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Endomorphism Algebras of $\mathscr{F}(\Delta)$ over Quasi-hereditary Algebras

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Let A be a finite-dimensional algebra over an algebraically closed field. If A is an $\mathscr{F}(\Delta)$-finite quasi-hereditary algebra, then the endomorphism algebra of the direct sum of all non-isomorphic indecomposable Δ-good modules over A is quasi-hereditary. Moreover, this endomorphism algebra is left QF-3 if and only if the injective direct summand of the characteristic module T cogenerates T.

1. INTRODUCTION AND THE RESULTS

Quasi-hereditary algebras have been defined by E. Cline, B. Parshall, and L. Scott to build the relationship between the representation theory of semisimple complex Lie-algebras and algebraic groups on the one hand and the representation theory of finite-dimensional associative algebras on the other hand [CPS]. Many important algebras such as algebras of global dimension two, algebras of the category $\mathscr{O}$, which is defined by Bernstein, Gelfand, and Gelfand in [BGG], and Schur algebras [G] are quasi-hereditary algebras. Recently, Ringel, Dlab and Ringel have conducted many remarkable investigations into quasi-hereditary algebras (see [DR1] [R2]). Ringel proved in [R2] that the full subcategory of the category of all finitely generated A-modules consisting of all Δ-good modules over a quasi-hereditary algebra has almost split sequences and has a characteristic module which is a minimal Ext-injective cogenerator for the Δ-good module category. In [DR2] one may find further study of this full subcategory. In the present paper we study the algebraic properties of this category $\mathscr{F}(\Delta)$, namely, the endomorphism algebra of this full subcategory

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in the case in which there are only finitely many non-isomorphic indecomposable objects. Usually this algebra is no longer an algebra of global dimension two (see Section 4). However, our study shows that this endomorphism algebra is again quasi-hereditary.

To state our result more precisely, let us first introduce some notation.

Let A be a finite-dimensional k-algebra over an algebraically closed field k, and we denote by A-mod the category of all finitely generated left A-modules and by A-ind a full subcategory of A-mod formed by choosing representatives of isomorphism classes of indecomposable modules in A-mod. If Θ is a class of A-modules (closed under isomorphisms), $\mathcal{F}(\Theta)$ stands for the class of all A-modules M which have a Θ-filtration, i.e., a filtration $M = M_0 \supset M_1 \supset \cdots \supset M_t \supset \cdots \supset M_m = 0$ such that all factor modules M_{t-1}/M_t, $1 \leq t \leq m$, belong to Θ. Also, we use add Θ to denote the full subcategory of A-mod whose objects are direct sums of modules in Θ.

Let $E(1), \ldots, E(n)$ be the simple A-modules (one from each isomorphism class), and note that we fix here a particular ordering of simple modules. Let $P(i)$ be the projective cover of $E(i)$, and $Q(i)$ denote the injective envelope of $E(i)$. By $\Delta(i)$ we denote the maximal factor module of $P(i)$ with composition factors of the form $E(j)$, where $j \leq i$; the modules $\Delta(i)$ are called the standard modules, and we set $\Delta = \{\Delta(i) \mid 1 \leq i \leq n\}$ and call the modules in $\mathcal{F}(\Delta)$ Δ-good modules. Similarly, we denote by $\nabla(i)$ the maximal submodule of $Q(i)$ with composition factors of the form $E(j)$ with $j \leq i$; in this way, we get a set $\nabla = \{\nabla(i) \mid 1 \leq i \leq n\}$ of costandard modules.

The algebra A, or, better, the pair (A, E) is called quasi-hereditary provided

(1) $\mathrm{End}_A(\Delta(i)) \cong k$ for all i, and

(2) every projective module belongs to $\mathcal{F}(\Delta)$.

For each quasi-hereditary algebra A with standard modules $\Delta(i)$, $i = 1, \ldots, n$, C. M. Ringel proved in [R2] that for each $i \in \{1, \ldots, n\}$, there is a unique indecomposable module $T(i)$ which lies in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ and that the direct sum T of all $T(i)$ cogenerates $\mathcal{F}(\Delta)$ and the endomorphism algebra of T is quasi-hereditary. The module T is called the characteristic module of (A, E).

A quasi-hereditary algebra A is said to be $\mathcal{F}(\Delta)$-finite if there are only finitely many non-isomorphism indecomposable Δ-good modules over A. In this case, we denote by $\mathrm{End}\,\mathcal{F}(\Delta)$ the endomorphism algebra of the module $\bigoplus X_i$, where X_i ranges over all non-isomorphic indecomposable modules in $\mathcal{F}(\Delta)$. Our main results are the following theorems.

THEOREM A. *Let A be a quasi-hereditary algebra. If A is $\mathscr{F}(\Delta)$-finite then* End $\mathscr{F}(\Delta)$ *is quasi-hereditary. In particular, the global dimension of* End $\mathscr{F}(\Delta)$ *is finite.*

We remark that a special case of this theorem is discussed in [LX].

THEOREM B. *Let A be an $\mathscr{F}(\Delta)$-finite quasi-hereditary algebra with the characteristic module $T = \bigoplus_{i=1}^{n} T(i)$. Suppose $T = T_0 \oplus T_1$ with T_0 an injective module and T_1 having no injective direct summand. Then* End $\mathscr{F}(\Delta)$ *is a left QF-3 algebra if and only if T_0 cogenerates T_1.*

The proofs of the theorems are given in Sections 2 and 3, and the last section contains some examples related to the main results.

Throughout the paper algebras always mean finite dimensional algebras over a fixed algebraically closed field k and modules mean finitely generated left modules. The composition of two homomorphisms $f\colon M \to N$ and $g\colon N \to L$ is denoted by fg.

2. PROOF OF THEOREM A

This section is devoted to the proof of the Theorem A. We need some preparations.

Let A be a quasi-hereditary algebra. Define $\mathscr{F}(\Delta)\text{-ind} =: \{X \in A\text{-ind} \mid X \in \mathscr{F}(\Delta)\}$, and $\mathscr{F}(\Delta)_0 = \varnothing$ and $\mathscr{F}(\Delta)_i =: \{X \in \mathscr{F}(\Delta)\text{-ind} \mid X \in \mathscr{F}(\Delta(n), \dots, \Delta(n-i+1))\}$. The following lemma is easy to prove by the definition of quasi-hereditary algebras (cf. [R2]).

2.1. LEMMA.

(1) $\mathscr{F}(\Delta)_{i+1} \supset \mathscr{F}(\Delta)_i$ *for all i.*

(2) *For any module $M \in \mathscr{F}(\Delta)_{i+1}$, there is a unique largest submodule M' of M such that $M' \in$ add $\mathscr{F}(\Delta)_i$ and $M/M' \in$ add $\Delta(n-i)$.*

(3) $\operatorname{Hom}_A(\mathscr{F}(\Delta(n), \dots, \Delta(n-i+1)), \mathscr{F}(\Delta(n-i), \dots, \Delta(1))) = 0.$

(4) $\operatorname{Ext}_A^1(\mathscr{F}(\Delta(n), \dots, \Delta(n-i+1)), \mathscr{F}(\Delta(n-i+1), \dots, \Delta(1))) = 0.$

Proof. Parts (1), (3), and (4) are obvious from [R2].

(2) The existence of such a submodule M' of M follows from the definition of a quasi-hereditary algebra. Now let M'' be another sub-

module of M with the property that $M' \in \operatorname{add}\mathscr{F}(\Delta)_i$ and $M/M' \in \operatorname{add}\Delta(n-i)$. Then we may form the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{\alpha} & M & \xrightarrow{\pi_M} & M/M' & \longrightarrow & 0 \\ & & \downarrow f' & & \| & & \downarrow f & & \\ 0 & \longrightarrow & M'' & \xrightarrow{\beta} & M & \xrightarrow{\pi_M} & M/M'' & \longrightarrow & 0 \end{array}$$

where α and β are the inclusion homomorphisms. Clearly, f' is injective. This implies $\dim_k M' \le \dim_k M''$. Similarly, we can show that $\dim_k M'' \le \dim_k M'$. Hence we deduce that $M' = M''$.

The following lemma will be used often in what follows.

2.2. LEMMA. *For each i, the full subcategory* $\operatorname{add}\Delta(i)$ *is an abelian category. In particular, for any homomorphism $f\colon X \to Y$ with $X, Y \in$* $\operatorname{add}\Delta(i)$, *the kernel* $\operatorname{Ker}(f)$ *and the cokernel* $\operatorname{Cok}(f)$ *of f belong to* $\operatorname{add}\Delta(i)$.

Proof. Since $\operatorname{Ext}_A^1(\Delta(i), \Delta(i)) = 0$, we have $\mathscr{F}(\Delta(i)) = \operatorname{add}\Delta(i)$. Thus the lemma follows directly from $\operatorname{End}_A(\Delta(i)) \cong k$.

For a module $M \in \mathscr{F}(\Delta)$ we denote by $[M : \Delta(i)]$ the number of factors isomorphic to $\Delta(i)$ in a Δ-filtration of M.

2.3. DEFINITION. Define $\mathscr{C}_i := \mathscr{F}(\Delta)_i \setminus \mathscr{F}(\Delta)_{i-1}$. Then $\mathscr{F}(\Delta)$-ind is a disjoint union of $\mathscr{C}_i$, $1 \le i \le n$. We define a relation $\le'$ on $\mathscr{C}_i$ for $i = 1, \ldots, n$ as follows:

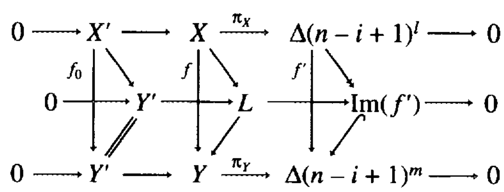
Suppose X, Y are in $\mathscr{C}_i$. We say $X \le' Y$ if and only if there is a homomorphism $f\colon Y \to X$ such that f cannot factor through a module in $\mathscr{T}_X' := \operatorname{add}(\mathscr{F}(\Delta)_{i-1} \cup \{Z \in \mathscr{C}_i \mid [Z : \Delta(n-i+1)] < [X : \Delta(n-i+1)]\})$, and we say $X \le Y$ if there are modules $X_0 = X, X_1, \ldots, X_m = Y$ in $\mathscr{C}_i$ such that $X_{j-1} \le' X_j$ for all j. We shall prove that with this relation $\le$ the set $\mathscr{C}_i$ is a partially ordered set.

2.4. LEMMA. *Suppose X and Y are two modules in $\mathscr{C}_i$. Let X' be the largest submodule of X such that $X' \in \mathscr{F}(\Delta)_{i-1}$ and $X/X' \in \mathscr{F}(\Delta(n-i+1))$. If $f\colon X \to Y$ is a homomorphism which cannot factor through a module in $\mathscr{T}_Y'$ then we have the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \longrightarrow & X & \xrightarrow{\pi_X} & \Delta(n-i+1)^{[X:\Delta(n-i+1)]} & \longrightarrow & 0 \\ & & \downarrow f_0 & & \downarrow f & & \downarrow f' & & \\ 0 & \longrightarrow & Y' & \longrightarrow & Y & \xrightarrow{\pi_Y} & \Delta(n-i+1)^{[Y:\Delta(n-i+1)]} & \longrightarrow & 0 \end{array}$$

with f' surjective.

Proof. The existence of f_0 follows from $\operatorname{Hom}_A(X', \Delta(n-i+1)) = 0$ by 2.1. Note that the image $\operatorname{Im}(f')$ of f' belongs to add $\Delta(n-i+1)$ according to 2.2. Thus if $\operatorname{Im}(f') \neq \Delta(n-i+1)^{[Y:\Delta(n-i+1)]}$ then we have the commutative diagram



where L is the pullback of π_Y and the canonical inclusion, and $l = [X : \Delta(n-i+1)]$ and $m = [Y : \Delta(n-i+1)]$. This means that f factors through the module L in $\mathcal{T}_Y$ since $\mathcal{F}(\Delta)$ is closed under extensions and $[\operatorname{Im}(f') : \Delta(n-i+1)] < m$, a contradiction. Hence $\operatorname{Im}(f') = \Delta(n-i+1)^m$ and the map f' is surjective.

2.5. LEMMA. *$(\mathscr{C}_i, \leq)$ is a partially ordered set.*

Proof.

(1) If $X \in \mathscr{C}_i$, then $X \leq X$. This follows from the fact that the identity map 1_X cannot factor through a module T in $\operatorname{add}(\mathcal{F}(\Delta)_{i-1} \cup \{Z \in \mathscr{C}_i \mid [Z : \Delta(n-i+1)] < [X : \Delta(n-i+1)]\})$ because for each indecomposable summand T' of T there holds $[T' : \Delta(n-i+1)] < [X : \Delta(n-i+1)]$.

(2) Suppose X, Y belong to $\mathscr{C}_i$ with $X \leq Y$ and $Y \leq X$. We want to show $X \cong Y$. By definition, we have modules $X = X_0, X_1, \ldots, X_m = Y$, $Y = Y_0, Y_1, \ldots, Y_t = X \in \mathscr{C}_i$ such that $X_{i+1} \leq' X_i$ and $Y_{j+1} \leq' Y_j$ for all i and j. Thus we have homomorphisms $f_i \colon X_i \to X_{i+1}$ and $g_{j+1} \colon Y_j \to Y_{j+1}$ such that f_i' and g_j' are surjective by 2.4. Since $\pi_X(f_0' \cdots f_{m-1}' g_0' \cdots g_{t-1}') = (f_0 \cdots f_{m-1} g_0 \cdots g_{t-1})\pi_X$ and $\pi_X(f_0' \cdots f_{m-1}' g_0' \cdots g_{t-1}')$ is surjective, we know that $f_0 \cdots f_{m-1} g_0 \cdots g_{m-1} \in \operatorname{End}_A(X)$ is not nilpotent. Here, π_X is defined by 2.4. Therefore it follows from the fact that $\operatorname{End}_A(X)$ is a local algebra that $f_0 \cdots f_{m-1} g_0 \cdots g_{t-1}$ is an isomorphism. Thus f_i, $0 \leq i \leq m-1$, are isomorphisms and $X \cong Y$, as desired.

(3) The transitivity of $\leq$ is obvious.

2.6. DEFINITION. Suppose A is an $\mathcal{F}(\Delta)$-finite quasi-hereditary algebra. Then $(\mathscr{C}_i, \leq)$ is a finite poset. Now we enumerate the elements in $\mathscr{C}_i$ as

$$X_{i1}, \ldots, X_{il_i}$$

so that if $X_{ij} \leq X_{ij'}$ then $j < j'$, where $l_i = |\mathscr{C}_i|$.

In order to prove the theorem, we use the following equivalent definition of quasi-hereditary algebras (for the proof of this fact see [CPS, R2]).

Let A be a finite-dimensional algebra. An ideal J of an algebra A is said to be a heredity ideal in A provided J is idempotent, $J(\mathrm{rad}(A))J = 0$ and ${}_AJ$ is a projective A-module. The algebra A is called quasi-hereditary if there is a finite chain

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_m = A$$

of ideals in A such that J_{i+1}/J_i, $0 \le i \le m-1$, are heredity ideals in A/J_i. Such a chain is called a heredity chain for A.

Proof of Theorem A. Let E be the endomorphism algebra of the module $\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{l_i} X_{ij}$ and let J_{it} be the ideal of E consisting of all endomorphisms f in E which factor through a module in $\mathcal{T}_{it} := \mathrm{add}(\oplus_{j=1}^{i-1} \bigoplus_{k=1}^{l_j} X_{jk} \oplus \bigoplus_{j=1}^{t} X_{ij}) = \mathrm{add}(\mathcal{F}(\Delta)_{i-1} \cup \{X_{i1}, \dots, X_{it}\})$. We shall prove that

$$0 = J_{00} \subseteq J_{11} \subseteq J_{21} \subseteq \cdots \subseteq J_{2l_2} \subseteq \cdots \subseteq J_{n1} \subseteq J_{n2} \subseteq \cdots \subseteq J_{nl_n} = E$$

is a heredity chain of E.

Let us introduce some further notation. Given a module $X \in \mathcal{F}(\Delta)$-ind, we denote by e_X the endomorphism in E which projects canonically $\bigoplus X_{ij}$ onto X. Thus, the elements e_X, $X \in \mathcal{F}(\Delta)$-ind, form a complete set of pairwise orthogonal primitive idempotents of E. Note that for $X, Y \in \mathcal{F}(\Delta)$-ind, we can identify $e_X E e_Y$ with $\mathrm{Hom}_A(X, Y)$. If N is the radical of E then $e_x N e_Y$ is the set of noninvertible maps in $\mathrm{Hom}_A(X, Y)$. Put $\bar{E} = E/J_{i,t-1}$ and $\bar{J}_{it} = J_{it}/J_{i,t-1}$. For an element $x \in E$, the residue class in $\bar{E}$ of x is denoted by $\bar{x}$ in what follows. For a module $X \in \mathcal{C}_i$ we define $\mathcal{T}_X := \{Z \in \mathcal{C}_i \mid [Z : \Delta(n-i+1)] < [X : \Delta(n-i+1)]\}$.

2.7. Lemma. *Every non-invertible homomorphism from X_{it} to X_{it} factors through a module in* $\mathrm{add}(\mathcal{F}(\Delta)_{i-1} \cup \{Z \in \mathcal{C}_i \mid Z <' X_{it}\})$. *In particular,* $\bar{e}_{X_{it}} \bar{E} \bar{e}_{X_{it}} \cong k$.

Proof. Indeed, if we take a map $f\colon X_{it} \to X_{it}$ which is not invertible, then f is nilpotent since $\mathrm{End}_A(X_{it})$ is a local algebra. Set $X = X_{it}$. Let X' be the maximal submodule of X with $X' \in \mathrm{add}\, \mathcal{F}(\Delta)_{i-1}$. Then we consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & \Delta(n-i+1)^{[X:\Delta(n-i+1)]} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \\
0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & \Delta(n-i+1)^{[Y:\Delta(n-i+1)]} & \longrightarrow & 0
\end{array}$$

with f' nilpotent. This means that the image of f' is proper contained in $\Delta(n-i+1)^{[X:\Delta(n-i+1)]}$ and $[\mathrm{Im}(f') : \Delta(n-i+1)] < [X : \Delta(n-i+1)]$.

As in the proof of 2.4 we see that f factors through a module $L \in \operatorname{add}(\mathscr{F}(\Delta)_{i-1} \cup \{Z \in \mathscr{C}_i \mid [Z:\Delta(n-i+1)] < [X:\Delta(n-i+1)]\})$. Suppose $L = L_0 \oplus L_1 \oplus \cdots \oplus L_s$ with $L_0 \in \operatorname{add}\mathscr{F}(\Delta)_{i-1}$ and $L_j \in \mathscr{T}_X = \{Z \in \mathscr{C}_i \mid [Z:\Delta(n-i+1)] < [X:\Delta(n-i+1)]\}$, and $f = (f_0,\dots,f_s)(g_0,\dots,g_s)^t$ with $f_j\colon X \to L_j$ and $g_j\colon L_j \to X$. Clearly, L_j is not isomorphic to X for all j. If f_j is a homomorphism such that $L_j <' X$ then we have what we wanted. So we may assume that all f_j factor through a module in $\operatorname{add}(\mathscr{F}(\Delta)_{i-1} \cup \mathscr{T}_{L_j})$, say $f_j = (f_{j0},\dots,f_{j,s(j)})(g_{j0},\dots,g_{j,s(j)})^t$ with $f_{ji}\colon X \to L_{ji}$ and $L_{ji} \in \mathscr{T}_{L_j}$. If f_{ji} is a homomorphism such that $L_{ji} <' X$ then we have what we desired. If not, we decompose f_{ji} into $f_{ji} = (f_{ji0},\dots,f_{ji,s(ji)})(g_{ji0},\dots,f_{ji,s(ji)})^t$ with $f_{jik}\colon X \to L_{jik}$ and $L_{jik} \in \mathscr{T}_{L_{ji}}$ and repeat the above discussion. If we continue this procedure we see that after finitely many steps we must stop since

$$\begin{aligned}\left[L_{jik}:\Delta(n-i+1)\right] < \left[L_{ji}:\Delta(n-i+1)\right] < \left[L_j:\Delta(n-i+1)\right]\\ < \left[X:\Delta(n-i+1)\right].\end{aligned}$$

this shows that f factors through a module in $\operatorname{add}(\mathscr{F}(\Delta)_{i-1} \cup \{Z \in \mathscr{C}_i \mid Z <' X\}$.

With the same argument as that in the proof of 2.7, we have the following

2.7′. LEMMA. *Suppose X, Y are modules in $\mathscr{C}_i$. If $X \not\leq' Y$, then every homomorphism $f\colon Y \to X$ factors through a module in* $\operatorname{add}(\mathscr{F}(\Delta)_{i-1} \cup \{Z \in \mathscr{C}_i \mid Z <' Y\})$.

Proof. Since $X \not\leq' Y$, by definition the homomorphism f factors over a module $L_0 \oplus L_1 \oplus \cdots \oplus L_s$ with $L_0 \in \operatorname{add}\mathscr{F}(\Delta)_{i-1}$ and $L_j \in \mathscr{T}_X$ for all $j > 0$, say $f = (f_0,\dots,f_s)(g_0,\dots,g_s)^t$ with $f_j\colon Y \to L_j$. If f_j is a map such that $L_j <' Y$, then we have done what we wanted. If $Y \cong L_j$, then f_j can be assumed to be nilpotent. (Otherwise we can omit this direct summand.) By 2.7, this f_j factors over a desired module. So we may assume that all L_j are not isomorphic to Y and all f_j factor through a module in $\operatorname{add}(\mathscr{F}(\Delta)_{i-1} \cup \mathscr{T}_{L_j})$. Let us decompose f_j as $(f_{j0},\dots,f_{j,s(j)})(g_{j0},\dots,g_{j,s(j)})^t$ with $f_{ji}\colon Y \to L_{ji}$ and $L_{ji} \in \mathscr{T}_{L_j}$ for $j \neq 0$ and $L_{j0} \in \operatorname{add}\mathscr{F}(\Delta)_{i-1}$. As above we may assume that all L_{ji} are not isomorphic to Y and all f_{ji} for $i \neq 0$ factor through a module in $\operatorname{add}(\mathscr{T}_{L_{ji}} \cup \mathscr{F}(\Delta)_{i-1})$, namely, $f_{ji} = (f_{ji0},\dots,f_{ji,s(ji)})(g_{ji0},\dots,g_{ji,s(ji)})^t$ for all j, i with $f_{jik}\colon Y \to L_{jik}$ and $L_{jik} \in \mathscr{T}_{L_{ji}}$. Since $[L_{jik}:\Delta(n-i+1)] < [L_{ji}:\Delta(n-i+1)] < [L_j:\Delta(n-i+1)] < [X:\Delta(n-i+1)]$, we see that this procedure must stop after finitely many steps, and then we have that f factors through a module in $\operatorname{add}(\mathscr{F}(\Delta)_{i-1} \cup \{Z \in \mathscr{C}_i \mid Z <' Y\})$.

2.8. LEMMA. *$\bar{J}_{i,t}$ is a projective left $\bar{E}$-module.*

Proof. We use a result of [DR1] which says that an idempotent ideal AeA of a given algebra A generated by an idempotent e with $e(\mathrm{rad}(A))e = 0$ is a projective left A-module if and only if the multiplication map

$$Ae \underset{eAe}{\otimes} eA \to AeA$$

is bijective. Hence it is enough to show that the multiplication map

$$\bar{e}_Y \bar{E} \bar{e}_X \underset{\bar{e}_X \bar{E} \bar{e}_X}{\otimes} \bar{e}_X \bar{E} \bar{e}_Z \to \bar{e}_Y \bar{E} \bar{e}_X \bar{E} \bar{e}_Z \qquad (*)$$

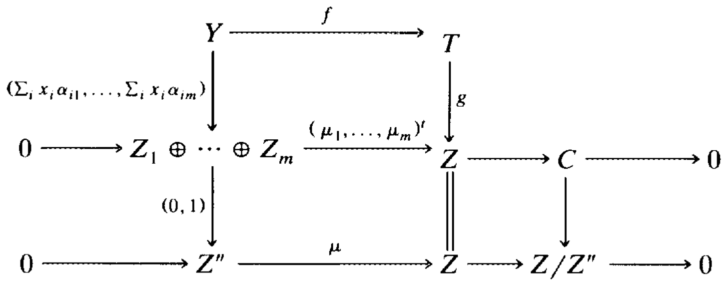
is bijective for all $Z, Y \in \mathcal{F}(\Delta)$-ind. (Recall that here $X = X_{it}$.)

By Lemma 2.7, $\bar{e}_X \bar{E} \bar{e}_X \cong k$, so the elements of $\bar{e}_Y \bar{E} \bar{e}_X \otimes_{\bar{e}_X \bar{E} \bar{e}_X} \bar{e}_X \bar{E} \bar{e}_Z$ are of the form $\Sigma_{i=1}^{c} \bar{x}_i \otimes_k \bar{\delta}_i$, where $x_i \in \mathrm{Hom}_A(Y, X)$ and $\delta_i \in \mathrm{Hom}_A(X, Z)$. Suppose there is an element $u = \Sigma_{i=1}^{c} \bar{x}_i \otimes \bar{\delta}_i$ such that $\Sigma \bar{x}_i \bar{\delta}_i = 0$. We may assume that $\bar{x}_i \neq 0 \neq \bar{\delta}_i$ for all i. Then $Z, Y \in \{X_{i,t}, \ldots, X_{il_i}\} \cup \mathscr{C}_{i+1} \cup \cdots \cup \mathscr{C}_n$. Let X' be the maximal submodule of X such that $X' \in \mathcal{F}(\Delta)_{i-1}$ and $X/X' \in \mathrm{add}\, \Delta(n-i+1)$. Similarly, let Z'' be the maximal submodule of Z such that $Z'' \in \mathrm{add}\, \mathcal{F}(\Delta)_i$ and $Z/Z'' \in \mathcal{F}(\Delta(1), \ldots, \Delta(n-i))$. Since $\mathrm{Hom}_A(\mathcal{F}(\Delta(n-i+1), \Delta(n-i+2), \ldots, \Delta(n)), \mathcal{F}(\Delta(1), \ldots, \Delta(n-i))) = 0$, we see that there is a homomorphism from X' to Z'' and then a homomorphism δ_i'' from $\Delta(n-i+1)^{[X:\Delta(n-i+1)]}$ to Z/Z'' such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X' & \longrightarrow & X & \xrightarrow{\pi_x} & \Delta(n-i+1)^{[X:\Delta(n-i+1)]} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \delta_i} & & \downarrow{\scriptstyle \delta_i''} & & \\
0 & \longrightarrow & Z'' & \xrightarrow{\mu} & Z & \longrightarrow & Z/Z'' & \longrightarrow & 0
\end{array}$$

Again by Lemma 2.1, $\delta_i'' = 0$ holds. Thus there exists a homomorphism γ_i: $X \to Z''$ such that $\delta_i = \gamma_i \mu$. Now suppose $Z'' = Z_0 \oplus Z_1 \oplus \cdots \oplus Z_m$ with $Z_0 \in \mathrm{add}\, \mathcal{F}(\Delta)_{i-1}$ and $Z_j \in \mathscr{C}_i$ for all $j > 0$, and $\gamma_i = (\gamma_{i0}, \ldots, \gamma_{im})$ as well as $\mu = (\mu_0, \ldots, \mu_m)^t$, then $\bar{\delta}_i = \bar{\gamma}_i \bar{\mu} = \Sigma_{j=0} \bar{\gamma}_{ij} \bar{\mu}_j$. Since $\bar{\delta}_i \neq 0$, we may assume that $\bar{\gamma}_{ij} \neq 0$ for $j \neq 0$. If there is some Z_j such that $Z_j \nleq X$, then $Z_j \nleq {}'X$. By Lemma 2.7', γ_{ij} factors through a module in $\mathrm{add}(\mathcal{F}(\Delta)_{i-1} \cup \{Z \in \mathscr{C}_i \mid Z < {}'X\})$; in particular, γ_{ij} factors through a module in $\mathrm{add}(\mathcal{F}(\Delta)_{i-1} \cup \{Z \in \mathscr{C}_i \mid Z < {}'X\})$; in particular, γ_{ij} factors through a module in $\mathcal{T}_{i,t-1}$ and we would have $\bar{\gamma}_{ij} = 0$. Thus we can assume that $Z_j \leq X$ for all $j > 0$. Hence we can identify all Z_j with X and regard γ_{ij} as an endomorphism of Z_j. According to 2.7, one can write

$\bar{\gamma}_{ij} = \bar{\alpha}_{ij}$ for some $\alpha_{ij} \in k$. Since $\sum x_i \delta_i = \sum x_i(\gamma_i \mu) = \sum_{i=1}^{c}\sum_{j=1}^{m} x_i \gamma_{ij} \mu_j$ and $0 = \sum \bar{x}_i \bar{\gamma}_i = \sum_{i,j} \bar{x}_i \bar{\gamma}_{ij} \bar{\mu}_j = \sum_i \bar{x}_i \bar{\alpha}_i \bar{\mu}_j = \sum_{j=1}(\sum_i \bar{x}_i \alpha_{ij})\bar{\mu}_j$, one finds that there is a module $T \in \mathscr{T}_{i,t-1}$ such that $\sum_{ij} x_i \alpha_{ij} \mu_j$ factors through the module T, say $\sum_j(\sum_i x_i \alpha_{ij})\mu_j = fg$. Now consider the commutative diagram



where C is the cokernel of $(\mu_1, \ldots, \mu_m)'$. Since $T \in \operatorname{add}\mathscr{F}(\Delta)_i = \mathscr{F}(\Delta(n-i+1), \ldots, \Delta(n))$ and $Z/Z'' \in \mathscr{F}(\Delta(1), \ldots, \Delta(n-i))$, again by $\operatorname{Hom}_A(\mathscr{F}(\Delta(n-i+1), \ldots, \Delta(n)), \mathscr{F}(\Delta(1), \ldots, \Delta(n-i))) = 0$ one obtains a homomorphism $\varphi\colon T \to Z''$ such that $g = \varphi\mu$. This implies that $f\varphi = (\sum_i x_i \alpha_{i1}, \ldots, \sum_i x_i \alpha_{im})(0, 1)$ because μ is injective. If we decompose φ into $(\varphi_0, \varphi_1, \ldots, \varphi_m)$, then $\sum x_i \alpha_{ij} = f\varphi_j$ for all $j \geq 1$. This means that $\sum_i x_i \alpha_{ij}$ factors through the module T in $\mathscr{T}_{i,t-1}$, and therefore, $\sum_i \bar{x}_i \bar{\alpha}_{ij} = 0$ for all j and

$$\begin{aligned}
\sum \bar{x}_i \otimes_k \bar{\delta}_i &= \sum \bar{x}_i \otimes_k \bar{\gamma}_i \bar{\mu} = \sum_{i=1}^{c} \bar{x}_i \otimes_k \sum_{j=1}^{m} \bar{\gamma}_{ij} \bar{\mu}_j \\
&= \sum_{ij} \bar{x}_i \otimes_k \bar{\gamma}_{ij} \bar{\mu}_j = \sum_{ij} \bar{x}_i \otimes_k \bar{\alpha}_{ij} \bar{\mu}_j \\
&= \sum_{ij} \bar{x}_i \alpha_{ij} \otimes_k \bar{\mu}_j = \sum_{ij} \bar{x}_i \bar{\alpha}_{ij} \otimes_k \bar{\mu}_j \\
&= \sum_j \Big(\sum_i \bar{x}_i \bar{\alpha}_{ij}\Big) \otimes_k \bar{\mu}_j = \sum_j 0 \otimes_k \bar{\mu}_j \\
&= 0.
\end{aligned}$$

Hence $u = 0$ and the multiplication map $(*)$ is bijective. This finishes the proof of the theorem.

3. PROOF OF THEOREM B

A well-known result of M. Auslander says that if an algebra is representation-finite then the endomorphism algebra of the direct sum of all non-isomorphic indecomposable modules is always a left QF-3 algebra. In this section we study a similar question for $\mathcal{F}(\Delta)$ of quasi-hereditary algebras. As the example in Section 4 shows, for $\mathcal{F}(\Delta)$ the situation is very different, though the endomorphism algebra $\mathrm{End}_A(\mathcal{F}(\Delta))$ is quasi-hereditary. Even if the quasi-hereditary algebra itself is a left QF-3 algebra, the endomorphism algebra may not be a left QF-3 algebra (see 4.3 below).

Let us first recall the definition of left QF-3 algebras.

3.1. Definition. An algebra A is called a left QF-3 algebra if there is a faithful left projective, injective A-module.

This definition is equivalent to that in [T, pp. 40–42]: An algebra is called a left QF-algebra if it has a minimal faithful left module.

For a finite additive k-category $\mathcal{X}$ we denote by $\mathrm{End}(\mathcal{X})$ the endomorphism algebra of the direct sum of all non-isomorphic indecomposable objects in $\mathcal{X}$.

3.2. Theorem. *Let A be an $\mathcal{F}(\Delta)$-finite quasi-hereditary algebra with the characteristic module $T = \oplus_{i=1}^{n} T(i)$. Suppose $T = T_0 \oplus T_1$ with T_0 an injective module and T_1 having no injective direct summand. Then $\mathrm{End}_A(\mathcal{F}(\Delta))$ is a left QF-3 algebra if and only if T_0 cogenerates T_1.*

The theorem follows from the following more general fact.

3.3. Proposition. *Let A be an algebra and $\mathcal{X} = \{X_1, \dots, X_m\}$ a finite class of indecomposable A-modules such that ${}_AA \in \mathrm{add}\,\mathcal{X}$. Then $\mathrm{End}_A(\mathcal{X})$ is a left QF-3 algebra if and only if there is an injective module U in $\mathrm{add}\,\mathcal{X}$ such that U cogenerates $\mathcal{X}$.*

Proof. Let $X = \oplus_{i=1}^{m} X_i$ and $X_i \not\cong X_j$ for $i \neq j$. Suppose $E := \mathrm{End}_A(X)$ is a left QF-3 algebra. Then, by definition, there is a minimal faithful module which is of the form Ee with e an idempotent, say $Ee = \mathrm{Hom}_A(X, X_1) \oplus \cdots \oplus \mathrm{Hom}_A(X, X_s)$ with $s \leq m$. We shall show first that $U := \oplus_{i=1}^{s} X_i$ cogenerates $\mathcal{X}$. In fact, given a module $X_l \in \mathcal{X}$, there is a natural number t and an injective E-homomorphism φ such that

$$0 \to \mathrm{Hom}_A(X, X_l) \xrightarrow{\varphi} \mathrm{Hom}_A(X, U^t)$$

is an exact sequence. Since $\mathrm{add}\,\mathcal{X}$ and the full subcategory consisting of all projective E-modules are equivalent, φ is induced by an A-homomorphism $\alpha \colon X_l \to U^t$ such that $\mathrm{Hom}_A(X, \alpha) = \varphi$. We claim that α is an injective map. Let P be the projective cover of the kernel $\ker(\alpha)$ of α. Then we have the diagram

$$\begin{array}{ccccccc} & & \mathrm{P} & \overset{\pi}{\searrow} & & & \\ & & \downarrow{\scriptstyle \pi'} & & & & \\ 0 & \longrightarrow & \ker(\alpha) & \xrightarrow{\mu} & \mathrm{X} & \xrightarrow{\alpha} & U^t \end{array}$$

and apply $\mathrm{Hom}_A(X, -)$ to it, we get $\mathrm{Hom}_A(X, \pi)\varphi = 0$. Thus $\mathrm{Hom}_A(X, \pi) = 0$ and $\mathrm{Hom}_A(X, \pi') = 0$. On the other hand, since X contains a copy of each indecomposable direct summand of P, we must have $\pi' = 0$. This implies that α is injective.

To finish the proof, it suffices to show that all X_j, $1 \le j \le s$, are injective A-modules. Toward this goal we require the following lemma in [T, p. 51]:

3.4. Lemma. *Let R be a left* QF-3 *algebra and Re a minimal faithful left ideal with e an idempotent. If fR is a faithful projective right ideal in R with f an idempotent, then ${}_{fRf}fRe$ is injective.*

By e_M, $M \in \{X_1, \dots, X_m\}$, we denote the endomorphism in E which projects X canonically onto M. Let ${}_AA = P_1 \oplus \cdots \oplus P_n$ with P_j indecomposable and $f = e_{P_1} + \cdots + e_{P_n}$ and $e = e_{X_1} + \cdots + e_{X_s}$. Then

(1) $Ee = \mathrm{Hom}_A(X, U)$ is a minimal faithful left ideal in E, and

(2) $fE = \mathrm{Hom}_A({}_AA, X)$ is a projective right ideal of E and faithful.

Hence the hypotheses of Lemma 3.4 are satisfied, and so the fEf-module fEe is injective. Since $fEf \cong A$ and $fEe \cong U$ as A-modules, we have the injectivity of the module U.

Conversely, suppose there is an injective module $U = \bigoplus_{j=1}^{s} X_j$ such that U cogenerates $\mathscr{X}$. Put $e = \sum_{j=1}^{s} e_{X_j}$. We claim that Ee is a faithful projective, injective left ideal in E. It is clear that $Ee = \mathrm{Hom}_A(X, U)$ is a faithful E-module since U cogenerates $\mathscr{X}$. For each X_j with $1 \le j \le s$, we may write $X_j = D(e'_jA)$ with e'_j a primitive idempotent of A and $D = \mathrm{Hom}_k(-, k)$. It follows now from

$$\begin{aligned}\mathrm{Hom}_A({}_AX_E, X_j) &= \mathrm{Hom}_A\big({}_AX_E, D(e'_jA)\big) = \mathrm{Hom}_A\big(X, \mathrm{Hom}_k(e'_jA, k)\big)\\ &\cong \mathrm{Hom}_k\big(e'_jA \otimes_A X_E, k\big) = \mathrm{Hom}_k\big(e'_jX_E, k\big)\\ &= \mathrm{Hom}_k\big(\mathrm{Hom}_A(Ae'_j, X_E), k\big)\\ &= D\,\mathrm{Hom}_A(Ae'_j, X)\end{aligned}$$

that Ee_{X_j} is an injective E-module. Hence Ee is an injective E-module. This finishes the proof.

3.5. Corollary. *Suppose A is an $\mathscr{F}(\Delta)$-finite quasi-hereditary algebra. If* $\mathrm{End}_A(\mathscr{F}(\Delta))$ *is a left QF-3 algebra then so is* $\mathrm{End}_A(T)$.

Proof. This follows from Theorem B and 3.3.

4. EXAMPLES

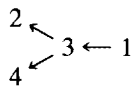
In this section we give some examples related to the results in this paper. For the terminology on quivers we refer to [R1, Chap. 2].

4.1. The algebra End $\mathcal{F}(\Delta)$ may have arbitrary finite global dimension. Let A be the algebra with radical-square-zero given by the quiver

$$2 \leftarrow 1 \leftarrow 3 \leftarrow \cdots \leftarrow n-1 \leftarrow n, \qquad n \geq 4.$$

Then A is quasi-hereditary and $\Delta(i) = E(i)$ for $i = 1, 2$ and $\Delta(i) = P(i)$ for $3 \leq i \leq n$. One can easily write out the Auslander–Reiten quiver of A and then see that the global dimension of $\mathrm{End}_A(\mathcal{F}(\Delta))$ is $n-2$.

4.2. The converse of 3.5 is not true if one considers the quasi-hereditary algebra given by the quiver

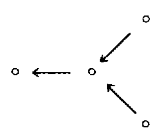


If one computes $\mathrm{End}_A(T)$ then it is given by the quiver $2 \xleftarrow{\gamma} 4 \xleftarrow{\beta} 3 \xleftarrow{\alpha} 1$ with the relation $\alpha\beta\gamma = 0$. It is clear that $\mathrm{End}_A(T)$ is a left QF-3 algebra but the maximal injective direct summand of T cannot cogenerate T.

4.3. Let A be the hereditary algebra given by the quiver

$$2 \leftarrow 1 \leftarrow 3.$$

Then A is a left QF-3 algebra and an $\mathcal{F}(\Delta)$-finite quasi-hereditary algebra. An easy computation shows that $\mathrm{End}_A(\mathcal{F}(\Delta))$ is given by the quiver



with only one zero-relation. It is obvious that this algebra is not a left QF-3 algebra.

4.4. One can easily see that there do exist quasi-hereditary algebras that are not of the form $\mathrm{End}_A \mathcal{F}(\Delta)$ for any $\mathcal{F}(\Delta)$-finite quasi-hereditary algebra A. For instance, the algebra given by the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

with relation $\alpha\beta = 0$ is clearly not of the form $\mathrm{End}_A(\mathcal{F}(\Delta))$ for any quasi-hereditary algebra A.

It would be interesting to determine which quasi-hereditary algebras are of this form.

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