1.4. 分块矩阵

给定矩阵

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 11 & 123 & 1 & 0 \\ 453 & 987 & 0 & 1 \end{pmatrix},$$

如何计算 $\boldsymbol{A}^{1\,000}$

$$\boldsymbol{A} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 11 & 123 & 1 & 0 \\ 453 & 987 & 0 & 1 \end{array}\right),$$

如果令

$$\boldsymbol{X} = \begin{pmatrix} 11 & 123 \\ 543 & 987 \end{pmatrix}$$

那么 $\boldsymbol{A}$ 由四个矩阵拼接而成:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{X} & \boldsymbol{I} \end{pmatrix}.$$

下面带问号的等式成立吗?

$$
\begin{aligned}
&\begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{X} & \boldsymbol{I} \end{pmatrix}\begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{X} & \boldsymbol{I} \end{pmatrix} \\
\overset{?}{=}&\begin{pmatrix} \boldsymbol{II}+\boldsymbol{OX} & \boldsymbol{IO}+\boldsymbol{OI} \\ \boldsymbol{XI}+\boldsymbol{IX} & \boldsymbol{XO}+\boldsymbol{II} \end{pmatrix} \\
=&\begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \\ 2\boldsymbol{X} & \boldsymbol{I} \end{pmatrix}
\end{aligned}
$$

如果成立的话，重复这个过程可以得到

$$
\boldsymbol{A}^{1\,000} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \\ 1\,000\boldsymbol{X} & \boldsymbol{I} \end{pmatrix}.
$$

实际上，上面的等式的确是成立的！

定义

在一个矩阵 $\boldsymbol{A}$ 的行和列之间横平竖直地加上一些线，将其分成很多小块矩阵的形式称为分块矩阵(partitioned matrix)。一个矩阵可以根据需要进行不同的分块。分块矩阵的一般形式如下.

$$\boldsymbol{A} = \begin{array}{c c} & \begin{array}{cccc} n_1 & n_2 & \cdots & n_r \end{array} \\ \begin{array}{c} m_1 \\ m_2 \\ \vdots \\ m_s \end{array} & \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \cdots & \boldsymbol{A}_{1r} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \cdots & \boldsymbol{A}_{2r} \\ \vdots & \vdots & & \vdots \\ \boldsymbol{A}_{s1} & \boldsymbol{A}_{s2} & \cdots & \boldsymbol{A}_{sr} \end{pmatrix} \end{array},$$

其中矩阵 $\boldsymbol{A}_{ij}$ 是 m_i 行 n_j 列矩阵。

分块矩阵的乘法

对于分块矩阵

$$A = \begin{array}{c} \\ m_1 \\ m_2 \\ \vdots \\ m_s \end{array} \begin{array}{c} \begin{array}{cccc} n_1 & n_2 & \dots & n_r \end{array} \\ \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1r} \\ A_{21} & A_{22} & \dots & A_{2r} \\ \vdots & \vdots & & \vdots \\ A_{s1} & A_{s2} & \dots & A_{sr} \end{pmatrix} \end{array}, B = \begin{array}{c} \\ n_1 \\ n_2 \\ \vdots \\ n_r \end{array} \begin{array}{c} \begin{array}{cccc} l_1 & l_2 & \dots & l_t \end{array} \\ \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1t} \\ B_{21} & B_{22} & \dots & B_{2t} \\ \vdots & \vdots & & \vdots \\ B_{r1} & B_{r2} & \dots & B_{rt} \end{pmatrix} \end{array}$$

其中 A 的列分法与 B 的行分法一致.

那么

$$\boldsymbol{AB} = \begin{array}{c} \\ m_1 \\ m_2 \\ \vdots \\ m_s \end{array} \begin{array}{c} \begin{array}{cccc} l_1 & l_2 & \dots & l_t \end{array} \\ \begin{pmatrix} \boldsymbol{C}_{11} & \boldsymbol{C}_{12} & \dots & \boldsymbol{C}_{1t} \\ \boldsymbol{C}_{21} & \boldsymbol{C}_{22} & \dots & \boldsymbol{C}_{2t} \\ \vdots & \vdots & & \vdots \\ \boldsymbol{C}_{s1} & \boldsymbol{C}_{s2} & \dots & \boldsymbol{C}_{st} \end{pmatrix} \end{array},$$

其中

$$\boldsymbol{C}_{ij} = \sum_{k=1}^{r} \boldsymbol{A}_{ik}\boldsymbol{B}_{kj}.$$

对矩阵进行分块，

$$\boldsymbol{A}=\left(\begin{array}{cc|cc}1&0&3&0\\0&1&0&3\\\hline 0&0&4&2\end{array}\right)=\begin{pmatrix}\boldsymbol{I}_2&3\boldsymbol{I}_2\\\boldsymbol{O}&\boldsymbol{A}_{22}\end{pmatrix},\quad \boldsymbol{A}_{22}=\begin{pmatrix}4&2\end{pmatrix};$$

$$\boldsymbol{B}=\left(\begin{array}{c|cc}1&1&2\\1&3&1\\\hline 0&1&0\\0&0&1\end{array}\right)=\begin{pmatrix}\boldsymbol{B}_{11}&\boldsymbol{B}_{12}\\\boldsymbol{O}&\boldsymbol{I}_2\end{pmatrix},\quad \boldsymbol{B}_{11}=\begin{pmatrix}1\\1\end{pmatrix},\quad \boldsymbol{B}_{12}=\begin{pmatrix}1&2\\3&1\end{pmatrix}.$$

$$
\boldsymbol{AB} = \begin{pmatrix} \boldsymbol{I}_2 & 3\boldsymbol{I}_2 \\ \boldsymbol{O} & \boldsymbol{A}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{O} & \boldsymbol{I}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_2\boldsymbol{B}_{11} & \boldsymbol{I}_2\boldsymbol{B}_{12} + 3\boldsymbol{I}_2\boldsymbol{I}_2 \\ \boldsymbol{O} & \boldsymbol{A}_{22}\boldsymbol{I}_2 \end{pmatrix}
$$

$$
= \begin{pmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} + 3\boldsymbol{I}_2 \\ \boldsymbol{O} & \boldsymbol{A}_{22} \end{pmatrix}.
$$

由此得到

$$
\boldsymbol{AB} = \left(\begin{array}{c|cc} 1 & 4 & 2 \\ 1 & 3 & 4 \\ \hline 0 & 4 & 2 \end{array}\right).
$$

分块矩阵的加法、数乘和转置

对于两个**分块方式相同**的分块矩阵

$$\boldsymbol{A} = \begin{array}{c c} & \begin{array}{cccc} n_1 & n_2 & \cdots & n_r \end{array} \\ \begin{array}{c} m_1 \\ m_2 \\ \vdots \\ m_s \end{array} & \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \cdots & \boldsymbol{A}_{1r} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \cdots & \boldsymbol{A}_{2r} \\ \vdots & \vdots & & \vdots \\ \boldsymbol{A}_{s1} & \boldsymbol{A}_{s2} & \cdots & \boldsymbol{A}_{sr} \end{pmatrix} \end{array}$$

$$\boldsymbol{B} = \begin{array}{c c} & \begin{array}{cccc} n_1 & n_2 & \cdots & n_r \end{array} \\ \begin{array}{c} m_1 \\ m_2 \\ \vdots \\ m_s \end{array} & \begin{pmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} & \cdots & \boldsymbol{B}_{1r} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} & \cdots & \boldsymbol{B}_{2r} \\ \vdots & \vdots & & \vdots \\ \boldsymbol{B}_{s1} & \boldsymbol{B}_{s2} & \cdots & \boldsymbol{B}_{sr} \end{pmatrix} \end{array}$$

分块矩阵的加法

$$\boldsymbol{A}+\boldsymbol{B}=\begin{pmatrix} \boldsymbol{A}_{11}+\boldsymbol{B}_{11} & \boldsymbol{A}_{12}+\boldsymbol{B}_{12} & \cdots & \boldsymbol{A}_{1r}+\boldsymbol{B}_{1r} \\ \boldsymbol{A}_{21}+\boldsymbol{B}_{21} & \boldsymbol{A}_{22}+\boldsymbol{B}_{22} & \cdots & \boldsymbol{A}_{2r}+\boldsymbol{B}_{2r} \\ \vdots & \vdots & & \vdots \\ \boldsymbol{A}_{s1}+\boldsymbol{B}_{s1} & \boldsymbol{A}_{s2}+\boldsymbol{B}_{s2} & \cdots & \boldsymbol{A}_{sr}+\boldsymbol{B}_{sr} \end{pmatrix}$$

分块矩阵的数乘

对任意数 λ,

$$\lambda \boldsymbol{A} = \begin{pmatrix} \lambda \boldsymbol{A}_{11} & \lambda \boldsymbol{A}_{12} & \cdots & \lambda \boldsymbol{A}_{1r} \\ \lambda \boldsymbol{A}_{21} & \lambda \boldsymbol{A}_{22} & \cdots & \lambda \boldsymbol{A}_{2r} \\ \vdots & \vdots & & \vdots \\ \lambda \boldsymbol{A}_{s1} & \lambda \boldsymbol{A}_{s2} & \cdots & \lambda \boldsymbol{A}_{sr} \end{pmatrix}$$

分块矩阵的转置

$$A^{\mathrm{T}} = \begin{pmatrix} A_{11}^{\mathrm{T}} & A_{21}^{\mathrm{T}} & \cdots & A_{s1}^{\mathrm{T}} \\ A_{12}^{\mathrm{T}} & A_{22}^{\mathrm{T}} & \cdots & A_{s2}^{\mathrm{T}} \\ \vdots & \vdots & & \vdots \\ A_{1r}^{\mathrm{T}} & A_{2r}^{\mathrm{T}} & \cdots & A_{sr}^{\mathrm{T}} \end{pmatrix}$$

分块对角矩阵

一个分块对角矩阵形如

$$\begin{pmatrix} \boldsymbol{A}_1 & & & \\ & \boldsymbol{A}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{A}_s \end{pmatrix},$$

其中 $\boldsymbol{A}_i$ 是 n_i 阶方阵, $i = 1, 2, \cdots, s$, 其他位置上的分块都是零矩阵.

分块对角矩阵的乘法

如果

$$\boldsymbol{B}=\begin{pmatrix} \boldsymbol{B}_1 & & & \\ & \boldsymbol{B}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{B}_s \end{pmatrix}$$

且 $\boldsymbol{B}_i$ 也为 n_i 阶方阵，$i=1,2,\cdots,s$，那么

$$\boldsymbol{AB}=\begin{pmatrix} \boldsymbol{A}_1\boldsymbol{B}_1 & & & \\ & \boldsymbol{A}_2\boldsymbol{B}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{A}_s\boldsymbol{B}_s \end{pmatrix},$$

$$\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} \boldsymbol{A}_1 + \boldsymbol{B}_1 & & & \\ & \boldsymbol{A}_2 + \boldsymbol{B}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{A}_s + \boldsymbol{B}_s \end{pmatrix}.$$