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A Self-injective Cellular Algebra Is Weakly Symmetric

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1. INTRODUCTION

Textbooks on representation theory of finite groups or related topics often use the following hierarchy of finite-dimensional algebras:

$$\begin{array}{c} \{\textit{semisimple}\} \\ \cap \\ \{\textit{symmetric}\} \\ \cap \\ \{\textit{weakly symmetric}\} \\ \cap \\ \{\textit{quasi-Frobenius} = \textit{self-injective}\} \end{array}$$

Here "self-injective" means that each projective module is injective as well, whereas "weakly symmetric" says that the projective cover of any

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given simple module is the injective envelope of the same simple module. That is, the permutation $\mathrm{top}(P) \mapsto \mathrm{soc}(P)$ (for P indecomposable projective–injective) is the identity. It is well known that in general all these inclusions are proper.

The main result of this note states that inside the class of cellular algebras (which contains group algebras of symmetric groups, various kinds of Hecke algebras, Brauer algebras, Temperley–Lieb algebras, etc.; see [2]) the third one of these inclusions is an identity.

THEOREM 1.1. *Let A be a cellular algebra over a field. If A is self-injective, then it is weakly symmetric.*

We show by examples that the other inclusions given above still are proper, if one restricts one's attention to cellular algebras.

As a consequence of the theorem we obtain

COROLLARY 1.2. *Let A be a self-injective cellular algebra over a field. Let P be an indecomposable projective A-module. Then either P is simple or it has a cell filtration of length at least two. If a cell ($=$ standard) module Δ is projective, then it must be simple.*

In Section 2 we recall the definition of cellular algebras, and then we prove Theorem 1.1. Moreover, we show by examples that the other inclusions given above are proper. In Section 3 we discuss some other properties of self-injective cellular algebras. This includes a proof of Corollary 1.2.

2. PROOF OF THEOREM AND SOME EXAMPLES

We first have to recall the two equivalent definitions of cellular algebras given in [2] and (later) in [3].

For simplicity we stick to the ground ring being an (arbitrary) field k. By algebra we always mean a finite-dimensional associative algebra with unit.

DEFINITION 2.1 (Graham and Lehrer, [2]). An associative k-algebra A is called a *cellular algebra* with cell datum (I, M, C, i) if the following conditions are satisfied:

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has a k-basis $C^{\lambda}_{S,T}$ where (S, T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is a k-linear antiautomorphism of A with $i^2 = id$, which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^{\lambda}_{S,T}$ can be written as $(\Sigma_{U \in M(\lambda)} r_a(U, S)C^{\lambda}_{U,T}) + r'$, where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $r_a(U, S) \in k$ do not depend on T.

In the following we shall call a k-linear antiautomorphism i of A with $i^2 = id$ an *involution of* A. In [3] it has been shown that the previous definition is equivalent to the following one.

DEFINITION 2.2 ([3]). Let A be a k-algebra. Assume there is an antiautomorphism i on A with $i^2 = id$. A two-sided ideal J in A is called a *cell ideal* if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that Δ has finite k-dimension and that there is an isomorphism of A-bimodules α: $J \simeq \Delta \otimes_k i(\Delta)$ (where $i(\Delta) \subset J$ is the i-image of Δ), making the following diagram commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_k i(\Delta) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle x \otimes y \mapsto i(y) \otimes i(x)} \\ J & \xrightarrow{\alpha} & \Delta \otimes_k i(\Delta) \end{array}$$

The algebra A (with the involution i) is called *cellular* if and only if there is a vector space decomposition $A = J'_1 \oplus J'_2 \oplus \cdots \oplus J'_n$ (for some n) with $i(J'_j) = J'_j$ for each j and such that setting $J_j = \oplus_{l=1}^{j} J'_l$ gives a chain of two-sided ideals of A: $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ (each of them fixed by i) and for each j ($j = 1, \ldots, n$) the quotient $J'_j = J_j/J_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1}.

The modules $\Delta(j)$, $1 \leq j \leq n$, are called *standard* modules of the cellular algebra A, and the above chain in A is called a *cell chain*. (Standard modules are called *cell modules* in [2].)

Before we can prove Theorem 1.1 we have to demonstrate a lemma.

LEMMA 2.1. *Let A be an algebra (over a field) with cell ideal J (with respect to some involution i). Then all indecomposable projective–injective left A-ideals intersecting nontrivially with J are isomorphic.*

Proof. Let P_a be an indecomposable projective–injective left A-ideal which intersects J nontrivially. We can write P as Ae for some idempotent $e = e^2 \in A$. By definition, J has the form $J \simeq \Delta \otimes_k i(\Delta)$. Hence the intersection $Ae \cap J$ is isomorphic to $\Delta \otimes_k i(\Delta)e$, which is a direct sum of

copies of Δ. But the socle L of P is simple. Hence $J \cap Ae$ has precisely one direct summand Δ, and this module must have a simple socle which is isomorphic to L as well. Thus if Δ is a submodule of an injective module, say I, then I must be isomorphic to P. ∎

Proof of 1.1. We fix a self-injective cellular algebra A over a field k. We fix a cell chain $0 = J_0 \subset J_1 \subset \cdots \subset J_n = A$, and we fix the involution i. We write the identity as a complete sum $1 = \sum_j e_j$ of pairwise orthogonal nonequivalent idempotents, and we decompose the algebra into $A = \bigoplus_{j,l} e_j A e_l$. (Note that the idempotents e_j are not required to be primitive. We only have to separate the equivalence classes of idempotents from each other.)

We denote the indecomposable projective left A-ideals by $P_1, \ldots, P_n$, where the ordering is compatible with the partial order in the cell structure in the following way: let P_a and P_b be two projective left ideals, and let J be the smallest ideal in the cell chain intersecting P_a nontrivially. Then if J intersects P_b nontrivially, we must have $b \leq a$. In fact, let $J = J_s$ be the smallest ideal in the cell chain with the property that $J \cap P_a \neq 0$ and consider the factor algebra $\bar{A} := A/J_{s-1}$. We denote by $\bar{J}$ (respectively, $\bar{a}$) the image of J (respectively, $a \in A$) under the canonical surjective mapping from A to A/J_{s-1}. Thus $\bar{J}$ is a cell ideal in $\bar{A}$, and it gives us a standard module Δ. Since $P_a \cap J_{s-1} = 0$, the projective A-module $P_a = Ae_a$ is also a projective-injective $\bar{A}$-module, where e_a is a primitive idempotent element in A. Thus we are in the situation of Lemma 2.1, which implies Δ cannot be a submodule of any other injective $\bar{A}$-module. This implies that if $P_b, b \neq a$, is another indecomposable projective–injective A-module with $P_b \cap J \neq 0$, then $P_b \cap J_{s-1} \neq 0$, and hence the required compatibility is possible.

By assumption, each P_a is injective, i.e., $P_a \simeq I_{\sigma(a)}$, where the index $\sigma(a)$ is given by some permutation σ and $I_{\sigma(a)}$ is the injective hull of the simple module $top(P_a)$. We want to show that $a = \sigma(a)$ for every a. We proceed inductively.

Induction start: $a = 1$. By assumption $J = J_1$ intersects P_1 nontrivially. Thus Δ_1 has a simple socle $L_{\sigma(1)}$, and this socle is generated by an element x_1 which is a path from 1 to $\sigma(1)$. This means that x_1 can be written as a product $y_{\sigma(1), l_1} \cdots y_{l_j, l_{j+1}} \cdots y_{l_m, 1}$, where each element $y_{l_j, l_{j+1}}$ is in

$$e_{l_j} \cdot \left(\mathrm{rad}(A) - \mathrm{rad}^2(A)\right) \cdot e_{l_{j+1}}.$$

(Note that the "arrows" $y_{l_j, l_{j+1}}$ in such a path start and end at equivalence classes of primitive idempotents, not at primitive idempotents. Note, moreover, that the order in the product is chosen in such a way that the

starting point 1 is at the very right and the ending point $\sigma(1)$ is at the very left.) It is known ([3], Proposition 5.1) that the involution i sends an idempotent to an idempotent in the same equivalence class. Thus the i-image $i(x_1)$ is a path from $\sigma(1)$ to 1 in the same sense. But J intersects all other projective (left or right) ideals trivially; hence $fJ = Jf = 0$ for all primitive idempotents f with $Af \not\simeq P_1$, since, for example, $fJ \neq 0$ would imply $J \cap fA \neq 0$, a contradiction. Since J is fixed by i, the element $i(x_1)$ must be in J again. This shows that the socle of J has a direct summand isomorphic to L_1, the simple quotient of P_1. But we know already that the socle of J is a direct sum of copies of $L_{\sigma(1)}$ since $P_1 \simeq I_{\sigma(1)}$. This implies $1 = \sigma(1)$.

Induction step: We have to consider the case of P_j. We know already that top $P_i = \mathrm{soc}(P_i)$ for $1 \leq i < j$. Hence $\sigma(j)$ must be greater than or equal to j. Let J_s be the smallest ideal in the cell chain with $P_j \cap J_s \neq 0$. Let $\bar{A} = A/J_{s-1}$. As above we get that the standard module Δ corresponding to the cell ideal $\bar{J} = J_s/J_{s-1}$ can be considered as a submodule of P_j with a simple socle $L_{\sigma(j)}$ and that $\bar{J}$ intersects the (projective–injective) modules $P_{j+1}, \ldots, P_n$ trivially. (Note that all the P_t, with $t \geq j+1$, have trivial intersection with J_s, so they are $\bar{A}$-modules as well.) The socle L of P_j is generated by an element x which is a path (as defined above) from j to $\sigma(j)$.

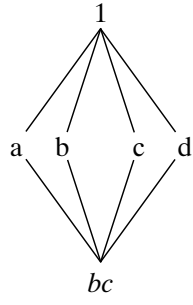
We claim that multiplying x on the right by any radical element gives zero. In fact, such a right multiplication defines a homomorphism φ from P_j to $\bar{A}$, which has an image in $\mathrm{rad}(\bar{A})$. If φ is an injective map, then its image (being isomorphic to the injective module P_j) splits off. This contradicts the choice of φ. Hence φ cannot be injective, and its kernel must contain the socle of P_j, which proves the claim. (Of course, multiplication on the left also gives zero, since x is in the socle.)

Hence multiplying $i(x)$ on the left by any radical element of $\bar{A}$ also gives zero, which means that it must be in the socle of $\bar{A}$. Since $\bar{J}$ is fixed by i, the element $i(x)$ stays inside J, the socle of which is a direct sum of copies of L. But this implies that $i(x)$ is a path ending at $\sigma(j)$, whereas by construction the path should end at j. Therefore j equals $\sigma(j)$. ∎

The following examples show that the theorem is optimal. More precisely, the other two inclusions discussed in the Introduction are proper. Furthermore, the class of self-injective cellular algebras, which are not semisimple, is properly contained in the class of cellular algebras of infinite global dimension.

Group algebras of symmetric groups provide examples of cellular algebras, which are self-injective but not semisimple.

EXAMPLE 1. Let k be a field and let $\lambda \in k$. Let $A = k\langle a, b, c, d\rangle/I$, where I is generated by $a^2, b^2, c^2, d^2, ab, ac, ba, bd, ca, cd, db, dc, cb - \lambda bc, ad - bc, da - bc$. As a left module over itself this algebra looks as follows:



Then this algebra is symmetric if and only if $\lambda = 1$. Indeed, suppose f: $A \to k$ is a nonzero symmetric linear from whose kernel does not contain any nonzero left ideal in A. Since the socle of A is one-dimensional and is spanned by bc, we have $0 \neq f(bc) = f(cb) = \lambda f(bc)$ and $\lambda = 1$. Conversely, if $\lambda = 1$, then we define a nondegenerate linear form f by $f(1) = f(a) = f(b) = f(c) = f(d) = 0$ and $f(bc) = 1$, and this makes A a symmetric algebra for $\lambda = 1$.

Moreover, there is an involution i on A, namely, i fixes a and d, but interchanges b and c. With respect to this involution the algebra A is a cellular algebra. Let us exhibit a cell chain $J_1 \subseteq J_2 \subseteq J_3 = A$ of A by giving a basis in square form:

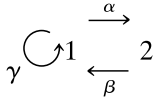
$$J_1\colon bc; \qquad J_2/J_1\colon \begin{matrix} a, & b, \\ c, & d; \end{matrix} \qquad J_3/J_2\colon 1.$$

Note that we can use a result in [4] to verify that this chain is really a cell chain of A.

If we choose λ different from 1, then we get a cellular algebra which is weakly symmetric but not symmetric.

EXAMPLE 2. Now, we give an example of a cellular algebra that has infinite global dimension but is not self-injective.

Consider the algebra A, which is given by the quiver



with relations $\gamma^2 = \alpha\beta$, $\beta\alpha = 0$, $\gamma\alpha = 0$, $\beta\gamma = 0$.

Thus A has the following decomposition as a left module over itself:

$$ {}_AA = \begin{matrix} & 1 & \\ 1 & & 2 \\ & 1 & \end{matrix} \oplus \begin{matrix} 2. \\ \\ 1 \end{matrix} $$

Clearly, this algebra has infinite global dimension and is cellular, but it is not a self-injective algebra.

3. PROPERTIES OF SELF-INJECTIVE CELLULAR ALGEBRAS

Let us first prove Corollary 1.2.

PROPOSITION 3.1. *Let A be a cellular algebra over a field k (with respect to an involution i), let P be an indecomposable projective–injective A-ideal with simple top L, and let J be a cell ideal with associated standard module Δ. Suppose the intersection $J \cap P$ is not zero. Then the composition multiplicity $[\Delta : L]$ equals one and $\mathrm{End}_A(L) = k$.*

Proof. We can write the bimodule J as $J \simeq \Delta \otimes_k i(\Delta)$ and $P = Ae$ for some primitive idempotent $e \in A$. Then $J \cap P$ equals Je, which is isomorphic (as left module) to $\Delta \otimes_k i(\Delta)e$. Denote by l the k-dimension of $e\Delta$, which equals that of $i(\Delta)e$ (by [3], Proposition 5.1). Then $J \cap P$ is a direct sum of l copies of Δ. But being contained in the injective module P, it must have a simple socle. Hence l equals one. By definition l also equals $[\Delta : L] \cdot \dim_k(\mathrm{End}_A(L))$. ∎

Proof of Corollary 1.2. Let A and P be as in the statement of Corollary 1.2. Denote by L the simple top of P. By Theorem 1.1, L is isomorphic to the socle of P. Since A is cellular, its projective module P has a cell filtration. Assume this filtration has length one, that is, P equals a standard module, say Δ. Then by Proposition 3.1, the composition multiplicity $[\Delta : L]$ equals one. But L occurs both as top and as socle; thus P must be simple. ∎

For a cellular self-injective algebra, we also have the following property.

PROPOSITION 3.2. *Let $\{C^\lambda_{s,t} | s, t \in M(\lambda)\}$ be a cell basis of a cellular algebra A. If A is self-injective, then $AC^\lambda_{s,t}$ is indecomposable for a minimal element λ and $s, t \in M(\lambda)$.*

Proof. Since A is self-injective, we have that $\mathrm{Hom}_A(Ax, I) = x\mathrm{A} \cap I$ for any left ideal I in A and $x \in A$. In particular, $\mathrm{End}_A(AC^\lambda_{s,s}) \simeq C^\lambda_{s,s}A \cap AC^\lambda_{s,s} \simeq kC^\lambda_{s,s}$, since λ is minimal. This shows that $AC^\lambda_{s,s}$ is

indecomposable. Now our proposition follows immediately from the following fact, which holds true for all cell ideals.

LEMMA 3.3. *Let A be a k-algebra with an involution i. Suppose J is a cell ideal in A, with a cell basis $\{C_{s,t} | 1 \le s, t \le n\}$. Then $AC_{j,s} \simeq AC_{j,t}$ for all j, s, and t.*

Here, a cell basis of J is that subset of the basis $C^{\lambda}_{s,t}$ given by Definition 2.1, which spans J. That is, $\lambda \in I$ is chosen minimal.

Proof. We define $r_j = (C_{j,1}, C_{j,2}, \ldots, C_{j,n})^{tr}$, the column vector of the elements $C_{j,1}, C_{j,2}, \ldots, C_{j,n}$. Then for any element $a \in A$ we have

$$ar_j = \begin{pmatrix} aC_{j,1} \\ aC_{j,2} \\ \vdots \\ aC_{j,n} \end{pmatrix} = r_a(1,j)r_1 + r_a(2,j)r_2 + \cdots + r_a(n,j)r_n,$$

where $r_a(j,l)$ is an element in k for all j, l. Now we define a mapping

$$\varphi\colon AC_{j,s} \to AC_{j,t}, \qquad \sum_l r_a(l,j)C_{l,s} \mapsto \sum_l r_a(l,j)C_{j,t}.$$

Since J is a cell ideal, this mapping is well defined. Clearly, it is injective and surjective. Moreover, φ is also an A-homomorphism by the definition of a cell ideal. ∎

Under the assumption of A being self-injective, we can say more about this situation.

PROPOSITION 3.4. *If A is a self-injective algebra with an involution i and J is a cell ideal with a cell basis $\{C_{s,t}\}$ such that $i(C_{s,t}) = C_{t,s}$, then each nonzero homomorphism from $AC_{s,s}$ to $AC_{t,t}$ is an isomorphism.*

Proof. Since A is self-injective, we have that $\mathrm{Hom}_A(AC_{s,s}, AC_{t,t}) \cong C_{s,s}A \cap AC_{t,t} \subset kC_{s,t}$. Let f_0 be a nonzero homomorphism from $AC_{s,s}$ to $AC_{t,t}$. Then $C_{s,s}A \cap AC_{t,t} = kC_{s,t}$ and there are two elements $a, b \in A$ such that $C_{s,s}a = C_{s,t} = bC_{t,t}$. We define an A-homomorphism $f\colon AC_{s,s} \to AC_{t,t}$ by $xC_{s,s} \mapsto xC_{s,s}a$. It is clear that f is an A-homomorphism and that f_0 is a scalar multiple of f. Thus it is sufficient to show that f is an isomorphism. Now we define $g\colon AC_{t,t} \to AC_{s,s}$ by $yC_{t,t} \mapsto yC_{t,t}i(b)$. One can check that g is well defined and is obviously an A-homomorphism. Since $C_{s,t} = bC_{t,t} = \sum_l r_b(l,t)C_{l,t}$, we have that $r_b(s,t) = 1$, $r_b(l,t) = 0$ for $l \neq s$ and $bC_{t,t}i(b) = C_{s,t}i(b) = \sum_j r_b(j,t)C_{s,j} = r_b(s,t)C_{s,s}$. This implies

that the composition of f and g is a nonzero endomorphism of $AC_{s,s}$. Thus the statement follows from $\mathrm{End}_A(AC_{s,s}) \cong k$. ∎

Many known examples of quasi-hereditary algebras arise as endomorphism rings of sums of modules satisfying certain ordering conditions. This leads us to the following problem.

Question. Let A be a weakly symmetric cellular algebra. When is the endomorphism algebra $\mathrm{End}(\oplus_{\lambda \in \Lambda, S \in M(\lambda)} AC^{\lambda}_{S,S})$ a quasi-hereditary algebra?

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