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GLOBAL DIMENSIONS OF DUAL EXTENSION ALGEBRAS

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Dedicated to the memory of Maurice Auslander

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Introduction

Motivated by the study of quasi–hereditary algebras introduced by Cline, Parshall and Scott in [CPS], a class of finite–dimensional algebras is constructed in [X], namely the class of dual extension algebras (for the definition see 1.7 below). Some nice properties of these algebras have been developed in [DX1] and [DX2]. In the present paper we are going to investigate the relationship of homological dimensions between a given algebra and its dual extension, here we do not assume that the resulting dual extension algebra is quasi–hereditary. The main result is the following

Theorem. Let C be a finite–dimensional basic k–algebra and A its dual extension. Then gl.dim (A) = 2 gl.dim (C), where gl.dim (A) denote the global dimension of the algebra A.

Thus, comparing with the construction of a family of algebras with large global dimensions in [Y] (see also [G]), the dual extension provides us a more convenient way to obtain families of algebras with a fixed number of simple modules and large global dimensions.

To prove this result, we need some preparations which are done in section one under a more general setting. Section two is devoted to the proof of the main result.

Throughout this paper, algebras mean always finite-dimensional k-algebras over a fixed field k and modules mean finitely generated (left) modules.

This note is stimulated by the preprint [K2], and I would like to thank the author for sending me the preprint.

1. Basic properties and definitions

In this section, we assume that we are given a basic algebra A with $1 = \sum e_i$, where $\{e_i\}$ is a complete set of orthogonal primitive idempotents of A and that A has two subalgebras C and B such that $S = \oplus k e_i$ is a maximal semisimple subalgebra of A, B and C and $B \cap C = S$. Throughout the section we suppose that we have an isomorphism of bimodules $\varphi : \ {}_C C \otimes_S B_B \cong_C A_B$ given by

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multiplication, and we denote by ψ the inverse of φ. We shall develop some properties of the algebra A.

1.1 Proposition. (1) A_B is a projective right B–module.
(2) $_CA$ is a projective C–module

Proof. We prove only the first statement. Since $B = \oplus e_iB$ and $A_B \cong C\otimes_S B_B$, we have

$$A_B \cong C\otimes_S (\bigoplus e_iB) = \bigoplus_i C\otimes_S e_iB = \bigoplus_i Ce_i\otimes_S e_iB \cong \bigoplus_{i,j} e_jCe_i\otimes_S e_iB$$

If $e_j\alpha_1e_i, ..., e_j\alpha_me_i$ are a k–basis of e_jCe_i, then from $e_j\alpha_le_i\otimes_S e_iB \cong e_j\alpha_le_iB \cong e_iB$ we see that $e_j\alpha_le_i\otimes_S e_iB$ is projective right B–module. Hence A_B is projective.

1.2 Lemma. (1) $Ae_i \cong C\otimes_S Be_i$ as C–modules.
(2) $_CA\otimes_B E(i) \cong {}_CCe_i$, where $E(i)$ is the simple B–module $Be_i/rad(B)e_i$.

Proof. (1) is clear. It follows from 1.1 that $A\otimes_B E(i) \cong Ae_i/(A\otimes_B rad(B)e_i) \cong (C\otimes_S Be_i)/(C\otimes_S rad(B)e_i)$. Since S is semisimple, C_S is projective. Thus $(C\otimes_S Be_i)/(C\otimes_S rad(B)\,e_i) \cong C\otimes_S E(i) \cong Ce_i$.

1.3 Lemma. If $rad(B)C \subset Crad(B)$, then $A/<rad(B)>\cong C$, where $<rad(B)>$ stands for the ideal of A generated by rad(B) in A.

Proof. Note that $A = CB = C(S\oplus rad(B)) = C\oplus C\,rad(B)$ since $C\otimes_S(S\oplus rad(B))$ is mapped under φ to $C\oplus C\,rad(B)$. Thus each element $a\in A$ has the expression

$$a = c_a + \sum c_ib_i, \quad c_a\in C, c_i\in C, b_i\in \mathrm{rad}(B).$$

Define $\sigma : A/<rad(B)>\longrightarrow C$ by $a+<rad(B)>\longrightarrow c_a$. It is clear, that σ is a well–defined k–linear map. Let $a' = c_{a'} + \sum c_j'b_j'$ be another element of A. Then $a+a'$ is mapped to $c_a + c_{a'}$. Moreover, since

$$aa' = c_ac_{a'} + \sum c_ib_ic_{a'} + \sum c_ac_j'b_j' + \sum c_ib_ic_j'b_j'$$

and rad(B) $C\subset C$ rad(B), one has $\sigma(a+a') = c_ac_{a'} = \sigma(a)\sigma(a')$. Thus σ is an algebra homomorphism. It is easy to see that σ is bijective. Hence the lemma 1.3 follows.

Similarly, we have $A/<rad(C)>\cong B$ if $B\,rad(C)\subset rad(C)B$.

1.4 Lemma. Suppose $rad(B)C\subset Crad(B)$. Then every C–module can be regarded as A–module by σ in 1.3 and the isomorphism in 1.2 (2) is an A–module isomorphism.

The following lemma gives a condition to guarantee the truth of $rad(B)C\subset Crad(B)$ and $Brad(C)\subset rad(C)B$, and this condition is satisfied by dual extensions and other interesting algebras (see [D]).

1.5 Lemma. If $rad(B)rad(C)\subset rad(C)rad(B)$, then

(1) $rad(B)C \subset C\, rad(B)$ and $Brad(C) \subset rad(C)B$.
(2) $rad(A) = rad(C)B + Crad(B)$

The proof of this lemma is straightout, we omit it.

1.6 Lemma. Let $C(A)$ denote the Cartan–matrix of the algebra A. Then $C(A) = C(C)C(B)$.

Proof. Since ${}_CA_B \cong C \otimes_S B$, it follows that $e_iAe_j \cong e_iC\otimes_S Be_j = \bigoplus_t e_iCe_t \otimes_k e_tBe_j$ and $\dim_k e_iAe_j = \sum_t(\dim_k e_iCe_t)(\dim_k e_tBe_j)$. Thus we have the lemma 1.6.

1.7 In the rest of this section we shall present a class of algebras which satisfy all conditions that we have assumed. This special class of algebras is constructed in [X] (see also [K1]). Let us now recall the construction.

Let C and B be two finite–dimensional basic algebras over a field k. As usual, we suppose that C and B are given by a quiver $Q = (Q_0, Q_1)$ with relations $\{\rho_i \mid i \in I_C\}$ and a quiver $\Gamma = (\Gamma_0, \Gamma_1)$ with relations $\{\rho_i \mid i \in I_B\}$, respectively, that is, we consider the algebras $kQ^*/ < \{\rho_i^* \mid i \in I_C\} >$ and $k\Gamma^*/ < \{\rho_i^* \mid i \in I_B\} >$, where Q^* is the opposite quiver of Q and the multiplication $\alpha\beta$ of two arrows α and β means that α comes first and then β follows (for the notation see [R, Chapt.2] for details).

Now we assume that $Q_0 = \Gamma_0$ and define a new k–algebra A given by the quiver $\overline{Q} = (Q_0, Q_1\dot{\cup}\Gamma_1)$ with relations $\{\rho_i \mid i \in I_C\} \cup \{\rho_j \mid j \in I_B\} \cup \{\alpha\beta \mid \alpha \in Q_1$ and $\beta \in \Gamma_1\}$. Then A is a finite–dimensional k–algebra with the maximal semisimple subalgebra $S = kQ_0$. It is clear that B and C are subalgebras of A with $C \cap B = S$. We call A the dual extension of C and B, denoted by $\mathcal{A}(C, B)$. In case B is the opposite algebra C^{op} of C, we simply say that A is the dual extension of C, denoted by $\mathcal{A}(C)$.

The following lemma collects some properties of the algebra A. (For the definition of quasi–hereditary algebras and BGG–algebras we refer to [CPS] and [I]).

Lemma. (1) $rad(B)rad(C) = 0$.
(2) ${}_CA_B \cong {}_CC \otimes_S B_B$.
(3) If C is a quasi–hereditary algebra with the weight poset $(Q_0, \leq)$ and B has no oriented cycle in its quiver, then $A = \mathcal{A}(C, B)$ is quasi– hereditary. Moreover, the dual extension of B is always a BGG–algebra.

Proof. (1) and (2) follow from the definiton of the algebra A. (3) is proved in [X, 1.6] and [K1].

2. Global dimension

In this section we shall use the properties in section one to estimate the global dimension of the algebra A of the form ${}_CA_B \cong {}_CC \otimes_S B_B$ and give a formula to compute the global dimension of dual extensions.

Throughout this section we keep the assumptions on A, B, C and S at the beginning in section one.

2.1 Theorem. Suppose $rad(B)C \subset Crad(B)$. Then

$$max\{\text{gl.dim}(B), \text{gl.dim}(C)\} \leq \text{gl.dim}(A) \leq \text{gl.dim}(B) + \text{gl.dim}(C).$$

Proof. The first inequality follows immediately from [CE, Chap. VI, Prop. 4.1.3] since simple A-modules and simple B–modules (or simple C–modules) coincide. To prove the second inequality we may assume that gl.dim (C) and gl.dim (B) are finite. Let $E(i)$ be a simple A–module. As a C–module $E(i)$ has the projective dimension at most gl.dim(C). By 1.2 and 1.4, we have $Ce_i \cong A \otimes_B E(i)$ as A–modules. This means that proj.dim ${}_ACe_i \leq$ gl.dim (B). Thus proj.dim ${}_AC \leq$ gl.dim (B). By 1.3 and the change of rings, we have

$$\text{proj.dim}_A E(i) \leq \text{proj.dim}_C E(i) + \text{proj.dim}_A C \leq \text{gl.dim}(C) + \text{gl.dim}(B)$$

for each simple A–module $E(i)$. Thus Theorem 2.1 follows.

2.2 Lemma. Suppose $rad(B)C \subset Crad(B)$. Let M be a C–module. Then M can be regarded as an A–module via the canonical surjective map $A \longrightarrow A/Crad(B) \cong C$ and there is an exact sequence of A–modules

$$0 \longrightarrow Crad(B) \otimes_C M \longrightarrow A \otimes_C M \xrightarrow{\mu} M \longrightarrow 0$$

where $\mu : A \otimes_C M \longrightarrow M$ is given by $\sum a_i \otimes x_i \longrightarrow \sum a_i x_i$.

Proof. Since $Crad(B)$ is the ideal in A generated by rad(B) and there is the following exact sequence

$$0 \longrightarrow Crad(B) \longrightarrow A \longrightarrow C \longrightarrow 0,$$

the lemma follows evidently from tensoring the sequence by ${}_CM$.

2.3 Lemma. Suppose that rad(B)rad$(C) = 0$. Let M be a C–module and $g : P_C(M) \longrightarrow {}_CM$ a projective cover of the C–module M. We denote by $\Omega_C(M)$ the kernel of g. Then the composition map $\mu g : A \otimes_C P_C(M) \longrightarrow M$ of $\mu : A \otimes_C P_C(M) \longrightarrow P_C(M)$ and g is a projective cover of the A–module M and $\Omega_C(M)$ is a direct summand of the kernel of μg.

Proof. Since $Ae_i \cong A \otimes_C Ce_i$, we know that $A \otimes_C P_C(M)$ has the same top as the A–module $P_C(M)$. Thus the composition map μg is a projective cover of the A–module M. To prove the last statement, we consider the following commutative diagram of A–modules:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & Crad(B)\otimes_C \Omega_C(M) & \xrightarrow{\alpha'} & A\otimes_C \Omega_C(M) & \longrightarrow & \Omega_C(M) & \longrightarrow & 0\\
 & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle 1\otimes\iota} & & \downarrow & & \\
0 & \longrightarrow & Crad(B)\otimes_C P_C(M) & \xrightarrow{\beta} & A\otimes_C P_C(M) & \xrightarrow{\mu} & P_C(M) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & Crad(B)\otimes_C M & \longrightarrow & A\otimes_C M & \longrightarrow & M & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where i is the inclusion map $\Omega_C(M) \longrightarrow P_C(M)$.

A diagram chase shows that we have the following exact sequence

$$0 \longrightarrow \mathrm{Im}(\alpha\beta) \longrightarrow \mathrm{Ker}(\mu g) \longrightarrow \Omega_C(M) \oplus Crad(B) \otimes_C M \longrightarrow 0.$$

Note that $\mathrm{Im}(\alpha\beta) = \mathrm{Im}(\alpha'(1 \otimes i))$ and α' is the inclusion map in 2.2. Since $\Omega_C(M) \subset rad(C)P_C(M)$, the map i is a composition of the canonical inclusion maps $i_1 : \Omega_C(M) \longrightarrow rad(C)P_C(M)$ and $i_2 : rad(C)P_C(M) \longrightarrow P_C(M)$. Thus $1 \otimes i = (1 \otimes i_1)(1 \otimes i_2)$. Suppose $x \in \Omega_C(M)$, then $x = ry$ with $r \in rad(C)$ and $y \in P_C(M)$. This implies that for each $a \in Crad(B)$,

$$\begin{aligned}(a \otimes x)\alpha'(1 \otimes i) &= (a \otimes x)(1 \otimes i) = (a \otimes ry)(1 \otimes i_1)(1 \otimes i_2) \\ &= (a \otimes ry)(1 \otimes i_2) = a \otimes ry = ar \otimes y = 0 \otimes y = 0\end{aligned}$$

since $\mathrm{rad}(B)\mathrm{rad}(C)=0$. Thus $\mathrm{Im}(\alpha\beta)=0$ and the lemma follows.

2.4 Lemma. For any (finitely generated) A-module M and any simple A-module E, the following two numbers coincide: the multiplicity of the projective cover of E in the n-th term of the minimal projective resolution of M; the dimension of $\mathrm{Ext}^n_A(M, E)$ over the skewfield $\mathrm{End}_A(E)$.

Now we can prove our main result.

2.5 Theorem. Let C be a basic algebra with gl.dim $(C)= m$. Then the global dimension of the dual extension of C is $2m$.

Proof. By 2.1, we may assume that gl.dim $(C)=m < \infty$. Suppose $\mathrm{proj.dim}_C E(i) = m$. Then there exists a simple module $E(j)$ such that $\mathrm{Ext}^m_C(E(i), E(j)) \neq 0$. This implies that $\mathrm{inj.dim}_C E(j) = m$ and $\mathrm{proj.dim}_{C^{op}} E(j) = m$. Suppose we are given a minimal projective resolution of the C-module $E(i)$:

$$0 \rightarrow P'_m(E(i)) \rightarrow \ldots \rightarrow P'_0(E(i)) \rightarrow E(i) \rightarrow 0.$$

Then $P_C(j) := P_C(E(j))$, the projective cover of the C-moddule $E(j)$, is a direct summand of $P'_m(E(i))$ by 2.4.

Consider a minimal projective resolution of the A-module $E(i)$:

$$0 \rightarrow P_{2m} \rightarrow \ldots \rightarrow P_m \rightarrow \ldots \rightarrow P_1 \rightarrow P_0 \rightarrow E(i) \rightarrow 0.$$

We have to show that $P_{2m} \neq 0$. According to 2.3, we know that $P'_m(E(i))$ is a direct summand of the kernel of the map $P_{m-1} \longrightarrow P_{m-2}$. Thus $P_C(j)$ is a direct summand of the Kernel. But it follows from 1.1, 1.2 and 1.4 that $P_C(j) \cong A \otimes_{C^{op}} E(j)$ and proj.dim ${}_A P_C(j) = m$. This means that the A-projective resolution of $E(i)$ contains the A-projective resolution of $P_C(j)$, and this resolution begins at the step $m+1$. Thus $P_{2m} \neq 0$.

2.6 Recently, K.Yamagata gives in [Y] a construction of algebras with large global dimensions, which generalizes an example of Green in [G], his construction depends upon the decomposition of the starting algebra into a direct sum of indecomposable projective modules. In fact, the dual extension provides us another construction of algebras with a fixed number of simple modules and large global dimensions. In our case, the global dimensions increase exponentially.

Let A be a basic algebra. We define $A_0 = A$, $A_1 = \mathcal{A}(A_0)$ and inductively, $A_n = \mathcal{A}(A_{n-1})$. Then we have the follwing

Proposition. (1) gl.dim $(A_n)=2^n\cdot$ gl.dim (A).

(2) Let $C(A_n)$ be the Cartan-matrix of A_n, then $C(A_n) = (C(A)C(A)^T)^{2^{n-1}}$ is symmetric for $n \geq 1$, where T denote the transpose of a matrix.

(3) det $C(A_n) = (\det C(A))^{2^n}$.

Proof. (1) follows from 2.5 and, (2) and (3) follows from 1.6.

2.7 Remark. (1) Theorem 2.5 shows that the upper bound in 2.1 can be attained. If one takes C or B to be the semisimple algebra S in 1.7 then the lower bound in 2.1 can be obtained. However, one can not hope gl.dim (A) = gl.dim (B) + gl.dim (C) in general as the following counterexample shows: If we take in 1.8 the algebras B and C to be the path algebra of the quiver $\circ \longleftarrow \circ$, respectively, then the extension algebra of C and B is just the Kronecker algebra. Thus the global dimensions of A, B and C are 1 and gl.dim $(A) \neq$ gl.dim (B) + gl.dim (C).

(2) Let us finally remark that if C has no oriented cycle in its quiver then the dual extension algebra A is quasi-hereditary. In this case, Theorem 2.5 can be deduced from [K2], using the quasi-heredity of A.

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