Leavitt path algebras, B_∞-algebras and Keller's conjecture for singular Hochschild cohomology

陈小伍, 中国科学技术大学

第三届代数与表示论前沿进展研讨会
北京师范大学, 2020.9.19–20

Overview

- Keller's conjecture links the singular Hochschild cohomology to the Hochschild cohomology of the dg singularity category, on the B_∞-level

Overview

- Keller's conjecture links the singular Hochschild cohomology to the Hochschild cohomology of the dg singularity category, on the B_∞-level
- Confirm Keller's conjecture for finite dimensional algebras with radical square zero, via Leavitt path algebras (which are usually infinite dimensional)

Overview

- Keller's conjecture links the singular Hochschild cohomology to the Hochschild cohomology of the dg singularity category, on the B_∞-level
- Confirm Keller's conjecture for finite dimensional algebras with radical square zero, via Leavitt path algebras (which are usually infinite dimensional)
- joint with 李换换（安徽大学），汪正方（斯图加特大学）

The content

- An introduction to the singularity category
- Singular Hochschild cohomology and Keller's conjecture
- The main results
- Main ingredients of the proof

The convention and notation

- We work over a fixed field k.
- A = a finite dimensional associative k-algebra with unit
- A-mod = the abelian category of finite dimensional left A-modules
- A-proj = the full subcategory of finite dimensional projective A-modules

The derived category

- $\mathbf{D}^b(A\text{-mod})$ = the bounded derived category of A-mod
- $\mathbf{K}^b(A\text{-proj})$ = the bounded homotopy category of A-proj
- View $\mathbf{K}^b(A\text{-proj}) \subseteq \mathbf{D}^b(A\text{-mod})$ a full triangulated subcategory

The derived category

- $\mathbf{D}^b(A\text{-mod})$ = the bounded derived category of A-mod
- $\mathbf{K}^b(A\text{-proj})$ = the bounded homotopy category of A-proj
- View $\mathbf{K}^b(A\text{-proj}) \subseteq \mathbf{D}^b(A\text{-mod})$ a full triangulated subcategory

Lemma

$\mathbf{K}^b(A\text{-proj}) = \mathbf{D}^b(A\text{-mod})$ *if and only if* $\mathrm{gl.dim}(A) < \infty$.

The singularity category

Definition (Buchweitz 1987/Orlov 2004)

The *singularity category* of A is the Verdier quotient category

$$\mathbf{D}_{\mathrm{sg}}(A) = \mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(A\text{-proj}).$$

The singularity category

Definition (Buchweitz 1987/Orlov 2004)

The *singularity category* of A is the Verdier quotient category

$$\mathbf{D}_{\mathrm{sg}}(A) = \mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(A\text{-proj}).$$

- $\mathbf{D}_{\mathrm{sg}}(A)$ vanishes if and only if $\mathrm{gl.dim}(A) < \infty$

The singularity category

Definition (Buchweitz 1987/Orlov 2004)

The *singularity category* of A is the Verdier quotient category

$$\mathbf{D}_{\mathrm{sg}}(A) = \mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(A\text{-proj}).$$

- $\mathbf{D}_{\mathrm{sg}}(A)$ vanishes if and only if $\mathrm{gl.dim}(A) < \infty$
- $\mathbf{D}_{\mathrm{sg}}(A)$ is a homological invariant for algebras with infinite global dimension

The singularity category

Definition (Buchweitz 1987/Orlov 2004)

The *singularity category* of A is the Verdier quotient category

$$\mathbf{D}_{\rm sg}(A) = \mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(A\text{-proj}).$$

- $\mathbf{D}_{\rm sg}(A)$ vanishes if and only if $\mathrm{gl.dim}(A) < \infty$
- $\mathbf{D}_{\rm sg}(A)$ is a homological invariant for algebras with infinite global dimension
- $\mathbf{D}_{\rm sg}(A)$ is invariant under derived equivalences

Aspects of singularity categories

- in mathematical physics, singularity categories are viewed as the B-side in the homological mirror symmetry of LG models

Aspects of singularity categories

- in mathematical physics, singularity categories are viewed as the B-side in the homological mirror symmetry of LG models
- in commutative algebra, it relates to matrix factorizations and classical singularities of equations

Aspects of singularity categories

- in mathematical physics, singularity categories are viewed as the B-side in the homological mirror symmetry of LG models
- in commutative algebra, it relates to matrix factorizations and classical singularities of equations
- in noncommutative geometry, its graded version relates to the bounded derived category of sheaves over noncommutative projective schemes

Aspects of singularity categories

- in mathematical physics, singularity categories are viewed as the B-side in the homological mirror symmetry of LG models
- in commutative algebra, it relates to matrix factorizations and classical singularities of equations
- in noncommutative geometry, its graded version relates to the bounded derived category of sheaves over noncommutative projective schemes
- in homological algebra, it relates to Gorenstein projective modules, and Tate-Vogel cohomology
- ......

Singularity category via stablization

- The *stable module category* $A\text{-}\underline{\mathrm{mod}} = A\text{-mod}/[A\text{-proj}]$: killing morphisms factoring through projectives

Singularity category via stablization

- The *stable module category* $A\text{-}\underline{\mathrm{mod}} = A\text{-mod}/[A\text{-proj}]$: killing morphisms factoring through projectives
- The stable module category $A\text{-}\underline{\mathrm{mod}}$ is *left triangulated*

Singularity category via stablization

- The *stable module category* $A\text{-}\underline{\mathrm{mod}} = A\text{-mod}/[A\text{-proj}]$: killing morphisms factoring through projectives
- The stable module category $A\text{-}\underline{\mathrm{mod}}$ is *left triangulated*
- The syzygy functor $\Omega\colon A\text{-}\underline{\mathrm{mod}} \longrightarrow A\text{-}\underline{\mathrm{mod}}$ (usually not an equivalence!)

Singularity category via stablization

- The *stable module category* $A\text{-}\underline{\mathrm{mod}} = A\text{-mod}/[A\text{-proj}]$: killing morphisms factoring through projectives
- The stable module category $A\text{-}\underline{\mathrm{mod}}$ is *left triangulated*
- The syzygy functor $\Omega\colon A\text{-}\underline{\mathrm{mod}} \longrightarrow A\text{-}\underline{\mathrm{mod}}$ (usually not an equivalence!)
- Short exact sequences induce exact triangles:

$$\begin{array}{ccccc} \Omega(N) & \longrightarrow & P(N) & \longrightarrow & N \\ \downarrow & & \downarrow & & \| \\ L & \longrightarrow & M & \longrightarrow & N \end{array}$$

Singularity category via stablization, continued

- The stablization $\mathcal{S}(A\text{-}\underline{\mathrm{mod}})$ is obtained from A-$\underline{\mathrm{mod}}$ by formally inverting Ω!

Singularity category via stablization, continued

- The stablization $\mathcal{S}(A\text{-}\underline{\mathrm{mod}})$ is obtained from A-$\underline{\mathrm{mod}}$ by formally inverting Ω!
- More precisely, the objects are (M, n), with an A-module M and $n \in \mathbb{Z}$; the morphisms are given $\mathrm{Hom}((M, n), (L, m)) = \mathrm{colim}\ \underline{\mathrm{Hom}}_A(\Omega^{i-n}(M), \Omega^{i-m}(L))$

Singularity category via stablization, continued

- The stablization $\mathcal{S}(A\text{-}\underline{\mathrm{mod}})$ is obtained from A-$\underline{\mathrm{mod}}$ by formally inverting Ω!
- More precisely, the objects are (M, n), with an A-module M and $n \in \mathbb{Z}$; the morphisms are given $\mathrm{Hom}((M, n), (L, m)) = \mathrm{colim}\ \underline{\mathrm{Hom}}_A(\Omega^{i-n}(M), \Omega^{i-m}(L))$
- Ω now becomes $M = (M, 0) \mapsto (M, -1)$, an automorphism!

Singularity category via stablization, continued

- The stablization $\mathcal{S}(A\text{-}\underline{\mathrm{mod}})$ is obtained from $A\text{-}\underline{\mathrm{mod}}$ by formally inverting Ω!
- More precisely, the objects are (M, n), with an A-module M and $n \in \mathbb{Z}$; the morphisms are given $\mathrm{Hom}((M, n), (L, m)) = \mathrm{colim}\ \underline{\mathrm{Hom}}_A(\Omega^{i-n}(M), \Omega^{i-m}(L))$
- Ω now becomes $M = (M, 0) \mapsto (M, -1)$, an automorphism!
- The stabilization $\mathcal{S}(A\text{-}\underline{\mathrm{mod}})$ is naturally triangulated.

Theorem (Keller-Vossieck 1987/Beligiannis 2000)

There is a triangle equivalence

$$\mathcal{S}(A\text{-}\underline{\mathrm{mod}}) \simeq \mathbf{D}_{\mathrm{sg}}(A).$$

The dg singularity category

The dg quotient [Keller 1999/Drinfeld 2004] enhances the Verdier quotient

The dg singularity category

The dg quotient [Keller 1999/Drinfeld 2004] enhances the Verdier quotient

for example, $\mathbf{D}^b_{\mathrm{dg}}(A\text{-}\mathrm{mod})$ = the bounded dg derived category: a dg category with $H^0(\mathbf{D}^b_{\mathrm{dg}}(A\text{-}\mathrm{mod})) = \mathbf{D}^b(A\text{-}\mathrm{mod})$

The dg singularity category

The dg quotient [Keller 1999/Drinfeld 2004] enhances the Verdier quotient

for example, $\mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})$ = the bounded dg derived category: a dg category with $H^0(\mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})) = \mathbf{D}^b(A\text{-mod})$

Definition (Keller 2018)

The *dg singularity category* of A is given by the dg quotient

$$\mathbf{S}_{\mathrm{dg}}(A) = \mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})/\mathbf{C}^b_{\mathrm{dg}}(A\text{-proj}).$$

The dg singularity category

The dg quotient [Keller 1999/Drinfeld 2004] enhances the Verdier quotient

for example, $\mathbf{D}^b_{\mathrm{dg}}(A\text{-}\mathrm{mod})$ = the bounded dg derived category: a dg category with $H^0(\mathbf{D}^b_{\mathrm{dg}}(A\text{-}\mathrm{mod})) = \mathbf{D}^b(A\text{-}\mathrm{mod})$

Definition (Keller 2018)

The *dg singularity category* of A is given by the dg quotient

$$\mathbf{S}_{\mathrm{dg}}(A) = \mathbf{D}^b_{\mathrm{dg}}(A\text{-}\mathrm{mod})/\mathbf{C}^b_{\mathrm{dg}}(A\text{-}\mathrm{proj}).$$

- $\mathbf{S}_{\mathrm{dg}}(A)$ is a finer invariant as $H^0(\mathbf{S}_{\mathrm{dg}}(A)) = \mathbf{D}_{\mathrm{sg}}(A)$

The dg singularity category

The dg quotient [Keller 1999/Drinfeld 2004] enhances the Verdier quotient

for example, $\mathbf{D}^b_{\rm dg}(A\text{-mod})$ = the bounded dg derived category: a dg category with $H^0(\mathbf{D}^b_{\rm dg}(A\text{-mod})) = \mathbf{D}^b(A\text{-mod})$

Definition (Keller 2018)

The *dg singularity category* of A is given by the dg quotient

$$\mathbf{S}_{\rm dg}(A) = \mathbf{D}^b_{\rm dg}(A\text{-mod})/\mathbf{C}^b_{\rm dg}(A\text{-proj}).$$

- $\mathbf{S}_{\rm dg}(A)$ is a finer invariant as $H^0(\mathbf{S}_{\rm dg}(A)) = \mathbf{D}_{\rm sg}(A)$
- There are various "realizations" of $\mathbf{S}_{\rm dg}(A)$; cf. [C-Li-Wang]

The content

- An introduction to the singularity category
- Singular Hochschild cohomology and Keller's conjecture
- The main results
- Main ingredients of the proof

The singular Hochschild cohomology

- the *enveloping algebra* $A^e = A \otimes A^{\mathrm{op}}$: A-A-bimodules = left A^e-modules

The singular Hochschild cohomology

- the *enveloping algebra* $A^e = A \otimes A^{\mathrm{op}}$: A-A-bimodules = left A^e-modules
- The Hochschild cohomology of A

$$\mathrm{HH}^*(A, A) = \mathrm{Hom}_{\mathbf{D}^b(A^e\text{-}\mathrm{mod})}(A, \Sigma^*(A))$$

The singular Hochschild cohomology

- the *enveloping algebra* $A^e = A \otimes A^{\rm op}$: A-A-bimodules = left A^e-modules
- The Hochschild cohomology of A

$$\mathrm{HH}^*(A, A) = \mathrm{Hom}_{\mathbf{D}^b(A^e\text{-mod})}(A, \Sigma^*(A))$$

- The *singular Hochschild cohomology* of A

$$\mathrm{HH}^*_{\rm sg}(A, A) = \mathrm{Hom}_{\mathbf{D}_{\rm sg}(A^e)}(A, \Sigma^*(A))$$

The singular Hochschild cohomology

- the *enveloping algebra* $A^e = A \otimes A^{\mathrm{op}}$: A-A-bimodules = left A^e-modules
- The Hochschild cohomology of A
$$\mathrm{HH}^*(A, A) = \mathrm{Hom}_{\mathbf{D}^b(A^e\text{-}\mathrm{mod})}(A, \Sigma^*(A))$$
- The *singular Hochschild cohomology* of A
$$\mathrm{HH}^*_{\mathrm{sg}}(A, A) = \mathrm{Hom}_{\mathbf{D}_{\mathrm{sg}}(A^e)}(A, \Sigma^*(A))$$
- Both are graded-commutative algebras

The singular Hochschild cohomology

- the *enveloping algebra* $A^e = A \otimes A^{\rm op}$: A-A-bimodules = left A^e-modules
- The Hochschild cohomology of A

$$\mathrm{HH}^*(A, A) = \mathrm{Hom}_{\mathbf{D}^b(A^e\text{-mod})}(A, \Sigma^*(A))$$

- The *singular Hochschild cohomology* of A

$$\mathrm{HH}^*_{\rm sg}(A, A) = \mathrm{Hom}_{\mathbf{D}_{\rm sg}(A^e)}(A, \Sigma^*(A))$$

- Both are graded-commutative algebras
- The Hochschild cohomology are well known to relate to deformation theory and noncommutative differential geometry...

Keller's theorem, the background

- HH^* is defined naturally for any dg categories: the deformation theory of categories

Keller's theorem, the background

- HH^* is defined naturally for any dg categories: the deformation theory of categories
- Set $\mathbf{D}^b = \mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})$ the bounded dg derived category

Keller's theorem, the background

- HH^* is defined naturally for any dg categories: the deformation theory of categories
- Set $\mathbf{D}^b = \mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})$ the bounded dg derived category
- There is a canonical triangle embedding

$$\mathbf{D}^b(A^e\text{-mod}) \hookrightarrow \mathbf{D}(\mathbf{D}^b), \quad X \mapsto \mathbb{R}\mathrm{Hom}_A(-, X \otimes^{\mathbb{L}}_A -)$$

inducing an isomorphism

$$\mathrm{HH}^*(A, A) \simeq \mathrm{HH}^*(\mathbf{D}^b, \mathbf{D}^b).$$

Keller's theorem, the background

- HH^* is defined naturally for any dg categories: the deformation theory of categories
- Set $\mathbf{D}^b = \mathbf{D}^b_{\mathrm{dg}}(A\text{-mod})$ the bounded dg derived category
- There is a canonical triangle embedding

$$\mathbf{D}^b(A^e\text{-mod}) \hookrightarrow \mathbf{D}(\mathbf{D}^b), \quad X \mapsto \mathbb{R}\mathrm{Hom}_A(-, X \otimes^{\mathbb{L}}_A -)$$

inducing an isomorphism

$$\mathrm{HH}^*(A, A) \simeq \mathrm{HH}^*(\mathbf{D}^b, \mathbf{D}^b).$$

- Lowen-Van den Bergh 2005: this isomorphism lifts to B_∞-level

Keller's theorem

Theorem (Keller 2018)

Assume that $A/\mathrm{rad}(A)$ is separable over k. Then there is an canonical isomorphism of graded algebras

$$\Phi\colon \mathrm{HH}^*_{\mathrm{sg}}(A, A) \longrightarrow \mathrm{HH}^*(\mathbf{S}_{\mathrm{dg}}(A), \mathbf{S}_{\mathrm{dg}}(A)).$$

Keller's theorem

Theorem (Keller 2018)

Assume that $A/\mathrm{rad}(A)$ is separable over k. Then there is an canonical isomorphism of graded algebras

$$\Phi\colon \mathrm{HH}^*_{\mathrm{sg}}(A, A) \longrightarrow \mathrm{HH}^*(\mathbf{S}_{\mathrm{dg}}(A), \mathbf{S}_{\mathrm{dg}}(A)).$$

- The isomorphism is induced by a triangle functor

$$\mathbf{D}_{\mathrm{sg}}(A^e) \longrightarrow \mathbf{D}(\mathbf{S}_{\mathrm{dg}}(A)).$$

Keller's theorem

Theorem (Keller 2018)

Assume that $A/\mathrm{rad}(A)$ is separable over k. Then there is an canonical isomorphism of graded algebras

$$\Phi\colon \mathrm{HH}^*_{\mathrm{sg}}(A,A) \longrightarrow \mathrm{HH}^*(\mathbf{S}_{\mathrm{dg}}(A), \mathbf{S}_{\mathrm{dg}}(A)).$$

- The isomorphism is induced by a triangle functor

$$\mathbf{D}_{\mathrm{sg}}(A^e) \longrightarrow \mathbf{D}(\mathbf{S}_{\mathrm{dg}}(A)).$$

- It is compatible with the previous isomorphism.

Keller's theorem

Theorem (Keller 2018)

Assume that $A/\mathrm{rad}(A)$ is separable over k. Then there is an canonical isomorphism of graded algebras

$$\Phi\colon \mathrm{HH}^*_{\mathrm{sg}}(A,A) \longrightarrow \mathrm{HH}^*(\mathbf{S}_{\mathrm{dg}}(A), \mathbf{S}_{\mathrm{dg}}(A)).$$

- The isomorphism is induced by a triangle functor

$$\mathbf{D}_{\mathrm{sg}}(A^e) \longrightarrow \mathbf{D}(\mathbf{S}_{\mathrm{dg}}(A)).$$

- It is compatible with the previous isomorphism.
- It plays an essential role in Keller-Hua's work on Donovan-Wemyss's conjecture.

Kelle's conjecture

Conjecture (Keller 2018)

The isomorphism Φ lifts to B_∞-level, in particular, Φ preserves the Gerstenhaber structures.

Kelle's conjecture

Conjecture (Keller 2018)

The isomorphism Φ lifts to B_∞-level, in particular, Φ preserves the Gerstenhaber structures.

To be more precise,

- The Hochschild cochain complex $C^*(\mathbf{S}_{\mathrm{dg}}(A), \mathbf{S}_{\mathrm{dg}}(A))$, lifting $\mathrm{HH}^*(\mathbf{S}_{\mathrm{dg}}(A), \mathbf{S}_{\mathrm{dg}}(A))$, is a B_∞-algebra, with the cup product and brace operations

Kelle's conjecture

Conjecture (Keller 2018)

The isomorphism Φ lifts to B_∞-level, in particular, Φ preserves the Gerstenhaber structures.

To be more precise,

- The Hochschild cochain complex $C^*(\mathbf{S}_{\mathrm{dg}}(A), \mathbf{S}_{\mathrm{dg}}(A))$, lifting $\mathrm{HH}^*(\mathbf{S}_{\mathrm{dg}}(A), \mathbf{S}_{\mathrm{dg}}(A))$, is a B_∞-algebra, with the cup product and brace operations
- The singular Hochschild cochain complex $C^*_{\mathrm{sg}}(A, A)$, lifting $\mathrm{HH}^*_{\mathrm{sg}}(A, A)$, is also a B_∞-algebra, with the cup product and brace operations [Wang 2018]

The singular Hochschild cochain complex

- Following [Cuntz-Quillen 1995], in the bar resolution, we have

$$\Omega^p = (s\bar{A})^{\otimes p} \otimes A$$

the (graded) *bimodule of noncommutative differential p-forms*

The singular Hochschild cochain complex

- Following [Cuntz-Quillen 1995], in the bar resolution, we have

$$\Omega^p = (s\bar{A})^{\otimes p} \otimes A$$

the (graded) *bimodule of noncommutative differential p-forms*

- using stabilization

$$\mathrm{HH}^*_{\mathrm{sg}}(A, A) = \mathrm{colim}\ \mathrm{HH}^*(A, \Omega^p)$$

The singular Hochschild cochain complex

- Following [Cuntz-Quillen 1995], in the bar resolution, we have

$$\Omega^p = (s\bar{A})^{\otimes p} \otimes A$$

the (graded) *bimodule of noncommutative differential p-forms*

- using stabilization

$$\mathrm{HH}^*_{\mathrm{sg}}(A, A) = \mathrm{colim}\ \mathrm{HH}^*(A, \Omega^p)$$

- There are natural cochain maps

$$\theta_p \colon C^*(A, \Omega^p) \longrightarrow C^*(A, \Omega^{p+1}), \quad f \mapsto \mathrm{Id}_{s\bar{A}} \otimes f$$

between the Hochschild cochain complexes.

The singular Hochschild cochain complex

- Following [Cuntz-Quillen 1995], in the bar resolution, we have

$$\Omega^p = (s\bar{A})^{\otimes p} \otimes A$$

the (graded) *bimodule of noncommutative differential p-forms*

- using stabilization

$$\mathrm{HH}^*_{\mathrm{sg}}(A, A) = \mathrm{colim}\ \mathrm{HH}^*(A, \Omega^p)$$

- There are natural cochain maps

$$\theta_p \colon C^*(A, \Omega^p) \longrightarrow C^*(A, \Omega^{p+1}), \quad f \mapsto \mathrm{Id}_{s\bar{A}} \otimes f$$

between the Hochschild cochain complexes. Therefore, taking the colimit, we obtain $C^*_{\mathrm{sg}}(A, A)$, called the *singular Hochschild cochain complex* of A; it computes $\mathrm{HH}^*_{\mathrm{sg}}(A, A)$.

The singular Hochschild cochain complex

- Following [Cuntz-Quillen 1995], in the bar resolution, we have

$$\Omega^p = (s\bar{A})^{\otimes p} \otimes A$$

the (graded) *bimodule of noncommutative differential p-forms*

- using stabilization

$$\mathrm{HH}^*_{\mathrm{sg}}(A, A) = \mathrm{colim}\ \mathrm{HH}^*(A, \Omega^p)$$

- There are natural cochain maps

$$\theta_p \colon C^*(A, \Omega^p) \longrightarrow C^*(A, \Omega^{p+1}), \quad f \mapsto \mathrm{Id}_{s\bar{A}} \otimes f$$

between the Hochschild cochain complexes. Therefore, taking the colimit, we obtain $C^*_{\mathrm{sg}}(A, A)$, called the *singular Hochschild cochain complex* of A; it computes $\mathrm{HH}^*_{\mathrm{sg}}(A, A)$.

Wang's theorem

Theorem (Wang 2018)

There is a natural B_∞-algebra structure on $C^*_{\mathrm{sg}}(A, A)$.

- It is compatible with the inclusion $C^*(A, A) \hookrightarrow C^*_{\mathrm{sg}}(A, A)$.

Wang's theorem

Theorem (Wang 2018)

There is a natural B_∞-algebra structure on $C^*_{\mathrm{sg}}(A, A)$.

- It is compatible with the inclusion $C^*(A, A) \hookrightarrow C^*_{\mathrm{sg}}(A, A)$.
- Two versions of $C^*_{\mathrm{sg}}(A, A)$, *right* and *left*; there is a nontrivial B_∞-duality between them.

A few words on B_∞-algebras

1. The notion of a B_∞-algebra is due to [Getzler-Jones, 1994].

A few words on B_∞-algebras

1. The notion of a B_∞-algebra is due to [Getzler-Jones, 1994].
2. Roughly speaking, a B_∞-algebra B is a graded Poisson algebra up to homotopy; its cohomology $H^*(B)$ is a Gerstenhaber algebra.

A few words on B_∞-algebras

1. The notion of a B_∞-algebra is due to [Getzler-Jones, 1994].
2. Roughly speaking, a B_∞-algebra B is a graded Poisson algebra up to homotopy; its cohomology $H^*(B)$ is a Gerstenhaber algebra.
3. a B_∞-algebra is an A_∞-algebra with $\mu_{p,q}\colon B^{\otimes p} \otimes B^{\otimes q} \to B$ with $p, q \geq 0$.

A few words on B_∞-algebras

1. The notion of a B_∞-algebra is due to [Getzler-Jones, 1994].
2. Roughly speaking, a B_∞-algebra B is a graded Poisson algebra up to homotopy; its cohomology $H^*(B)$ is a Gerstenhaber algebra.
3. a B_∞-algebra is an A_∞-algebra with $\mu_{p,q}\colon B^{\otimes p} \otimes B^{\otimes q} \to B$ with $p, q \geq 0$.
4. Our concern: *brace B_∞-algebra*, with dg algebra and $\mu_{p,q} = 0$ for $p > 1$;

A few words on B_∞-algebras

1. The notion of a B_∞-algebra is due to [Getzler-Jones, 1994].
2. Roughly speaking, a B_∞-algebra B is a graded Poisson algebra up to homotopy; its cohomology $H^*(B)$ is a Gerstenhaber algebra.
3. a B_∞-algebra is an A_∞-algebra with $\mu_{p,q}\colon B^{\otimes p} \otimes B^{\otimes q} \to B$ with $p, q \geq 0$.
4. Our concern: *brace B_∞-algebra*, with dg algebra and $\mu_{p,q} = 0$ for $p > 1$; more precisely, a dg algebra with brace operations subject to the higher pre-Jacobi identity, the distributivity, and the higher homotopy.

Keller's conjecture, revisited

- Two (brace) B_∞-algebras for A: the classical one $C^*(\mathbf{S}_{\rm dg}(A), \mathbf{S}_{\rm dg}(A))$, and the singular one $C^*_{\rm sg}(A, A)$

Keller's conjecture, revisited

- Two (brace) B_∞-algebras for A: the classical one $C^*(\mathbf{S}_{\mathrm{dg}}(A), \mathbf{S}_{\mathrm{dg}}(A))$, and the singular one $C^*_{\mathrm{sg}}(A, A)$
- Keller's theorem says that they have the same cohomology

Keller's conjecture, revisited

- Two (brace) B_∞-algebras for A: the classical one $C^*(\mathbf{S}_{\mathrm{dg}}(A), \mathbf{S}_{\mathrm{dg}}(A))$, and the singular one $C^*_{\mathrm{sg}}(A, A)$
- Keller's theorem says that they have the same cohomology

Conjecture (Keller 2018)

There is an isomorphism in the homotopy category of B_∞-algebras

$$C^*(\mathbf{S}_{\mathrm{dg}}(A), \mathbf{S}_{\mathrm{dg}}(A)) \simeq C^*_{\mathrm{sg}}(A, A).$$

In particular, the isomorphism on the cohomology respects the Gerstenhaber structures.

Keller's conjecture, revisited

- Two (brace) B_∞-algebras for A: the classical one $C^*(\mathbf{S}_{\mathrm{dg}}(A), \mathbf{S}_{\mathrm{dg}}(A))$, and the singular one $C^*_{\mathrm{sg}}(A, A)$
- Keller's theorem says that they have the same cohomology

Conjecture (Keller 2018)

There is an isomorphism in the homotopy category of B_∞-algebras

$$C^*(\mathbf{S}_{\mathrm{dg}}(A), \mathbf{S}_{\mathrm{dg}}(A)) \simeq C^*_{\mathrm{sg}}(A, A).$$

In particular, the isomorphism on the cohomology respects the Gerstenhaber structures.

- The stronger version: the above isomorphism is required to be compatible with the canonical isomorphism Φ.
- We treat the above slightly weakened form.

The content

- An introduction to the singularity category
- Singular Hochschild cohomology and Keller's conjecture
- The main results
- Main ingredients of the proof

An invariance theorem

Theorem (C.-Li-Wang)

Keller's conjecture is invariant under one-point (co)extensions and singular equivalences with levels.

An invariance theorem

Theorem (C.-Li-Wang)

Keller's conjecture is invariant under one-point (co)extensions and singular equivalences with levels.

- We can remove the sinks and sources from the quiver of A.

An invariance theorem

Theorem (C.-Li-Wang)

Keller's conjecture is invariant under one-point (co)extensions and singular equivalences with levels.

- We can remove the sinks and sources from the quiver of A.
- Keller's conjecture is invariant under derived equivalences.

The proof of the invariance theorem

- It is well known that one-point (co)extensions and singular equivalences with level preserve singularity categories [C. 2011], [Wang 2015].

The proof of the invariance theorem

- It is well known that one-point (co)extensions and singular equivalences with level preserve singularity categories [C. 2011], [Wang 2015]. These equivalences lift to the dg singularity categories.

The proof of the invariance theorem

- It is well known that one-point (co)extensions and singular equivalences with level preserve singularity categories [C. 2011], [Wang 2015]. These equivalences lift to the dg singularity categories.
- For the invariance of $C^*_{\rm sg}(A, A)$ under one-point (co)extension, one constructs explicit B_∞-quasi-isomorphisms;

The proof of the invariance theorem

- It is well known that one-point (co)extensions and singular equivalences with level preserve singularity categories [C. 2011], [Wang 2015]. These equivalences lift to the dg singularity categories.
- For the invariance of $C^*_{\mathrm{sg}}(A, A)$ under one-point (co)extension, one constructs explicit B_∞-quasi-isomorphisms; for the invariance of $C^*_{\mathrm{sg}}(A, A)$ under singular equivalences with level, one modifies an argument by [Keller 2013], using a triangular matrix algebra.

Keller's conjecture for algebras with radical square zero

- Q = a finite quiver without sinks
- $A_Q = kQ/J^2$ the algebra with radical square zero
- $L(Q)$ = the Leavitt path algebra

Keller's conjecture for algebras with radical square zero

- Q = a finite quiver without sinks
- $A_Q = kQ/J^2$ the algebra with radical square zero
- $L(Q)$ = the Leavitt path algebra

Theorem (C.-Li-Wang)

Then there are isomorphisms in the homotopy category of B_∞-algebras

$$C^*_{\rm sg}(A_Q, A_Q) \xrightarrow{\Upsilon} C^*(L(Q), L(Q)) \xrightarrow{\Delta} C^*(\mathbf{S}_{\rm dg}(A_Q), \mathbf{S}_{\rm dg}(A_Q)).$$

Keller's conjecture for algebras with radical square zero

- Q = a finite quiver without sinks
- $A_Q = kQ/J^2$ the algebra with radical square zero
- $L(Q)$ = the Leavitt path algebra

Theorem (C.-Li-Wang)

Then there are isomorphisms in the homotopy category of B_∞-algebras

$$C^*_{\mathrm{sg}}(A_Q, A_Q) \xrightarrow{\Upsilon} C^*(L(Q), L(Q)) \xrightarrow{\Delta} C^*(\mathbf{S}_{\mathrm{dg}}(A_Q), \mathbf{S}_{\mathrm{dg}}(A_Q)).$$

- Keller's conjecture holds for any kQ/J^2 (iterated one-point coextensions), and also for gentle algebras (singular equivalence with level).

Keller's conjecture for algebras with radical square zero

- Q = a finite quiver without sinks
- $A_Q = kQ/J^2$ the algebra with radical square zero
- $L(Q)$ = the Leavitt path algebra

Theorem (C.-Li-Wang)

Then there are isomorphisms in the homotopy category of B_∞-algebras

$$C^*_{\rm sg}(A_Q, A_Q) \xrightarrow{\Upsilon} C^*(L(Q), L(Q)) \xrightarrow{\Delta} C^*(\mathbf{S}_{\rm dg}(A_Q), \mathbf{S}_{\rm dg}(A_Q)).$$

- Keller's conjecture holds for any kQ/J^2 (iterated one-point coextensions), and also for gentle algebras (singular equivalence with level).
- We use the Leavitt path algebra $L(Q)$ as a bridge!

The content

- An introduction to the singularity category
- Singular Hochschild cohomology and Keller's conjecture
- The main results
- Main ingredients of the proof

To be explained

- What is Leavitt path algebra $L(Q)$?
- How does $A_Q = kQ/J^2$ relate to $L(Q)$?
- The categorical proof of
  $\Delta\colon C^*(L(Q), L(Q)) \to C^*(\mathbf{S}_{\mathrm{dg}}(A_Q), \mathbf{S}_{\mathrm{dg}}(A_Q))$
- The combinatorial proof of
  $\Upsilon\colon C^*_{\mathrm{sg}}(A_Q, A_Q) \to C^*(L(Q), L(Q))$

Reminders on quivers

- $Q = (Q_0, Q_1; s, t\colon Q_1 \to Q_0)$ a finite *quiver* (= oriented graph)
- Q_0 = the set of vertices, Q_1 = the set of arrows
- visualize an arrow α as $s(\alpha) \xrightarrow{\alpha} t(\alpha)$
- a vertex i is called a *sink*, if $s^{-1}(i) = \emptyset$;
- We assume that Q has no sinks.

Reminders on path algebras

- a finite *path* in Q is $p = \alpha_n \cdots \alpha_2 \alpha_1$ of length n

$$\cdot \xrightarrow{\alpha_1} \cdot \xrightarrow{\alpha_2} \cdot \cdots \cdot \xrightarrow{\alpha_n} \cdot$$

  In this case, we set $s(p) = s(\alpha_1)$ and $t(p) = t(\alpha_n)$.

- paths of length one = arrows; paths of length zero = vertices (for $i \in Q_0$, we associate a *trivial* path e_i.)
- The *path algebra* kQ: k-basis = paths in Q, the multiplication = concatenation of paths.

Reminders on path algebras

- a finite *path* in Q is $p = \alpha_n \cdots \alpha_2\alpha_1$ of length n

$$\cdot \xrightarrow{\alpha_1} \cdot \xrightarrow{\alpha_2} \cdots\cdots \xrightarrow{\alpha_n} \cdot$$

  In this case, we set $s(p) = s(\alpha_1)$ and $t(p) = t(\alpha_n)$.

- paths of length one = arrows; paths of length zero = vertices (for $i \in Q_0$, we associate a *trivial* path e_i.)
- The *path algebra* kQ: k-basis = paths in Q, the multiplication = concatenation of paths. More precisely, for two paths p and q in Q, $p \cdot q = pq$ if $s(p) = t(q)$, otherwise, $p \cdot q = 0$.
  For example, $e_ie_j = \delta_{i,j}e_i$, $e_ip = \delta_{i,t(p)}p$, $pe_i = \delta_{s(p),i}p$.

Reminders on path algebras, continued

- Q_n = the set of paths in Q of length n; then $kQ = \bigoplus_{n\geq 0} kQ_n$ is naturally $\mathbb{N}$-graded.
- The unit $1_{kQ} = \sum_{i\in Q_0} e_i$ has a decomposition into pairwise orthogonal idempotents.
- Set $J = \bigoplus_{n\geq 1} kQ_n$, the two-sided ideal of kQ generated by arrows.
- The algebra $A_Q = kQ/J^2$ with radical square zero is finite dimensional. Indeed, A_Q has a basis $\{e_i \mid i \in Q_0\} \cup \{\alpha \mid \alpha \in Q_1\}$, the multiplication rule is given by $e_i e_j = \delta_{i,j} e_i$, $e_i\alpha = \delta_{i,t(\alpha)}\alpha$, $\beta e_j = \delta_{s(\beta),j}\beta$, $\alpha\beta = 0$.

What is Leavitt path algebra?

$\bar{Q}$ = the *double quiver* of Q, that is, for each arrow $\alpha\colon i \to j$ in Q, we add a new arrow $\alpha^*\colon j \to i$.

What is Leavitt path algebra?

$\bar{Q}$ = the *double quiver* of Q, that is, for each arrow $\alpha\colon i \to j$ in Q, we add a new arrow $\alpha^*\colon j \to i$.

Definition (Abrams-Aranda Pino 2005/Ara-Moreno-Pardo 2007)

The *Leavitt path algebra* $L(Q)$ of Q is the quotient algebra of $k\bar{Q}$ by the two-sided ideal generated by the following elements

What is Leavitt path algebra?

$\bar{Q}$ = the *double quiver* of Q, that is, for each arrow $\alpha\colon i \to j$ in Q, we add a new arrow $\alpha^*\colon j \to i$.

Definition (Abrams-Aranda Pino 2005/Ara-Moreno-Pardo 2007)

The *Leavitt path algebra* $L(Q)$ of Q is the quotient algebra of $k\bar{Q}$ by the two-sided ideal generated by the following elements

- (CK1) $\alpha\beta^* - \delta_{\alpha,\beta} e_{t(\alpha)}$, for all $\alpha, \beta \in Q_1$;

What is Leavitt path algebra?

$\bar{Q}$ = the *double quiver* of Q, that is, for each arrow $\alpha\colon i \to j$ in Q, we add a new arrow $\alpha^*\colon j \to i$.

Definition (Abrams-Aranda Pino 2005/Ara-Moreno-Pardo 2007)

The *Leavitt path algebra* $L(Q)$ of Q is the quotient algebra of $k\bar{Q}$ by the two-sided ideal generated by the following elements

- (CK1) $\alpha\beta^* - \delta_{\alpha,\beta} e_{t(\alpha)}$, for all $\alpha, \beta \in Q_1$;
- (CK2) $\sum_{\{\alpha \in Q_1 \mid s(\alpha)=i\}} \alpha^*\alpha - e_i$, for all $i \in Q_0$.

What is Leavitt path algebra?

$\bar{Q}$ = the *double quiver* of Q, that is, for each arrow $\alpha : i \to j$ in Q, we add a new arrow $\alpha^* : j \to i$.

Definition (Abrams-Aranda Pino 2005/Ara-Moreno-Pardo 2007)

The *Leavitt path algebra* $L(Q)$ of Q is the quotient algebra of $k\bar{Q}$ by the two-sided ideal generated by the following elements

- (CK1) $\alpha\beta^* - \delta_{\alpha,\beta} e_{t(\alpha)}$, for all $\alpha, \beta \in Q_1$;
- (CK2) $\sum_{\{\alpha \in Q_1 \mid s(\alpha)=i\}} \alpha^*\alpha - e_i$, for all $i \in Q_0$.

Here, CK stands for Cuntz-Krieger.

Example: The Leavitt algebra

Example

Let Q be the rose quiver with two petals. Then we have an isomorphism

$$L(Q) \simeq \frac{k\langle x_1, x_2, y_1, y_2\rangle}{\langle x_i y_j - \delta_{i,j}, y_1 x_1 + y_2 x_2 - 1\rangle}.$$

Example: The Leavitt algebra

Example

Let Q be the rose quiver with two petals. Then we have an isomorphism

$$L(Q) \simeq \frac{k\langle x_1, x_2, y_1, y_2\rangle}{\langle x_i y_j - \delta_{i,j}, y_1 x_1 + y_2 x_2 - 1\rangle}.$$

The latter algebra is called the *Leavitt algebra* L_2 of order two,

Example: The Leavitt algebra

Example

Let Q be the rose quiver with two petals. Then we have an isomorphism

$$L(Q) \simeq \frac{k\langle x_1, x_2, y_1, y_2\rangle}{\langle x_i y_j - \delta_{i,j}, y_1 x_1 + y_2 x_2 - 1\rangle}.$$

The latter algebra is called the *Leavitt algebra* L_2 of order two, studied by W. Leavitt in 1958, related to the non-IBN property.

Nice properties of the Leavitt path algebra

- The Leavitt path algebra $L(Q)$ is naturally $\mathbb{Z}$-graded as $L(Q) = \bigoplus_{n\in\mathbb{Z}} L(Q)_n$ with $e_i \in L(Q)_0$, $\alpha \in L(Q)_1$ and $\alpha^* \in L(Q)_{-1}$.

Nice properties of the Leavitt path algebra

- The Leavitt path algebra $L(Q)$ is naturally $\mathbb{Z}$-graded as $L(Q) = \bigoplus_{n\in\mathbb{Z}} L(Q)_n$ with $e_i \in L(Q)_0$, $\alpha \in L(Q)_1$ and $\alpha^* \in L(Q)_{-1}$.
- $L(Q)_n \cdot L(Q)_m = L(Q)_{n+m}$, that is, $L(Q)$ is *strongly graded*.

Nice properties of the Leavitt path algebra

- The Leavitt path algebra $L(Q)$ is naturally $\mathbb{Z}$-graded as $L(Q) = \bigoplus_{n\in\mathbb{Z}} L(Q)_n$ with $e_i \in L(Q)_0$, $\alpha \in L(Q)_1$ and $\alpha^* \in L(Q)_{-1}$.
- $L(Q)_n \cdot L(Q)_m = L(Q)_{n+m}$, that is, $L(Q)$ is *strongly graded*.
- The zeroth component subalgebra $L(Q)_0$ is a direct limit of finite products of full matrix algebras; in particular, it is von Neumann regular.

Nice properties of the Leavitt path algebra

- The Leavitt path algebra $L(Q)$ is naturally $\mathbb{Z}$-graded as $L(Q) = \bigoplus_{n\in\mathbb{Z}} L(Q)_n$ with $e_i \in L(Q)_0$, $\alpha \in L(Q)_1$ and $\alpha^* \in L(Q)_{-1}$.
- $L(Q)_n \cdot L(Q)_m = L(Q)_{n+m}$, that is, $L(Q)$ is *strongly graded*.
- The zeroth component subalgebra $L(Q)_0$ is a direct limit of finite products of full matrix algebras; in particular, it is von Neumann regular.
- The subalgebra $\bigoplus_{i\in Q_0} e_i L(Q) e_i$ is related to *parallel paths* in Q, and also to an explicit colimit (namely, $(p,q) \mapsto q^* p \in L(Q)$; very useful to us, later!).

Some consequences

Consider the category $L(Q)\text{-grproj}$ of finitely generated $\mathbb{Z}$-graded projective $L(Q)$-modules.

Proposition

The category $L(Q)\text{-grproj}$ is a semisimple abelian category.

Some consequences

Consider the category $L(Q)\text{-grproj}$ of finitely generated $\mathbb{Z}$-graded projective $L(Q)$-modules.

Proposition

The category $L(Q)\text{-grproj}$ is a semisimple abelian category.

The proof: strongly gradation implies that

$$L(Q)\text{-grproj} \simeq L(Q)_0\text{-proj}.$$

Now, use the von Neumann regularity of $L(Q)_0$.

- We will consider the degree-shift (1) on $L(Q)\text{-grproj}$.

Some consequences

Consider the category $L(Q)\text{-grproj}$ of finitely generated $\mathbb{Z}$-graded projective $L(Q)$-modules.

Proposition

The category $L(Q)\text{-grproj}$ is a semisimple abelian category.

The proof: strongly gradation implies that

$$L(Q)\text{-grproj} \simeq L(Q)_0\text{-proj}.$$

Now, use the von Neumann regularity of $L(Q)_0$.

- We will consider the degree-shift (1) on $L(Q)\text{-grproj}$.
- $$(L(Q)e_i)(1) \simeq \bigoplus_{\{\alpha\in Q_1 \mid s(\alpha)=i\}} L(Q)e_{t(\alpha)}$$

How does A_Q relate to $L(Q)$?

Theorem (Smith 2012)

There is an equivalence (of triangulated categories)

$$\mathbf{D}_{\mathrm{sg}}(A_Q) \simeq L(Q)\text{-}\mathrm{grproj}$$

sending the simple A_Q-module S_i to $L(Q)e_i$, with Σ^{-1} corresponding to (1).

How does A_Q relate to $L(Q)$?

Theorem (Smith 2012)

There is an equivalence (of triangulated categories)

$$\mathbf{D}_{\mathrm{sg}}(A_Q) \simeq L(Q)\text{-}\mathrm{grproj}$$

sending the simple A_Q-module S_i to $L(Q)e_i$, with Σ^{-1} corresponding to (1).

The idea: the degree-shift functor (1) on $L(Q)$-grproj behaves similarly as the syzygy functor Ω on A_Q-$\underline{\mathrm{mod}}$. Now use stabilization as in [C. 2011].

Enhancing Smith's equivalence

The dg level contains more rigid information, for example, the Hochschild cohomology.

Enhancing Smith's equivalence

The dg level contains more rigid information, for example, the Hochschild cohomology. Enhancements:

$$\mathbf{D}_{\mathrm{sg}}(A_Q) \rightsquigarrow \mathbf{S}_{\mathrm{dg}}(A_Q) \text{ and } L(Q)\text{-grproj} \rightsquigarrow \mathbf{per}_{\mathrm{dg}}(L(Q)^{\mathrm{op}})$$

Enhancing Smith's equivalence

The dg level contains more rigid information, for example, the Hochschild cohomology. Enhancements:

$$\mathbf{D}_{\mathrm{sg}}(A_Q) \rightsquigarrow \mathbf{S}_{\mathrm{dg}}(A_Q) \text{ and } L(Q)\text{-grproj} \rightsquigarrow \mathbf{per}_{\mathrm{dg}}(L(Q)^{\mathrm{op}})$$

Proposition (C.-Li-Wang)

There is a zigzag of quasi-equivalences between

$$\mathbf{S}_{\mathrm{dg}}(A_Q) \simeq \mathbf{per}_{\mathrm{dg}}(L(Q)^{\mathrm{op}}).$$

Taking H^0, we recover Smith's equivalence.

Enhancing Smith's equivalence

The dg level contains more rigid information, for example, the Hochschild cohomology. Enhancements:

$$\mathbf{D}_{\mathrm{sg}}(A_Q) \rightsquigarrow \mathbf{S}_{\mathrm{dg}}(A_Q) \text{ and } L(Q)\text{-grproj} \rightsquigarrow \mathbf{per}_{\mathrm{dg}}(L(Q)^{\mathrm{op}})$$

Proposition (C.-Li-Wang)

There is a zigzag of quasi-equivalences between

$$\mathbf{S}_{\mathrm{dg}}(A_Q) \simeq \mathbf{per}_{\mathrm{dg}}(L(Q)^{\mathrm{op}}).$$

Taking H^0, we recover Smith's equivalence.

The idea: enhance a result of [Krause 2005] and use H. Li's injective Leavitt complex [Li 2018], which gives an explicit compact generator to realize a triangle equivalence in [C.-Yang 2015].

The categorical proof of Δ

Proposition

There is an isomorphism in the homotopy category of B_∞-algebras

$$C^*(L(Q), L(Q)) \xrightarrow{\Delta} C^*(\mathbf{S}_{\mathrm{dg}}(A_Q), \mathbf{S}_{\mathrm{dg}}(A_Q)).$$

The categorical proof of Δ

Proposition

There is an isomorphism in the homotopy category of B_∞-algebras

$$C^*(L(Q), L(Q)) \xrightarrow{\Delta} C^*(\mathbf{S}_{\mathrm{dg}}(A_Q), \mathbf{S}_{\mathrm{dg}}(A_Q)).$$

Recall the fact that $C^*(-,-)$ is invariant under Morita morphisms between dg categories [Keller 2013] (eg. quasi-equivalences or $L(Q) \hookrightarrow \mathbf{per}_{\mathrm{dg}}(L(Q)^{\mathrm{op}})$). Then use the above enhancement of Smith's equivalence.

Towards Υ: $C^*_{\mathrm{sg}}(A_Q, A_Q) \to C^*(L(Q), L(Q))$

We introduce two new and explicit B_∞-algebras:

Towards Υ: $C^*_{\mathrm{sg}}(A_Q, A_Q) \to C^*(L(Q), L(Q))$

We introduce two new and explicit B_∞-algebras:

(1) the *combinatorial B_∞-algebra* $C^*_{\mathrm{sg}}(Q, Q)$, via parallel paths in Q (appearing in the relative bar resolution!), and taking colimits (as in $C^*_{\mathrm{sg}}(A_Q, A_Q)$)

Towards Υ: $C^*_{\rm sg}(A_Q, A_Q) \to C^*(L(Q), L(Q))$

We introduce two new and explicit B_∞-algebras:

(1) the *combinatorial* B_∞*-algebra* $C^*_{\rm sg}(Q, Q)$, via parallel paths in Q (appearing in the relative bar resolution!), and taking colimits (as in $C^*_{\rm sg}(A_Q, A_Q)$)

(2) the *Leavitt* B_∞*-algebra* $\widehat{C}^*(L, L)$, whose algebra structure is a trivial extension of a subalgebra of $L = L(Q)$

$$\bigoplus_{i\in Q_0} e_i L e_i \oplus s^{-1} \bigoplus_{i\in Q_0} e_i L e_i$$

Towards Υ: $C^*_{\mathrm{sg}}(A_Q, A_Q) \to C^*(L(Q), L(Q))$

We introduce two new and explicit B_∞-algebras:

(1) the *combinatorial B_∞-algebra* $C^*_{\mathrm{sg}}(Q, Q)$, via parallel paths in Q (appearing in the relative bar resolution!), and taking colimits (as in $C^*_{\mathrm{sg}}(A_Q, A_Q)$)

(2) the *Leavitt B_∞-algebra* $\widehat{C}^*(L, L)$, whose algebra structure is a trivial extension of a subalgebra of $L = L(Q)$

$$\bigoplus_{i \in Q_0} e_i L e_i \oplus s^{-1} \bigoplus_{i \in Q_0} e_i L e_i$$

So, we have

$$C^*_{\mathrm{sg}}(A_Q, A_Q) \xrightarrow{\kappa} C^*_{\mathrm{sg}}(Q, Q) \xrightarrow{\rho} \widehat{C}^*(L, L)$$

strict B_∞-isomorphisms, where ρ sends a parallel path (p, q) to $q^* p \in L$!

Towards Υ: $C^*_{\mathrm{sg}}(A_Q, A_Q) \to C^*(L(Q), L(Q))$, continued

- an explicit bimodule projective resolution P of $L = L(Q)$, together with a homotopy deformation retract (in particular, L is *quasi-free* in the sense of [Cuntz-Quillen 1995]);

Towards Υ: $C^*_{\mathrm{sg}}(A_Q, A_Q) \to C^*(L(Q), L(Q))$, continued

- an explicit bimodule projective resolution P of $L = L(Q)$, together with a homotopy deformation retract (in particular, L is *quasi-free* in the sense of [Cuntz-Quillen 1995]); moreover, we have $\widehat{C}^*(L, L) = \mathrm{Hom}_{L^e}(P, L)$.

Towards Υ: $C^*_{\mathrm{sg}}(A_Q, A_Q) \to C^*(L(Q), L(Q))$, continued

- an explicit bimodule projective resolution P of $L = L(Q)$, together with a homotopy deformation retract (in particular, L is *quasi-free* in the sense of [Cuntz-Quillen 1995]); moreover, we have $\widehat{C}^*(L, L) = \mathrm{Hom}_{L^e}(P, L)$.
- the homotopy transfer theorem for dg algebras yields an A_∞-quasi-isomorphism

$$(\Phi_1, \Phi_2, \cdots)\colon \widehat{C}^*(L, L) \longrightarrow C^*(L, L)$$

Towards Υ: $C^*_{\mathrm{sg}}(A_Q, A_Q) \to C^*(L(Q), L(Q))$, continued

- an explicit bimodule projective resolution P of $L = L(Q)$, together with a homotopy deformation retract (in particular, L is *quasi-free* in the sense of [Cuntz-Quillen 1995]); moreover, we have $\widehat{C}^*(L, L) = \mathrm{Hom}_{L^e}(P, L)$.
- the homotopy transfer theorem for dg algebras yields an A_∞-quasi-isomorphism
$$(\Phi_1, \Phi_2, \cdots)\colon \widehat{C}^*(L, L) \longrightarrow C^*(L, L)$$
- each Φ_i is explicit; by manipulation on brace B_∞-algebras, we eventually verify that it is a B_∞-morphism.

The combinatorial proof of Υ

In summary, we have

Proposition

There is an isomorphism in the homotopy category of B_∞-algebras

$$C^*_{\mathrm{sg}}(A_Q, A_Q) \xrightarrow{\Upsilon} C^*(L(Q), L(Q)).$$

The combinatorial proof of Υ

In summary, we have

Proposition

There is an isomorphism in the homotopy category of B_∞-algebras

$$C^*_{\mathrm{sg}}(A_Q, A_Q) \xrightarrow{\Upsilon} C^*(L(Q), L(Q)).$$

It is given by the following composition:

$$\begin{array}{ccc} C^*_{\mathrm{sg}}(A_Q, A_Q) & \overset{\Upsilon}{\dashrightarrow} & C^*(L, L) \\ \kappa \downarrow & & \uparrow (\Phi_1, \Phi_2, \cdots) \\ C^*_{\mathrm{sg}}(Q, Q) & \xrightarrow{\rho} & \widehat{C}^*(L, L) \end{array}$$

G. Abrams, and G. Aranda Pino, *The Leavitt path algebra of a graph*, J. Algebra **293** (2) (2005), 319–334.

P. Ara, M.A. Moreno, and E. Pardo, *Nonstable K-theory for graph algebras*, Algebr. Represent. Theory **10** (2) (2007), 157–178.

R.O. Buchweitz, Maximal Cohen-Macaulay modules and Tate cohomology over Gorenstein rings, unpublished manuscript, 1987.

X.W. Chen, *The singularity category of an algebra with radical square zero*, Doc. Math. **16** (2011), 921–936.

X.W. Chen, H. Li, and Z. Wang, *Leavitt path algebras, B_∞-algebras and Keller's conjecture for singular Hochschild cohomology*, arXiv:2007.06895v1, 2020.

- X.W. Chen, and D. Yang, *Homotopy categories, Leavitt path algebras and Gorenstein projective modules*, Int. Math. Res. Not. IMRN **10** (2015), 2597–2633.
- J. Cuntz, and D. Quillen, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc. **8** (2) (1995), 251–289.
- E. Getzler, and J.D.S. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*, arXiv:hep-th/9403055, 1994.
- B. Keller, *Derived invariance of higher structures on the Hochschild complex*, available at https://webusers.imj-prg.fr/bernhard.keller/publ/index.html.

- B. Keller, *Singular Hochschild cohomology via the singularity category*, C. R. Math. Acad. Sci. Paris **356** (11-12) (2018), 1106–1111. *Corrections*, C. R. Math. Acad. Sci. Paris **357** (6) (2019), 533–536. See also arXiv:1809.05121v9, 2020.
- H. Krause, *The stable derived category of a noetherian scheme*, Compositio Math. **141** (2005), 1128–1162.
- H. Li, *The injective Leavitt complex*, Algebr. Represent. Theor. 21 (2018), 833-858.
- W. Lowen, and M. Van den Bergh, *Hochschild cohomology of abelian categories and ringed spaces*, Adv. Math. **198** (1) (2005), 172–221.
- D. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, Trudy Steklov Math. Institute **204** (2004), 240–262.

- S.P. Smith, *Equivalence of categories involving graded modules over path algebras of quivers*, Adv. Math. **230** (2012), 1780–1810.
- Z. Wang, *Gerstenhaber algebra and Deligne' s conjecture on Tate-Hochschild cohomology*, arXiv:1801.07990v1, Trans. Amer. Math. Soc., to appear, 2020.

Thank You!

http://home.ustc.edu.cn/~xwchen