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# Higher Algebraic $K$-theory of Ring Epimorphisms

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**Abstract** Given a homological ring epimorphism from a ring $R$ to another ring $S$, we show that if the left $R$-module $S$ has a finite-type resolution, then the algebraic $K$-group $K_n(R)$ of $R$ splits as the direct sum of the algebraic $K$-group $K_n(S)$ of $S$ and the algebraic $K$-group $K_n(\mathbf{R})$ of a Waldhausen category $\mathbf{R}$ determined by the ring epimorphism. This result is then applied to endomorphism rings, matrix subrings, rings with idempotent ideals, and universal localizations which appear often in representation theory and algebraic topology.

**Keywords** Algebraic $K$-theory · Matrix subring · Universal localization · Ring epimorphism · Waldhausen category

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## 1 Introduction

A ring epimorphism $R \to S$ between rings with identity is said to be *homological* if the derived module category of the ring $S$ can be regarded as a full subcategory of the derived module category of the ring $R$ by restriction. This is equivalent to $\mathrm{Tor}_j^R(S, S) = 0$ for all $j \geq 1$ (see [6]). Such epimorphisms appear often in localizations of commutative rings as well as in representation theory (see [3, 13]).

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In algebraic $K$-theory, the precise calculation of higher algebraic $K$-groups of rings is rare and very hard, and any progress in this direction would be exciting. For a homological universal localization $\lambda : R \to S$ of rings, Neeman and Ranicki have discovered a remarkable long exact sequence of algebraic $K$-groups in [9]:

$$\cdots \longrightarrow K_{n+1}(S) \longrightarrow K_n(\mathbf{R}) \longrightarrow K_n(R) \longrightarrow K_n(S) \longrightarrow K_{n-1}(\mathbf{R}) \longrightarrow$$

$$\cdots \longrightarrow K_0(\mathbf{R}) \longrightarrow K_0(R) \longrightarrow K_0(S)$$

for all $n \in \mathbb{N}$, where $\mathbf{R}$ is a Waldhausen category determined by $\lambda$. This result extends many results in the literature (see [8]). A further consideration has been given in [7] for homological ring epimorphisms such that the chain map lifting problem has a positive solution. In general, the long exact sequence does not have to split into a series of short exact sequences. However, for calculation of algebraic $K$-groups, it is certainly of interest to know when the above long exact sequence could split. So, it is natural to ask the following question:

**Question** Given an arbitrary homological ring epimorphism $\lambda : R \to S$, under which conditions on $\lambda$ does the above long exact sequence of algebraic $K$-groups break into a series of split short exact sequences?

In this paper, we shall give an answer to the above question, that is, we prove that under a finite-type condition the algebraic $K$-groups of $R$ can be described completely by the ones of $S$ and the category $\mathbf{R}$. Further, we establish reduction formulas for calculation of algebraic $K$-groups of universal localizations, endomorphism rings, matrix subrings and rings with idempotent elements.

Before stating our result precisely, we first recall some definitions and introduce some notation.

Throughout the paper, for a given small exact category $\mathscr{E}$, we denote by $K(\mathscr{E})$ the $K$-theory space of $\mathscr{E}$, and by $K_n(\mathscr{E})$ the $n$-th homotopy group of $K(\mathscr{E})$ in the sense of Quillen (see [10]). For a ring $R$ with identity, we denote by $K(R)$ the $K$-theory space of the exact category of finitely generated projective $R$-modules, and by $K_n(R)$ the $n$-th algebraic $K$-group of $R$ for each $n \in \mathbb{N}$ (see [10]).

For a ring homomorphism $\lambda : R \to S$, we denote by $\mathbf{W}(R, \lambda)$ the complicial biWaldhausen subcategory (see [15]) of $\mathscr{C}^b(R\text{-proj})$ consisting of those complexes $X^\bullet$ in $\mathscr{C}^b(R\text{-proj})$ such that $S \otimes_R X^\bullet$ is acyclic. The cofibrations of $\mathbf{W}(R, \lambda)$ are by definition the chain maps which are split monomorphism in each degree, and the weak equivalences are the homotopy equivalences. Further, let $K(R, \lambda)$ be the $K$-theory space of the Waldhausen category $\mathbf{W}(R, \lambda)$ (see [16]), and let $K_n(R, \lambda) := K_n(\mathbf{W}(R, \lambda))$, the $n$-th homotopy group of $K(R, \lambda)$.

An $R$-module $M$ is said to have a *finite-type resolution* if it has a finite projective resolution by finitely generated projective $R$-modules, that is, there is an exact sequence $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ for some $n \in \mathbb{N}$ such that all $R$-modules $P_j$ are projective and finitely generated. By add$(M)$ we denote the category of modules which are direct summands of direct sums of finitely many copies of $M$.

Now, our result on algebraic $K$-theory of homological ring epimorphisms can be stated as follows:

**Theorem 1.1** *Let $\lambda : R \to S$ be a homological ring epimorphism. Suppose that ${}_RS$ admits a finite-type resolution. Then*

$$K_n(R) \simeq K_n(S) \oplus K_n(R, \lambda) \quad \textit{for all } n \in \mathbb{N}.$$

Observe that a special case of finite-type condition is that ${}_RS$ is a finitely generated projective $R$-module. A lot of examples of this type can be founded in Section 3 below (see also [17]).

As a natural consequence, we first apply Theorem 1.1 to universal localizations which generalize the usual localizations of commutative rings ([4], [13]).

Let $\Sigma$ be a set of homomorphisms between finitely generated projective $R$-modules. Let $\lambda_\Sigma : R \to R_\Sigma$ be the universal localization of $R$ at $\Sigma$ (see Lemma 2.3 for definition). Note that universal localizations are the "noncommutative localizations" in terminology of [9]. For methods to get homological universal localizations, we refer the reader to a recent paper [3].

Associated with $\Sigma$, there is a small Waldhausen category $\mathbf{R}$ defined in [9, Definition 0.4]. More precisely, the category $\mathbf{R}$ is the smallest full subcategory of $\mathscr{C}^b(R\text{-proj})$ which

(i) contains all the complexes in $\Sigma$,
(ii) contains all acyclic complexes,
(iii) is closed under the formation of mapping cones and shifts,
(iv) contains all direct summands of any of its objects.

Note that, in the category $\mathbf{R}$, the cofibrations are injective chain maps which are degreewise split, and the weak equivalences are homotopy equivalences.

According to [8, Theorem 0.5], if all the maps in $\Sigma$ are injective, then the $K$-theory space $K(\mathbf{R})$ of $\mathbf{R}$ in the sense of Waldhausen is homotopy equivalent to the $K$-theory space $K(\mathscr{E})$ of the exact category $\mathscr{E}$ of $(R, \Sigma)$-torsion modules. Recall that an $R$-module $M$ is called an $(R, \Sigma)$-torsion module if it is finitely presented, and of projective dimension at most 1 such that $R_\Sigma \otimes_R M = 0 = \mathrm{Tor}_1^R(R_\Sigma, M)$.

Applying Theorem 1.1 to universal localizations, we have the following corollary.

**Corollary 1.2** *Suppose that the universal localization $\lambda_\Sigma : R \to R_\Sigma$ of $R$ at $\Sigma$ is homological and that the left $R$-module $R_\Sigma$ has a finite-type resolution. Then*

$$K_n(R) \simeq K_n(R_\Sigma) \oplus K_n(\mathbf{R}) \quad \textit{for all} \;\; n \in \mathbb{N}.$$

*If in addition all the maps in $\Sigma$ are injective, then*

$$K_n(R) \simeq K_n(R_\Sigma) \oplus K_n(\mathscr{E}) \quad \textit{for all} \;\; n \in \mathbb{N}$$

*where $\mathscr{E}$ is the exact category of $(R, \Sigma)$-torsion modules.*

Note that both [9, Theorem 0.5]) and [7, Theorem 14.9] provide us only with a long exact sequence of algebraic $K$-groups, while both Theorem 1.1 and Corollary 1.2 go one step further, they describe the algebraic $K$-groups of $R$ as the direct sum of algebraic $K$-groups of a related ring and a category naturally determined by the ring epimorphism. In many cases, Corollary 1.2 gives us a handy way to estimate the algebraic $K$-groups of $R$. For example, we have the following decomposition formula for algebraic $K$-groups of rings with idempotent elements, which seems to be stronger than the excision theorem (see [14]).

**Corollary 1.3** *Let $R$ be a ring with identity, and let $e^2 = e$ be an idempotent element in $R$ such that there is an exact sequence*

$$0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow {}_RReR \longrightarrow 0$$

*with $P_j \in add(Re)$ for $0 \leq j \leq m$. Then*

$$K_n(R) \simeq K_n(R/ReR) \oplus K_n(eRe)$$

*for all $n \geq 0$.*

Next, we apply Corollary 1.3 to the endomorphism rings of objects in an additive category.

Let us first introduce the notion of covariant morphisms in an additive category, which is related to a wide variety of concepts, such as traces of modules, Auslander-Reiten sequences and GV-ideals.

Let $\mathcal{C}$ be an additive category, and let $X$ and $Y$ be objects in $\mathcal{C}$. A morphism $\lambda : Y \to X$ in $\mathcal{C}$ is said to be *$X$-covariant* if the induced map $\mathrm{Hom}_{\mathcal{C}}(X, \lambda) : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, X)$ is a split monomorphism of $\mathrm{End}_{\mathcal{C}}(X)$-modules; and *covariant* if the induced map $\mathrm{Hom}_{\mathcal{C}}(X, \lambda) : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, X)$ is injective and the induced map $\mathrm{Hom}_{\mathcal{C}}(Y, \lambda) : \mathrm{Hom}_{\mathcal{C}}(Y, Y) \to \mathrm{Hom}_{\mathcal{C}}(Y, X)$ is a split epimorphism of $\mathrm{End}_{\mathcal{C}}(Y)$-modules. For example, if $\mathcal{C}$ is the module category of a unitary ring $R$ and if $X$ is an $R$-module, then, for every submodule $Y$ of $X$ with $\mathrm{Hom}_R(Y, X/Y) = 0$, the inclusion map is covariant. In particular, for an idempotent ideal $I$ in $R$, the inclusion from $I$ into $R$ is covariant.

Let $\mathrm{End}_{\mathcal{C},Y}(X)$ denote the quotient ring of the endomorphism ring $\mathrm{End}_{\mathcal{C}}(X)$ of the object $X$ modulo the ideal generated by all those endomorphisms of $X$ which factorize through the object $Y$.

**Corollary 1.4** *Let $\mathcal{C}$ be an additive category and $f : Y \to X$ be a morphism of objects in $\mathcal{C}$.*

(1) *If $f$ is covariant, then $K_n\big(End_{\mathcal{C}}(X \oplus Y)\big) \simeq K_n\big(End_{\mathcal{C},Y}(X)\big) \oplus K_n\big(End_{\mathcal{C}}(Y)\big)$ for all $n \in \mathbb{N}$.*

(2) *If $f$ is $X$-covariant, then $K_n\big(End_{\mathcal{C}}(X \oplus Y)\big) \simeq K_n\big(End_{\mathcal{C}}(X)\big) \oplus K_n\big(End_{\mathcal{C},X}(Y)\big)$ for all $n \in \mathbb{N}$.*

As a concrete example of applications of Corollary 1.4, we have the following result.

**Corollary 1.5** *If $I$ is an idempotent ideal in a ring $R$, then $K_n(End_R(R \oplus I)) \simeq K_n(R/I) \oplus K_n(End_R(I))$ for all $n \geq 0$. In particular, if the idempotent ideal ${}_RI$ is projective and finitely generated, then $K_n(R) \simeq K_n(R/I) \oplus K_n(End_R(I))$ for all $n \geq 0$.*

Our results also apply to matrix subrings. In this case, we get several reduction formulas of algebraic $K$-groups for some classes of matrix subrings. For details, we refer the reader to the last section of this paper.

The proofs of Theorem 1.1 and its Corollaries 1.2 and 1.3 will be given in the next section where we first provide necessary materials needed in our proofs. For example, we recall the definitions of Waldhausen categories and universal localizations. In Section 3, we construct homological ring epimorphisms by using convariant morphisms, endomorphism rings and matrix subrings. As a consequence of our discussions, we present a proof of Corollary 1.4, and give reduction formulas for $K$-groups of matrix subrings and rings with idempotent ideals. At the end of this section, we display a simple example of universal localizations to illustrate the idea of Theorem 1.1.

## 2 Proofs of Theorem 1.1 and Corollaries 1.2-1.3

Let $R$ be an associative ring with identity. We denote by $R$-Mod the category of all left $R$-modules, and by $R$-mod the category of all finitely presented left $R$-modules. For an $R$-module $M$, we denote by add$(M)$ the full subcategory of $R$-Mod consisting of all direct summands of direct sums of finitely many copies of $M$. As usual, we denote by $R$-proj the category add$({}_RR)$, and by $\mathscr{C}(R)$, $\mathscr{K}(R)$ and $\mathscr{D}(R)$ the complex, homotopy and derived categories of $R$-Mod, respectively.

A complex $X^\bullet \in \mathscr{D}(R)$ is said to be *compact* if the functor $\mathrm{Hom}_{\mathscr{D}(R)}(X^\bullet, -)$ commutes with direct sums in $\mathscr{D}(R)$. It is shown that $X^\bullet$ is compact if and only if it is quasi-isomorphic to a bounded chain complex of finitely generated, projective $R$-modules (see, for example, [9, Corollary 4.4]). We denote by $\mathscr{D}^c(R)$ the full subcategory of $\mathscr{D}(R)$ consisting of compact objects. Note that $\mathscr{D}^c(R)$ is closed under triangles and direct summands (see [9, Remark 3.4]). It is known that the localization function $\mathscr{K}(R) \to \mathscr{D}(R)$ is restricted to a triangle equivalence from $\mathscr{K}^b(R\text{-proj})$ to $\mathscr{D}^c(R)$. Moreover, the intersection of $\mathscr{D}^c(R)$ with $R$-Mod, as a full subcategory of $R$-Mod, exactly consists of those $R$-modules with finite-type resolutions. This category is denoted by $\mathscr{P}^{<\infty}(R)$.

The category $R$-mod with short exact sequences forms an exact category in the sense of Quillen (see [10]), and its $K$-theory is denoted by $G_*(R)$. As usual, we denote by $K_*(R)$ the $K$-theory of $R$-proj with split exact sequences. If $R$ is left noetherian and has finite global dimension, then $K_*(R) \simeq G_*(R)$ for all $* \in \mathbb{N}$. In general, even for finite dimensional algebras over a field, the $G$-theory and $K$-theory are not isomorphic, though the former is reduced to the one of the endomorphism rings of simple modules.

Now we recall some elementary notion about the $K$-theory of Waldhausen categories (see [15, 16]).

By *a category with cofibrations* we mean a category $\mathcal{C}$ with a zero object 0, together with a chosen class co$(\mathcal{C})$ of morphisms in $\mathcal{C}$ satisfying the following three axioms:

(1) Every isomorphism in $\mathcal{C}$ is in co$(\mathcal{C})$.
(2) For any object $A$ in $\mathcal{C}$, the unique morphism $0 \to A$ is in co$(\mathcal{C})$.
(3) If $X \to Y$ is a morphism in co$(\mathcal{C})$, and $X \to Z$ is a morphism in $\mathcal{C}$, then the push-out $Y \cup_X Z$ exists in $\mathcal{C}$, and the canonical morphism $Z \to Y \cup_X Z$ is in co$(\mathcal{C})$. In particular. finite coproducts exist in $\mathcal{C}$.

A morphism in co$(\mathcal{C})$ is called a *cofibration*.

Following [15], a category $\mathcal{C}$ with cofibrations is called a *Waldhausen category* if $\mathcal{C}$ admits a class w$(\mathcal{C})$ of morphisms satisfying the following two axioms:

(1) Every isomorphism in $\mathcal{C}$ is in w$(\mathcal{C})$.
(2) Given a commutative diagram

$$\begin{array}{ccccc} B & \longleftarrow & A & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ B' & \longleftarrow & A' & \longrightarrow & C' \end{array}$$

in $\mathcal{C}$ with two morphisms $A \to B$ and $A' \to B'$ being cofibrations, and with $B \to B'$, $A \to A'$ and $C \to C'$ being in w$(\mathcal{C})$, then the induced morphism $B \cup_A C \to B' \cup_{A'} C'$ is in w$(\mathcal{C})$.

The morphisms in $w(\mathcal{C})$ are called *weak equivalences*. Thus a Waldhausen category consists of a triple data: A category, cofibrations and weak equivalences.

A functor between Waldhausen categories is called an *exact* functor if it preserves zero, cofibrations, weak equivalence classes and pushouts along the cofibrations.

A typical example of Waldhausen categories can be obtained from complexes of modules over rings in the following manner:

Let $R$ be a ring with identity. Let $\mathscr{C}^b(R\text{-proj})$ be the small category consisting of all bounded complexes of finitely generated projective $R$-modules. This is a Waldhausen category, where the weak equivalences are the homotopy equivalences, and the cofibrations are the degreewise split monomorphisms.

For a small Waldhausen category $\mathcal{C}$, a $K$-theory space $K(\mathcal{C})$ was defined in [16]. For each $n \in \mathbb{N}$, the $n$-th homotopy group of $K(\mathcal{C})$ is called the *$n$-th algebraic $K$-group* of $\mathcal{C}$, denoted by $K_n(\mathcal{C})$. In particular, for the small Waldhausen category $\mathscr{C}^b(R\text{-proj})$, it is shown by a theorem of Gillet-Waldhausen that its $K$-theory is the same as the $K$-theory of $R$ in the sense of Quillen. That is, $K_n(R) \simeq K_n(\mathscr{C}^b(R\text{-proj}))$ for all $n \geq 0$.

In this paper, we assume that all Waldhausen categories considered are small, complicial biWaldhausen categories in the sense of Thomason and Trobaugh (see [15]). That is, the Waldhausen category $\mathcal{C}$ is assumed to be a full subcategory of the category $\mathscr{C}(\mathcal{A})$ of chain complexes over some abelian category $\mathcal{A}$, the cofibrations are maps of complexes which are split monomorphisms in each degree, and the weak equivalences contain the quasi-isomorphisms. Note that if $\lambda : R \to S$ is a ring homomorphism, then $S \otimes_R - : \mathscr{C}^b(R\text{-proj}) \to \mathscr{C}^b(S\text{-proj})$ is an exact functor of complicial biWaldhausen categories.

The following result can be concluded from [7, Theorem 14.9], which generalizes a result on algebraic $K$-groups for homological universal localizations in [9, Theorem 0.5] to the one of algebraic $K$-groups for arbitrary homological ring epimorphisms.

**Lemma 2.1** *Let $\lambda : R \to S$ be a homological ring epimorphism. Suppose that every map $f$ in $\mathscr{K}^b(R\text{-proj})$ with $S \otimes_R f = 0$ factorizes through some $X^\bullet$ in $\mathscr{K}^b(R\text{-proj})$ such that $S \otimes_R X^\bullet = 0$. Then the inclusion $F : \mathbf{W}(R, \lambda) \to \mathscr{C}^b(R\text{-proj})$ and the functor $S \otimes_R - : \mathscr{C}^b(R\text{-proj}) \to \mathscr{C}^b(S\text{-proj})$ induce a long exact sequence of algebraic $K$-groups:*

$$\cdots \longrightarrow K_{n+1}(S) \longrightarrow K_n(R, \lambda) \xrightarrow{K_n(F)} K_n(R) \xrightarrow{K_n(S\otimes_R -)} K_n(S) \longrightarrow K_{n-1}(R, \lambda) \longrightarrow$$
$$\cdots \longrightarrow K_0(R, \lambda) \longrightarrow K_0(R) \longrightarrow K_0(S)$$

*for all $n \in \mathbb{N}$.*

The following classical 'resolution theorem' on algebraic $K$-theory of exact categories is well known (see, for example, [10, Section 4]).

**Lemma 2.2** *Let $\mathscr{E}'$ be a full subcategory of a small exact category $\mathscr{E}$. Assume that the following two conditions hold:*

(*a*) *If $X \rightarrowtail Y \twoheadrightarrow Z$ is a conflation in $\mathscr{E}$ with $Z \in \mathscr{E}'$, then $Y \in \mathscr{E}'$ if and only if $X \in \mathscr{E}'$.*

(*b*) *For any object $M \in \mathscr{E}$, there is an exact sequence in $\mathscr{E}$ :*

$$0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

*such that $M_i \in \mathscr{E}'$ for all $0 \leq i \leq n$.*

*Then the inclusion $\mathscr{E}' \subseteq \mathscr{E}$ of exact categories induces a homotopy equivalence of $K$-theory space*

$$K(\mathscr{E}') \xrightarrow{\sim} K(\mathscr{E}).$$

*Proof* of Theorem 1.1.

We first show that the ring homomorphism $\lambda : R \to S$ in Theorem 1.1 satisfies the assumptions of Lemma 2.1.

Note that the localization functor $\mathscr{K}(R) \to \mathscr{D}(R)$ is restricted to a triangle equivalence $\mathscr{K}^b(R\text{-proj}) \xrightarrow{\simeq} \mathscr{D}^c(R)$. So, it is sufficient to show that if $f : P^\bullet \to Q^\bullet$ is a homomorphism in $\mathscr{D}^c(R)$ such that $S\otimes^L_R f = 0$, then $f$ factorizes through some object $X^\bullet \in \mathscr{D}^c(R)$ such that $S\otimes^L_R X^\bullet = 0$, where $S\otimes^L_R - : \mathscr{D}(R) \to \mathscr{D}(S)$ denotes the total left-derived tensor functor of $S \otimes_R - : \mathscr{K}(R) \to \mathscr{K}(S)$.

Let $D(\lambda_*) : \mathscr{D}(S) \to \mathscr{D}(R)$ be the restriction functor induced from $\lambda$. Then $\big(S\otimes^L_R -, D(\lambda_*)\big)$ is an adjoint pair of functors. Furthermore, let

$$\varepsilon : Id_{\mathscr{D}(R)} \longrightarrow D(\lambda_*)(S\otimes^L_R -)$$

be the unit adjunction. Then we have a canonical morphism $\varepsilon_{Q^\bullet} : Q^\bullet \to {}_R S\otimes^L_R Q^\bullet$ in $\mathscr{D}(R)$. Now, we extend $\varepsilon_{Q^\bullet}$ to a triangle $X^\bullet \longrightarrow Q^\bullet \xrightarrow{\varepsilon_{Q^\bullet}} S\otimes^L_R Q^\bullet \longrightarrow X^\bullet[1]$ in $\mathscr{D}(R)$, and claim that $X^\bullet$ is a desired object.

Actually, since $\lambda$ is a homological ring epimorphism, the functor $D(\lambda_*)$ is fully faithful. This implies that the counit adjunction $(S\otimes^L_R -)D(\lambda_*) \to Id_{\mathscr{D}(S)}$ is a natural isomorphism. Consequently, the morphism

$$S\otimes^L_R \varepsilon_{Q^\bullet} : S\otimes^L_R Q^\bullet \longrightarrow S\otimes^L_R (S\otimes^L_R Q^\bullet)$$

is an isomorphism in $\mathscr{D}(S)$. Thus $S\otimes^L_R X^\bullet = 0$. Note that the unit adjunction gives rise to the following commutative diagram in $\mathscr{D}(R)$:

$$\begin{tikzcd} P^\bullet \arrow[r, "f"] \arrow[d, "\varepsilon_{P^\bullet}"] & Q^\bullet \arrow[d, "\varepsilon_{Q^\bullet}"] \\ S\otimes^{\mathbf{L}}_R P^\bullet \arrow[r, "S\otimes^{\mathbf{L}}_R f"] & S\otimes^{\mathbf{L}}_R Q^\bullet \end{tikzcd}$$

Since $S\otimes^L_R f = 0$, we have $f\varepsilon_{Q^\bullet} = 0$. Note that the functor $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, -)$ is homological. Thus it follows from the triangle that $f$ factorizes through $X^\bullet$.

It remains to show $X^\bullet \in \mathscr{D}^c(R)$. Indeed, by assumption, we have ${}_RS \in \mathscr{D}^c(R)$. Since each compact object of $\mathscr{D}^c(S)$ is quasi-isomorphic to a bounded complex of finitely generated, projective $S$-modules, the functor $D(\lambda_*)$ preserves compact objects. In other words, $D(\lambda_*)$ is restricted to a functor $\mathscr{D}^c(S) \to \mathscr{D}^c(R)$. Clearly, the tensor functor $S\otimes^L_R -$ always preserves compact objects. Since $Q^\bullet \in \mathscr{D}^c(R)$, we have ${}_RS\otimes^L_R Q^\bullet \in \mathscr{D}^c(R)$. Note that $\mathscr{D}^c(R)$ is closed under triangles, it follows that $X^\bullet \in \mathscr{D}^c(R)$.

Thus, we have shown that $\lambda$ satisfies the assumptions of Lemma 2.1. Therefore, by Lemma 2.1, we have a long exact sequence

$$\cdots \longrightarrow K_{n+1}(S) \longrightarrow K_n(R,\lambda) \xrightarrow{K_n(F)} K_n(R) \xrightarrow{K_n(S\otimes_R -)} K_n(S) \longrightarrow K_{n-1}(R,\lambda) \longrightarrow$$
$$\cdots \longrightarrow K_0(R,\lambda) \longrightarrow K_0(R) \longrightarrow K_0(S)$$

for all $n \in \mathbb{N}$.

To show the isomorphisms in Theorem 1.1, it is enough to verify that $K_n(S \otimes_R -) : K_n(R) \to K_n(S)$ is a split surjection for each $n \in \mathbb{N}$.

Let

$$\mathscr{X} := \{M \in \mathscr{P}^{<\infty}(R) \mid \mathrm{Tor}^R_n(S, M) = 0 \text{ for all } n > 0\}.$$

Then $\mathcal{X}$ is a fully exact subcategory of $\mathscr{P}^{<\infty}(R)$. Note that both $R$-proj and $\mathcal{X}$ are closed under isomorphisms, extensions and kernels of surjective homomorphisms in $\mathscr{P}^{<\infty}(R)$, and that each module in $\mathscr{P}^{<\infty}(R)$ has a finite projective resolution with its terms belonging to $R$-proj. By Lemma 2.2, the inclusions $R\text{-proj} \subseteq \mathcal{X} \subseteq \mathscr{P}^{<\infty}(R)$ induce homotopy equivalences

$$K(R) \xrightarrow{\sim} K(\mathcal{X}) \xrightarrow{\sim} K(\mathscr{P}^{<\infty}(R)).$$

Since $\lambda$ is homological and ${}_RS \in \mathscr{P}^{<\infty}(R)$, the following two functors

$$\mathscr{P}^{<\infty}(S) \xrightarrow{\lambda_*} \mathcal{X} \quad \text{and} \quad \mathcal{X} \xrightarrow{S\otimes_R -} \mathscr{P}^{<\infty}(S)$$

are well defined and exact as functors of exact categories, such that the composition functor

$$(S \otimes_R -)(\lambda_*) : \mathscr{P}^{<\infty}(S) \longrightarrow \mathscr{P}^{<\infty}(S)$$

is naturally isomorphic to the identity functor $Id_{\mathscr{P}^{<\infty}(S)}$. By passing to $K$-theory spaces, we see that the composite of the following two maps

$$K(\lambda_*) : K(\mathscr{P}^{<\infty}(S)) \longrightarrow K(\mathcal{X}) \quad \text{and} \quad K(S \otimes_R -) : K(\mathcal{X}) \longrightarrow K(\mathscr{P}^{<\infty}(S))$$

is homotopic to the identity map $Id_{K(\mathscr{P}^{<\infty}(S))}$ of the $K$-theory space $K(\mathscr{P}^{<\infty}(S))$. In view of $n$-th algebraic $K$-groups, this implies that the homomorphism $K_n(S \otimes_R -) : K_n(\mathcal{X}) \to K_n(\mathscr{P}^{<\infty}(S))$ is a split surjection for all $n \geq 0$. It follows from the following diagram

$$\begin{tikzcd}
K_n(R) \arrow[r, "K_n(S\otimes_R -)"] \arrow[d, "\simeq"] & K_n(S) \arrow[d, "\simeq"] \\
K_n(\mathcal{X}) \arrow[r, "K_n(S\otimes_R -)"] & K_n(\mathscr{P}^{<\infty}(S))
\end{tikzcd}$$

that $K_n(S \otimes_R -) : K_n(R) \to K_n(S)$ is a split surjection for $n \geq 0$. This finishes the proof of Theorem 1.1. □

Before starting with the proof of Corollary 1.2, we first recall the definition of universal localizations and collect some of their basic properties in the following lemma.

**Lemma 2.3** ([4], [13, Theorem 4.1]) *Let $R$ be a ring and $\Sigma$ a set of homomorphisms between finitely generated projective $R$-modules. Then there is a ring $R_\Sigma$ and a homomorphism $\lambda : R \to R_\Sigma$ of rings with the following properties:*

(1) *$\lambda$ is $\Sigma$-inverting, that is, if $\alpha : P \to Q$ belongs to $\Sigma$, then $R_\Sigma \otimes_R \alpha : R_\Sigma \otimes_R P \to R_\Sigma \otimes_R Q$ is an isomorphism of $R_\Sigma$-modules, and*

(2) *$\lambda$ is universal $\Sigma$-inverting, that is, if $S$ is a ring such that there exists a $\Sigma$-inverting homomorphism $\varphi : R \to S$, then there exists a unique homomorphism $\psi : R_\Sigma \to S$ of rings such that $\varphi = \lambda\psi$.*

(3) *The homomorphism $\lambda : R \to R_\Sigma$ is a ring epimorphism with $\mathrm{Tor}_1^R(R_\Sigma, R_\Sigma) = 0$.*

The map $\lambda : R \to R_\Sigma$ in Lemma 2.3 is called the *universal localization* of $R$ at $\Sigma$. Clearly, if the left $R$-module $R_\Sigma$ is flat,or has projective dimension at most 1, then $\lambda$ is homological. But it is not always the case (see [9]).

By abuse of notation, we always identify a homomorphism $\alpha : P \to Q$ in $\Sigma$ with the two-term complex $0 \to P \xrightarrow{\alpha} Q \to 0$ in $\mathscr{C}^b(R\text{-proj})$, where $P$ and $Q$ are in degrees $-1$ and 0, respectively.

*Proof* of Corollary 1.2.

By Theorem 1.1, to show the first part of Corollary 1.2, it suffices to show $\mathbf{R} = \mathbf{W}(R, \lambda_\Sigma)$ as Waldhausen categories. Since cofibrations and weak equivalences in the two categories are defined in the same way, it is sufficient to show $\mathbf{R} = \mathbf{W}(R, \lambda_\Sigma)$ as full subcategories of $\mathscr{C}^b(R\text{-proj})$.

In fact, since $\lambda_\Sigma : R \to R_\Sigma$ is the universal localization of $R$ at $\Sigma$, we see from Lemma 2.3 that $R_\Sigma \otimes_R f$ is an isomorphism for any $f : P_1 \to P_0$ in $\Sigma$. This implies that $\Sigma \subseteq \mathbf{W}(R, \lambda_\Sigma)$. Note that $\mathbf{W}(R, \lambda_\Sigma)$ is the same as the full subcategory of $\mathscr{C}^b(R\text{-proj})$ consisting of all those complexes $X^\bullet$ such that $R_\Sigma \otimes_R^L X^\bullet = 0$ in $\mathscr{D}(R_\Sigma)$, where $R_\Sigma \otimes_R^L - : \mathscr{D}(R) \to \mathscr{D}(R_\Sigma)$ is the total left-derived functor of $R_\Sigma \otimes_R -$. This implies that $\mathbf{W}(R, \lambda_\Sigma)$ satisfies the conditions (i)-(iv), and therefore $\mathbf{R} \subseteq \mathbf{W}(R, \lambda_\Sigma)$. In the following, we show the converse inclusion: $\mathbf{W}(R, \lambda_\Sigma) \subseteq \mathbf{R}$.

Observe that $\mathbf{R}$ has the following additional properties:

(v) The category $\mathbf{R}$ is closed under finite direct sums in $\mathscr{C}^b(R\text{-proj})$.

(vi) If $N^\bullet \in \mathbf{R}$ and $M^\bullet \in \mathscr{C}^b(R\text{-proj})$ such that, in $\mathscr{K}^b(R\text{-proj})$, $M^\bullet$ is a direct summand of $N^\bullet$, then $M^\bullet \in \mathbf{R}$. In particular, $\mathbf{R}$ is closed under isomorphisms in $\mathscr{K}^b(R\text{-proj})$.

Actually, these two properties can be deduced from (ii)-(iv) with the help of the following two general facts:
Let $X^\bullet, Y^\bullet \in \mathscr{C}^b(R\text{-proj})$. Then

(1) $X^\bullet \oplus Y^\bullet$ is exactly the mapping cone of the zero map from $X^\bullet[-1]$ to $Y^\bullet$.
(2) $X^\bullet \simeq Y^\bullet$ in $\mathscr{K}^b(R\text{-proj})$ if and only if there are two acyclic complexes $U^\bullet, V^\bullet \in \mathscr{C}^b(R\text{-proj})$ such that $X^\bullet \oplus U^\bullet \simeq Y^\bullet \oplus V^\bullet$ in $\mathscr{C}^b(R\text{-proj})$.

Let $\mathscr{P}$ be the full subcategory of $\mathscr{K}^b(R\text{-proj})$ consisting of all objects of $\mathbf{W}(R, \lambda_\Sigma)$, and let $\mathscr{R}$ be the full subcategory of $\mathscr{K}^b(R\text{-proj})$ consisting of all objects of $\mathbf{R}$. Recall that both $\mathbf{W}(R, \lambda_\Sigma)$ and $\mathbf{R}$ are full subcategories of $\mathscr{C}^b(R\text{-proj})$. Here, both $\mathscr{P}$ and $\mathscr{R}$ are full subcategories of $\mathscr{K}^b(R\text{-proj})$. Then $\mathscr{P}$ is the kernel of the triangle functor $R_\Sigma \otimes_R - : \mathscr{K}^b(R\text{-proj}) \to \mathscr{K}^b(R_\Sigma\text{-proj})$, and thus a full triangulated subcategory of $\mathscr{K}^b(R\text{-proj})$ closed under direct summands. Moreover, due to (i)-(vi), we see that $\mathscr{R}$ is a full triangulated subcategory of $\mathscr{K}^b(R\text{-proj})$ containing $\Sigma$ and being closed under direct summands. It follows from (vi) that $\mathbf{W}(R, \lambda_\Sigma) \subseteq \mathbf{R}$ if and only if $\mathscr{P} \subseteq \mathscr{R}$. To check the latter, it is enough to show that $\mathscr{P}$ is exactly the smallest full triangulated subcategory of $\mathscr{K}^b(R\text{-proj})$ which contains $\Sigma$ and is closed under direct summands. Actually, this was shown in [8, Theorem 0.11]. Hence $\mathbf{W}(R, \lambda_\Sigma) \subseteq \mathbf{R}$.

Thus $\mathbf{R} = \mathbf{W}(R, \lambda_\Sigma)$. Now, Theorem 1.1 implies the first part of Corollary 1.2. Note that if all the maps in $\Sigma$ are injective, then $K_n(\mathbf{R}) \simeq K_n(\mathscr{E})$ for each $n \in \mathbb{N}$ due to [8, Theorem 0.5]. Hence the second part of Corollary 1.2 follows. □

*Proof* of Corollary 1.3.

We show that Corollary 1.3 is a consequence of Corollary 1.2. First, we note that the universal localization of $R$ at the injective map $0 \to Re$ is exactly the canonical surjective map $\lambda : R \to R/ReR$. In the following, we first prove that $\lambda$ is homological. For this purpose, we point out the following general fact.

**Lemma 2.4** *Let $R$ be a ring with identity, and let $J = ReR$ for $e^2 = e \in R$. Suppose that $M$ is an $R$-module with the following two properties:*

*(1)* $Tor_j^R(R/J, M) = 0$ *for all* $j \geq 0$, *and*

*(2)* $M$ *has a finite-type resolution, that is, there is an exact sequence* $0 \longrightarrow P'_n \xrightarrow{\epsilon'_n} \cdots \longrightarrow P'_1 \xrightarrow{\epsilon'_1} P'_0 \xrightarrow{\epsilon'_0} M \to 0$ *with all* $P'_j$ *finitely generated projective* $R$*-modules.*

*Then there is an exact sequence of* $R$*-modules:*

$$0 \to P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

*such that all* $P_j$ *lie in* $add(Re)$.

□

*Proof* This result is known for modules over Artin algebras, where one may use minimal projective resolutions (see [1]). For general rings, projective covers of modules may not exist. For convenience of the reader, we include here a proof.

Given such a sequence in (2), we define $N'_i = \mathrm{Ker}(\epsilon'_{i-1})$ for $1 \leq i \leq n$. Then $N'_i$ is finitely generated.

It follows from $\mathrm{Tor}_0^R(R/J, M) = 0$ that $JM = M$. Since the trace of $Re$ in the module $M$ is just $JM$ and since $M$ is finitely generated, there is a finite index set $I_0$ and a surjective homomorphism $P_0 := \bigoplus_{i \in I_0} Re \xrightarrow{\epsilon_0} M$. We define $N_1 = \mathrm{Ker}(\epsilon_0)$. Then, by Schanuel's Lemma, we have $N_1 \oplus P'_0 \simeq N'_1 \oplus P_0$, and therefore $N_1$ is finitely generated. It follows from $\mathrm{Tor}_1^R(R/J, M) = 0$ that the sequence

$$0 \longrightarrow N_1/JN_1 \longrightarrow P_0/JP_0 \longrightarrow M/JM \longrightarrow 0$$

is exact. This means that $JN_1 = N_1$ because $JP_0 = P_0$. Observe that $\mathrm{Tor}_1^R(R/J, N_1) = \mathrm{Tor}_2^R(R/J, M) = 0$. So, for $N_1$, we can do the same procedure as we did above and get a surjective homomorphism $P_1 := \bigoplus_{j \in I_1} Re \xrightarrow{\epsilon_1} N_1$ with $I_1$ a finite set, such that $N_2 := \mathrm{Ker}(\epsilon_1)$ is finitely generated and that $JN_2 = N_2$ and $\mathrm{Tor}_1^R(R/J, N_2) = 0$. Hence, by using the generalized Schanuel's Lemma, we can iterate this procedure. Since the projective dimension of $M$ is at most $n$, we must stop after $n$ steps and reach at a desired sequence mentioned in the lemma. □

*Remark 2.5 The above proof shows that for an* $R$*-module* $M$*, the condition (1) is equivalent to*

*(2') There is a projective resolution* $\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ *such that* $P_j \in Add(Re)$*, where* $Add(Re)$ *is the full subcategory of* $R$*-Mod consisting of all those* $R$*-modules which are direct summands of direct sums of copies of* $Re$.

Thus the canonical surjection $R \to R/J$ is homological if and only if such a sequence (2') for ${}_RJ$ exists. So, under the assumption of Corollary 1.3, all conditions of Corollary 1.2 are fulfilled. By Corollary 1.2, we have

$$K_n(R) \simeq K_n(R/J) \oplus K_n(\mathscr{E}) \ \text{ for } \ n \geq 0,$$

where $\mathscr{E}$ is the exact category of $(R, \Sigma)$-torsion modules with $\Sigma := \{0 \to Re\}$.

To finish the proof of Corollary 1.3, we have to show $K_n(\mathscr{E}) \simeq K_n(eRe)$. However, this follows from the following lemma.

**Lemma 2.6** *Suppose that* $R$ *is a ring with identity and that* $e$ *is an idempotent element in* $R$. *Let* $\mathscr{E}$ *be the exact category of* $(R, \Sigma)$*-torsion modules with* $\Sigma := \{0 \to Re\}$. *Then* $K(\mathscr{E})$ *and* $K(eRe)$ *are homotopy equivalent.*

*Proof* Let $J := ReR$. Note that the universal localization of $R$ at $\Sigma$ is exactly the canonical surjection $R \to R/J$. Thus an $R$-module $M$ is $(R, \Sigma)$-torsion if and only if it is finitely presented, and of projective dimension at most 1 such that $R/J \otimes_R M = 0 = \mathrm{Tor}_1^R(R/J, M)$. By Lemma 2.4, the latter is also equivalent to saying that there is an exact sequence of $R$-modules:

$$0 \to P_1 \longrightarrow P_0 \longrightarrow M \to 0$$

such that $P_0$ and $P_1$ lie in $\mathrm{add}(Re)$. Now, we regard $\mathrm{add}(Re)$ as a full subcategory of the exact category $\mathscr{E}$. Clearly, the pair $(\mathrm{add}(Re), \mathscr{E})$ satisfies the conditions in Lemma 2.2. It follows that the inclusion $\mathrm{add}(Re) \subseteq \mathscr{E}$ induces a homotopy equivalence $K(\mathrm{add}(Re)) \xrightarrow{\sim} K(\mathscr{E})$. Since the exact functor $eR \otimes_R - : \mathrm{add}(Re) \to \mathrm{add}(eRe)$ is an equivalence, we know that $K\big(\mathrm{add}(Re)\big)$ is homotopy equivalent to $K(eRe)$. Thus $K(\mathscr{E})$ and $K(eRe)$ are homotopy equivalent. □

This completes the proof of Corollary 1.3.

## 3 Examples: Proof of Corollary 1.4

In this section, we demonstrate methods of producing homological ring epimorphisms of the form $R \to R/J$ with $J$ an idempotent ideal. Here, forming endomorphism rings and matrix subrings will enter into our play. As a consequence of our discussions, we have a proof of Corollary 1.4.

### 3.1 Endomorphism rings

Let $R$ be a ring with identity and $X$ be an $R$-module. A submodule $Y$ of $X$ is called a *trace* in $X$ if $\mathrm{Hom}_R(Y, X/Y) = 0$; and a *weak trace* in $X$ if the inclusion from $Y$ to $X$ induces an isomorphism $\mathrm{Hom}_R(Y, Y) \to \mathrm{Hom}_R(Y, X)$ of abelian groups. For example, every idempotent ideal of $R$ is a trace in the regular $R$-module ${}_RR$, and every GV-ideal $J$ of $R$ is a weak trace in ${}_RR$ (see [17, Section 7] for definition). Also, the socle of any finite-dimensional algebra $A$ over a field is a weak trace in ${}_AA$. In particular, the socle of the ring $R := \mathbb{Q}[X]/(X^2)$ is a weak trace in $R$, but not a trace in $R$.

In general, for any $R$-modules $X$ and $Y$, there is a recipe for getting weak trace submodules of $X$. Let $t_Y(X)$ be the sum of all images of homomorphisms from $Y$ to $X$ of $R$-modules. Then $t_Y(X)$ is a weak trace of $X$.

Motivated by weak trace submodules, we introduce the following notion.

**Definition 3.1** Let $\mathcal{C}$ be an additive category. A morphism $\lambda : Y \to X$ of objects in $\mathcal{C}$ is said to be covariant if

(1) the induced map $\mathrm{Hom}_{\mathcal{C}}(X, \lambda) : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, X)$ is injective, and
(2) the induced map $\mathrm{Hom}_{\mathcal{C}}(Y, \lambda) : \mathrm{Hom}_{\mathcal{C}}(Y, Y) \to \mathrm{Hom}_{\mathcal{C}}(Y, X)$ is a split epimorphism of $\mathrm{End}_{\mathcal{C}}(Y)$-modules.

Dually, a morphism $\beta : N \to M$ in $\mathcal{C}$ is said to be contravariant if

(1') $\mathrm{Hom}_{\mathcal{C}}(\beta, N) : \mathrm{Hom}_{\mathcal{C}}(M, N) \to \mathrm{Hom}_{\mathcal{C}}(N, N)$ is injective, and
(2') $\mathrm{Hom}_{\mathcal{C}}(\beta, M) : \mathrm{Hom}_{\mathcal{C}}(M, M) \to \mathrm{Hom}_{\mathcal{C}}(N, M)$ is a split epimorphism of right $\mathrm{End}_{\mathcal{C}}(M)$-modules

Clearly, if $Y$ is a weak trace submodule of an $R$-module $X$, then the inclusion map is covariant. Another example of covariant homomorphisms is the following: If $0 \to Z \to Y \xrightarrow{g} X \to 0$ is an Auslander-Reiten sequence in $R$-mod with $\mathrm{Hom}_R(Y, Z) = 0$, then the homomorphism $g$ is covariant.

For covariant morphisms, we have the following properties.

**Lemma 3.2** *Let $\mathcal{C}$ be an additive category, and let $\lambda : Y \to X$ be a covariant morphism of objects in $\mathcal{C}$. We define $\Lambda := End_{\mathcal{C}}(X \oplus Y)$, and let $e_Y$ be the idempotent element of $\Lambda$ corresponding to the projection onto $Y$. Then*

(1) ${}_\Lambda \Lambda e_Y \Lambda$ *is a finitely generated projective $\Lambda$-module.*
(2) *The composition map $\mu : Hom_{\mathcal{C}}(X, Y) \otimes_{End_{\mathcal{C}}(Y)} Hom_{\mathcal{C}}(Y, X) \to End_{\mathcal{C}}(X)$ is injective, and the cokernel of $\mu$ is isomorphic to $End_{\mathcal{C},Y}(X)$.*

To prove this lemma, we use the following observation.

**Lemma 3.3** *Let $S$ be a ring with identity, and let $e$ be an idempotent element in $S$. Then ${}_S SeS$ (respectively, $SeS_S$) is projective and finitely generated if and only if $eS(1-e)$ (respectively, $(1-e)Se$) is projective and finitely generated as an $eSe$-module (respectively, a right $eSe$-module), and the multiplication map $\mu : (1-e)Se \otimes_{eSe} eS(1-e) \to (1-e)S(1-e)$ is injective.*

*Proof* Suppose that $eS(1-e)$ is a finitely generated projective $eSe$-module and that the multiplication map $(1-e)Se \otimes_{eSe} eS(1-e) \xrightarrow{\mu} (1-e)S(1-e)$ is injective. Then it is easy to see that the multiplication map $Se \otimes_{eSe} eS \to SeS$ is an isomorphism of $S$-$S$-bimodules. Since $eS = eSe \oplus eS(1-e)$, we know that ${}_S SeS$ is projective and finitely generated.

Conversely, suppose that ${}_S SeS$ is projective and finitely generated. One the one hand, since ${}_S SeS$ is projective, we can show that the multiplication map $\mu : Se \otimes_{eSe} eS \to SeS$ is injective (see [5, Statement 7]). This implies that the map $\mu : (1-e)Se \otimes_{eSe} eS(1-e) \to (1-e)S(1-e)$ is injective. On the other hand, since ${}_S SeS$ is finitely generated, there is a finite subset $\{x_i \mid i \in I\}$ of $S$ such that the map $\bigoplus_{i \in I} Se \to SeS$, defined by $(a_i)_{i \in I} \mapsto \sum_{i \in I} a_i x_i$, is surjective. This shows that ${}_S SeS$ is a direct summand of a direct sum of finitely many copies of $Se$. Thus $eS$ is a direct summand of a free $eSe$-module of finite rank. This implies that the $eSe$-module $eS(1-e)$ is projective and finitely generated.

The same arguments applies to the right module $SeS_S$. □

*Proof of Lemma 3.2*. Clearly, $\Lambda = \begin{pmatrix} \mathrm{End}_{\mathcal{C}}(X) & \mathrm{Hom}_{\mathcal{C}}(X, Y) \\ \mathrm{Hom}_{\mathcal{C}}(Y, X) & \mathrm{End}_{\mathcal{C}}(Y) \end{pmatrix}$. Let $e := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $f := 1 - e$. Thus $e = e_Y$, $e\Lambda e \simeq \mathrm{End}_{\mathcal{C}}(Y)$, $f\Lambda f \simeq \mathrm{End}_{\mathcal{C}}(X)$, $f\Lambda e \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $e\Lambda f \simeq \mathrm{Hom}_{\mathcal{C}}(Y, X)$, where the left $\mathrm{End}_{\mathcal{C}}(Y)$-module structure of $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ is induced from the right $\mathrm{End}_{\mathcal{C}}(Y)$-module structure of $Y$. In the following we will often use these identifications without further references. Since $\lambda$ is a covariant homomorphism, the induced map

$$\lambda^* = \mathrm{Hom}_{\mathcal{C}}(Y, \lambda) : \mathrm{Hom}_{\mathcal{C}}(Y, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

is a split epimorphism of $\mathrm{End}_{\mathcal{C}}(Y)$-modules. Thus there is a homomorphism $\gamma : \mathrm{Hom}_{\mathcal{C}}(Y, X) \to \mathrm{Hom}_{\mathcal{C}}(Y, Y)$ such that $\gamma\lambda^* = id$. This means that $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ is a direct summand of the regular $\mathrm{End}_{\mathcal{C}}(Y)$-module. Thus $e\Lambda f$ is a finitely generated projective $e\Lambda e$-module since a direct summand of a finitely generated module is finitely generated.

Now we show that the multiplication map $f\Lambda e\otimes_{e\Lambda e}e\Lambda f\to f\Lambda f$ is injective. This is equivalent to showing that the composition map

$$\mu:\quad \mathrm{Hom}_{\mathcal{C}}(X,Y)\otimes_{\mathrm{End}_{\mathcal{C}}(Y)}\mathrm{Hom}_{\mathcal{C}}(Y,X)\longrightarrow \mathrm{End}_{\mathcal{C}}(X),$$

given by $x\otimes f\mapsto xf$ for $x\in\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $f\in\mathrm{Hom}_{\mathcal{C}}(Y,X)$, is injective. However, the injectivity of $\mu$ follows from the injectivity of $\mathrm{Hom}_{\mathcal{C}}(X,\lambda):\mathrm{Hom}_{\mathcal{C}}(X,Y)\to\mathrm{Hom}_{\mathcal{C}}(X,X)$ together with the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y)\otimes_{\mathrm{End}_{\mathcal{C}}(Y)}\mathrm{Hom}_{\mathcal{C}}(Y,Y) \arrow[r, "\simeq", "\mu'"'] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(X,Y)\otimes\lambda^*"] & \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(X,\lambda)"] \\
\mathrm{Hom}_{\mathcal{C}}(X,Y)\otimes_{\mathrm{End}_{\mathcal{C}}(Y)}\mathrm{Hom}_{\mathcal{C}}(Y,X) \arrow[r, "\mu"] & \mathrm{Hom}_{\mathcal{C}}(X,X)
\end{tikzcd}$$

since the bottom $\mu$ is a composite of three injective maps, that is, $\mu=\big(\mathrm{Hom}_{\mathcal{C}}(X,Y)\otimes\gamma\big)\mu'\,\mathrm{Hom}_{\mathcal{C}}(X,\lambda)$. Here, we use the identity $\gamma\lambda^*=id$. Thus, by Lemma 3.3, we see that ${}_\Lambda\Lambda e\Lambda$ is a finitely generated projective $\Lambda$-module.

Now, the second statement of Lemma 3.2 also becomes clear. $\square$

Dually, for contravariant morphisms, we have the following statement.

**Lemma 3.4** *Let $\mathcal{C}$ be an additive category, and let $\lambda:Y\to X$ be a contravariant morphism of objects in $\mathcal{C}$. We define $\Lambda:=End_{\mathcal{C}}(X\oplus Y)$, and let $e_X$ be the idempotent element of $\Lambda$ corresponding to the projection onto $X$. Then*

(1) *$\Lambda e_X\Lambda_\Lambda$ is a finitely generated projective right $\Lambda$-module.*
(2) *The composition map $\mu:Hom_{\mathcal{C}}(Y,X)\otimes_{End_{\mathcal{C}}(X)}Hom_{\mathcal{C}}(X,Y)\to End_{\mathcal{C}}(Y)$ is injective. Thus the cokernel of $\mu$ is isomorphic to $End_{\mathcal{C},X}(Y)$.*

For convenience, we introduce the following definition of $X$-covariant morphisms. Observe that the condition in this definition strengthens only the first and does not involve the second condition in the definition of covariant or contravariant morphisms.

**Definition 3.5** A morphism $f:Y\to X$ in an additive category $\mathcal{C}$ is said to be

(1) $X$-covariant if the induced map $Hom_{\mathcal{C}}(X,f)$ is a split monomorphism of $\mathrm{End}_{\mathcal{C}}(X)$-modules.
(2) $Y$-contravariant if the induced map $Hom_{\mathcal{C}}(f,Y)$ is a split monomorphism of right $\mathrm{End}_{\mathcal{C}}(Y)$-modules.

Clearly, if $f:Y\to X$ is covariant, then the inclusion from $\mathrm{Ker}(f)$ into $Y$ is $Y$-covariant. Dually, if $f:Y\to X$ is contravariant, then the canonical surjection from $X$ to $\mathrm{Coker}(f)$ is $X$-contravariant. For $X$-covariant and $Y$-contravariant morphisms, we have the following properties.

**Lemma 3.6** *Let $\mathcal{C}$ be an additive category, and let $\lambda:Y\to X$ be a morphism of objects in $\mathcal{C}$. We define $\Lambda:=End_{\mathcal{C}}(X\oplus Y)$, and let $e_X$ and $e_Y$ be the idempotent elements of $\Lambda$ corresponding to the projection onto $X$ and $Y$, respectively.*

(1) *If $\lambda$ is $X$-covariant, then ${}_\Lambda\Lambda e_X\Lambda$ is a finitely generated projective $\Lambda$-module. In this case, $\Lambda/\Lambda e_X\Lambda\simeq End_{\mathcal{C},X}(Y)$.*

(2) *If $\lambda$ is $Y$-contravariant, then $\Lambda e_Y\Lambda_\Lambda$ is a finitely generated projective $\Lambda$-module. In this case, $\Lambda/\Lambda e_Y\Lambda \simeq End_{\mathcal{C},Y}(X)$.*

*Proof* (1) The proof is similar to that of Lemma 3.2. Here, we only outline its main points.

Since $\lambda^* = \mathrm{Hom}_{\mathcal{C}}(X,\lambda) : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,X)$ is a split monomorphism of $\mathrm{End}_{\mathcal{C}}(X)$-modules, we see that

(a) $e_X\Lambda e_Y$ is a finitely generated projective $e_X\Lambda e_X$-module, and
(b) $\mathrm{Hom}_{\mathcal{C}}(Y,X)\otimes\lambda^* : \mathrm{Hom}_{\mathcal{C}}(Y,X)\otimes_{\mathrm{End}_{\mathcal{C}}(X)}\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(Y,X)\otimes_{\mathrm{End}_{\mathcal{C}}(X)}\mathrm{Hom}_{\mathcal{C}}(X,X)$ is a split monomorphism.

To see that the multiplication map $\mu : e_Y\Lambda e_X\otimes_{e_X\Lambda e_X} e_X\Lambda e_Y \to e_Y\Lambda e_Y$ is injective, we consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Y,X)\otimes_{\mathrm{End}_{\mathcal{C}}(X)}\mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(Y,X)\otimes\lambda^*"] & \mathrm{Hom}_{\mathcal{C}}(Y,Y) \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(Y,\lambda)"] \\
\mathrm{Hom}_{\mathcal{C}}(Y,X)\otimes_{\mathrm{End}_{\mathcal{C}}(X)}\mathrm{Hom}_{\mathcal{C}}(X,X) \arrow[r, "\simeq"] & \mathrm{Hom}_{\mathcal{C}}(Y,X)
\end{tikzcd}$$

where the horizontal maps are composition maps. This implies that $\mu$ is injective, and therefore $\Lambda/\Lambda e_X\Lambda \simeq \mathrm{End}_{\mathcal{C},X}(Y)$. Now (1) follows immediately from Lemma 3.3.

(2) The proof is left to the reader. □

*Proof* of Corollary 1.4.

(1) Assume that $\lambda : Y \to X$ is a covariant morphism of objects in $\mathcal{C}$. Set $\Lambda := \mathrm{End}_{\mathcal{C}}(X\oplus Y)$, and let $J$ be the ideal of $\Lambda$ generated by the projection $e$ from $X\oplus Y$ onto $Y$. Then $e\Lambda e \simeq \mathrm{End}_{\mathcal{C}}(Y)$ and $\Lambda/J$ is isomorphic to the quotient ring of $\mathrm{End}_{\mathcal{C}}(X)$ modulo the ideal generated by those endomorphisms of $X$ which factorize through the object $Y$, that is, $\Lambda/J \simeq \mathrm{End}_{\mathcal{C},Y}(X)$ by Lemma 3.2(2). Since ${}_\Lambda J$ is projective and finitely generated by Lemma 3.2(1), we can apply Corollary 1.3 to $\Lambda$ and $J$. In this case, we see that
$$K_n(\Lambda) \simeq K_n\big(\mathrm{End}_{\mathcal{C},Y}(X)\big)\oplus K_n\big(\mathrm{End}_{\mathcal{C}}(Y)\big)$$
for all $n\in\mathbb{N}$.
(2) Similarly, we can use Lemma 3.6 and Corollary 1.3 to show (2).

□

The dual of Corollary 1.4 can be stated as follows. We leave its proof to the interested reader.

**Corollary 3.7** *Let $\mathcal{C}$ be an additive category and $f : Y \to X$ a morphism of objects in $\mathcal{C}$.*

(1) *If $f$ is contravariant, then $K_*\big(End_{\mathcal{C}}(X\oplus Y)\big) \simeq K_*\big(End_{\mathcal{C}}(X)\big)\oplus K_*\big(End_{\mathcal{C},X}(Y)\big)$ for all $*\in\mathbb{N}$.*
(2) *If $f$ is $Y$-contravariant, then $K_*\big(End_{\mathcal{C}}(X\oplus Y)\big) \simeq K_*\big(End_{\mathcal{C},Y}(X)\big)\oplus K_*\big(End_{\mathcal{C}}(Y)\big)$ for all $*\in\mathbb{N}$.*

Let us now mention a few consequences of Corollary 1.4.

**Corollary 3.8** *Let $R$ be a ring with identity and let $I$ be an idempotent ideal of $R$. Then*

$$K_n\big(End_R(R \oplus I)\big) \simeq K_n(R/I) \oplus K_n\big(End_R(I)\big) \ \textit{for all} \ \ n \in \mathbb{N}.$$

*In particular, if the idempotent ideal $I$ is projective and finitely generated as a left $R$-module, then $K_n(R) \simeq K_n(R/I) \oplus K_n\big(End_R(I)\big)$ for all $n \in \mathbb{N}$.*

*Proof* Since $I^2 = I$, we have $\mathrm{Hom}_R(I, R/I) = 0$. This implies that $I$ is a trace in ${}_RR$. So the inclusion $I \hookrightarrow R$ is covariant. Note that $\mathrm{End}_{R,I}(R) \simeq R/I$. Thus the first statement of Corollary 3.8 follows from Corollary 1.4.

Now assume further that ${}_RI$ is projective and finitely generated. Then the $R$-module $R \oplus I$ is a progenerator for $R$-Mod, and therefore $R$ and $\Lambda := \mathrm{End}_R(R \oplus I)$ are Morita equivalent. Thus $K_n(R) \simeq K_n(\Lambda) \simeq K_n\big(\mathrm{End}_R(I)\big) \oplus K_n(R/I)$ for all $n \in \mathbb{N}$. $\square$

At this point, let us give some comments on the conditions in Corollaries 1.3 and 3.8.

(1) The assumption on projective resolution in Corollary 1.3 cannot be weakened to a finite-type resolution. For example, let $k$ be a field and let $R$ be the quotient algebra of the path algebra of the quiver

$$\begin{tikzcd} 1\bullet \arrow[r, bend left, "\alpha"] & \bullet 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

modulo the ideal generated by $\beta\alpha$. Note that the global dimension of this algebra is 2. By applying Corollary 1.3 to the idempotent element corresponding to the vertex 2, we see that $K_n(R) \simeq K_n(k) \oplus K_n(k)$. Let $e_1$ be the idempotent element of $R$ corresponding to the vertex 1. Since $K_1(k[x]/(x^2)) \simeq k \oplus \ k^\times$ and $K_1(k) = k^\times$, we have $K_1(R) \not\simeq K_1(e_1Re_1) \oplus K_1(R/Re_1R)$. It is easy to check that ${}_RRe_1R$ does not admit a finite projective resolution with all terms in $\mathrm{add}({}_RRe_1)$. This implies that the assumption on the projective resolution of ${}_RReR$ in Corollary 1.3 cannot be weakened to a finite-type resolution.

(2) The projectivity on the idempotent ideal $I$ in the second statement of Corollary 3.8 cannot be dropped.

In the above example, the ideal $I := Re_1R$ is finitely generated but not projective. In fact, we have ${}_RI \simeq Re_1 \oplus S_1$, where $S_1$ is the simple $R$-module corresponding to the vertex 1. Moreover, $e_1Re_1 \simeq k[x]/(x^2)$, $R/I \simeq k$ and $\mathrm{End}_R(I) \simeq R$ as rings. It follows that $K_n(R) \simeq K_n\big(\mathrm{End}_A(I)\big)$. Thus the second statement of Corollary 3.8 may fail if the ideal $I$ is not projective.

(3) The projectivity of $I$ in Corollary 3.8 cannot be relaxed to that the canonical surjection $R \to R/I$ is homological.

If we modify the above example slightly and just consider the algebra $S$ given by the above quiver but with the relation $\beta\alpha\beta = 0$, then the ideal $J_1 := Se_1S$, as a left $S$-module, admits a projective resolution of infinite length with each term belonging to $\mathrm{add}({}_SSe_1)$. Thus the canonical surjection $S \to S/J_1$ is homological. However, since $S/J_1 \simeq k$ and $\mathrm{End}_S(J_1)$ is isomorphic to the ring $R$ in (1), we see that $K_0(S) \not\simeq K_0(S/J_1) \oplus K_0(\mathrm{End}_S(J_1))$. In fact, $K_n(S) \simeq K_n(k) \oplus K_n(k[x]/(x^2))$ for all $n$. This follows from Corollary 1.3 since the ideal $J_2 := Se_2S$ is projective and finitely generated and since $S/J_2 \simeq k$ and $e_2Se_2 \simeq k[x]/(x^2)$ as rings.

(4) The assumption that $I$ is idempotent in Corollary 3.8 cannot be removed. We consider the triangular matrix ring $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ over a field $k$. Then $K_n(R) \simeq K_n(k) \oplus K_n(k)$. If we

take $I = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$, then $R/I \simeq k \oplus k$ and $\mathrm{End}_R(I) \simeq k$ as rings. Moreover, the module ${}_R I$ is projective and finitely generated. It follows that

$$K_n(R/I) \oplus K_n(\mathrm{End}_R(I)) \simeq K_n(k) \oplus K_n(k) \oplus K_n(k) \not\simeq K_n(R) \simeq K_n(\mathrm{End}_R(R \oplus I)).$$

In fact, in this example, we have $I^2 = 0$. Thus Corollary 3.8 may fail if $I^2 \neq I$.

**Corollary 3.9** *Let $R$ be a ring with identity, and let $soc(R)$ be the socle of ${}_R R$. Then*

$$K_n\big(End_R(R \oplus soc(R))\big) \simeq K_n\big(End_R(soc(R))\big) \oplus K_n\big(R/soc(R)\big)$$

*for all $n \in \mathbb{N}$.*

*Proof* Recall that for an $R$-module $M$, the socle of $M$ is the sum of all simple submodules of $M$. Thus $\mathrm{soc}(R)$ is a direct sum of minimal left ideals of $R$, and therefore it is actually an ideal in $R$. Since $\mathrm{soc}(R)$ is a weak trace submodule of ${}_R R$ by the definition of socles, we can apply Corollary 1.4 and get

$$K_n\big(\mathrm{End}_R(R \oplus \mathrm{soc}(R))\big) \simeq K_n\big(\mathrm{End}_R(\mathrm{soc}(R))\big) \oplus K_n(R/\mathrm{soc}(R))$$

for all $n \in \mathbb{N}$. □

For Auslander-Reiten sequences, we have the following result.

**Corollary 3.10** *Let $A$ be an Artin algebra, and let $0 \to Z \xrightarrow{g} Y \xrightarrow{f} X \to 0$ be an Auslander-Reiten sequence in $A$-mod. If $Hom_A(Y, Z) = 0$, then*

$$K_n\big(End_A(Y \oplus X)\big) \simeq K_n\big(End_A(Y)\big) \oplus K_n\big(End_A(X)/rad(End_A(X))\big) \simeq K_n\big(End_A(Y)\big) \oplus K_n\big(End_A(Z)/rad(End_A(Z))\big)$$

*for all $n \in \mathbb{N}$, where rad stands for the Jacobson radical of rings.*

*Proof* Note that $\mathrm{Hom}_A(Y, Z) = 0$ if and only if the induced surjective map $\mathrm{Hom}_A(Y, Y) \to \mathrm{Hom}_A(Y, X)$ is an isomorphism of $\mathrm{End}_A(Y)$-modules. Since $f$ is surjective, it follows also from $\mathrm{Hom}_A(Y, Z) = 0$ that $\mathrm{Hom}_A(X, Z) = 0$. Thus $f : Y \to X$ is a covariant homomorphism and, by Corollary 1.4, we have $K_n\big(\mathrm{End}_A(Y \oplus X)\big) \simeq K_n\big(\mathrm{End}_A(Y)\big) \oplus K_n\big(\mathrm{End}_{A,Y}(X)\big)$ for all $n \in \mathbb{N}$.

For an Auslander-Reiten sequence, we know that $\mathrm{rad}(\mathrm{End}_A(X))$ is the image of the map $\mathrm{Hom}_A(X, f) : \mathrm{Hom}_A(X, Y) \to \mathrm{Hom}_A(X, X)$. Thus $\mathrm{End}_{A,Y}(X) \simeq \mathrm{End}_A(X)/\mathrm{rad}(\mathrm{End}_A(X))$ which is a division ring and isomorphic to $\mathrm{End}_A(Z)/\mathrm{rad}(\mathrm{End}_A(Z))$. Hence

$$K_n\big(\mathrm{End}_A(Y \oplus X)\big) \simeq K_n\big(\mathrm{End}_A(Y)\big) \oplus K_n\big(\mathrm{End}_A(X)/\mathrm{rad}(\mathrm{End}_A(X)\big) \simeq K_n\big(\mathrm{End}_A(Y)\big) \oplus K_n\big(\mathrm{End}_A(Z)/\mathrm{rad}(\mathrm{End}(Z))\big)$$

for all $n \in \mathbb{N}$. □

### 3.2 Matrix subrings

In the following, we apply our results in the previous sections to establish formulas for algebraic $K$-groups of certain matrix subrings which have been used in universal algebraic geometry (see [2]) and representation theory (see [11, Chapter 39]).

The following result extends the result [17, Proposition 5.3] for $K_1$ to a result for higher algebraic $K$-groups.

**Corollary 3.11** *Let $R$ be a ring with identity, and let $J$ and $I_{ij}$, with $1 \leq i < j \leq n$, be arbitrary ideals of $R$ such that $I_{ij+1}J \subseteq I_{i\,j}$, $JI_{i\,j} \subseteq I_{i+1\,j}$ and $I_{i\,j}I_{jk} \subseteq I_{ik}$ for $j < k \leq n$. Define a ring*

$$S := \begin{pmatrix} R & I_{12} & \cdots & \cdots & I_{1n} \\ J & R & \ddots & \ddots & \vdots \\ J^2 & J & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & R & I_{n-1\,n} \\ J^{n-1} & \cdots & J^2 & J & R \end{pmatrix}_{n\times n}.$$

*If ${}_RJ$ is projective and finitely generated, then $K_*(S) \simeq K_*(R) \oplus \bigoplus_{j=1}^{n-1} K_*(R/I_{j\,j+1}J)$.*

*Proof* We use induction on $n$ to prove this corollary.

Now let $e_i$ be the idempotent element of $S$ with $1_R$ at the $(i,i)$-entry and zero at all other entries, and $e := e_2 + \cdots + e_n$. As $J$ is a projective $R$-module, we have $I_{ij} \otimes_R J \simeq I_{ij}J$. Thus

$$SeS = \begin{pmatrix} I_{12}J & I_{12} & \cdots & \cdots & I_{1n} \\ J & R & \ddots & \cdots & I_{2n} \\ J^2 & J & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & R & I_{n-1n} \\ J^{n-1} & J^{n-2} & \cdots & J & R \end{pmatrix} \simeq Se \oplus Se_2 \otimes_{e_2Se_2} J.$$

Here, we identify $R$ with $e_2Se_2$. Since ${}_RJ$ is projective and finitely generated, we infer that the $S$-module $SeS$ is also projective and finitely generated. Clearly, $S/SeS$ is isomorphic to $R/I_{12}J$. It follows from Corollary 1.3 that $K_*(S) \simeq K_*(R/I_{12}J) \oplus K_*(eSe)$. By induction, we know that $K_*(eSe) \simeq K_*(R) \oplus \bigoplus_{j=2}^{n-1} K_*\big(R/I_{j\,j+1}J\big)$. Thus

$$K_*(S) \simeq K_*(R) \oplus \bigoplus_{j=1}^{n-1} K_*(R/I_{j\,j+1}J).$$

This finishes the proof. □

As a consequence of Corollary 3.11, we can prove the following corollary.

**Corollary 3.12** *Let $R$ be a ring with identity, and let $r$ be a regular element of $R$ with $Rr = rR$. If $I_{ij}$ is an ideal of $R$ for $1 \leq i < j \leq n$ such that $I_{ij+1}r \subseteq I_{i\,j}$, $rI_{ij} \subseteq I_{i+1\,j}$ and $I_{ij}I_{jk} \subseteq I_{ik}$ for $j < k \leq n$, then, for the matrix ring*

$$T := \begin{pmatrix} R & I_{12} & I_{13} & \cdots & I_{1\,n} \\ Rr & R & I_{23} & \cdots & I_{2\,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ Rr^{n-2} & \cdots & Rr & R & I_{n-1\,n} \\ Rr^{n-1} & \cdots & Rr^2 & Rr & R \end{pmatrix},$$

*we have $K_*(T) \simeq K_*(R) \oplus \bigoplus_{j=1}^{n-1} K_*(R/I_{j\,j+1}r)$ for all $n \in \mathbb{N}$.*

By a *regular element* we mean an element of $R$, which is not a zero-divisor of $R$.
Now, we point out the following result.

**Proposition 3.13** *Let $R$ be a commutative ring with identity, and let $x, y \in R$ such that $Rx + Ry = R$ and $Rx \cap Ry = Rxy$ (for example, $R$ is a principle integral domain with $x$ and $y$ coprime in $R$). Suppose that $y$ is invertible in an extension ring $R'$ of $R$. Then, for the ring*

$$S := \begin{pmatrix} R & Rx & \cdots & Rx \\ Ry & \ddots & \ddots & \vdots \\ \vdots & \ddots & R & Rx \\ Ry & \cdots & Ry & R \end{pmatrix}_{n\times n},$$

*we have $K_n(S) \simeq K_n(R) \oplus (n-1)K_n(R/Rx) \oplus (n-1)K_n(R/Ry)$ for all $n \in \mathbb{N}$.*

*Proof* Let $\sigma$ be the diagonal matrix with the (1, 1)-entry $y$ and all other diagonal entries 1. Then $\sigma$ is invertible in $M_n(R')$, the $n$ by $n$ full matrix ring of $R'$. Let $B := \sigma S \sigma^{-1}$. Thus $S \simeq B$ and $B$ is of the form

$$B := \begin{pmatrix} R & Rxy & Rxy & \cdots & Rxy \\ R & R & Rx & \cdots & Rx \\ R & Ry & R & \ddots & \vdots \\ \vdots & \vdots & \ddots & R & Rx \\ R & Ry & \cdots & Ry & R \end{pmatrix}_{n\times n}.$$

Define $A := M_n(R)$. Then $B$ is a subring of $A$ with the same identity. Moreover, ${}_BA$ is isomorphic to the direct sum of $n$ copies of $Be_1$ where $e_1$ is the diagonal matrix $\mathrm{diag}(1, 0, \cdots, 0)$ of $B$. Thus ${}_BA$ is a finitely generated projective $B$-module. Hence, by [17, Lemma 3.1], $B$ is derived equivalent to $\mathrm{End}_B(B \oplus A/B)$. Clearly, the latter is Morita equivalent to $\mathrm{End}_B(Be_1 \oplus Q_2 \oplus \cdots \oplus Q_n)$, where $Q_j$ is given by the exact sequence

$$0 \longrightarrow Be_j \longrightarrow Be_1 \longrightarrow Q_j \longrightarrow 0, \quad 2 \leq j \leq n.$$

As in [17, Section 3], we can show that $\mathrm{End}_B(Be_1 \oplus Q_2 \oplus \cdots \oplus Q_n)$ is isomorphic to the following ring

$$C := \begin{pmatrix} R & R/Rxy & R/Rxy & \cdots & R/Rxy \\ 0 & R/Rxy & Rx/Rxy & \cdots & Rx/Rxy \\ 0 & Ry/Rxy & R/Rxy & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & Rx/Rxy \\ 0 & Ry/Rxy & \cdots & Ry/Rxy & R/Rxy \end{pmatrix}.$$

From the Chinese remainder theorem we know that $R/Rxy \simeq R/Rx \oplus R/Ry$ as rings. Moreover, it follows from the assumptions that the $R/Rxy$-bimodules $Rx/Rxy$ and $Ry/Rxy$ are isomorphic to $R/Ry$ and $R/Rx$, respectively. Let $D$ be the lower right corner

$(n-1)\times(n-1)$-submatrix of $C$. Then the ring $D$ is actually a direct sum of the following two rings:

$$D=\begin{pmatrix} R/Ry & R/Ry & \cdots & R/Ry \\ 0 & R/Ry & \ddots & \vdots \\ \vdots & \ddots & R/Ry & R/Ry \\ 0 & \cdots & 0 & R/Ry \end{pmatrix}_{n-1} \bigoplus \begin{pmatrix} R/Rx & 0 & \cdots & 0 \\ R/Rx & R/Rx & \ddots & \vdots \\ \vdots & \ddots & R/Ry & 0 \\ R/Rx & \cdots & R/Rx & R/Rx \end{pmatrix}_{n-1}.$$

Since derived equivalences preserve algebraic $K_n$-groups, it follows that $K_n(S)\simeq K_n(C)\simeq K_n(R)\oplus K_n(D)\simeq K_n(R)\oplus(n-1)K_n(R/Rx)\oplus(n-1)K_n(R/Ry)$ for all $n\in\mathbb{N}$. $\square$

*Remark* For $n=2$, we can remove the conditions "$Rx+Ry=R$ and $Rx\cap Ry=Rxy$" in Proposition 3.13, and get $K_*(S)\simeq K_*(R)\oplus K_*(R/Rxy)$ for all $*\in\mathbb{N}$.

Related to calculation of algebraic $K$-groups of the rings in the proof of Proposition 3.13, the following result may be of interest.

**Corollary 3.14** *Let $R$ be a ring with identity, and let $I$ and $J$ be ideals in $R$ with $JI=0$. If ${}_RI$ (or $J_R$) is projective and finitely generated, then, for the ring*

$$S:=\begin{pmatrix} R & I & \cdots & I \\ J & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & I \\ J & \cdots & J & R \end{pmatrix}_{n\times n},$$

*we have $K_*(S)\simeq nK_*(R)$ for all $*\in\mathbb{N}$.*

*Proof* We assume that the $R$-module ${}_RI$ is projective and finitely generated. Let $e:=e_1\in S$. Then

$$SeS:=\begin{pmatrix} R & I & \cdots & I \\ J & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ J & 0 & \cdots & 0 \end{pmatrix}.$$

Since ${}_RI$ is projective, we have $J\otimes_R I\simeq JI=0$ and $Se\otimes_R I\simeq SeSe_j$ for $2\le j\le n$. Here we identify $eSe$ with $R$. Since ${}_RI$ is projective and finitely generated, we know that ${}_SSeSe_j$ is projective and finitely generated for $j=2,\cdots,n$, and therefore the $S$-module ${}_SSeS\simeq Se\oplus SeSe_2\oplus\cdots\oplus SeSe_n$ is a finitely generated projective module. Thus, by Corollary 1.3 and induction on $n$, we have $K_*(S)\simeq nK_*(R)$ for all $*\in\mathbb{N}$.

The proof for the case that $J_R$ is projective and finitely generated can be done similarly. $\square$

*Remark* If $R$ is an arbitrary ring with $I$, $J$ ideals in $R$ such that $IJ=JI=0$, then the ring $S$ in Corollary 3.14 is the trivial extension of $R\times R\times\cdots\times R$ by the bimodule $L$, where

$$L:=\begin{pmatrix} 0 & I & \cdots & I \\ J & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & I \\ J & \cdots & J & 0 \end{pmatrix}.$$

Thus we always have $K_n(S) \simeq nK_n(R) \oplus K_n(S, L)$ for all $n \in \mathbb{N}$, where $K_n(S, L)$ is the $n$-th relative $K$-group of $S$ with respect to the ideal $L$ (see [12] for definition). This is due to the split epimorphism $K_n(S) \to K_n(S/L)$ of abelian groups, which is induced from the split surjection $S \to S/L$.

Observe that rings of the form in Corollaries 3.12 or Proposition 3.13 occur in terminal orders over smooth projective surfaces. For example, if we take $R$ to be the power series ring $k[[z]]$ over a field $k$ in one variable $z$, $x = z$ and $y = 1$, then the ring $S$ in Proposition 3.13 is related to the completion of a closed point in a quasi-projective surface (see [2]).

Finally, we give an example of universal localizations to illustrate Theorem 1.1.

Let $k$ be a field, and let $R$ be a $k$-algebra with the $2 \times 2$ matrix ring $M_2(k)$ over $k$ as its vector space, and with the multiplication given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' & ab' + bd' \\ ca' + dc' & dd' \end{pmatrix}$$

for $a, a', b, b', c, c', d, d' \in k$. Note that $R$ can be depicted as the following quiver algebra with relations

$$(\sharp) \qquad \begin{tikzcd} 1 \bullet \arrow[r, bend left, "\alpha"] & \bullet 2 \arrow[l, bend left, "\beta"] \end{tikzcd}, \quad \alpha\beta = \beta\alpha = 0.$$

Let $e_i$ be the idempotent element of $R$ corresponding to the vertex $i$ for $i = 1, 2$. We consider the universal localization $\lambda : R \to S$ of $R$ at the homomorphism $\varphi : Re_2 \to Re_1$ induced by $\alpha$. This means that, to work out the new algebra $S$, we need to add a new arrow $\alpha^{-1} : 2 \to 1$ and two new relations $\alpha\alpha^{-1} = e_1$ and $\alpha^{-1}\alpha = e_2$ to the quiver ($\sharp$). Thus $\beta = e_2\beta = \alpha^{-1}\alpha\beta = 0$ in $S$ since $\alpha\beta = 0$. In other words, $S$ can be expressed as the following quiver algebra with relations:

$$\begin{tikzcd} 1 \bullet \arrow[r, bend left, "\alpha"] & \bullet 2 \arrow[l, bend left, "\alpha^{-1}"] \end{tikzcd}, \quad \alpha\alpha^{-1} = e_1 \text{ and } \alpha^{-1}\alpha = e_2,$$

which is isomorphic to the usual matrix ring $M_2(k)$ over $k$. Moreover, the ring homomorphism $\lambda : R \to S$ can be given explicitly by

$$e_1 \mapsto e_1,\ e_2 \mapsto e_2,\ \alpha \mapsto \alpha,\ \beta \mapsto 0.$$

It is easy to see that $Se_2 \simeq Se_1 \simeq Re_1$ and $S \simeq Se_1 \oplus Se_2 \simeq Re_1 \oplus Re_1$ as $R$-modules. In particular, ${}_RS$ is finitely generated and projective. Thus $\lambda$ is a homological ring epimorphism with ${}_RS \in \mathscr{P}^{<\infty}(R)$. For more examples of homological ring epimorphisms, we refer the reader to [3, 17].

Let $\mathbf{W}(R, \lambda)$ be the complicial biWaldhausen subcategory of $\mathscr{C}^b(R\text{-proj})$ consisting of those complexes $X^\bullet$ in $\mathscr{C}^b(R\text{-proj})$ such that $S \otimes_R X^\bullet$ is acyclic, and set $K_n(R, \lambda) := K_n\big(\mathbf{W}(R, \lambda)\big)$. It follows from Theorem 1.1 that

$$K_n(R) \simeq K_n(S) \oplus K_n(R, \lambda) \quad \text{for each} \quad n \in \mathbb{N}.$$

Now, we point out that $\mathbf{W}(R, \lambda)$ is equal to the full subcategory of $\mathscr{C}^b(R\text{-proj})$ consisting of those complexes $X^\bullet$ such that $H^i(X^\bullet) \in \mathrm{add}(S_1)$ for all $i \in \mathbb{Z}$, where $S_1$ is the simple $R$-module corresponding to the vertex 1.

In fact, since $S_R \simeq e_1S \oplus e_2S \simeq e_2R \oplus e_2R$ as right $R$-modules, we have

$$H^i\big(S \otimes_R X^\bullet\big) \simeq S \otimes_R H^i(X^\bullet) \simeq e_2R \otimes_R H^i(X^\bullet) \oplus e_2R \otimes_R H^i(X^\bullet) \simeq e_2H^i(X^\bullet) \oplus e_2H^i(X^\bullet).$$

Thus $H^i\left(S \otimes_R X^\bullet\right) = 0$ if and only if $e_2 H^i(X^\bullet) = 0$. This is also equivalent to $H^i(X^\bullet) \in \mathrm{add}(S_1)$ since $R/Re_2R \simeq k$ as algebras.

In this example, $K_0(R) \simeq \mathbb{Z} \oplus \mathbb{Z}$, $K_0(S) \simeq \mathbb{Z}$, $K_1(R) = k^\times \oplus k^\times$ and $K_1(S) = k^\times$. Thus $K_0(R, \lambda) \simeq \mathbb{Z}$ and $K_1(R, \lambda) = k^\times$.

Related to the results in this note, we mention the following open questions.

**Questions** (1) Let $R$ be a ring with identity, and let $e$ be an idempotent element in $R$. Suppose that there is an infinite exact sequence

$$\cdots \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow {}_R ReR \longrightarrow 0$$

such that $P_j \in \mathrm{add}(Re)$ for all $j$. Then we conjecture that $K_n(R) \simeq K_n(R/ReR) \oplus K_n(eRe)$ for every $n \in \mathbb{N}$. (Compare this with Corollary 1.3, Corollary 3.8 and its comment (3)).

(2) Let $R$ be a ring with identity and $I$ be an ideal of $R$ with $I^2 = 0$. We define a ring $S := \begin{pmatrix} R & I \\ I & R \end{pmatrix}$. How is the algebraic $K$-group $K_n(S)$ of $S$ related to the $K_n$-groups of rings produced from $R$ and $I$ for $n \geq 2$?

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## References

1. Auslander, M., Platzeck, M.I., Todorov, G.: Homological theory of idempotent ideals. Trans. Amer. Math. Soc. **332**, 667–692 (1992)
2. Chan, D., Ingalls, C.: Minimal model program for orders over surfaces. Invent. Math. **161**, 427–452 (2005)
3. Chen, H.X., Xi, C.C.: Recollements of derived categories I: Exact contexts, Preprint, arXiv:1203.5168
4. Cohn, P.M.: Free rings and their relations. Academic Press (1971)
5. Dlab, V., Ringel, C.M.: Quasi-hereditary algebras. Ill. J. Math. **33**, 280–291 (1989)
6. Geigle, W., Lenzing, H.: Perpendicular categories with applications to representations and sheaves. J. Algebra **144**, 273–343 (1991)
7. Krause, H., Cohomological quotients and smashing localizations: Amer. J. Math. **127**, 1191–1246 (2005)
8. Neeman, A.: Noncommutative localization in algebraic $K$-theory II. Adv. Math. **213**, 785–819 (2007)
9. Neeman, A., Ranicki, A.: Noncommutative localization in algebraic $K$-theory I. Geom. Topol. **8**, 1385–1425 (2004)
10. Quillen, D.: Higher algebraic K-theory, I. In: Algebraic K-theory, I: Higher K-theories (Seattle, 1972), Lecture Notes in Mathematics 341, 85–147. Springer-Verlag, Berlin (1973)
11. Reiner, I.: Maximal orders. Academic Press (1975)
12. Rosenberg, J.: Algebraic K-theory and its applications, Graduate Texts in Mathematics 147. Springer-Verlag, Berlin (1994)
13. Schofield, A.: Representations of rings over skew fields, London Mathematical Social Lecture Note Series 2. Cambridge University Press, Cambridge (1985)
14. Suslin, A.A., Wodzicki, M.: Excision in algebraic $K$-theory. Ann. Math. **136**, 51–122 (1992)
15. Thomason, R.W., Trobaugh, T.: Higher Algebraic K-theory of Schemes and of Derived Categories. In: The Grothendieck Festschrift, Vol. III. Progress in Mathematics, vol. 88, pp. 247–435. Boston, MA, Birkhäuser (1990)
16. Waldhausen, F.: Algebraic K-theory of Spaces. In: Algebraic and geometric topology (New Brunswick, N.J., 1983). Lecture Notes in Mathematics 1126, 318-419. Springer-Verlag, Berlin (1985)
17. Xi, C.C.: Higher algebraic $K$-groups and $\mathcal{D}$-split sequences. Math. Z **273**, 1025–1052 (2013)