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QUASI-HEREDITY OF ALGEBRAS AND THEIR FACTOR ALGEBRAS

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Dedicated to Professor Tiande Lei on the occasion of his 65th birthday

ABSTRACT. Let A be a finite-dimensional algebra over an algebraically closed field and denote by N the Jacobson radical of A. If there is an integer $i \geq 2$ such that A/N^i is quasi-hereditary, then A is quasi-hereditary.

Let A be a finite-dimensional algebra over an algebraically closed field k. By N we denote the Jacobson radical of A. An ideal of A is called a heredity ideal of A if it satisfies (1) $J^2 = J$, (2) $JNJ = 0$, and (3) J is a projective left A-module. We recall that the algebra A is said to be quasi-hereditary provided there is a chain

$$0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$$

of ideals of A such that J_i/J_{i-1} is a heredity ideal of A/J_{i-1} for all $i = 1, \ldots, n$. Some basic properties on quasi-hereditary algebras may be found in [DR]. The aim of this note is to show the following: If the algebra A is not quasi-hereditary, then, for any $i \geq 2$, the factor algebra A/N^i never becomes a quasi-hereditary algebra.

Throughout this note all algebras are finite-dimensional k-algebras with 1, module means finitely generated left module. By $\bar{a}$ (or $\overline{J}$) we denote the image of $a \in A$ (or $J \subseteq A$) under the canonical map $A \to A/I$, where I is an ideal of A.

The above-mentioned result may be reformulated as the following theorem.

Theorem 1. *Let A be a basic connected algebra with Jacobson radical N. If A/N^i is quasi-hereditary for some $i \geq 2$, then A is quasi-hereditary.*

To prove this result we need some preparations.

Lemma 2. *Let A be a basic algebra and e be a primitive idempotent such that $J = AeA$ is a heredity ideal of A. Then $eAe \cong k$.*

Proof. Since the field k is algebraically closed and the Jacobson radical of eAe is eNe, it follows from the definition of a heredity ideal that $eNe = 0$ and $eAe \cong k$.

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Lemma 3. *Let A be a basic connected algebra with radical N such that $\overline{A} = A/N^i$ is quasi-hereditary for some $i \geq 2$. Let e be a primitive idempotent of A such that $\overline{A}\bar{e}\overline{A}$ is a heredity ideal of $\overline{A}$. Then $J = AeA$ is a heredity ideal of A.*

Proof. By [DR] an idempotent ideal AeA of A with $eNe = 0$ is a heredity ideal of A if and only if the multiplication map

$$Ae \otimes_{eAe} eA \to AeA$$

is bijective. Since $\overline{A}\bar{e}\overline{A}$ is a heredity ideal of $\overline{A}$, the multiplication map

$$\overline{N^{i-1}e} \otimes_k \overline{eN} \to \overline{N^{i-1}eN} = 0$$

is bijective. This implies that $\overline{N}^{i-1}\bar{e} = 0$ or $\bar{e}\overline{N} = 0$. Similarly, we consider the multiplication map

$$\overline{Ne} \otimes_k \overline{eN^{i-1}} \to \overline{NeN^{i-1}} = 0.$$

This gives us that either $\overline{Ne} = 0$ or $\overline{eN^{i-1}} = 0$. If $\overline{Ne} = 0$ or $\overline{eN} = 0$, then we get $Ne = 0$ or $eN = 0$. Thus $J = AeA$ is obviously a heredity ideal of A. Now let us assume $\overline{Ne} \neq 0$ and $\overline{eN} \neq 0$. Then $\overline{N}^{i-1}\bar{e} = 0$ and $\bar{e}\overline{N}^{i-1} = 0$. It follows from $\overline{N^{i-1}e} = 0$ that $N^{i-1}e = 0$, since $N^{i-1}e \neq 0$ yields that $NN^{i-1}e$ is a proper submodule of $N^{i-1}e$. Similarly, there holds $eN^{i-1} = 0$. In particular, we have $N^ie = 0$ and $eN^i = 0$, and therefore the canonical maps $Ae \to \overline{A}\bar{e}$, $eA \to \bar{e}\overline{A}$ are bijective. On the one hand, it follows from $eNe \subseteq N^i$ that $eNe = 0$. On the other hand, the canonical commutative diagram

$$\begin{array}{ccc} Ae \otimes_k eA & \xrightarrow{\mu} & AeA \\ \downarrow & & \downarrow \\ \overline{A}\bar{e} \otimes_k \bar{e}\overline{A} & \xrightarrow{\bar{\mu}} & \overline{A}\bar{e}\overline{A} \end{array}$$

shows that with $\bar{\mu}$ also μ is injective. Hence $J = AeA$ is a heredity ideal of A.

The following lemma is an easy observation.

Lemma 4. *Let A be an artin algebra, $N = \operatorname{rad} A$, $\overline{A} = A/N^i$. Let e be a primitive idempotent in A. Then $(A/AeA)/\operatorname{rad}^i(A/AeA) \cong \overline{A}/\overline{AeA}$.*

Proof. Let $J = AeA$. Note that $\overline{J} \cong (J + N^i)/N^i$. From

$$\begin{aligned} \overline{A}/\overline{J} &\cong (A/N^i)/((J + N^i)/N^i) \\ &\cong A/(J + N^i) \cong (A/J)/((J + N^i)/J) \\ &\cong (A/J)/((J + N)/J)^i \end{aligned}$$

the lemma follows.

Proof of the theorem. We choose a complete set of pairwise nonisomorphic orthogonal primitive idempotents, say $e_1, \ldots, e_n$, such that for $J_j = A(e_1 + \cdots + e_j)A$, the chain

$$0 \subset \overline{J}_1 \subset \overline{J}_2 \subset \cdots \subset \overline{J}_n = \overline{A}$$

of ideals of $\overline{A}$ is a heredity chain for $\overline{A}$. Using Lemmas 3 and 4 repeatedly, we then get a heredity chain

$$0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$$

of ideals of A. And therefore A is quasi-hereditary.

Remarks 5. (1) The converse of the theorem is false. The following simple example is a desired one. Let A be given by the quiver with the relation

$$1 \circ \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \circ 2, \qquad \alpha\beta = 0.$$

Then it is easy to verify that A is quasi-hereditary but A/N^2 is not.

(2) If one only assumes in the theorem that there exists an ideal $J \subset N^2$ such that A/J is quasi-hereditary then A may not be quasi-hereditary. Let A be given by the above quiver with relations $\alpha\beta\alpha = 0$ and $\beta\alpha\beta = 0$. Then A is not quasi-hereditary, but if one takes J to be the socle of the projective module corresponding to the vertex 1 then $J \subset N^2$ and A/J is isomorphic to the algebra displayed in (1), in particular, it is quasi-hereditary. Further examples may be found in [X].

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References

[DR] V. Dlab and C. M. Ringel, *Quasi-hereditary algebras*, Illinois J. Math. **33** (1989), 280–291.

[X] C. C. Xi, *The structure of Schur algebras* $S_k(n, p)$ *for* $n \geq p$, Canad. J. Math. **44** (1992), 665–672.

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