} \in \mathbb{C}^\times$:

$$[A_{\alpha,m}, A_{\beta,n}] = 2i \sin \hbar(m\beta - n\alpha) A_{\alpha+\beta,m+n} + m\delta_{\alpha+\beta,0}\delta_{m+n,0}\mathbf{c}$$

for $\alpha, \beta, m, n \in \mathbb{Z}$.

- For $\alpha \in \mathbb{Z}$, form a generating function

$$A_\alpha(z) = \sum_{n\in\mathbb{Z}} A_{\alpha,n} z^{-n-1}.$$

Then the defining relation of the trigonometric Lie algebra $\widehat{A_\hbar}$ can be written as

$$[A_\alpha(x), A_\beta(z)] = q^\alpha A_{\alpha+\beta}(q^\alpha z) x^{-1}\delta\left(\frac{q^{\alpha+\beta} z}{x}\right)$$
$$-q^{-\alpha} A_{\alpha+\beta}(q^{-\alpha} z) x^{-1}\delta\left(\frac{q^{-(\alpha+\beta)} z}{x}\right) + \delta_{\alpha+\beta,0} \frac{\partial}{\partial z} x^{-1}\delta\left(\frac{z}{x}\right)\mathbf{c}$$

for $\alpha, \beta \in \mathbb{Z}$.

Γ-local system

- Let Γ be a subgroup of $\mathbb{C}^*$. A subset U of $\mathcal{E}(W)$ is called Γ-local if for any $a(x), b(x) \in U$, there exists

$$p(x_1, x_2) \in \langle (x_1 - \alpha x_2) \mid \alpha \in \Gamma \rangle \subset \mathbb{C}[x_1, x_2]$$

such that

$$p(x_1, x_2)a(x_1)b(x_2) = p(x_1, x_2)b(x_2)a(x_1).$$

- When we consider $a(\alpha x)$ with $\alpha \in \mathbb{C}^*$, the vertex algebra comes with a group action.

Γ-vertex algebra

Definition 2.1

Let Γ be a group. A Γ-vertex algebra *is a vertex algebra V equipped with group homomorphism*

$$R : \Gamma \to \mathrm{GL}(V);\ g \longmapsto R_g,\ \text{and}\ \chi : \Gamma \to \mathbb{C}^{\times}$$

such that $R_g(\mathbf{1}) = \mathbf{1}$ *for* $g \in \Gamma$ *and*

$$R_g Y(v, x) R_g^{-1} = Y(R_g(v), \chi(g)^{-1} x) \quad \text{for } g \in \Gamma,\ v \in V.$$

H.-S. Li, A new construction of vertex algebras and quasi modules for vertex algebras, *Adv. Math.* **202** (2006) 232-286.

Γ-vertex algebra

Definition 2.2

Let V be a Γ-vertex algebra. An equivariant quasi V-module is a quasi module (W, Y_W) for V viewed as a vertex algebra, satisfying the following conditions that $Y_W(R_g(v), x) = Y_W(v, \chi(g)x)$ for $g \in \Gamma$, $v \in V$ and for $u, v \in V$, there exist $\alpha_1, \ldots, \alpha_k \in \chi(\Gamma) \subset \mathbb{C}^\times$ such that

$$(x_1 - \alpha_1 x_2) \cdots (x_1 - \alpha_k x_2)[Y_W(u, x_1), Y_W(v, x_2)] = 0.$$

- Let U be a Γ-local subset of $\mathcal{E}(W)$. Set $U_\Gamma = \{a(\alpha x) | a(x) \in U, \alpha \in \Gamma\}$, then U_Γ is also Γ-local.

Theorem 2.3 (Li, 2006)

For every Γ-local subset U of $\mathcal{E}(W)$, the vertex algebra $\langle U_\Gamma \rangle$ generated by U_Γ is a Γ-vertex algebra with W an equivariant quasi module.

Trigonometric Lie algebras and affine Lie algebras

- Now we relate $\widehat{A_\hbar}$ to an affine Lie algebra.
- $\mathfrak{gl}_\infty$: the associative algebra
- $\mathfrak{gl}_\infty$ is a Lie algebra with a basis $E_{m,n}$ ($m, n \in \mathbb{Z}$), where

$$[E_{m,n}, E_{p,q}] = \delta_{n,p}E_{m,q} - \delta_{q,m}E_{p,n}$$

  for $m, n, p, q \in \mathbb{Z}$.
- Equip $\mathfrak{gl}_\infty$ with the bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$\langle E_{m,n}, E_{r,s} \rangle = \mathrm{tr}(E_{m,n}E_{r,s}) = \delta_{m,s}\delta_{n,r}$$

  for $m, n, r, s \in \mathbb{Z}$.
- This bilinear form is non-degenerate, symmetric and associative
- For $r \in \mathbb{Z}$, define a linear operator σ_r on $\mathfrak{gl}_\infty$ by

$$\sigma_r(E_{m,n}) = E_{m+r,n+r} \quad \text{for } m, n \in \mathbb{Z}.$$

Trigonometric Lie algebras and affine Lie algebras

- Set $\mathcal{A} = \mathrm{span}\{E_{m,n} \mid m, n \in \mathbb{Z} \text{ with } m + n \in 2\mathbb{Z}\}$, which is an associative subalgebra of $\mathfrak{gl}_\infty$.
- For $\alpha, m \in \mathbb{Z}$, set $G_{\alpha,m} = E_{\alpha+m,m-\alpha} \in \mathcal{A}$. Then $G_{\alpha,m}$ (for $\alpha, m \in \mathbb{Z}$) form a basis of $\mathcal{A}$ and

$$\langle G_{\alpha,m}, G_{\beta,n}\rangle = \delta_{\alpha+\beta,0}\delta_{m,n}.$$

- $\mathcal{A}$ is stable under the action of $\mathbb{Z}$, where

$$\sigma_r(G_{\alpha,m}) = G_{\alpha,m+r} \quad \text{for } r, \alpha, m \in \mathbb{Z}.$$

- View $\mathcal{A}$ as a Lie algebra and equip $\mathcal{A}$ with the non-degenerate symmetric invariant bilinear form $\langle\cdot,\cdot\rangle$ defined above. Then we have an affine Lie algebra

$$\widehat{\mathcal{A}} = \mathcal{A} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k},$$

on which $\mathbb{Z}$ acts as an automorphism group.

- Define a linear character $\chi_q : \mathbb{Z} \to \mathbb{C}^{\times}$ by

$$\chi_q(n) = q^n \quad \text{for } n \in \mathbb{Z},$$

where $q = e^{i\hbar}$ is the same one as for $\widehat{A_\hbar}$. Then we have:

Proposition 2.4 (Li-Tan-W, 2020)

The trigonometric Lie algebra $\widehat{A_\hbar}$ is isomorphic to the $(\mathbb{Z}, \chi_q)$-covariant algebra $\widehat{\mathcal{A}}[\mathbb{Z}]$ of the affine Lie algebra $\widehat{\mathcal{A}}$ with $\mathbf{c} = \mathbf{k}$ and with

$$A_{\alpha,m} = \overline{G_{\alpha,0} \otimes t^m} \quad \text{for } \alpha, m \in \mathbb{Z}.$$

- Now we associate the trigonometric Lie algebras to vertex algebras.

Theorem 2.5 (Li-Tan-W, 2020)

Assume that q is not a root of unity and let $\ell \in \mathbb{C}$. Then for any restricted $\widehat{A}_{\hbar}$-module W of level ℓ, there exists an equivariant quasi $V_{\widehat{\mathcal{A}}}(\ell, 0)$-module structure $Y_W(\cdot, x)$ on W, which is uniquely determined by

$$Y_W(G_{\alpha,m}, x) = q^m A_\alpha(q^m x) \quad \text{for } \alpha, m \in \mathbb{Z}.$$

On the other hand, for any equivariant quasi $V_{\widehat{\mathcal{A}}}(\ell, 0)$-module (W, Y_W), W becomes a restricted $\widehat{A}_{\hbar}$-module of level ℓ with

$$A_\alpha(z) = Y_W(G_{\alpha,0}, z) \quad \text{for } \alpha \in \mathbb{Z}.$$

Simple vertex algebra $L_{\widehat{\mathcal{A}}}(\ell, 0)$ and its modules

- Let $L_{\widehat{\mathcal{A}}}(\ell, 0)$ be the simple quotient of $V_{\widehat{\mathcal{A}}}(\ell, 0)$

Proposition 2.6

Let ℓ be a complex number. Then $L_{\widehat{\mathcal{A}}}(\ell, 0)$ is an integrable $\widehat{\mathcal{A}}$-module if and only if ℓ is a nonnegative integer.

Theorem 2.7 (Li-Tan-W, 2020)

Let W be a restricted $\widehat{\mathcal{A}}$-module. Then W is an $L_{\widehat{\mathcal{A}}}(\ell, 0)$-module if and only if W is an integrable $\widehat{\mathcal{A}}$-module of level ℓ.

- Unitary quasifinite highest weight irreducible $\widehat{A_\hbar}$-modules were classified by Kac-Radul.

V. G. Kac and A. Radul, Quasifinite highest weight modules over the Lie algebra of differential operators on the circle, *Commun. Math. Phys.* **157** (1993) 429-457.

Then we have:

Theorem 2.8 (Li-Tan-W, 2020)

Let ℓ be a positive integer. Then every unitary quasifinite highest weight irreducible $\widehat{A_\hbar}$-module of level ℓ is an irreducible equivariant quasi $L_{\widehat{\mathcal{A}}}(\ell, 0)$-module.

Proposition 2.9

Lie algebra $\widehat{B}_\hbar$ is isomorphic to the $(\mathbb{Z}_2 \times \mathbb{Z}, \chi_q^B)$-covariant algebra $\widehat{\mathcal{A}}[\mathbb{Z}_2 \times \mathbb{Z}]$ of the affine Lie algebra $\widehat{\mathcal{A}}$ with $\mathbf{c} = \frac{1}{2}\mathbf{k}$ and with

$$B_{\alpha,m} = \overline{G_{\alpha,0} \otimes t^m} \quad \text{for } \alpha, m \in \mathbb{Z}, \tag{2.4}$$

where

$$B_{\alpha,m} := A_{\alpha,m} - (-1)^m A_{-\alpha,m} \tag{2.5}$$

for $\alpha, m \in \mathbb{Z}$ and $\mathbf{c}$ are generators of $\widehat{B}_\hbar$.

Proposition 2.10

Lie algebra $\widehat{C}_\hbar$ is isomorphic to $\widehat{B}_\hbar$, which is isomorphic to the $(\mathbb{Z}_2 \times \mathbb{Z}, \chi_q^B)$-covariant algebra $\widehat{\mathcal{A}}[\mathbb{Z}_2 \times \mathbb{Z}]$.

Theorem 2.11 (Li-Tan-W, 2020)

Assume that q is not a root of unity and let $\ell \in \mathbb{C}$. Then for any restricted $\widehat{B}_\hbar$-module W of level ℓ, there exists an equivariant quasi $V_{\widehat{\mathcal{A}}}(2\ell, 0)$-module structure $Y_W(\cdot, x)$ on W, which is uniquely determined by

$$Y_W(G_{\alpha,m}, x) = q^m B_\alpha(q^m x) \quad \text{for } \alpha, m \in \mathbb{Z}. \tag{2.6}$$

On the other hand, every equivariant quasi $V_{\widehat{\mathcal{A}}}(2\ell, 0)$-module W is a restricted $\widehat{B}_\hbar$-module of level ℓ with

$$B_\alpha(z) = Y_W(G_{\alpha,0}, z) \quad \text{for } \alpha \in \mathbb{Z}.$$

- Recall that the Lie algebra $\widehat{D_\hbar} = (\widehat{A_\hbar})^{\tau_D}$ is the τ_D-fixed points subalgebra of $\widehat{A_\hbar}$.
- For $\alpha, m \in \mathbb{Z}$, set

$$D_{\alpha,m} = A_{\alpha,m} - q^{2\alpha} A_{-\alpha,m}. \tag{2.7}$$

Then $\widehat{D_\hbar}$ is linearly spanned by $D_{\alpha,m}$ for $\alpha, m \in \mathbb{Z}$ and $\mathbf{c}$, and

$$\begin{aligned}[D_{\alpha,m}, D_{\beta,n}] &= 2i \sin \hbar(m\beta - n\alpha) D_{\alpha+\beta,m+n} \\ &+ 2iq^{2\beta} \sin \hbar(m\beta + n\alpha) D_{\alpha-\beta,m+n} + 2m(\delta_{\alpha+\beta,0} - q^{2\alpha}\delta_{\alpha-\beta,0})\delta_{m+n,0}\mathbf{c}.\end{aligned} \tag{2.8}$$

- Denote by $\mathcal{A}^\tau$ the Lie subalgebra of τ-fixed points in $\mathcal{A}$:

$$\mathcal{A}^\tau = \{a \in \mathcal{A} \mid \tau(a) = a\}, \tag{2.9}$$

where $\tau(E_{m,n}) = -E_{n,m}$ for $m, n \in \mathbb{Z}$.

Proposition 2.12

Lie algebra $\widehat{D_\hbar}$ is isomorphic to the covariant algebra $\widehat{\mathcal{A}^\tau}[\mathbb{Z}]$ with

$$D_{\alpha,m} = q^\alpha \overline{G^\tau_{\alpha,0} \otimes t^m} \quad \text{for } \alpha, m \in \mathbb{Z}$$

and with $\mathbf{c} = \mathbf{k}$, where $G^\tau_{\alpha,0} = G_{\alpha,0} - G_{-\alpha,0} \in \mathcal{A}^\tau$.

- The trigonometric Lie algebra $\widehat{D_\hbar}$ is isomorphic to the q-Virasoro algebra D_q (cf. [N]) in the study of lattice conformal theory.

[N] A. Nigro, A q-Virasoro algebra at roots of unity, Free Fermions and Temperley Lieb Hamiltonians, *J. Math. Phys.* **57(4)** (2016), 041702.

- The *q*-Virasoro algebra D_q is a Lie algebra with generators **c** and $D^{\alpha}(n)$ $(\alpha, n \in \mathbb{Z})$, subject to relations $D^{-\alpha}(n) = -D^{\alpha}(n)$, and

$$\begin{aligned}[D^{\alpha}(n), D^{\beta}(m)] &= (q-q^{-1})[\alpha m - \beta n]_q D^{\alpha+\beta}(m+n) \\ &\quad - (q-q^{-1})[\alpha m + \beta n]_q D^{\alpha-\beta}(m+n) \\ &\quad + ([m]_{q^{\alpha+\beta}} - [m]_{q^{\alpha-\beta}})\delta_{m+n,0}\mathbf{c}\end{aligned}$$

  for $\alpha, \beta, m, n \in \mathbb{Z}$, where **c** is a central element and $[n]_q$ is the *q*-integer defined by $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$.

[BC] A. Belov and K. Chaltikian, *Q*-deformation of Virasoro algebra and lattice conformal theories, *Modern Phys. Lett.* **8** (1993), 1233-1242.

Theorem 2.13 (Guo-Li-Tan-W, 2019)

Let q be a primitive $(2l+1)$-th root of unity, then q-Virasoro algebra D_q (i.e. the trigonometric Lie algebra $\widehat{D_\hbar}$) is isomorphic to the affine Kac-Moody algebra of type B_l, that is, $D_q \cong B_l^{(1)}$.

[GLTW1] H. Guo, H. Li, S. Tan and Q. Wang, *q*-Virasoro algebra and vertex algebras, *J. Pure and Appl. Algebra*, **219** (2015), 1258-1277.

[GLTW2] H. Guo, H. Li, S. Tan and Q. Wang, *q*-Virasoro algebra and affine Kac-Moody Lie algebras, *J. Algebra*, **534** (2019), 168-189.

Trigonometric Lie algebra of type D and vertex algebra

Theorem 2.14 (Guo-Li-Tan-W, 2019; Li-Tan-W, 2020)

Let $\ell \in \mathbb{C}$, for any restricted $\widehat{D}_\hbar$-module W of level ℓ, there exists an equivariant quasi $V_{\widehat{\mathcal{A}^\tau}}(\ell, 0)$-module structure $Y_W(\cdot, x)$ on W, which is uniquely determined by

$$Y_W(G^\tau_{\alpha,m}, x) = q^{-\alpha+m} D_\alpha(q^m x) \quad \text{for } \alpha, m \in \mathbb{Z}, \tag{2.10}$$

where $G^\tau_{\alpha,m} = G_{\alpha,m} - G_{-\alpha,m} \in \mathcal{A}^\tau$. On the other hand, every equivariant quasi $V_{\widehat{\mathcal{A}^\tau}}(\ell, 0)$-module (W, Y_W), one has a restricted $\widehat{D}_\hbar$-module structure of level ℓ on W such that $D_\alpha(x) = q^\alpha Y_W(G^\tau_{\alpha,0}, x)$ for $\alpha \in \mathbb{Z}$.

H. Guo, H. Li, S. Tan and Q. Wang, *q*-Virasoro algebra and affine Kac-Moody Lie algebras, *J. Algebra*, **534** (2019), 168-189.

H. Li, S. Tan, Q. Wang, Trigonometric Lie algebras, affine Lie algebras and vertex algebras, *Adv. Math.*, **363** (2020), 106985.

Thank You!