ON A CLASS OF BCI-ALGEBRAS

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Abstract. In this paper, we discuss the BCI-algebras satisfying $(x*y)*z \leq x*(y*z)$ and give some properties of such algebras.

In 1966, K. Iséki introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra, let us recall the definition:

Definition. *A BCI-algebra is an algebra* $(X, *, 0)$ *of type* $(2, 0)$ *with the following conditions:*

(1) $((x*y)*(x*z))*(z*y) = 0$,
(2) $(x*(x*y))*y = 0$,
(3) $x*x = 0$,
(4) $x*y = 0 = y*x$ *implies* $x = y$.

In a BCI-alegebra X, *the following was shown to be true:*

(5) $x*0 = x$,
(6) $(x*y)*z = (x*z)*y$,
(7) $x \leq y$ *implies* $x*z \leq y*z$ *and* $z*y \leq z*x$.

We will use the standard terminology and the facts mentioned in our joint paper [1].

In [2], Q. P. Hu and K. Iséki discussed the BCI-algebra satisfying $(x*y)*z = x*(y*z)$, which is called an associative BCI-algebra. In this note, we want to study the BCI-algebra satisfying $(x*y)*z \leq x*(y*z)$ for all x, y, z in the algebra, which is called a quasi-associative BCI-algebra.

First of all, we give some examples to show that quasi-associative BCI-algebras do exist.

Example.

1. *Ang associative BCI-algebra is quasi-associative.*
2. *Ang BCK-algebra is quasi-associative.*
3. *Let* $X = \{0, a, b\}$ *and* $*$ *be given by the table:*

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

Then X is quasi-associative, but not associative.

Theorem 1. *Let X be a BCI-algebra and $P(X)$ the p-radical of X (see [1]). Then the following conditions are equivalent:*

(i) *X is quasi-associative,*
(ii) *$0 * x = 0 * (0 * x)$ for each x in X,*
(iii) *$X/P(x)$ is associative.*

Proof. (i) implies (ii). Assume X is quasi-associative, we have

$$(0 * 0) * z \leq 0 * (0 * z),$$

that is, $0 * z \leq 0 * (0 * z)$. On the other hand, we also have

$$(0 * (0 * z)) * (0 * z) \leq 0 * ((0 * z) * (0 * z)) = 0$$

this means $0 * (0 * z) \leq 0 * z$, thus we have $0 * z = 0 * (0 * z)$.

(ii) implies (i). Assume that $0 * x = 0 * (0 * x)$ holds for all x in X. By (6), we have $(0 * x) * x = (0 * (0 * x)) * x = (0 * x) * (0 * x) = 0$. Thus

$$\begin{aligned} ((x * y) * z) * (x * (y * z)) &= ((x * y) * (x * (y * z))) * z \qquad \text{(by (6))} \\ &\leq ((y * z) * y) * z \qquad \text{(by (1) and (7))} \\ &= ((y * y) * z) * z \qquad \text{(by (6))} \\ &= (0 * z) * z = 0, \end{aligned}$$

therefore X is quasi-associative.

(ii) implies (iii). According to a theorem in [2], we only need to show that $C_0 * C_x = C_x$ holds in $X/P(X)$. From [1] it follows that $X/P(X)$ is p-semisimple. By the theorem 1 in [1], we have

$$C_0 * C_x = C_0 * (C_0 * C_x) = C_x,$$

thus the condition (iii) holds.

(iii) implies (ii). First, we prove the following lemma.

Lemma 1. *For ang BCI-algebra $(X, *, 0)$ we have*

$$0 * (x * y) = (0 * x) * (0 * y).$$

Proof. Since

$$\begin{aligned}
&((0*x)*(0*y))*(0*(x*y)) \\
=&((0*(0*y))*x)*(0*(x*y)) && \text{(by (6))} \\
=&(((0*(0*y))*(0*(x*y))*x && \text{(by (6))} \\
\le&((x*y)*(0*y))*x && \text{(by (1) and (7))} \\
=&((x*y)*x)*(0*y) && \text{(by (6))} \\
=&(0*y)*(0*y) && \text{(by (6) and (3))} \\
=&0,
\end{aligned}$$

we have $(0*x)*(0*y) \le 0*(x*y)$. On the other hand, since $(0*y)*(x*y) \le 0*x$, we have $0*(x*y) = ((0*y)*(x*y))*(0*y) \le (0*y)*(0*y)$. Hence we now get $0*(x*y) = (0*x)*(0*y)$ by the definition of a BCI-algebra.

Now assume that $X/P(X)$ is associative, according to [2], we have $C_0 * C_x = C_{0*x} = C_x$, that is, $(0*x)*x \in P(X)$ and $x*(0*x) \in P(X)$, and therefore it follows from lemma 1 that $(0*(0*x))*(0*x) = 0*((0*x)*x) = 0 = 0*(x*(0*x)) = (0*x)*(0*(0*x))$ holds, which means $0*x = 0*(0*x)$. Thus we have finished the proof of the theorem 1.

From the theorem 1 we have

Theorem 2. *Let X and Y be quasi-associative BCI-algebras and I an ideal of X, then*

(i) *X/I is also quasi-associative,*
(ii) *the product $X \times Y$ of X and Y is quasi-associative. (see [1])*

Theorem 3. *Let X be a BCI-algebra, then the following conditions are equivalent:*

(i) *X is quasi-associative,*
(ii) *$0*x \le x$ for all x in X,*
(iii) *$0*(x*y) = 0*(y*x)$ for all x, y in X,*
(iv) *$(0*x)*y = 0*(x*y)$,*
(v) *$(x*y)*(y*x) \in P(X)$ for all x, y in X.*

Proof. We first prove that

(8) $0*x \le x$ if and only if $0*x = 0*(0*x)$.

If $0*x \le x$, then we have $x*(0*x) \ge 0$ and $(0*x)*x = 0$, by the lemma 1, we get $0 = 0*((0*x)*x) = (0*(0*x))*(0*x)$ and $(0*x)*(0*(0*x)) = 0*(x*(0*x)) = 0$, thus $0*x = 0*(0*x)$ holds. The other direction is trivial.

From theorem 1 we have that (i) is equivalent to (ii). By [3, lemma 3. 2] (or direct to check), (ii) implies (iii)–(v). It is easy to see that (iii) resp. (iv) implies $0*x = 0*(0*x)$.

(v) implies (8). Let $x = 0$ in (v) we get $(0*y)*y \in P(X)$ and let y be 0 we get $x*(0*x) \in P(X)$ for all x, y in X, using the lemma 1 we see that $0*x = 0*(0*x)$ holds in X. Thus we have completed the proof.

Lemma 2. *For ang BCI-algebra X we have*

$$0 * (0 * (0 * x)) = 0 * x \quad \textit{for each} \quad x \quad \textit{in} \quad X.$$

Proof. From (2) it follows that $0 * (0 * (0 * x)) \leq 0 * x$; If we replace x by 0 and z by $0 * (0 * y)$ in (1), we get

$$(0 * y) * (0 * (0 * (0 * y))) \leq (0 * (0 * y)) * y = (0 * y) * (0 * y) = 0$$

for all y in X, this means that $0 * x \leq 0 * (0 * (0 * x))$ for all x in X. Therefore we have proved the lemma 2.

Theorem 4. *Every quasi-associative BCI-algebra contains an associative BCI-subalgebra $A(X)$ such that $X/P(X) \cong A(X)$.*

Every BCI-algebra X contains a p-semisimple subalgebra $S(X)$ such that $X/P(X) \cong S(X)$.

Proof. Put $A(X) = \{x \in X | 0 * x = x\}$, by lemma 1, it is an associative subalgebra. Define a homomorphism by

$$\Phi : A(X) \longrightarrow X/P(X), \quad x \longmapsto C_x.$$

Let $\Phi(x) = C_0$ for some x in $A(X)$, this means $C_x = C_0$ and $x \in P(X)$. Hence $x = 0*x = 0$, and Φ is mono. From theorem 1 we know that $X/P(X)$ is associative, therefore $C_x = C_0 * C_x = C_{0*x}$ holds for each x in $A(X)$, which shows that Φ is an epimorphism, because $0 * x \in A(X)$.

Put $S(X) = \{x \in X | 0 * (0 * x) = x\}$, by lemma 1, lemma 2 and a theorem in [1], we know that $S(X)$ is a p-semisimple subalgebra of X. Define

$$\Phi : S(X) \longrightarrow X/P(X), \quad x \longmapsto C_x.$$

Then Φ is mono, and moreover, Φ is also epimorphism, in fact, $X/P(X)$ is p-semisimple, thus we have $C_x = C_0 * (C_0 * C_x) = C_{0*(0*x)}$ in $X/P(X)$, which means Φ is epimorphism, because $0 * (0 * x) \in S(X)$ by the lemma 2.

Definition. *An ideal I is called quasi-associative if for each x in I we have $0 * x = 0 * (0 * x)$.*

Theorem 5. *Every BCI-algebra X contains a maximal quasi-associative ideal, which is also a subalgebra of X.*

Proof. Put $Q(X) = \{x \in X | 0 * x = 0 * (0 * x)\}$, then it is a subalgebra. We show that it is also an ideal of X. Assume that y, $x * y \in Q(X)$. Note that

$Q(X) = \{x \in X | 0 * x \leq x\}$ holds, thus we have $(0*x)*(0*y) = 0*(x*y) \leq x*y$ and $(0*x)*(x*y) \leq 0*y$. By (6) we get $(0*(x*y))*x \leq 0*y$, this also means

$$0 * x = ((0*(x*y))*x)*((0*(x*y))) \leq (0*y)*(0*(x*y))$$
$$\leq (x*y)*y. \qquad \text{(by (1))}$$

Hence $(0*x)*x \leq ((x*y)*y)*x = ((x*y)*x)*y = (0*y)*y = 0$, because $y \in Q(X)$ and $0*y \leq y$. This implies that $0*x \leq x$ and $x \in Q(X)$ hold. The proof is finished.

Remark. There is a BCI-algebra X such that $Q(X/Q(X)) \neq 0$, for example, let X be the adjoint algebra of a 2-Prüfer group, then X is a desired example.

References

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