Tensor products of $\mathbb{C}[K^{\pm 1}]$-free modules for $U_q(\mathfrak{sl}_2)$

Hongjia Chen

School of Mathematical Sciences
University of Science and Technology of China

第三届代数与表示论前沿进展研讨会

北京师范大学数学科学学院

2020年9月19-20日

Contents

§1. Motivation： $U(\mathfrak{h})$–free modules for Lie algebras

Lie algebras with triangular decompositions

Triangular decomposition

Let $\mathfrak{g}$ be a Lie algebra with decomposition:

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$$

where $\mathfrak{h}$ is maximal abelian and acts on $\mathfrak{g}$ semisimply.

Lie algebras with triangular decompositions

Triangular decomposition

Let $\mathfrak{g}$ be a Lie algebra with decomposition:

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$$

where $\mathfrak{h}$ is maximal abelian and acts on $\mathfrak{g}$ semisimply.

Examples

- finite-dimensional semi-simple algebras
- Kac-Moody algebras
- Virasoro algebra
- extended affine Lie algebras
- Witt algebras
- ......

Weight modules and $U(\mathfrak{h})$–free modules

Weight modules

$\mathfrak{h}$ acts on the modules semisimply.

Weight modules and $U(\mathfrak{h})$–free modules

Weight modules

$\mathfrak{h}$ acts on the modules semisimply.

$U(\mathfrak{h})$–free modules

$\mathfrak{h}$ acts on the modules freely

Weight modules and $U(\mathfrak{h})$–free modules

Weight modules

$\mathfrak{h}$ acts on the modules semisimply.

$U(\mathfrak{h})$–free modules

$\mathfrak{h}$ acts on the modules freely–an "opposite conditions" with weight modules.

Rank of the $U(\mathfrak{h})$–free module

Rank 1

A $\mathfrak{g}$-module M is a $U(\mathfrak{h})$–free module of rank 1 if

$$\mathrm{Res}^{U(\mathfrak{g})}_{U(\mathfrak{h})} M \cong {}_{U(\mathfrak{h})}U(\mathfrak{h}) = S(\mathfrak{h}).$$

History of $U(\mathfrak{h})$–free modules

The notation of $U(\mathfrak{h})$-free module was first introduced by J. Nilsson

Simple $\mathfrak{sl}_{n+1}$-module structures on $U(\mathfrak{h})$. *J. Algebra*, 424 (2015), 294–329. arXiv:1312.5499

History of $U(\mathfrak{h})$–free modules

The notation of $U(\mathfrak{h})$-free module was first introduced by J. Nilsson

Simple $\mathfrak{sl}_{n+1}$-module structures on $U(\mathfrak{h})$. *J. Algebra*, 424 (2015), 294–329. arXiv:1312.5499

Remark

- Tan-Zhao constructed a large class of $U(\mathfrak{h})$-free $\mathfrak{sl}_{n+1}$-modules from the $U(\mathfrak{h})$-free modules over Witt algebras W_n, without giving a classification.

  Irreducible modules over Witt algebras W_n and over $\mathfrak{sl}_{n+1}(\mathbb{C})$. *Algebra Represent. Theory*, 21 (2018), 787–806. arXiv:1312.5539.

History of $U(\mathfrak{h})$–free modules

The notation of $U(\mathfrak{h})$-free module was first introduced by J. Nilsson

Simple $\mathfrak{sl}_{n+1}$-module structures on $U(\mathfrak{h})$. *J. Algebra*, 424 (2015), 294–329. arXiv:1312.5499

Remark

- Tan-Zhao constructed a large class of $U(\mathfrak{h})$-free $\mathfrak{sl}_{n+1}$-modules from the $U(\mathfrak{h})$-free modules over Witt algebras W_n, without giving a classification.

Irreducible modules over Witt algebras W_n and over $\mathfrak{sl}_{n+1}(\mathbb{C})$. *Algebra Represent. Theory*, 21 (2018), 787–806. arXiv:1312.5539.

- The above mentioned $U(\mathfrak{h})$-free modules over W_n were first constructed from Weyl modules by Lv-Zhao.

Irreducible Virasoro modules from irreducible Weyl modules. *J. Algebra*, 414 (2014) 271–287.

History of $U(\mathfrak{h})$–free modules

The notation of $U(\mathfrak{h})$-free module was first introduced by J. Nilsson

Simple $\mathfrak{sl}_{n+1}$-module structures on $U(\mathfrak{h})$. *J. Algebra*, 424 (2015), 294–329. arXiv:1312.5499

Remark

- Tan-Zhao constructed a large class of $U(\mathfrak{h})$-free $\mathfrak{sl}_{n+1}$-modules from the $U(\mathfrak{h})$-free modules over Witt algebras W_n, without giving a classification.

Irreducible modules over Witt algebras W_n and over $\mathfrak{sl}_{n+1}(\mathbb{C})$. *Algebra Represent. Theory*, 21 (2018), 787–806. arXiv:1312.5539.

- The above mentioned $U(\mathfrak{h})$-free modules over W_n were first constructed from Weyl modules by Lv-Zhao.

Irreducible Virasoro modules from irreducible Weyl modules. *J. Algebra*, 414 (2014) 271–287.

- For the special case W_1, the $U(\mathfrak{h})$-free modules first given by Guo-Lv-Zhao.

Fraction representations and highest-weight-like representations of the Virasoro algebra. *J. Algebra*, 387 (2013) 68–86.

$U(\mathfrak{h})$-free module

- Nilsson showed that $\mathfrak{g}$ has nontrivial $U(\mathfrak{h})$-free module iff $\mathfrak{g}$ is of type A or C, and also gave a classification of $U(\mathfrak{h})$-free modules of rank 1.

$U(\mathfrak{h})$-free modules and coherent families. *J. Pure Appl. Algebra* , 220 (2016) 1475–1488.

$U(\mathfrak{h})$-free module

- Nilsson showed that $\mathfrak{g}$ has nontrivial $U(\mathfrak{h})$-free module iff $\mathfrak{g}$ is of type A or C, and also gave a classification of $U(\mathfrak{h})$-free modules of rank 1.

  $U(\mathfrak{h})$-free modules and coherent families. *J. Pure Appl. Algebra* , 220 (2016) 1475–1488.

- Martin-Prieto constructed a family of $U(\mathfrak{h})$-free $\mathfrak{sl}_2$ modules of arbitrary rank.

  Construction of simple non-weight $\mathfrak{sl}(2)$-modules of arbitrary rank. *J. Algebra*, 472 (2017) 172–194.

$U(\mathfrak{h})$–free modules over finite dimensional simple Lie algebras

$U(\mathfrak{h})$-free module

- Nilsson showed that $\mathfrak{g}$ has nontrivial $U(\mathfrak{h})$-free module iff $\mathfrak{g}$ is of type A or C, and also gave a classification of $U(\mathfrak{h})$-free modules of rank 1.

$U(\mathfrak{h})$-free modules and coherent families. *J. Pure Appl. Algebra* , 220 (2016) 1475–1488.

- Martin-Prieto constructed a family of $U(\mathfrak{h})$-free $\mathfrak{sl}_2$ modules of arbitrary rank.

Construction of simple non-weight $\mathfrak{sl}(2)$-modules of arbitrary rank. *J. Algebra*, 472 (2017) 172–194.

- Cai-He-Lv constructed a family of $U(\mathfrak{h})$-free $\mathfrak{sl}_n$ modules of arbitrary rank.

A class of new simple modules for $\mathfrak{sl}_{n+1}$ and the Witt algebra. *J. Algebra*, 541 (2020) 415–435.

$U(\mathfrak{h})$–free modules over finite dimensional simple Lie algebras

$U(\mathfrak{h})$-free module

- Nilsson showed that $\mathfrak{g}$ has nontrivial $U(\mathfrak{h})$-free module iff $\mathfrak{g}$ is of type A or C, and also gave a classification of $U(\mathfrak{h})$-free modules of rank 1.

  $U(\mathfrak{h})$-free modules and coherent families. *J. Pure Appl. Algebra* , 220 (2016) 1475–1488.

- Martin-Prieto constructed a family of $U(\mathfrak{h})$-free $\mathfrak{sl}_2$ modules of arbitrary rank.

  Construction of simple non-weight $\mathfrak{sl}(2)$-modules of arbitrary rank. *J. Algebra*, 472 (2017) 172–194.

- Cai-He-Lv constructed a family of $U(\mathfrak{h})$-free $\mathfrak{sl}_n$ modules of arbitrary rank.

  A class of new simple modules for $\mathfrak{sl}_{n+1}$ and the Witt algebra. *J. Algebra*, 541 (2020) 415–435.

Generalization

- Nilsson classified the $\mathfrak{sl}(V)$-modules whose restriction to $U(\mathfrak{n})$ is free of rank 1, where $\mathfrak{n}$ is a nilradical of some parabolic subalgebra with maximal dimension.

  Simple $\mathfrak{sl}(V)$-modules which are free over an abelian subalgebra. arXiv:1903.09431.

$U(\mathfrak{h})$–free modules over finite dimensional simple Lie algebras

Applications

- Cai-Liu-Nilsson-Zhao constructed a class of generalized Verma modules over $\mathfrak{sl}_{n+1}$ from rank 1 $U(\mathfrak{h})$-free $\mathfrak{sl}_n$-modules.

Generalized Verma modules over $\mathfrak{sl}_{n+2}$ induced from $U(\mathfrak{h}_n)$-free $\mathfrak{sl}_{n+1}$-modules. *J. Algebra*, 502 (2018) 146–162.

$U(\mathfrak{h})$–free modules over finite dimensional simple Lie algebras

Applications

- Cai-Liu-Nilsson-Zhao constructed a class of generalized Verma modules over $\mathfrak{sl}_{n+1}$ from rank 1 $U(\mathfrak{h})$-free $\mathfrak{sl}_n$-modules.

  Generalized Verma modules over $\mathfrak{sl}_{n+2}$ induced from $U(\mathfrak{h}_n)$-free $\mathfrak{sl}_{n+1}$-modules. *J. Algebra*, 502 (2018) 146–162.

- C-Wan classified the rank one free modules over the Euclidean Lie algebra $\mathfrak{e}_3$.

  Simple Lie algebra $\mathfrak{sl}_2$ and a class of modules over the Euclidean Lie algebra $\mathfrak{e}_3$.

$U(\mathfrak{h})$–free modules over Kac-Moody algebras

$U(\mathfrak{h})$-free module

- Cai-Tan-Zhao classified $U(\mathfrak{h})$-free modules of rank-1 for all Kac-Moody Lie algebras.

  Module structure on $U(\mathfrak{h})$ for Kac-Moody algebras. *Sci. China Math.*, 47 (2017) 1491–1514.

$U(\mathfrak{h})$–free modules over Kac-Moody algebras

$U(\mathfrak{h})$-free module

- Cai-Tan-Zhao classified $U(\mathfrak{h})$-free modules of rank-1 for all Kac-Moody Lie algebras.

  Module structure on $U(\mathfrak{h})$ for Kac-Moody algebras. *Sci. China Math.*, 47 (2017) 1491–1514.

- Cai-Tan-Zhao classified $U(\tilde{\mathfrak{h}})$-free modules of rank-1 for all affine Kac-Moody Lie algebras without containing the center.

  New representations of affine Kac-Moody algebras. *J. Algebra*, 547 (2020), Pages 95–115.

$U(\mathfrak{h})$–free modules over Kac-Moody algebras

$U(\mathfrak{h})$-free module

- Cai-Tan-Zhao classified $U(\mathfrak{h})$-free modules of rank-1 for all Kac-Moody Lie algebras.

  Module structure on $U(\mathfrak{h})$ for Kac-Moody algebras. *Sci. China Math.*, 47 (2017) 1491–1514.

- Cai-Tan-Zhao classified $U(\tilde{\mathfrak{h}})$-free modules of rank-1 for all affine Kac-Moody Lie algebras without containing the center.

  New representations of affine Kac-Moody algebras. *J. Algebra*, 547 (2020), Pages 95–115.

  In fact, such modules exist only for affine Lie algebras of type $A_n^{(1)}, n \geq 1$ and $C_n^{(1)}, n \geq 2$.

$U(\mathfrak{h})$–free modules over Kac-Moody algebras

$U(\mathfrak{h})$-free module

- Cai-Tan-Zhao classified $U(\mathfrak{h})$-free modules of rank-1 for all Kac-Moody Lie algebras.

  Module structure on $U(\mathfrak{h})$ for Kac-Moody algebras. *Sci. China Math.*, 47 (2017) 1491–1514.

- Cai-Tan-Zhao classified $U(\tilde{\mathfrak{h}})$-free modules of rank-1 for all affine Kac-Moody Lie algebras without containing the center.

  New representations of affine Kac-Moody algebras. *J. Algebra*, 547 (2020), Pages 95–115.

  In fact, such modules exist only for affine Lie algebras of type $A_n^{(1)}, n \geq 1$ and $C_n^{(1)}, n \geq 2$.

Generalization or applications

The Virasoro algebra

The Virasoro algebra is the universal central extension of the Witt algebra W_1, with a basis $\{L_n, C \mid n \in \mathbb{Z}\}$ and defining relations:

$$[L_m, L_n] = (n-m)L_{n+m} + \delta_{n,-m}\frac{m^3-m}{12}C, \quad [C, L_m] = 0$$

$U(\mathfrak{h})$–free modules over the Virasoro algebra

The Virasoro algebra

The Virasoro algebra is the universal central extension of the Witt algebra W_1, with a basis $\{L_n, C \mid n \in \mathbb{Z}\}$ and defining relations:

$$[L_m, L_n] = (n-m)L_{n+m} + \delta_{n,-m}\frac{m^3-m}{12}C, \quad [C, L_m] = 0$$

The module $\Omega(\lambda, \alpha)$

$$C.f(t) = 0, \quad L_m.f(t) = \lambda^m(t - m\alpha)f(t-m).$$

- First constructed by Guo-Lv-Zhao.

Fraction representations and highest-weight-like representations of the Virasoro algebra. *J. Algebra*, 387 (2013) 68–86.

Irreducible Virasoro modules from irreducible Weyl modules. *J. Algebra*, 414 (2014) 271–287.

$U(\mathfrak{h})$–free modules over the Virasoro algebra

The Virasoro algebra

The Virasoro algebra is the universal central extension of the Witt algebra W_1, with a basis $\{L_n, C \mid n \in \mathbb{Z}\}$ and defining relations:

$$[L_m, L_n] = (n-m)L_{n+m} + \delta_{n,-m}\frac{m^3-m}{12}C, \quad [C, L_m] = 0$$

The module $\Omega(\lambda, \alpha)$

$$C.f(t) = 0, \quad L_m.f(t) = \lambda^m(t - m\alpha)f(t-m).$$

- First constructed by Guo-Lv-Zhao.
  - Fraction representations and highest-weight-like representations of the Virasoro algebra. *J. Algebra*, 387 (2013) 68–86.
  - Irreducible Virasoro modules from irreducible Weyl modules. *J. Algebra*, 414 (2014) 271–287.
- Classified by Tan-Zhao.
  - $\mathcal{W}_n^+$ and $\mathcal{W}_n$-module structures on $U(\mathfrak{h})$. *J. Algebra*, 424 (2015), 357–375.

$U(\mathfrak{h})$–free modules over the Virasoro algebra

Infinite rank

- C-Guo constructed a class of $U(\mathfrak{h})$-free Vir-modules of infinite-rank.

A new family of modules over the Virasoro algebra. *J. Algebra*, 457 (2016), 73–105.

$U(\mathfrak{h})$–free modules over the Virasoro algebra

Infinite rank

- C-Guo constructed a class of $U(\mathfrak{h})$-free Vir-modules of infinite-rank.

A new family of modules over the Virasoro algebra. *J. Algebra*, 457 (2016), 73–105.

Applications

- Tensor products of $U(\mathfrak{h})$-free modules and modules with locally-finite action of positive part were studied by Tan-Zhao and Guo-Wang-Liu.

Irreducible Virasoro modules from tensor products. *Ark. Mat.*, 54 (2016), 181–200.

Irreducible Virasoro modules from tensor products (II). *J. Algebra*, 394 (2013), 357–373.

New irreducible tensor product modules for the Virasoro algebra. *Asian J. Math.*, accepted.

A new class of irreducible Virasoro modules from tensor product. *J. Algebra*, 541 (2020), 324–344.

$U(\mathfrak{h})$–free modules over other Lie algebras

H. Chen, X. Guo, Non-weight modules over the Heisenberg-Virasoro algebra and the W algebra $W(2,2)$. *J. Algebra Appl.*, 16 (2017), 1750097.

Q. Chen, Y. Cai, Modules over algebras related to the Virasoro algebra. *Int. J. Math.*, 26 (2015), 1550070.

J. Han, Q. Chen, Y. Su, Modules over the algebra Vir(a,b). *Linear Algebra Appl.* 515 (2017), 11–23.

Q. Chen, Y. Yao, Non-weight modules over algebras related to the Virasoro algebra *J. Geom. Phys.*, 134 (2018), 11–18.

Q. Chen, J. Han, Non-weight modules over the affine-Virasoro algebra of type A_1. *J. Math. Phys.*, 60 (2019), 071707.

J. Zhang, Non-weight representations of Cartan type S Lie algebras. *Comm. Algebra*, 46 (2018), 4243-4264.

J. Zhang, Non-weight representations of Cartan type H Lie algebras. *Comm. Algebra*, 47 (2019), 67-87.

♠ Liu-Guo: $U(\mathcal{B}(q))$, Wang-Zhang: Schrödinger-Virasoro algebras, Chen-Dai-Liu-Su: super-BMS$_3$ algebra, Yang-Yao-Xia: untwisted $N=2$ superconformal algebras ......

$U(\mathfrak{h})$–free modules over other algebras

- Y. Cai, K. Zhao, Module structure on $U(\mathfrak{h})$ for basic Lie superalgebras. *Toyama Math. J.*, 37 (2015), 55–72.
- H. Yang, Y. Yao, L. Xia, A family of non-weight modules over the super-Virasoro algebras. *J. Algebra*, 547 (2020), 538–555.
- Y. Cai, H. Chen, X. Guo, Y. Ma, M. Zhu, Tensor product modules of $U_q(\mathfrak{sl}_2)$ and Clebsch-Gordan type formulas. (**The rest part of this talk**.)
- Y. Mao, Research on non-weight modules over quantum $U_q(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_3)$. *Postdoctoral Fellowship Final Report*, USTC (2017), 51 pp.
- H. Chen, L. Wang, Simple $U_q(\mathfrak{so}_5)$-module structures on $\mathbb{C}[K_1^{\pm 1}, K_2^{\pm 1}]$. Preprint.
- H. Chen, L. Wang, A family of modules over a class of algebras similar to the enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$ (the Rueda's algebras).

………

§2. Classification of $\mathbb{C}[K^{\pm 1}]$-free $U_q(\mathfrak{sl}_2)$-modules.

Definition

The quantum group $U_q(\mathfrak{sl}_2)$ is the unital associative algebra over $\mathbb{C}$ with generators E, F, K and K^{-1} subject to the relations (see [Jantzen]):

(R1). $KK^{-1} = K^{-1}K = 1,$

(R2). $KEK^{-1} = q^2E,$

(R3). $KFK^{-1} = q^{-2}F,$

(R4). $[E, F] = EF - FE = \dfrac{K - K^{-1}}{q - q^{-1}}.$

The triangular decomposition

U admits a natural triangular decomposition:

$$U = U^+ U_0 U^-$$

where $U^+ = \mathbb{C}[E]$, $U_0 = \mathbb{C}[K^{\pm 1}]$ and $U^- = \mathbb{C}[F]$.

The triangular decomposition

U admits a natural triangular decomposition:

$$U = U^+ U_0 U^-$$

where $U^+ = \mathbb{C}[E]$, $U_0 = \mathbb{C}[K^{\pm 1}]$ and $U^- = \mathbb{C}[F]$.

Weight modules

U_0 acts diagonalized.

The triangular decomposition and subcategory $\mathcal{H}$

The triangular decomposition

U admits a natural triangular decomposition:

$$U = U^+ U_0 U^-$$

where $U^+ = \mathbb{C}[E]$, $U_0 = \mathbb{C}[K^{\pm 1}]$ and $U^- = \mathbb{C}[F]$.

Weight modules

U_0 acts diagonalized.

Subcategory $\mathcal{H}$

Let $\mathcal{H}$ be the full subcategory of U-Mod whose objects are free of rank 1 when restricted to $U_0 = \mathbb{C}[K^{\pm 1}]$, i.e.,

$$\mathcal{H} = \left\{ M \in U\text{-Mod} \mid \mathrm{Res}^U_{U_0} M \cong_{U_0} U_0 \right\}.$$

Module: M_a

For any $a \in \mathbb{C}$, define the $U_q(\mathfrak{sl}_2)$-module $M_a = \mathbb{C}[K^{\pm 1}]$ as follows:

$$\begin{aligned} K^{\pm 1} \cdot g(K) =& K^{\pm 1} g(K), \\ E \cdot g(K) =& g(q^{-2}K), \\ F \cdot g(K) =& -g(q^2K)\frac{q^2K^2 + 2aqK + 1}{qK(q-q^{-1})^2}. \end{aligned}$$

Construction of U_0-free modules for $U_q(\mathfrak{sl}_2)$

Module: M_a

For any $a \in \mathbb{C}$, define the $U_q(\mathfrak{sl}_2)$-module $M_a = \mathbb{C}[K^{\pm 1}]$ as follows:

$$\begin{aligned} K^{\pm 1} \cdot g(K) &= K^{\pm 1} g(K), \\ E \cdot g(K) &= g(q^{-2}K), \\ F \cdot g(K) &= -g(q^2K)\frac{q^2K^2 + 2aqK + 1}{qK(q-q^{-1})^2}. \end{aligned}$$

Module: M'_b

Similarly, for $b \neq 0$, define the $U_q(\mathfrak{sl}_2)$-module $M'_b = \mathbb{C}[K^{\pm 1}]$ as follows:

$$\begin{aligned} K^{\pm 1} \cdot g(K) &= K^{\pm 1} g(K), \\ E \cdot g(K) &= g(q^{-2}K)q^{-1}(K + qb), \\ F \cdot g(K) &= -g(q^2K)\frac{K + q^{-1}b^{-1}}{K(q-q^{-1})^2}. \end{aligned}$$

Irreducibility of the modules M_a and M'_b

Proposition

(1). The modules M_a are irreducible over $U_q(\mathfrak{sl}_2)$ for all $a \in \mathbb{C}$.

Irreducibility of the modules M_a and M'_b

Proposition

(1). The modules M_a are irreducible over $U_q(\mathfrak{sl}_2)$ for all $a \in \mathbb{C}$.

(2). The modules M'_b are reducible over $U_q(\mathfrak{sl}_2)$ if and only if $b = \pm q^{-m}$ for some positive integer m.

Irreducibility of the modules M_a and M'_b

Proposition

(1). The modules M_a are irreducible over $U_q(\mathfrak{sl}_2)$ for all $a \in \mathbb{C}$.

(2). The modules M'_b are reducible over $U_q(\mathfrak{sl}_2)$ if and only if $b = \pm q^{-m}$ for some positive integer m.

(3). For any $m \in \mathbb{N}$ and $\varepsilon \in \{\pm 1\}$, the module $M'_{\varepsilon q^{-m}}$ has a unique nonzero proper submodule

$$W_{m,\varepsilon} = \prod_{i=0}^{m-1} (K + \varepsilon q^{1-m+2i}) \mathbb{C}[K^{\pm 1}],$$

and the corresponding quotient module $M'_{\varepsilon q^{-m}}/W_{m,\varepsilon}$ is isomorphic to $L(m-1, -\varepsilon)$, the m-dimensional $U_q(\mathfrak{sl}_2)$-module.

Isomorphism classes

Proposition

Suppose $a_1, a_2, b_1, b_2 \in \mathbb{C}, b_1, b_2 \neq 0$.

(1). The modules M_{a_1} and M_{a_2} are isomorphic if and only if $a_1 = a_2$.

Isomorphism classes

Proposition

Suppose $a_1, a_2, b_1, b_2 \in \mathbb{C}, b_1, b_2 \neq 0$.

(1). The modules M_{a_1} and M_{a_2} are isomorphic if and only if $a_1 = a_2$.

(2). The modules M'_{b_1} and M'_{b_2} are isomorphic if and only if $b_1 = b_2$.

Isomorphism classes

Proposition

Suppose $a_1, a_2, b_1, b_2 \in \mathbb{C}, b_1, b_2 \neq 0$.

(1). The modules M_{a_1} and M_{a_2} are isomorphic if and only if $a_1 = a_2$.

(2). The modules M'_{b_1} and M'_{b_2} are isomorphic if and only if $b_1 = b_2$.

(3). The modules M_a and M'_b are not isomorphic for any $a, b \in \mathbb{C}, b \neq 0$.

Twisting with automorphisms

Twisted module: M^σ

For any $U_q(\mathfrak{sl}_2)$-module M and automorphism σ of $U_q(\mathfrak{sl}_2)$, we can define a new $U_q(\mathfrak{sl}_2)$-module structure on M as follows:

$$x \circ v = \sigma(x) \cdot v, \ \forall\, x \in U_q(\mathfrak{sl}_2), v \in M,$$

where $\circ$ means the new action and $\cdot$ means the old. Denote the $U_q(\mathfrak{sl}_2)$-module by M^σ.

Twisting with automorphisms

Twisted module: M^σ

For any $U_q(\mathfrak{sl}_2)$-module M and automorphism σ of $U_q(\mathfrak{sl}_2)$, we can define a new $U_q(\mathfrak{sl}_2)$-module structure on M as follows:

$$x \circ v = \sigma(x) \cdot v, \ \forall\, x \in U_q(\mathfrak{sl}_2), v \in M,$$

where $\circ$ means the new action and $\cdot$ means the old. Denote the $U_q(\mathfrak{sl}_2)$-module by M^σ.

Involution ω

Let ω be the automorphism of $U_q(\mathfrak{sl}_2)$ given by

$$\omega(K) = K^{-1}, \quad \omega(E) = F, \quad \omega(F) = E.$$

Twisting with automorphisms

Twisted module: M^σ

For any $U_q(\mathfrak{sl}_2)$-module M and automorphism σ of $U_q(\mathfrak{sl}_2)$, we can define a new $U_q(\mathfrak{sl}_2)$-module structure on M as follows:

$$x \circ v = \sigma(x) \cdot v, \ \forall\ x \in U_q(\mathfrak{sl}_2), v \in M,$$

where $\circ$ means the new action and $\cdot$ means the old. Denote the $U_q(\mathfrak{sl}_2)$-module by M^σ.

Involution ω

Let ω be the automorphism of $U_q(\mathfrak{sl}_2)$ given by

$$\omega(K) = K^{-1}, \quad \omega(E) = F, \quad \omega(F) = E.$$

Automorphism $\sigma_{\xi,n}$

For $0 \neq \xi \in \mathbb{C}$ and $n \in \mathbb{Z}$, we have another class of automorphisms $\sigma_{\xi,n}$:

$$\sigma_{\xi,n}(K) = K, \quad \sigma_{\xi,n}(E) = \xi K^n E, \quad \sigma_{\xi,n}(F) = \xi^{-1} F K^{-n}.$$

Characterization of rank-1 $\mathbb{C}[K^{\pm 1}]$-free $U_q(\mathfrak{sl}_2)$-modules

Theorem

Suppose that M is a $U_q(\mathfrak{sl}_2)$-module which is $\mathbb{C}[K^{\pm 1}]$-free of rank 1. Then M is isomorphic to one of the modules: $(M_a)^{\sigma_{\xi,n}}$, $(M_a)^{\omega\sigma_{\xi,n}}$ and $(M'_b)^{\sigma_{\xi,n}}$ for suitable $a, b, \xi \in \mathbb{C}, b, \xi \neq 0$ and $n \in \mathbb{Z}$.

Characterization of rank-1 $\mathbb{C}[K^{\pm 1}]$-free $U_q(\mathfrak{sl}_2)$-modules

Theorem

Suppose that M is a $U_q(\mathfrak{sl}_2)$-module which is $\mathbb{C}[K^{\pm 1}]$-free of rank 1. Then M is isomorphic to one of the modules: $(M_a)^{\sigma_{\xi,n}}$, $(M_a)^{\omega\sigma_{\xi,n}}$ and $(M'_b)^{\sigma_{\xi,n}}$ for suitable $a, b, \xi \in \mathbb{C}, b, \xi \neq 0$ and $n \in \mathbb{Z}$.

Corollary

Any irreducible $U_q(\mathfrak{sl}_2)$-modules which are $\mathbb{C}[K^{\pm 1}]$-free of rank 1 is isomorphic to one of the modules: $(M_a)^{\sigma_{\xi,n}}$, $(M_a)^{\omega\sigma_{\xi,n}}$ and $(M'_b)^{\sigma_{\xi,n}}$ for suitable $n \in \mathbb{Z}$ and $a, b, \xi \in \mathbb{C}, \xi, b \neq 0, b \neq \pm q^{-m}$ for any positive integer m.

§3. Tensor product with finite-dimensional modules.

Hopf algebra structrue of $U_q(\mathfrak{sl}_2)$

Hopf algebra $U_q(\mathfrak{sl}_2)$

The algebra $U_q(\mathfrak{sl}_2)$ is a Hopf algebra via the comultiplication Δ: $U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$, which is a homomorphism of $\mathbb{C}$-algebras defined by

$$\Delta(K) = K \otimes K,$$
$$\Delta(E) = E \otimes 1 + K \otimes E,$$
$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F.$$

Hopf algebra structrue of $U_q(\mathfrak{sl}_2)$

Hopf algebra $U_q(\mathfrak{sl}_2)$

The algebra $U_q(\mathfrak{sl}_2)$ is a Hopf algebra via the comultiplication Δ: $U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$, which is a homomorphism of $\mathbb{C}$-algebras defined by

$$\Delta(K) = K \otimes K,$$
$$\Delta(E) = E \otimes 1 + K \otimes E,$$
$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F.$$

Tensor product of modules

If M and N are $U_q(\mathfrak{sl}_2)$-modules, then we can make $M \otimes N$ into a $U_q(\mathfrak{sl}_2)$-module via Δ:

$$x(v \otimes w) = \Delta(x)(v \otimes w), \ \forall\ x \in U_q(\mathfrak{sl}_2), v \in M, w \in N.$$

Finite-dimensional modules over $U_q(\mathfrak{sl}_2)$

Module: $L(n,\varepsilon)$

For any $n \in \mathbb{Z}_+$ and $\varepsilon \in \{\pm\}$, as a vector space, $L(n,\varepsilon)$ has a basis $\{v_0, v_1, ..., v_n\}$ and the module action is given by

$$\begin{aligned} Kv_i =& \varepsilon q^{n-2i} v_i, \\ Fv_i =& v_{i+1}, \ Fv_n = 0, \\ Ev_i =& \varepsilon[i][n+1-i]v_{i-1}. \end{aligned}$$

where $[i] = \frac{q^i - q^{-i}}{q - q^{-1}}$ for any positive integer i and $[0] = 0$.

Finite-dimensional modules over $U_q(\mathfrak{sl}_2)$

Module: $L(n,\varepsilon)$

For any $n \in \mathbb{Z}_+$ and $\varepsilon \in \{\pm\}$, as a vector space, $L(n,\varepsilon)$ has a basis $\{v_0, v_1, ..., v_n\}$ and the module action is given by

$$\begin{aligned} Kv_i =& \varepsilon q^{n-2i} v_i, \\ Fv_i =& v_{i+1}, \ Fv_n = 0, \\ Ev_i =& \varepsilon [i][n+1-i] v_{i-1}. \end{aligned}$$

where $[i] = \frac{q^i - q^{-i}}{q - q^{-1}}$ for any positive integer i and $[0] = 0$.

Theorem (Jantzen)

The modules $L(n,+)$ and $L(n,-)$ are exactly the simple $U_q(\mathfrak{sl}_2)$-modules of dimension $n+1$ up to isomorphisms.

Structure of $M_a \otimes L(1,\varepsilon)$

(1). If $a \neq \pm 1$, then we have

$$M_a \otimes L(1,\varepsilon) \cong M_{\varepsilon a q + \frac{\varepsilon}{2}(a+\sqrt{a^2-1})(q^{-1}-q)} \oplus M_{\varepsilon a q + \frac{\varepsilon}{2}(a-\sqrt{a^2-1})(q^{-1}-q)}.$$

Structure of $M_a \otimes L(1,\varepsilon)$

Structure of $M_a \otimes L(1,\varepsilon)$

(1). If $a \neq \pm 1$, then we have

$$M_a \otimes L(1,\varepsilon) \cong M_{\varepsilon a q + \frac{\varepsilon}{2}(a+\sqrt{a^2-1})(q^{-1}-q)} \oplus M_{\varepsilon a q + \frac{\varepsilon}{2}(a-\sqrt{a^2-1})(q^{-1}-q)}.$$

(2). If $a = \pm 1$, then $M_a \otimes L(1,\varepsilon)$ has a unique submodule isomorphic to $M_{\frac{\varepsilon}{2}a(q+q^{-1})}$.

Structure of $M_a \otimes L(1,\varepsilon)$

(1). If $a \neq \pm 1$, then we have

$$M_a \otimes L(1,\varepsilon) \cong M_{\varepsilon a q + \frac{\varepsilon}{2}(a+\sqrt{a^2-1})(q^{-1}-q)} \oplus M_{\varepsilon a q + \frac{\varepsilon}{2}(a-\sqrt{a^2-1})(q^{-1}-q)}.$$

(2). If $a = \pm 1$, then $M_a \otimes L(1,\varepsilon)$ has a unique submodule isomorphic to $M_{\frac{\varepsilon}{2}a(q+q^{-1})}$.

(3). For $a = \pm 1$, we have the following nonsplit exact sequence:

$$0 \to M_{\frac{\varepsilon a}{2}(q+q^{-1})} \to M_a \otimes L(1,\varepsilon) \to M_{\frac{\varepsilon a}{2}(q+q^{-1})} \to 0.$$

Decomposition of $M_a \otimes L(n, \varepsilon)$

Theorem

Let n be a positive integer and $a \in \mathbb{C}, a \neq \pm\frac{1}{2}(q^i + q^{-i}), i = 0, 1, \ldots, n-1$, then we have

$$M_a \otimes L(n, \varepsilon) \cong \bigoplus_{i=0}^{n} M_{\varepsilon a_{n-2i}},$$

where

$$a_i = \frac{1}{2}(q^{-i} + q^i)a + \frac{1}{2}(q^{-i} - q^i)\sqrt{a^2 - 1}.$$

Decomposition of $M_a \otimes L(n, \varepsilon)$

Theorem

Let n be a positive integer and $a \in \mathbb{C}, a \neq \pm\frac{1}{2}(q^i + q^{-i}), i = 0, 1, \ldots, n-1$, then we have

$$M_a \otimes L(n, \varepsilon) \cong \bigoplus_{i=0}^{n} M_{\varepsilon a_{n-2i}},$$

where

$$a_i = \frac{1}{2}(q^{-i} + q^i)a + \frac{1}{2}(q^{-i} - q^i)\sqrt{a^2 - 1}.$$

Clebsch-Gordan formula

For any nonnegative integers $n \leq m$ and $\varepsilon, \varepsilon' \in \{\pm\}$, we have

$$L(m, \varepsilon) \otimes L(n, \varepsilon') = \bigoplus_{i=0}^{n} L(m + n - 2i, \varepsilon\varepsilon').$$

Submodules

Denote

$$N_b' = \left\{ \left(\frac{\varepsilon}{q-q^{-1}} g(q^2K),\ g(K) \right) \;\middle|\; g(K) \in \mathbb{C}[K^{\pm 1}] \right\},$$

$$N_b'' = \left\{ \left(\frac{\varepsilon(qK+b^{-1})}{q-q^{-1}} g(q^2K),\ (q^{-1}K+b)g(K) \right) \;\middle|\; g(K) \in \mathbb{C}[K^{\pm 1}] \right\}.$$

Submodules of $M_b' \otimes L(1,\varepsilon)$

Submodules

Denote

$$N_b' = \left\{ \left(\frac{\varepsilon}{q-q^{-1}} g(q^2K),\ g(K) \right) \,\middle|\, g(K) \in \mathbb{C}[K^{\pm 1}] \right\},$$

$$N_b'' = \left\{ \left(\frac{\varepsilon(qK+b^{-1})}{q-q^{-1}} g(q^2K),\ (q^{-1}K+b)g(K) \right) \,\middle|\, g(K) \in \mathbb{C}[K^{\pm 1}] \right\}.$$

Proposition

Let b be any nonzero complex number.

- Both N_b' and N_b'' are submodule of $M_b' \otimes L(1,\varepsilon)$.

Submodules of $M'_b \otimes L(1,\varepsilon)$

Submodules

Denote

$$N'_b = \left\{ \left(\frac{\varepsilon}{q-q^{-1}} g(q^2K),\ g(K) \right) \,\middle|\, g(K) \in \mathbb{C}[K^{\pm 1}] \right\},$$

$$N''_b = \left\{ \left(\frac{\varepsilon(qK+b^{-1})}{q-q^{-1}} g(q^2K),\ (q^{-1}K+b)g(K) \right) \,\middle|\, g(K) \in \mathbb{C}[K^{\pm 1}] \right\}.$$

Proposition

Let b be any nonzero complex number.

- Both N'_b and N''_b are submodule of $M'_b \otimes L(1,\varepsilon)$.
- $N'_b \cong (M'_{\varepsilon q^{-1} b})^{\sigma_{\varepsilon q,0}}$ and $N''_b \cong (M'_{\varepsilon q b})^{\sigma_{\varepsilon q^{-1},0}}$.

Structure of $M_b' \otimes L(1, \varepsilon)$

Proposition

- If $b \neq \varepsilon q^{-m}$ for any positive integer m, then N_b' and N_b'' are precisely the two nonzero proper submodules of $M_b' \otimes L(1, \varepsilon)$.

Structure of $M'_b \otimes L(1,\varepsilon)$

Proposition

- If $b \neq \varepsilon q^{-m}$ for any positive integer m, then N'_b and N''_b are precisely the two nonzero proper submodules of $M'_b \otimes L(1,\varepsilon)$.
- If $b \neq \pm 1$, we have $M'_b \otimes L(1,\varepsilon) = N'_b \oplus N''_b$ and in particular,

$$M'_b \otimes L(1,\varepsilon) \cong (M'_{\varepsilon q^{-1} b})^{\sigma_{\varepsilon q,0}} \oplus (M'_{\varepsilon q b})^{\sigma_{\varepsilon q^{-1},0}}.$$

Proposition

- If $b \neq \varepsilon q^{-m}$ for any positive integer m, then N'_b and N''_b are precisely the two nonzero proper submodules of $M'_b \otimes L(1,\varepsilon)$.
- If $b \neq \pm 1$, we have $M'_b \otimes L(1,\varepsilon) = N'_b \oplus N''_b$ and in particular,
$$M'_b \otimes L(1,\varepsilon) \cong (M'_{\mathrm{\varepsilon q^{-1}b}})^{\sigma_{\varepsilon q,0}} \oplus (M'_{\mathrm{\varepsilon qb}})^{\sigma_{\varepsilon q^{-1},0}}.$$
- If $b = \pm 1$, we have $N''_b \subsetneq N'_b$,
$$\big(M'_b \otimes L(1,\varepsilon)\big)/N'_b \cong (M'_{\mathrm{\varepsilon qb}})^{\sigma_{\varepsilon q^{-1},0}} \quad \text{and} \quad N'_b/N''_b \cong L(0,-\varepsilon b).$$

Structure of $M'_b \otimes L(1,\varepsilon)$

Proposition

- If $b \neq \varepsilon q^{-m}$ for any positive integer m, then N'_b and N''_b are precisely the two nonzero proper submodules of $M'_b \otimes L(1,\varepsilon)$.
- If $b \neq \pm 1$, we have $M'_b \otimes L(1,\varepsilon) = N'_b \oplus N''_b$ and in particular,
$$M'_b \otimes L(1,\varepsilon) \cong (M'_{\varepsilon \mathrm{q}^{-1}\mathrm{b}})^{\sigma_{\varepsilon q,0}} \oplus (M'_{\varepsilon \mathrm{qb}})^{\sigma_{\varepsilon q^{-1},0}}.$$
- If $b = \pm 1$, we have $N''_b \subsetneq N'_b$,
$$\big(M'_b \otimes L(1,\varepsilon)\big)/N'_b \cong (M'_{\varepsilon \mathrm{qb}})^{\sigma_{\varepsilon q^{-1},0}} \quad \text{and} \quad N'_b/N''_b \cong L(0,-\varepsilon b).$$
- For $b = \pm 1$, we have the following nonsplit exact sequence:
$$0 \to (M'_{\varepsilon \mathrm{q}^{-1}\mathrm{b}})^{\sigma_{\varepsilon q,0}} \to M'_b \otimes L(1,\varepsilon) \to (M'_{\varepsilon \mathrm{qb}})^{\sigma_{\varepsilon q^{-1},0}} \to 0.$$

Decomposition of $M'_b \otimes L(n,\varepsilon)$

At last, we obtain the Clebsch-Gordan type formulas for $M'_b \otimes L(n,\pm)$:

Theorem

Let n be a positive integer and $b \in \mathbb{C}, b \neq \pm q^i, i = 0, 1, \ldots, n-1$, then we have

$$M'_b \otimes L(n,\varepsilon) \cong \bigoplus_{i=0}^{n} M''_{\varepsilon,n-2i}, \text{ where } M''_{\varepsilon,i} = (M'_{\varepsilon q^{-i}b})^{\sigma_{\varepsilon q^i,0}}.$$

Decomposition of $M'_b \otimes L(n,\varepsilon)$

At last, we obtain the Clebsch-Gordan type formulas for $M'_b \otimes L(n,\pm)$:

Theorem

Let n be a positive integer and $b \in \mathbb{C}, b \neq \pm q^i, i = 0, 1, \ldots, n-1$, then we have

$$M'_b \otimes L(n,\varepsilon) \cong \bigoplus_{i=0}^{n} M''_{\varepsilon,n-2i}, \text{ where } M''_{\varepsilon,i} = (M'_{\varepsilon q^{-i}b})^{\sigma_{\varepsilon q^i,0}}.$$

Remark

We have similar results for the tensor product modules

$$L(n,\varepsilon) \otimes M_a$$

and

$$L(n,\varepsilon) \otimes M'_b.$$

Thanks for your attention !