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Good tilting modules and recollements of derived module categories, II

By Hongxing Chen and Changchang Xi

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Abstract. Homological tilting modules of finite projective dimension are investigated. They generalize both classical and good tilting modules of projective dimension at most one, and produce recollements of derived module categories of rings in which generalized localizations of rings are involved. To decide whether a good tilting module is homological, a sufficient and necessary condition is presented in terms of the internal properties of the given tilting module. Consequently, a class of homological, non-trivial, infinitely generated tilting modules of higher projective dimension is constructed, and the first example of an infinitely generated n-tilting module which is not homological for each $n \geq 2$ is exhibited. To deal with both tilting and cotilting modules consistently, the notion of weak tilting modules is introduced. Thus similar results for infinitely generated cotilting modules of finite injective dimension are obtained, though dual technique does not work for infinite-dimensional modules.

1. Introduction.

Infinite dimensional tilting theory has been of interest for a long time, but only recently it has increasingly attracted attention to understand derived categories of general rings and related topics (see [**4**], [**6**], [**7**], [**32**], [**11**], [**12**], [**34**]). In this theory, one of significant and fundamental problems is to understand relationships between the derived category of a given ring and the one of the endomorphism ring of a tilting module. For an infinitely generated tilting module, there exists a recollement of triangulated categories (see [**6**]), in which two of them are the derived module categories of rings as expected, but the third one, which is the kernel of the derived tensor functor of the tilting module, is known only to be a triangulated subcategory. So, to understand the derived category of the endomorphism ring of an infinite-dimensional tilting module, it is crucial to understand this kernel of the derived tensor functor. Once the kernel can be realized as a derived module category of a ring, one gets a recollement of derived module categories of rings, and may use general theory and properties of recollements to reduce the investigation of the derived or homological properties of the endomorphism ring of the tilting module to those of the two outside rings of the recollement: the given ring and the new ring. This kind of reduction by recollements is quite useful in geometry and in representation theory of algebras (see [**8**], [**14**], [**22**], [**13**]).

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In [**11**], we considered good tilting modules of projective dimension at most 1. In this case, it was proved that the triangulated subcategory can always be realized as the derived category of a new ring which is explicitly described as a homological universal localization of the endomorphism ring (see [**11**, Theorem 1.1]). An application of this result is that the Jordan–Hölder theorem fails for stratifications of derived module categories by derived module categories (see [**11**], [**12**]).

For a good, infinitely generated tilting module of projective dimension at least 2, it is fully open whether the kernel of the derived tensor functor of the titling module can be realized as the derived category of an ordinary ring. What one only knows is that the kernel can be formally characterized as the derived category of a differential graded ring (see [**23**], [**30**], [**34**], [**7**]). Furthermore, neither positive non-trivial examples nor negative examples are known before our present consideration.

In the present paper, we continue the project on studying infinitely generated good tilting modules in the context of derived categories. Roughly, we shall give a necessary and sufficient condition for good tilting modules of higher projective dimension to induce recollements of derived module categories via homological ring epimorphisms. Such tilting modules are called *homological tilting modules*. Our condition is presented in terms of the internal properties of given tilting modules, which are handy to be verified in practice. As a consequence, we obtain a class of recollements of derived module categories from good, infinitely generated tilting modules, and construct a class of non-trivial, homological, infinitely generated tilting module as well as the first example of a good tilting module of projective dimension n for each $n \geq 2$, such that it is not homological. As the dual argument of infinitely generated tilting modules does not work for infinitely generated cotilting modules, we introduce the notion of weak tilting modules to handle uniformly the tilting and cotilting cases. In this way, similar results for cotilting modules are obtained.

To state our main results precisely, we recall some definitions and notation.

Let A be an arbitrary ring with identity and let n be a natural number. A left A-module T is called an *n-tilting* A-module (see [**16**], [**2**]) if the following three conditions are satisfied:

$(T1)$ The projective dimension of T, denoted by $\text{proj.dim}(_AT)$, is at most n, that is, there exists an exact sequence

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\pi} T \longrightarrow 0$$

with all P_i projective A-modules.

$(T2)$ $\text{Ext}^j_A(T, T^{(\alpha)}) = 0$ for all $j \geq 1$ and nonempty sets α, where $T^{(\alpha)}$ denotes the direct sum of α copies of T.

$(T3)$ There is an exact sequence of A-modules

$$0 \longrightarrow {_AA} \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_n \longrightarrow 0$$

such that T_i is isomorphic to a direct summand of a direct sum of copies of T for all $0 \leq i \leq n$.

An n-tilting module T is said to be *good* if $(T3)$ is replaced by

$(T3)'$ There is an exact sequence of A-modules

$$0 \longrightarrow {}_AA \stackrel{\omega}{\longrightarrow} T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_n \longrightarrow 0$$

such that T_i is isomorphic to a direct summand of a finite direct sum of copies of T for all $0 \leq i \leq n$.

In the following, let T be a good n-tilting A-module and let B be the endomorphism ring of ${}_AT$. Then T is obviously an A-B-bimodule. By $\mathscr{D}(A)$ and $\mathscr{D}(B)$, we denote the derived module categories of A and B, respectively.

By definition, the third triangulated category in the recollement associated with T is the kernel of the left derived functor $T \otimes^{\mathbb{L}}_B - : \mathscr{D}(B) \to \mathscr{D}(A)$, denoted by $\mathrm{Ker}(T \otimes^{\mathbb{L}}_B -)$. This category measures the difference between $\mathscr{D}(B)$ and $\mathscr{D}(A)$. In fact, it vanishes if and only if the tilting A-module T is *classical* in the sense that ${}_AT$ has a projective resolution of finite length by finitely generated projective modules (see [**10**], [**21**], [**27**], [**20**], [**6**]).

A ring epimorphism $\lambda : B \to C$ is called a *homological ring epimorphism* if $\mathrm{Tor}^B_i(C, C) = 0$ for all $i > 0$ (see [**18**]). We say that ${}_AT$ is *homological* if there exists a homological ring epimorphism $\lambda : B \to C$ such that the restriction functor $D(\lambda_*) : \mathscr{D}(C) \to \mathscr{D}(B)$ induces an equivalence of triangulated categories:

$$\mathscr{D}(C) \stackrel{\simeq}{\longrightarrow} \mathrm{Ker}(T \otimes^{\mathbb{L}}_B -).$$

Thus, every classical tilting module and every good 1-tilting module are homological. A further natural question is about the case of higher projective dimension.

Question. Is every good n-tilting A-module T with $n \geq 2$ homological? If this is not the case, when is an n-tilting module homological?

To answer this question, we first characterize homological tilting modules in terms of vanishing cohomologies of complexes related to projective resolutions of the tilting modules.

Theorem 1.1. *Suppose that A is a ring and n is a natural number. Let T be a good n-tilting A-module, and let B be the endomorphism ring of ${}_AT$. Then ${}_AT$ is homological if and only if the m-th cohomology of the complex $\mathrm{Hom}_A(P^\bullet, A) \otimes_A T_B$ vanishes for all $m \geq 2$, where the complex $P^\bullet$ is a deleted projective resolution of ${}_AT$. In this case, there exists a recollement of derived module categories*:

$$\mathscr{D}(B_T) \xrightarrow{D(\lambda_*)} \mathscr{D}(B) \xrightarrow{{}_AT\otimes^{\mathbb{L}}_B -} \mathscr{D}(A)$$

where $\lambda : B \to B_T$ is the generalized localization of B at the right B-module T and $D(\lambda_*)$ stands for the restriction functor induced by λ.

Since the condition in Theorem 1.1 is given in terms of the projective resolution of a given tilting module, it is handy to be checked for applications. For example, we have the following result.

COROLLARY 1.2. *Suppose that A is a ring and n is a natural number. Let T be a good n-tilting A-module, and let B be the endomorphism ring of ${}_AT$.*

(1) *If ${}_AT$ decomposes into a direct sum of M and N such that* $\mathrm{proj.dim}({}_AM) \leq 1$ *and the first syzygy of N is finitely generated, then T is homological.*

(2) *If A is commutative and* $\mathrm{Hom}_A(T_{i+1}, T_i) = 0$ *for all T_i in $(T3)'$ with $1 \leq i \leq n-1$, then T is homological if and only if it is a 1-tilting module.*

As a consequence of Corollary 1.2, we can construct not only a class of homological, infinitely generated tilting modules, but also the first example of a good n-tilting module T for each $n \geq 2$, such that T is not homological (see Section 7).

Dually, we consider good, infinitely generated cotilting modules of finite injective dimension over arbitrary rings. The definition of cotilting modules uses injective cogenerators of module categories (see Definition 6.1), and there are many choices of injective cogenerators. Hence, there is no nice duality between infinitely generated tilting and cotilting modules, and we cannot get analogous results on infinitely generated cotilting modules by duality from the ones on infinitely generated tilting modules. Nevertheless, our methods in the paper can deal with cotilting modules over rings with certain "nice" injective cogenerators. For example, the following theorem is an analogy of [**11**, Theorem 1.1].

THEOREM 1.3. *Suppose that A is an Artin algebra with the usual duality functor D. Let U be a good 1-cotilting A-module with respect to the injective cogenerator $D(A)$. Set $R := \mathrm{End}_A(U)$ and $M := \mathrm{Hom}_A(U, D(A))$. Then the universal localization $\lambda : R \to R_M$ of R at the module ${}_RM$ is homological, and there exists a recollement of derived module categories*:

$$\mathscr{D}(R_M) \xrightarrow{D(\lambda_*)} \mathscr{D}(R) \longrightarrow \mathscr{D}(A)$$

where $D(\lambda_*)$ stands for the restriction functor induced by λ.

Theorem 1.1 extends [**11**, Theorem 1.1] from good 1-tilting modules to homological n-tilting modules, and provides a class of recollements of derived categories of rings, while the recollements in [**34**] and [**7**] involve derived categories of differential graded algebras instead of usual rings. Moreover, the proof of Theorem 1.1 is different from the one in [**11**]. Note that the proof in [**11**] deals with a two-term complex which automatically yields an abelian subcategory of the module category of the endomorphism ring. However, this is not always true for tilting modules of higher projective dimension. To attack this general case, we have to extend some methods in [**11**] and develop some new ideas (see Section 4). It seems that this is the first time to apply Mittag-Leffler conditions to discuss tilting modules at the level of derived categories.

The contents of this article are sketched as follows. In Section 2, we fix notation, recall some definitions and prove some homological formulas for later proofs. In Section 3, we discuss when bireflective subcategories are homological. In Section 4, we introduce weak tilting modules and give several characterizations of homological subcategories arising from weak tilting modules. In Section 5, we apply the results in the previous sections to show Theorem 1.1 and Corollary 1.2. In Section 6, we first apply Proposition 4.4 to good cotilting modules in a general setting (see Corollary 6.3), and then prove Theorem 1.3 for Artin algebras. In Section 7, we employ Corollary 1.2 to construct a class of infinitely generated, homological tilting modules over non-commutative Gorenstein rings (see Proposition 7.1), and examples of good n-tilting modules T over commutative Gorenstein rings for $n \geq 2$, such that T is not homological (see Proposition 7.2). Such examples are not known before our discussions in this paper.

2. Preliminaries.

In this section, we briefly recall some definitions, basic facts and notation used in this paper. For unexplained notation employed in this paper, we refer the reader to [**11**] and the references therein.

2.1. Notation.

Let $\mathcal{C}$ be an additive category.

Throughout the paper, a full subcategory $\mathcal{B}$ of $\mathcal{C}$ is always assumed to be closed under isomorphisms, that is, if $X \in \mathcal{B}$ and $Y \in \mathcal{C}$ with $Y \simeq X$, then $Y \in \mathcal{B}$.

Let X be an object in $\mathcal{C}$. We denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{C}$ consisting of all direct summands of finite coproducts of copies of M. If $\mathcal{C}$ admits small coproducts (that is, coproducts indexed over sets exist in $\mathcal{C}$), then we denote by $\mathrm{Add}(X)$ the full subcategory of $\mathcal{C}$ consisting of all direct summands of small coproducts of copies of X. Dually, if $\mathcal{C}$ admits small products, then $\mathrm{Prod}(X)$ denotes the full subcategory of $\mathcal{C}$ consisting of all direct summands of small products of copies of X.

Given two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, the composition of f and g is written as fg which is a morphism from X to Z. The induced morphisms $\mathrm{Hom}_{\mathcal{C}}(Z, f) : \mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Z, Y)$ and $\mathrm{Hom}_{\mathcal{C}}(f, Z) : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ are denoted by f^* and f_*, respectively.

For two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, the composition of F and G is denoted by GF which is a functor from $\mathcal{C}$ to $\mathcal{E}$. Let $\mathrm{Ker}(F)$ and $\mathrm{Im}(F)$ be the kernel and image of the functor F, respectively. In particular, $\mathrm{Ker}(F)$ is closed under isomorphisms in $\mathcal{C}$. In this note, we require that $\mathrm{Im}(F)$ is closed under isomorphisms in $\mathcal{D}$.

Suppose that $\mathcal{Y}$ is a full subcategory of $\mathcal{C}$. Let $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{Y}))$ be the left orthogonal subcategory with respect to $\mathcal{Y}$, that is, the full subcategory of $\mathcal{C}$ consisting of the objects X such that $\mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$ for all objects Y in $\mathcal{Y}$. Similarly, we can define the right orthogonal subcategory $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(\mathcal{Y}, -))$ of $\mathcal{C}$ with respect to $\mathcal{Y}$.

Let $\mathscr{C}(\mathcal{C})$ be the category of all complexes over $\mathcal{C}$ with chain maps, and $\mathscr{K}(\mathcal{C})$ the homotopy category of $\mathscr{C}(\mathcal{C})$. As usual, we denote by $\mathscr{C}^b(\mathcal{C})$ the category of bounded complexes over C, and by $\mathscr{K}^b(\mathcal{C})$ the homotopy category of $\mathscr{C}^b(\mathcal{C})$. When $\mathcal{C}$ is abelian, the derived category of $\mathcal{C}$ is denoted by $\mathscr{D}(\mathcal{C})$, which is the localization of $\mathscr{K}(\mathcal{C})$ at all quasi-isomorphisms. Remark that both $\mathscr{K}(\mathcal{C})$ and $\mathscr{D}(\mathcal{C})$ are triangulated categories. For a triangulated category, its shift functor is denoted by [1] universally.

If $\mathcal{T}$ is a triangulated category with small coproducts, then, for an object U in $\mathcal{T}$, we denote by $\mathrm{Tria}(U)$ the smallest full triangulated subcategory of $\mathcal{T}$ containing U and being closed under small coproducts.

A *hereditary torsion pair* $(\mathcal{X}, \mathcal{Y})$ of a triangulated category $\mathcal{T}$ consists of two full triangulated subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(\mathcal{X}, \mathcal{Y}) = 0$ and for each object $M \in \mathcal{T}$, there is a triangle $X \to M \to Y \to X[1]$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ (see [**9**, Chapter I.2]).

Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are triangulated categories with small coproducts. If $F : \mathcal{T} \to \mathcal{T}'$ is a triangle functor which commutes with small coproducts, then $F(\mathrm{Tria}(U)) \subseteq \mathrm{Tria}(F(U))$ for every object U in $\mathcal{T}$.

2.2. Homological formulas in the derived categories of rings.

In this paper, all rings considered are associative and with identity, and all ring homomorphisms preserve identity. Unless stated otherwise, all modules are referred to left modules.

Let R be a ring. We denote by R-Mod the category of all unitary left R-modules, and by Ω_R^n the n-th syzygy operator of R-Mod for $n \in \mathbb{N}$. We regard Ω_R^0 as the identity operator on R-Mod. Note that Ω_R^n depends on the choice of projective resolutions of a module, but is unique up to projective summand.

If M is an R-module and I is a nonempty set, then $M^{(I)}$ and M^I denote the direct sum and product of I copies of M, respectively, $\mathrm{proj.dim}({}_RM)$ and $\mathrm{inj.dim}({}_RM)$ the projective and injective dimensions of M, respectively, and $\mathrm{End}_R(M)$ the endomorphism ring of M. Up to projective module, we denote by $\Omega^n(M)$ the n-th syzygy of a projective resolution of M for $n \geq 0$.

As usual, we simply write $\mathscr{C}(R)$, $\mathscr{K}(R)$ and $\mathscr{D}(R)$ for $\mathscr{C}(R\text{-Mod})$, $\mathscr{K}(R\text{-Mod})$ and $\mathscr{D}(R\text{-Mod})$, respectively, and identify R-Mod with the subcategory of $\mathscr{D}(R)$ consisting of all stalk complexes concentrated in degree zero. Let $\mathscr{C}(R\text{-proj})$ be the full subcategory of $\mathscr{C}(R)$ consisting of those complexes such that all of their terms are finitely generated projective R-modules.

For each $n \in \mathbb{Z}$, we denote by $H^n(-) : \mathscr{D}(R) \to R\text{-Mod}$ the n-th cohomology functor. A complex $X^\bullet$ is said to be *acyclic* (*or exact*) if $H^n(X^\bullet) = 0$ for all $n \in \mathbb{Z}$.

In the following, we shall recall some definitions and basic facts about derived functors (see [**33**], [**23**]).

Recall that $\mathscr{K}(R)_P$ (respectively, $\mathscr{K}(R)_I$) denotes the smallest full triangulated subcategory of $\mathscr{K}(R)$ which

(i) contains all the bounded-above (respectively, bounded-below) complexes of projective (respectively, injective) R-modules, and

(ii) is closed under arbitrary direct sums (respectively, direct products).

Let $\mathscr{K}(R)_C$ be the full subcategory of $\mathscr{K}(R)$ consisting of all acyclic complexes. Then $(\mathscr{K}(R)_P, \mathscr{K}(R)_C)$ forms a hereditary torsion pair in $\mathscr{K}(R)$. In particular, for each $X^\bullet \in \mathscr{K}(R)$, there exists a quasi-isomorphism $\alpha_{X^\bullet} : {}_pX^\bullet \to X^\bullet$ in $\mathscr{K}(R)$ such that ${}_pX^\bullet \in \mathscr{K}(R)_P$. The complex ${}_pX^\bullet$ is called the *projective resolution* of $X^\bullet$ in $\mathscr{D}(R)$. For example, if X is an R-module, then ${}_pX$ can be chosen as a deleted projective resolution of ${}_RX$.

Dually, $(\mathscr{K}(R)_C, \mathscr{K}(R)_I)$ is a hereditary torsion pair in $\mathscr{K}(R)$. In particular, for each $X^\bullet$ in $\mathscr{D}(R)$, there exists a quasi-isomorphism $\beta_{X^\bullet} : X^\bullet \to {}_iX^\bullet$ in $\mathscr{K}(R)$ with

${}_iX^\bullet \in \mathscr{K}(R)_I$. The complex ${}_iX^\bullet$ is called the *injective resolution* of $X^\bullet$ in $\mathscr{D}(R)$.

The localization functor $q : \mathscr{K}(R) \to \mathscr{D}(R)$ induces an isomorphism $\mathrm{Hom}_{\mathscr{K}(R)}(X^\bullet, Y^\bullet) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{D}(R)}(X^\bullet, Y^\bullet)$ whenever either $X^\bullet \in \mathscr{K}(R)_P$ or $Y^\bullet \in \mathscr{K}(R)_I$. Moreover, q restricts to equivalences of triangulated categories: $\mathscr{K}(R)_P \xrightarrow{\simeq} \mathscr{D}(R)$ and $\mathscr{K}(R)_I \xrightarrow{\simeq} \mathscr{D}(R)$.

Now, let S be another ring. For a triangle functor $F : \mathscr{K}(R) \to \mathscr{K}(S)$, its *left derived functor* $\mathbb{L}F : \mathscr{D}(R) \to \mathscr{D}(S)$ is defined by $X^\bullet \mapsto F({}_pX^\bullet)$, and its *right derived functor* $\mathbb{R}F : \mathscr{D}(R) \to \mathscr{D}(S)$ is defined by $X^\bullet \mapsto F({}_iX^\bullet)$. Further, if $F(X^\bullet)$ is acyclic whenever $X^\bullet$ is acyclic, then F induces a triangle functor $D(F) : \mathscr{D}(R) \to \mathscr{D}(S)$, $X^\bullet \mapsto F(X^\bullet)$. In this case, up to natural isomorphism, $\mathbb{L}F = \mathbb{R}F = D(F)$, and $D(F)$ is called the *derived functor* of F.

Let $M^\bullet$ be a complex of R-S-bimodules. We denote by $M^\bullet \otimes_S^{\mathbb{L}} -$ the left derived functor of $M^\bullet \otimes_S^\bullet -$, and by $\mathbb{R}\mathrm{Hom}_R(M^\bullet, -)$ the right derived functor of $\mathrm{Hom}_R^\bullet(M^\bullet, -)$. Note that $(M^\bullet \otimes_S^{\mathbb{L}} -, \mathbb{R}\mathrm{Hom}_R(M^\bullet, -))$ is an adjoint pair of triangle functors. If $Y \in S$-Mod and $X \in R$-Mod, we denote $M^\bullet \otimes_S^\bullet Y$ and $\mathrm{Hom}_R^\bullet(M^\bullet, X)$ simply by $M^\bullet \otimes_S Y$ and $\mathrm{Hom}_R(M^\bullet, X)$, respectively.

Let $\mathcal{L}_F$ denote the full subcategory of $\mathscr{K}(R)$ consisting of all complexes $X^\bullet$ such that the chain map $F(\alpha_{X^\bullet}) : F({}_pX^\bullet) \to F(X^\bullet)$ is a quasi-isomorphism in $\mathscr{K}(S)$, and $\mathcal{R}_F$ the full subcategory of $\mathscr{K}(R)$ consisting of all complexes $X^\bullet$ such that the chain map $F(\beta_{X^\bullet}) : F(X^\bullet) \to F({}_iX^\bullet)$ is a quasi-isomorphism in $\mathscr{K}(S)$.

The following result on $\mathcal{L}_F$ and $\mathcal{R}_F$ is freely used, but not stated explicitly in the literature. Here, we arrange it as a lemma for later reference. For the idea of its proof, we refer to [**33**, Generalized Existence Theorem 10.5.9].

LEMMA 2.1. *Let $F : \mathscr{K}(R) \to \mathscr{K}(S)$ be a triangle functor. Then the following hold*:

(1) *There exists a commutative diagram of triangle functors*:

$$\begin{array}{ccc} \mathscr{K}(R)_P & \xrightarrow{\simeq} & \mathscr{D}(R) \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\mathbb{L}F} \\ \mathcal{L}_F/(\mathcal{L}_F \cap \mathscr{K}(R)_C) & \xrightarrow{D(F)} & \mathscr{D}(S) \end{array}$$

where $\mathcal{L}_F/(\mathcal{L}_F \cap \mathscr{K}(R)_C)$ denotes the Verdier quotient of $\mathcal{L}_F$ by $\mathcal{L}_F \cap \mathscr{K}(R)_C$, and where $D(F)$ is defined by $X^\bullet \mapsto F(X^\bullet)$ for $X^\bullet \in \mathcal{L}_F$.

(2) *There exists a commutative diagram of triangle functors*:

$$\begin{array}{ccc} \mathscr{K}(R)_I & \xrightarrow{\simeq} & \mathscr{D}(R) \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\mathbb{R}F} \\ \mathcal{R}_F/(\mathcal{R}_F \cap \mathscr{K}(R)_C) & \xrightarrow{D(F)} & \mathscr{D}(S) \end{array}$$

where $\mathcal{R}_F/(\mathcal{R}_F \cap \mathscr{K}(R)_C)$ denotes the Verdier quotient of $\mathcal{R}_F$ by $\mathcal{R}_F \cap \mathscr{K}(R)_C$, and where $D(F)$ is defined by $X^\bullet \mapsto F(X^\bullet)$ for $X^\bullet \in \mathcal{R}_F$.

Note that if F commutes with arbitrary direct sums, then $\mathcal{L}_F$ is closed under arbitrary direct sums in $\mathscr{K}(R)$. Dually, if F commutes with arbitrary direct products, then $\mathcal{R}_F$ is closed under arbitrary direct products in $\mathscr{K}(R)$.

From Lemma 2.1, up to natural isomorphism, the action of the functor $\mathbb{L}F$ (respectively, $\mathbb{R}F$) on a complex $X^\bullet$ in $\mathcal{L}_F$ (respectively, $\mathcal{R}_F$) is the same as that of the functor F on $X^\bullet$.

Corollary 2.2. *Let R and S be two rings. Suppose that (F, G) is a pair of adjoint triangle functors with $F : \mathscr{K}(S) \to \mathscr{K}(R)$ and $G : \mathscr{K}(R) \to \mathscr{K}(S)$. Let $\theta : FG \to \mathrm{Id}_{\mathscr{K}(R)}$ and $\varepsilon : (\mathbb{L}F)(\mathbb{R}G) \to \mathrm{Id}_{\mathscr{D}(R)}$ be the counit adjunctions. If $X^\bullet \in \mathcal{R}_G$ and $G(X^\bullet) \in \mathcal{L}_F$, then there exists a commutative diagram in $\mathscr{D}(R)$:*

$$\begin{array}{ccc} (\mathbb{L}F)(\mathbb{R}G)(X^\bullet) & \xrightarrow{\varepsilon_{X^\bullet}} & X^\bullet \\ \simeq\downarrow & & \Vert \\ FG(X^\bullet) & \xrightarrow{\theta_{X^\bullet}} & X^\bullet \end{array}$$

2.3. Relative Mittag-Leffler modules.

Now, we recall the definition of Mittag-Leffler modules (see [**19**], [**3**]).

Definition 2.3. Let $\mathcal{X}$ be a class of R-modules. A right R-module M is *$\mathcal{X}$-Mittag-Leffler* if for any nonempty set $\{X_i \mid i \in I\}$ of modules in $\mathcal{X}$ the canonical map

$$\rho_I : \ M \otimes_R \prod_{i\in I} X_i \longrightarrow \prod_{i\in I} M \otimes_R X_i, \quad m \otimes (x_i)_{i\in I} \mapsto (mx_i)_{i\in I} \ \text{for} \ m \in M, \ x_i \in X_i,$$

is injective. If $\mathcal{X}$ just consists of a single module X, then we say that M is *X-Mittag-Leffler.*

A right R-module M is *strongly R-Mittag-Leffler* if the m-th syzygy of M is R-Mittag-Leffler for every $m \geq 0$.

By [**19**, Theorem 1], a right R-module M is R-Mittag-Leffler if and only if, for any finitely generated submodule X of M_R, the inclusion $X \to M$ factorizes through a finitely presented right R-module. This implies that if M is finitely presented, then it is R-Mittag-Leffler. Actually, for such a module M, the above map ρ_I is always bijective (see [**17**, Theorem 3.2.22]). Further, if the ring R is right noetherian, then each right R-module is R-Mittag-Leffler since each finitely generated right R-module is finitely presented.

Lemma 2.4. *Let R be a ring and M a right R-module.*

(1) *If M is R-Mittag-Leffler, then so is each module in* $\mathrm{Add}(M_R)$. *In particular, each projective right R-module is R-Mittag-Leffler.*

(2) *The first syzygy of M in* R^{op}-Mod *is R-Mittag-Leffler if and only if* $\mathrm{Tor}_1^R(M, R^I) = 0$ *for every nonempty set I.*

(3) *M is strongly R-Mittag-Leffler if and only if M is R-Mittag-Leffler and* $\mathrm{Tor}_i^R(M, R^I) = 0$ *for each $i \geq 1$ and every nonempty set I.*

(4) *If M is finitely generated, then M is strongly R-Mittag-Leffler if and only if M has a finitely generated projective resolution.*

PROOF. (1) follows from the fact that tensor functors commute with direct sums.

(2) Note that the first syzygy $\Omega_{R^{\mathrm{op}}}(M)$ of M depends on the choice of projective presentations of M_R. However, the "R-Mittag-Leffler" property of $\Omega_{R^{\mathrm{op}}}(M)$ is independent of the choice of projective presentations of M_R. This is due to (1) and Schanuel's Lemma in homological algebra.

We choose an exact sequence

$$0 \longrightarrow K_1 \xrightarrow{f} P_1 \longrightarrow M \longrightarrow 0$$

of right R-modules with P_1 projective, and shall show that K_1 is R-Mittag-Leffler if and only if $\mathrm{Tor}_1^R(M, R^I) = 0$ for any nonempty set I. Obviously, we can construct the exact commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Tor}_1^R(M,R^I) & \longrightarrow & K_1 \otimes_R R^I & \xrightarrow{f\otimes 1} & P_1 \otimes_R R^I & \longrightarrow & M \otimes_R R^I & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_1} & & \downarrow & & \\ & & 0 & \longrightarrow & {K_1}^I & \xrightarrow{f^I} & {P_1}^I & \longrightarrow & M^I & \longrightarrow & 0 \end{array}$$

where ρ_i, $1 \le i \le 2$, are the canonical maps (see Definition 2.3). Since the projective module P_1 is R-Mittag-Leffler by (1), the map ρ_1 is injective. This means that ρ_2 is injective if and only if so is $f \otimes 1$. Clearly, the former is equivalent to saying that K_1 is R-Mittag-Leffler, while the latter is equivalent to $\mathrm{Tor}_1^R(M, R^I) = 0$. This finishes the proof of (2).

(3) For each $i \ge 0$ and each nonempty set I, we have $\mathrm{Tor}_{i+1}^R(M, R^I) \simeq \mathrm{Tor}_1^R(\Omega^i_{R^{\mathrm{op}}}(M), R^I)$. Now (3) follows immediately from (2).

(4) follows from the fact that a finitely generated right R-module is finitely presented if and only if it is R-Mittag-Leffler (see [**19**]). □

A special class of strongly Mittag-Leffler modules is the class of tilting modules.

LEMMA 2.5. *If M is a tilting right R-module, then M is strongly R-Mittag-Leffler.*

PROOF. Let $\mathcal{L} := \{Y \in R^{\mathrm{op}}\text{-Mod} \mid \mathrm{Ext}^i_{R^{\mathrm{op}}}(M, Y) = 0 \text{ for all } i \ge 1\}$ and $\mathcal{M} := \{X \in R^{\mathrm{op}}\text{-Mod} \mid \mathrm{Ext}^1_{R^{\mathrm{op}}}(X, Y) = 0 \text{ for all } Y \in \mathcal{L}\}$. Then $(\mathcal{M}, \mathcal{L})$ is the tilting cotorsion pair in R^{op}-Mod induced by M. Let $\mathcal{C} := \{Z \in R\text{-Mod} \mid \mathrm{Tor}_1^R(X, Z) = 0 \text{ for all } X \in \mathcal{M}\}$. It is shown in [**3**, Corollary 9.8] (see also [**3**, Theorem 9.5]) that each module $X \in \mathcal{M}$ is strict $\mathcal{L}$-stationary (see [**3**, Section 8] for definition) and thus $\mathcal{C}$-Mittag-Leffler. Since $\mathcal{M}$ contains M_R and is closed under taking syzygies in R^{op}-Mod, $\Omega^m_{R^{\mathrm{op}}}(M) \in \mathcal{M}$ for $m \ge 0$. Consequently, $\Omega^m_{R^{\mathrm{op}}}(M)$ is $\mathcal{C}$-Mittag-Leffler and therefore R-Mittag-Leffler because ${}_RR$ belongs to $\mathcal{C}$. Thus M is strongly R-Mittag-Leffler. □

By Lemma 2.5 and Lemma 2.4(4), a tilting right R-module is classical if and only if it is finitely generated.

3. Homological subcategories of derived module categories.

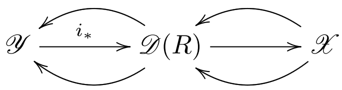
In this section, we shall give the definitions of bireflective and homological subcategories of derived module categories. In particular, we shall establish a few applicable criterions for bireflective subcategories to be homological.

Let $\lambda : R \to S$ be a homomorphism of rings. We denote by $\lambda_* : S\text{-Mod} \to R\text{-Mod}$ the restriction functor induced by λ, and by $D(\lambda_*) : \mathscr{D}(S) \to \mathscr{D}(R)$ the derived functor of the exact functor λ_*.

Recall that λ is a *ring epimorphism* if $\lambda_* : S\text{-Mod} \to R\text{-Mod}$ is fully faithful, or equivalently, the multiplication map $S \otimes_R S \to S$ is an isomorphism. A homomorphism $\lambda : R \to S$ of rings is *homological* (see [**18**]) if and only if the functor $D(\lambda_*) : \mathscr{D}(S) \to \mathscr{D}(R)$ is fully faithful, or equivalently $S \otimes^{\mathbb{L}}_R S \simeq S$ in $\mathscr{D}(S)$. Note that $D(\lambda_*)$ has a left adjoint $S \otimes^{\mathbb{L}}_R -$ and a right adjoint $\mathbb{R}\mathrm{Hom}_R(S, -)$.

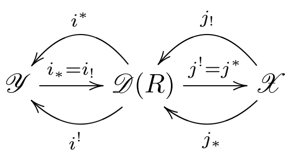
Let $\mathscr{Y}$ be a full triangulated subcategory of $\mathscr{D}(R)$. Then $\mathscr{Y}$ is said to be *bireflective* if the inclusion $\mathscr{Y} \to \mathscr{D}(R)$ admits both a left adjoint and a right adjoint; and *homological* if there is a homological ring epimorphism $\lambda : R \to S$ such that $D(\lambda_*)$ induces a triangle equivalence from $\mathscr{D}(S)$ to $\mathscr{Y}$.

If $\mathscr{Y}$ is homological in $\mathscr{D}(R)$, then it is bireflective, while bireflective categories are closely related to recollements, that is, a full triangulated subcategory $\mathscr{Y}$ of $\mathscr{D}(R)$ is bireflective if and only if there exists a recollement of triangulated categories of the form



where i_* is the inclusion functor (see [**28**, Chapter 9], [**9**, Chapter I.2], [**30**, Section 2.1] or [**11**, Section 2.3]).

Recollements were first introduced by Beilinson, Bernstein and Deligne in [**8**] to study the triangulated categories of perverse sheaves over singular spaces, and later were used by Cline, Parshall and Scott in [**14**] to stratify the derived categories of quasi-hereditary algebras. For our purpose in this section, we will focus on a special class of recollements of triangulated categories. Here, by a *recollement* of triangulated categories, we mean that there are six triangle functors between triangulated categories in the diagram:



such that

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs,

(2) i_*, j_* and $j_!$ are fully faithful functors,

(3) $i^! j_* = 0$ (and thus also $j^! i_! = 0$ and $i^* j_! = 0$), and

(4) for each object $X \in \mathscr{D}(R)$, there are two canonical distinguished triangles in $\mathscr{D}(R)$:

$$i_! i^!(X) \longrightarrow X \longrightarrow j_* j^*(X) \longrightarrow i_! i^!(X)[1], \qquad j_! j^!(X) \longrightarrow X \longrightarrow i_* i^*(X) \longrightarrow j_! j^!(X)[1],$$

where $i_!i^!(X) \to X$ and $j_!j^!(X) \to X$ are counit adjunction morphisms, and where $X \to j_*j^*(X)$ and $X \to i_*i^*(X)$ are unit adjunction morphisms.

From now on, we assume that $\mathscr{Y}$ is a *bireflective subcategory* of $\mathscr{D}(R)$, and define $\mathcal{E} := \mathscr{Y} \cap R\text{-Mod}$.

It is easy to see that $\mathscr{Y}$ is closed under isomorphisms, arbitrary direct sums and products in $\mathscr{D}(R)$. This implies that $\mathcal{E}$ also has the above properties in R-Mod. Moreover, $\mathcal{E}$ always admits the "2 out of 3" property: For an arbitrary short exact sequence in R-Mod, if any two of its three terms belong to $\mathcal{E}$, then the third one belongs to $\mathcal{E}$. Thus $\mathcal{E}$ is an abelian subcategory of R-Mod if and only if $\mathcal{E}$ is closed under kernels (respectively, cokernels) in R-Mod. Since $\mathcal{E}$ is closed under isomorphisms, direct sums and products in R-Mod, it follows from [**1**, Theorem 2.4] that $\mathcal{E}$ is an abelian subcategory of R-Mod if and only if there exists a unique ring epimorphism $\lambda : R \to S$ (up to equivalence) such that $\mathcal{E}$ is equal to $\mathrm{Im}(\lambda_*)$.

Let $i_* : \mathscr{Y} \to \mathscr{D}(R)$ be the inclusion functor with $i^* : \mathscr{D}(R) \to \mathscr{Y}$ as its left adjoint. Define $\Lambda := \mathrm{End}_{\mathscr{D}(R)}(i^*(R))$. Then, associated with $\mathscr{Y}$, there is a ring homomorphism defined by

$$\delta :\ R \longrightarrow \Lambda, \quad r \mapsto i^*(\cdot r) \ \text{ for } \ r \in R,$$

where $\cdot r : R \to R$ is the right multiplication by r map. This induces a functor $\delta_* : \Lambda\text{-Mod} \to R\text{-Mod}$, called the restriction functor.

Lemma 3.1. (1) *For each $Y^\bullet \in \mathscr{Y}$, we have $H^n(Y^\bullet) \in \mathrm{Im}(\delta_*)$ for all $n \in \mathbb{Z}$. In particular, $H^n(i^*(R))$ is an R-Λ-bimodule for all $n \in \mathbb{Z}$.*

(2) *Let $\eta_R : R \to i_*i^*(R)$ be the unit adjunction morphism with respect to the adjoint pair (i^*, i_*). Then there exists an isomorphism $\psi : \Lambda \to H^0(i^*(R))$ of R-Λ-bimodules such that $\delta\psi = H^0(\eta_R)$.*

(3) *If $H^0(i^*(R)) \in \mathscr{Y}$, then $H^n(i^*(R)) = 0$ for all $n \geq 1$, the map δ is a ring epimorphism and*

$$\mathscr{Y} = \{Y^\bullet \in \mathscr{D}(R) \mid H^m(Y^\bullet) \in \mathrm{Im}(\delta_*) \textit{ for all } m \in \mathbb{Z}\}.$$

Proof. The proof of Lemma 3.1 is motivated by the one in [**29**, Section 6 and Section 7] where $\mathscr{Y}$ is related to a set of two-term complexes in $\mathscr{C}(R\text{-proj})$.

By our convention, the full subcategory $\mathrm{Im}(\delta_*)$ of R-Mod is required to be closed under isomorphisms in R-Mod.

(1) Let $Y^\bullet \in \mathscr{Y}$. Then the following isomorphisms hold for each $n \in \mathbb{Z}$:

$$\mathrm{Hom}_{\mathscr{D}(R)}(i^*(R), Y^\bullet[n]) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{D}(R)}(R, i_*(Y^\bullet)[n]) = \mathrm{Hom}_{\mathscr{D}(R)}(R, Y^\bullet[n]) \simeq H^n(Y^\bullet),$$

where the first isomorphism is given by $\mathrm{Hom}_{\mathscr{D}(R)}(\eta_R, Y^\bullet[n])$, which is actually an isomorphism of R-modules. Since $\mathrm{Hom}_{\mathscr{D}(R)}(i^*(R), Y^\bullet[n])$ is a left Λ-module, $H^n(Y^\bullet) \in \mathrm{Im}(\delta_*)$. If $Y^\bullet = i^*(R)$, then the composition of the isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathscr{D}(R)}(i^*(R), i^*(R)[n]) &\simeq \mathrm{Hom}_{\mathscr{D}(R)}(R, i_*i^*(R)[n]) \\ &= \mathrm{Hom}_{\mathscr{D}(R)}(R, i^*(R)[n]) \simeq H^n(i^*(R))\end{aligned} \qquad (*)$$

is an isomorphism of R-Λ-bimodules. Thus $H^n(i^*(R))$ is an R-Λ-bimodule.

(2) In $(*)$, taking $n = 0$ yields an isomorphism $\psi : \Lambda \to H^0(i^*(R))$ of R-Λ-bimodules. Note that there is the commutative diagram of R-modules:

$$\begin{array}{ccc} \mathrm{Hom}_R(R,R) & \xrightarrow{i^*} & \mathrm{Hom}_{\mathscr{D}(R)}(i^*(R), i^*(R)) \\ & \searrow^{\mathrm{Hom}_{\mathscr{D}(R)}(R,\eta_R)} & \downarrow \simeq \\ & & \mathrm{Hom}_{\mathscr{D}(R)}(R, i_*i^*(R)) \end{array}$$

Now, identifying $\mathrm{Hom}_R(R,R)$, $\mathrm{Hom}_{\mathscr{D}(R)}(R, i_*i^*(R))$ and $\mathrm{Hom}_{\mathscr{D}(R)}(R, \eta_R)$ with R, $H^0(i^*(R))$ and $H^0(\eta_R)$, respectively, we get $\delta\psi = H^0(\eta_R)$.

(3) Define

$$\mathscr{Y}' := \{Y^\bullet \in \mathscr{D}(R) \mid H^m(Y^\bullet) \in \mathrm{Im}(\delta_*) \text{ for all } m \in \mathbb{Z}\}.$$

It follows from (1) that $\mathscr{Y} \subseteq \mathscr{Y}'$.

Suppose $H^0(i^*(R)) \in \mathscr{Y}$. We shall prove $\mathscr{Y}' \subseteq \mathscr{Y}$, and consequently, $\mathscr{Y} = \mathscr{Y}'$.

In fact, from (2) we have $\Lambda \simeq H^0(i^*(R))$ as R-modules, and so ${}_R\Lambda \in \mathscr{Y}$. Note that the derived functor $D(\delta_*) : \mathscr{D}(\Lambda) \to \mathscr{D}(R)$ admits a right adjoint, and therefore it commutes with arbitrary direct sums. Since $\mathscr{Y}$ is a full triangulated subcategory of $\mathscr{D}(R)$ closed under arbitrary direct sums in $\mathscr{D}(R)$, it follows from $\mathscr{D}(\Lambda) = \mathrm{Tria}({}_\Lambda\Lambda)$ and ${}_R\Lambda \in \mathscr{Y}$ that $\mathrm{Im}(D(\delta_*)) \subseteq \mathscr{Y}$. In particular, $\mathrm{Im}(\delta_*) \subseteq \mathscr{Y}$.

Now, we show $\mathscr{Y}' \subseteq \mathscr{Y}$.

Recall that $\mathscr{Y}$ is a full triangulated subcategory of $\mathscr{D}(R)$ closed under arbitrary direct sums and direct products in $\mathscr{D}(R)$. Therefore it is closed under taking homotopy limits and homotopy colimits in $\mathscr{D}(R)$. Observe that each complex can be obtained from bounded complexes by taking homotopy limits and homotopy colimits, while each bounded complex can be generated by its cohomologies via canonical truncations (see the proof of [**4**, Lemma 4.6]). Since $\mathrm{Im}(\delta_*) \subseteq \mathscr{Y}$, we have $\mathscr{Y}' \subseteq \mathscr{Y}$. Thus $\mathscr{Y} = \mathscr{Y}'$, as desired.

Next, we show $H^n(i^*(R)) = 0$ for all $n \geq 1$. The idea of the proof given here is essentially taken from [**29**, Lemma 6.4].

On the one hand, from the adjoint pair (i^*, i_*), we obtain a triangle in $\mathscr{D}(R)$:

$$X^\bullet \longrightarrow R \xrightarrow{\eta_R} i^*(R) \longrightarrow X^\bullet[1].$$

Evidently, the unit η_R induces an isomorphism $\mathrm{Hom}_{\mathscr{D}(R)}(i^*(R), Y^\bullet[n]) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(R, Y^\bullet[n])$ for each $Y^\bullet \in \mathscr{Y}$ and $n \in \mathbb{Z}$. This implies that $\mathrm{Hom}_{\mathscr{D}(R)}(X^\bullet, Y^\bullet[n]) = 0$ for $Y^\bullet \in \mathscr{Y}$ and $n \in \mathbb{Z}$.

On the other hand, by the canonical truncation at degree 0, we obtain a distinguished triangle of the form:

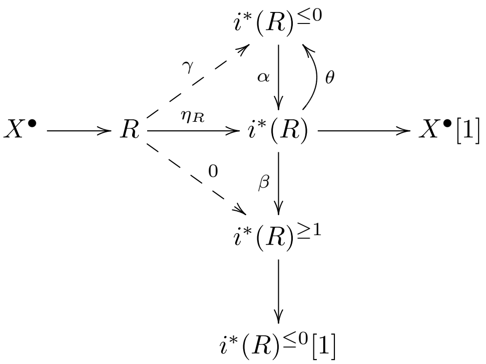
$$i^*(R)^{\leq 0} \xrightarrow{\alpha} i^*(R) \xrightarrow{\beta} i^*(R)^{\geq 1} \longrightarrow i^*(R)^{\leq 0}[1]$$

in $\mathscr{D}(R)$ such that

$$H^s\big(\,i^*(R)^{\leq 0}\big) \simeq \begin{cases} 0 & \text{if } s \geq 1, \\ H^s(i^*(R)) & \text{if } s \leq 0, \end{cases} \quad \text{and}$$

$$H^t\big(\,i^*(R)^{\geq 1}\big) \simeq \begin{cases} 0 & \text{if } t \leq 0, \\ H^t(i^*(R)) & \text{if } t \geq 1. \end{cases}$$

It follows that $\eta_R\beta = 0$ and there exists a homomorphism $\gamma : R \to i^*(R)^{\leq 0}$ with $\gamma\,\alpha = \eta_R$. Due to $i^*(R) \in \mathscr{Y} = \mathscr{Y}'$, we get $i^*(R)^{\leq 0} \in \mathscr{Y}$ and $\mathrm{Hom}_{\mathscr{D}(R)}(X^\bullet, i^*(R)^{\leq 0}) = 0$. Consequently, there exists a homomorphism $\theta : i^*(R) \to i^*(R)^{\leq 0}$ such that $\gamma = \eta_R\,\theta$. So, we have the diagram in $\mathscr{D}(R)$:



with $\eta_R\theta\,\alpha = \gamma\,\alpha = \eta_R$. It follows from the isomorphism

$$\mathrm{Hom}_{\mathscr{D}(R)}(\eta_R, i^*(R)) : \mathrm{Hom}_{\mathscr{D}(R)}(i^*(R), i^*(R)) \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(R, i^*(R))$$

(see $(*)$ in (1)) that $\theta\alpha = \mathrm{Id}_{i^*(R)}$ and

$$H^n(\theta\alpha) = H^n(\theta)H^n(\alpha) = \mathrm{Id}_{H^n(i^*(R))} \quad \text{for any} \quad n \in \mathbb{Z}.$$

This means that $H^n(\theta) : H^n(i^*(R)) \to H^n\big(\,i^*(R)^{\leq 0}\big)$ is injective. Observe that $H^n\big(\,i^*(R)^{\leq 0}\big) = 0$ for $n \geq 1$. Hence $H^n(i^*(R)) = 0$ for $n \geq 1$.

Finally, we prove that $\delta : R \to \Lambda$ is a ring epimorphism.

In fact, δ is a ring epimorphism if and only if $\Lambda \otimes_R \mathrm{Coker}(\delta) = 0$ (see [**31**, Proposition 1.2, p.225]) if and only if $\mathrm{Hom}_\Lambda(\Lambda \otimes_R \mathrm{Coker}(\delta), M) \simeq \mathrm{Hom}_R(\mathrm{Coker}(\delta), M) = 0$ for every Λ-module M if and only if $\mathrm{Hom}_R(\delta, M)$ is injective for all Λ-module M. To prove that $\mathrm{Hom}_R(\delta, M)$ is injective, we shall use the commutative diagram in (2) and show that the induced map

$$\mathrm{Hom}_R\big(H^0(\eta_R), M\big) : \mathrm{Hom}_R\big(H^0(i^*(R)), M\big) \longrightarrow \mathrm{Hom}_R(R, M)$$

is injective. That is, we have to prove that if $f_j : H^0(i^*(R)) \to M$, with $j = 1, 2$, are two homomorphisms in R-Mod such that $H^0(\eta_R)f_1 = H^0(\eta_R)f_2$, then $f_1 = f_2$.

Now, we describe the map $H^0(\eta_R)$. Recall that $H^n(i^*(R)) = 0$ for all $n \geq 1$. So, without loss of generality, we can assume that $i^*(R)$ is of the form (up to isomorphism in $\mathscr{D}(R)$)

$$i^*(R): \quad \cdots \longrightarrow V^{-n} \xrightarrow{d^{-n}} V^{-n+1} \longrightarrow \cdots \longrightarrow V^{-1} \xrightarrow{d^{-1}} V^0 \longrightarrow 0 \longrightarrow \cdots$$

By canonically truncating the above sequence, we obtain the distinguished triangle in $\mathscr{D}(R)$:

$$V^{\bullet\,\leq -1} \longrightarrow i^*(R) \xrightarrow{\pi} H^0(i^*(R)) \longrightarrow V^{\bullet\,\leq -1}[1] \qquad (\star)$$

where $V^{\bullet\,\leq -1}$ is of the form:

$$\cdots \longrightarrow V^{-n} \longrightarrow V^{-n+1} \longrightarrow \cdots \longrightarrow V^{-2} \longrightarrow \mathrm{Ker}(d^{-1}) \longrightarrow 0 \longrightarrow \cdots$$

and π is the chain map induced by the canonical surjection $V^0 \to H^0(i^*(R)) = \mathrm{Coker}(d^{-1})$. Applying $H^0(-) = \mathrm{Hom}_{\mathscr{D}(R)}(R, -)$ to $(\star)$, we see that $H^0(\eta_R) = \eta_R\,\pi$ in $\mathscr{D}(R)$ and that $H^0(\pi)$ is an isomorphism of R-modules.

Suppose that $H^0(\eta_R)f_1 = H^0(\eta_R)f_2 : R \to M$ with $f_i : H^0(i^*(R)) \to M$ for $j = 1, 2$. Then $\eta_R\pi f_1 = \eta_R\pi f_2$ in $\mathscr{D}(R)$. Note that M is a Λ-module and $\mathrm{Im}(\delta_*) \subseteq \mathscr{Y}$. Thus ${}_RM \in \mathscr{Y}$. Since the unit $\eta_R : R \to i_*i^*(R) = i^*(R)$ induces an isomorphism $\mathrm{Hom}_{\mathscr{D}(R)}(i^*(R), M) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(R, M)$, we obtain $\pi\, f_1 = \pi\, f_2$ and $H^0(\pi)f_1 = H^0(\pi)f_2$. It follows from the isomorphism of $H^0(\pi)$ that $f_1 = f_2$. Thus $\mathrm{Hom}_R(H^0(\eta_R), M)$ is injective and δ is a ring epimorphism. This finishes the proof of (3). □

When a bireflective subcategory in a derived module category is homological was discussed in the literature, for example, see [**4**, Proposition 1.7], [**11**, Proposition 3.6] and [**7**, Theorem 6.1]. In the following, we provide a slightly more general characterization of homological subcategories. This will be applied in Section 4.

Lemma 3.2. *Let $\mathscr{Y}$ be a bireflective subcategory of $\mathscr{D}(R)$, and let $i^* : \mathscr{D}(R) \to \mathscr{Y}$ be a left adjoint of the inclusion $\mathscr{Y} \hookrightarrow \mathscr{D}(R)$. Then the following are equivalent*:

(1) *$\mathscr{Y}$ is homological.*

(2) *$H^m(i^*(R)) = 0$ for any $m \neq 0$.*

(3) *$H^0(i^*(R)) \in \mathscr{Y}$ and $H^m(i^*(R)) = 0$ for any $m < 0$.*

(4) *$H^0(i^*(R)) \in \mathscr{Y}$ and $\delta : R \to \mathrm{End}_{\mathscr{D}(R)}(i^*(R))$ is a homological ring epimorphism.*

(5) *There is a ring epimorphism $\lambda : R \to S$ such that ${}_RS \in \mathscr{Y}$ and $i^*(R)$ is isomorphic in $\mathscr{D}(R)$ to a complex $Z^\bullet := (Z^n)_{n\in\mathbb{Z}}$ with $Z^i \in S$-*Mod *for $i = 0, 1$.*

(6) *$\mathcal{E} := \mathscr{Y} \cap R$-*Mod *is an abelian subcategory of* R-Mod *such that $i^*(R)$ is isomorphic in $\mathscr{D}(R)$ to a complex $Z^\bullet := (Z^n)_{n\in\mathbb{Z}}$ with $Z^i \in \mathcal{E}$ for $i = 0, 1$.*

In particular, if one of the above conditions is fulfilled, then $\mathscr{Y}$ can be realized as the derived category of the ring $\mathrm{End}_{\mathscr{D}(R)}(i^*(R))$ *via δ.*

Proof. It follows from the proof of [**4**, Proposition 1.7] that (1) and (2) are equivalent, and that (2) implies both (3) and (4). By Lemma 3.1(3), (3) implies (2).

Now, we show that (4) implies (1). Since $H^0(i^*(R)) \in \mathscr{Y}$, it follows from Lemma 3.1(3) that $\mathscr{Y} = \{Y^\bullet \in \mathscr{D}(R) \mid H^m(Y^\bullet) \in \mathrm{Im}(\delta_*) \text{ for all } m \in \mathbb{Z}\}$, where $\delta : R \to \Lambda := \mathrm{End}_{\mathscr{D}(R)}(i^*(R))$ is the associated homomorphism of rings. By assumption, δ is a homological ring epimorphism, and therefore $D(\delta_*) : \mathscr{D}(\Lambda) \to \mathscr{D}(R)$ is fully faithful. Furthermore, we know from [**4**, Lemma 4.6] that $\mathrm{Im}(D(\delta_*)) = \{Y^\bullet \in \mathscr{D}(R) \mid H^m(Y^\bullet) \in$

$\mathrm{Im}(\delta_*)$ for all $m \in \mathbb{Z}\}$. Thus $\mathscr{Y} = \mathrm{Im}(D(\delta_*)) \subseteq \mathscr{D}(R)$, that is, $\mathscr{Y}$ is homological by definition. Hence (4) implies (1).

Consequently, all conditions (1)–(4) are equivalent.

Note that (5) and (6) are equivalent because $\mathcal{E}$ is an abelian subcategory of R-Mod if and only if there is a ring epimorphism $\lambda : R \to S$ such that $\mathcal{E} = \mathrm{Im}(\lambda_*)$ (for example, see [**1**, Theorem 2.4]).

In the following, we shall prove that (1) implies (5), and that (5) implies (2).

Suppose that $\mathscr{Y}$ is homological, that is, there exists a homological ring epimorphism $\lambda : R \to S$ such that the functor $D(\lambda_*) : \mathscr{D}(S) \to \mathscr{D}(R)$ induces a triangle equivalence from $\mathscr{D}(S)$ to $\mathscr{Y}$. Thus $\mathscr{Y} = \mathrm{Im}(D(\lambda_*))$. Since $i^*(R) \in \mathscr{Y}$, $i^*(R) \in \mathrm{Im}(D(\lambda_*))$. It follows that there exists a complex $Z^\bullet := (Z^n)_{n\in\mathbb{Z}} \in \mathscr{C}(S)$ such that $i^*(R) \simeq Z^\bullet$ in $\mathscr{D}(R)$. This shows (5).

Now, we show that (5) implies (2). The idea of the proof arises essentially from the proof of [**11**, Proposition 3.6].

Let $\lambda : R \to S$ be a ring epimorphism satisfying (5). We may identify $\mathrm{Im}(\lambda_*)$ with S-Mod since $\lambda_* : S\text{-Mod} \to R\text{-Mod}$ is fully faithful. Let $Z^\bullet$ be a complex in $\mathscr{C}(R)$ such that $Z^\bullet \simeq i^*(R)$ in $\mathscr{D}(R)$. We may assume $Z^\bullet := (Z^n, d^n)_{n\in\mathbb{Z}}$ such that $Z^n \in S$-Mod for $n = 0, 1$, and define $\varphi = \mathrm{Hom}_{\mathscr{D}(R)}(\lambda, Z^\bullet) : \mathrm{Hom}_{\mathscr{D}(R)}(S, Z^\bullet) \to \mathrm{Hom}_{\mathscr{D}(R)}(R, Z^\bullet)$. We claim that the map φ is surjective.

In fact, there is a commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{K}(R)}(S, Z^\bullet) & \xrightarrow{q_1} & \mathrm{Hom}_{\mathscr{D}(R)}(S, Z^\bullet) \\ \big\downarrow{\scriptstyle \varphi'} & & \big\downarrow{\scriptstyle \varphi} \\ \mathrm{Hom}_{\mathscr{K}(R)}(R, Z^\bullet) & \xrightarrow{q_2} & \mathrm{Hom}_{\mathscr{D}(R)}(R, Z^\bullet), \end{array}$$

where $\varphi' = \mathrm{Hom}_{\mathscr{K}(R)}(\lambda, Z^\bullet)$ and where q_1 and q_2 are induced by the localization functor $q : \mathscr{K}(R) \to \mathscr{D}(R)$. Note that q_2 is a bijection. So, to prove that φ is surjective, it is sufficient to show that φ' is surjective.

Let $\bar{f^\bullet} := \overline{(f^i)} \in \mathrm{Hom}_{\mathscr{K}(R)}(R, Z^\bullet)$ with $(f^i)_{i\in\mathbb{Z}}$ a chain map from R to $Z^\bullet$. Then $f^i = 0$ for any $i \neq 0$ and $f^0 d^0 = 0$. Since Z^0 is an S-module, we can define $g : S \to Z^0$ by $s \mapsto s\,(1)f^0$ for $s \in S$. Then g is a homomorphism of R-modules with $f^0 = \lambda g$, as is shown in the visual diagram:

$$\begin{array}{ccccccccc} & & & & R & \xrightarrow{\lambda} & S & & \\ & & & & \big\downarrow{\scriptstyle f^0} & \swarrow{\scriptstyle g} & & & \\ \cdots & \longrightarrow & Z^{-1} & \xrightarrow{d^{-1}} & Z^0 & \xrightarrow{d^0} & Z^1 & \xrightarrow{d^1} & Z^2 \longrightarrow \cdots \end{array}$$

Since $\lambda : R \to S$ is a ring epimorphism and Z^1 is an S-module, the induced map $\mathrm{Hom}_R(\lambda, Z^1) : \mathrm{Hom}_R(S, Z^1) \to \mathrm{Hom}_R(R, Z^1)$ is a bijection. Thus, from this bijection together with $\lambda g d^0 = f^0 d^0 = 0$, it follows that $g d^0 = 0$. Now, we define a morphism $\bar{g^\bullet} := \overline{(g^i)} \in \mathrm{Hom}_{\mathscr{K}(R)}(S, Z^\bullet)$, where $(g^i)_{i\in\mathbb{Z}}$ is the chain map with $g^0 = g$ and $g^i = 0$ for all $i \neq 0$. Then $\bar{f^\bullet} = \lambda \bar{g^\bullet}$. This shows that φ' is surjective. Consequently, the map φ is surjective, and the induced map $\mathrm{Hom}_{\mathscr{D}(R)}(\lambda, i^*(R)) : \mathrm{Hom}_{\mathscr{D}(R)}(S, i^*(R)) \to \mathrm{Hom}_{\mathscr{D}(R)}(R, i^*(R))$ is surjective since $Z^\bullet \simeq i^*(R)$ in $\mathscr{D}(R)$.

Finally, if $i^*(R) \simeq S$ in $\mathscr{D}(R)$, then $H^m(i^*(R)) \simeq H^m(S) = 0$ for all $m \neq 0$ and (2) follows. So, it suffices to prove $i^*(R) \simeq S$ in $\mathscr{D}(R)$.

Indeed, let $i_* : \mathscr{Y} \to \mathscr{D}(R)$ be the inclusion and $\eta_R : R \to i_*i^*(R)$ the unit with respect to the adjoint pair (i^*, i_*). Clearly, $i^*(R) = i_*i^*(R)$ in $\mathscr{D}(R)$. Since we have proved that $\mathrm{Hom}_{\mathscr{D}(R)}(\lambda, i^*(R))$ is surjective, there exists a homomorphism $v : S \to i_*i^*(R)$ in $\mathscr{D}(R)$ such that $\eta_R = \lambda\, v$. Furthermore, since ${}_RS$ belongs to $\mathscr{Y}$ by assumption, $\mathrm{Hom}_{\mathscr{D}(R)}(\eta_R, S) : \mathrm{Hom}_{\mathscr{D}(R)}(i^*(R), S) \to \mathrm{Hom}_{\mathscr{D}(R)}(R, S)$ is an isomorphism. Thus there exists a homomorphism $u : i_*i^*(R) \to S$ in $\mathscr{D}(R)$ such that $\lambda = \eta_R\, u$. This yields the commutative diagram in $\mathscr{D}(R)$:

$$\begin{array}{ccccc} R & = & R & = & R \\ {\scriptstyle \eta_R}\downarrow & & {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \eta_R} \\ i_*i^*(R) & \overset{u}{\dashrightarrow} & S & \overset{v}{\dashrightarrow} & i_*i^*(R) \end{array}$$

which shows $\eta_R = \eta_R uv$ and $\lambda = \lambda vu$. Since $\mathrm{Hom}_{\mathscr{D}(R)}(\eta_R, i^*(R))$: $\mathrm{Hom}_{\mathscr{D}(R)}(i^*(R), i^*(R)) \to \mathrm{Hom}_{\mathscr{D}(R)}(R, i_*i^*(R))$ is an isomorphism, $uv = 1_{i_*i^*(R)}$. Note that $\mathrm{Hom}_R(\lambda, S) : \mathrm{Hom}_R(S, S) \to \mathrm{Hom}_R(R, S)$ is bijective since $\lambda : R \to S$ is a ring epimorphism. It follows from $\lambda = \lambda vu$ that $vu = 1_S$. Thus the map u is an isomorphism in $\mathscr{D}(R)$, and $i^*(R) = i_*i^*(R) \simeq S$ in $\mathscr{D}(R)$. This shows that (5) implies (2).

Hence all statements in Lemma 3.2 are equivalent. □

Now, we recall the definition of generalized localizations at complexes of projective modules, which were first discussed in [**25**] under the terminology "homological localizations". Here, we restate it without any restriction on complexes.

DEFINITION 3.3. Let R be a ring, and let Σ be a set of complexes in $\mathscr{C}(R)$. A homomorphism $\lambda_\Sigma : R \to R_\Sigma$ of rings is called a *generalized localization* of R at Σ provided that

(1) λ_Σ is Σ-exact, that is, if $P^\bullet$ belongs to Σ, then $R_\Sigma \otimes_R P^\bullet$ is exact as a complex over R_Σ, and

(2) λ_Σ is universally Σ-exact, that is, if S is a ring together with a Σ-exact homomorphism $\varphi : R \to S$, then there exists a unique ring homomorphism $\psi : R_\Sigma \to S$ such that $\varphi = \lambda_\Sigma \psi$.

If Σ consists of complexes in $\mathscr{C}^b(R\text{-proj})$, then, for each $P^\bullet \in \Sigma$, the complex $R_\Sigma \otimes_R P^\bullet$ is actually split exact as a complex over R_Σ since $R_\Sigma \otimes_R P^i$ is a projective R_Σ-module for each i. If Σ consists only of two-term complexes in $\mathscr{C}^b(R\text{-proj})$, then the generalized localization of R at Σ is nothing else than the *universal localization* of R at Σ in the sense of Cohn (see [**15**]), which appears often in algebraic K-theory and topology [**29**]. Note that universal localizations may not be homological, but always exist, while generalized localizations may not exist in general (see [**25**, Example 15.2]).

Suppose that $\mathcal{U}$ is a set of R-modules each of which possesses a finitely generated projective resolution of finite length. For each $U \in \mathcal{U}$, we choose such a projective resolution ${}_pU$ of finite length, and set $\Sigma := \{{}_pU \mid U \in \mathcal{U}\} \subseteq \mathscr{C}^b(R\text{-proj})$, and let $R_{\mathcal{U}}$ be the generalized localization of R at Σ. It is known that $R_{\mathcal{U}}$ does not depend on the choice of projective resolutions of U. Thus, we may speak of the *generalized localization of R at $\mathcal{U}$ if exists.*

LEMMA 3.4. *Let Σ be a set of complexes in $\mathscr{C}^b(R\text{-proj})$. Suppose that the generalized localization $\lambda_\Sigma : R \to R_\Sigma$ of R at Σ exists. Then:*

(1) *For any homomorphism $\varphi : R_\Sigma \to S$ of rings, the ring homomorphism $\lambda_\Sigma\varphi : R \to S$ is Σ-exact.*

(2) *The ring homomorphism λ_Σ is a ring epimorphism.*

(3) *Define $\Sigma^* := \{\mathrm{Hom}_R(P^\bullet, R) \mid P^\bullet \in \Sigma\}$. Then λ_Σ is also the generalized localization of R at the set Σ^*. In particular, $R_{\Sigma^*} \simeq R_\Sigma$ as rings.*

PROOF. (1) For $P^\bullet \in \Sigma$, we have the isomorphisms of complexes of S-modules:

$$S \otimes_R P^\bullet \simeq (S \otimes_{R_\Sigma} R_\Sigma) \otimes_R P^\bullet \simeq S \otimes_{R_\Sigma} (R_\Sigma \otimes_R P^\bullet).$$

Since $R_\Sigma \otimes_R P^\bullet$ is split exact in $\mathscr{C}(R_\Sigma)$, $S \otimes_R P^\bullet$ is split exact in $\mathscr{C}(S)$. This means that the ring homomorphism $\lambda_\Sigma\varphi$ is Σ-exact.

(2) Assume that $\varphi_i : R_\Sigma \to S$ is a ring homomorphism for $i = 1, 2$, such that $\lambda_\Sigma\varphi_1 = \lambda_\Sigma\varphi_2$. It follows from (1) that $\lambda_\Sigma\varphi_i$ is Σ-exact. By Definition 3.3(2), we obtain $\varphi_1 = \varphi_2$. Thus λ_Σ is a ring epimorphism.

(3) Note that $P^\bullet$ is in $\mathscr{C}^b(R\text{-proj})$. It follows that, for any homomorphism $R \to S$ of rings, there are the isomorphisms of complexes:

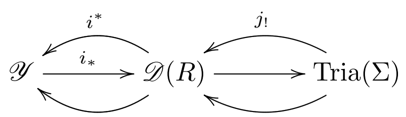
$$\mathrm{Hom}_R(P^\bullet, R) \otimes_R S \simeq \mathrm{Hom}_R(P^\bullet, S) \simeq \mathrm{Hom}_R(P^\bullet, \mathrm{Hom}_S({}_SS_R, S)) \simeq \mathrm{Hom}_S(S \otimes_R P^\bullet, S).$$

This implies that the complex $\mathrm{Hom}_R(P^\bullet, R) \otimes_R S$ is (split) exact in $\mathscr{C}(S^{\mathrm{op}})$ if and only if so is the complex $S \otimes_R P^\bullet$ in $\mathscr{C}(S)$. Now, (3) follows immediately from the definition of generalized localizations. □

Finally, we mention the following known result (see, for example, [**9**, Chapter III, Theorem 2.3; Chapter IV, Proposition 1.1]). It is related to generalized localizations.

LEMMA 3.5. *Let Σ be a set of complexes in $\mathscr{C}^b(R\text{-proj})$. Define $\mathscr{Y} := \mathrm{Ker}(\mathrm{Hom}_{\mathscr{D}(R)}(\mathrm{Tria}(\Sigma), -))$. Then $\mathscr{Y}$ is bireflective and equal to the full subcategory of $\mathscr{D}(R)$ consisting of complexes $Y^\bullet$ with $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, Y^\bullet[n]) = 0$ for every $P^\bullet \in \Sigma$ and $n \in \mathbb{Z}$.*

In the rest of this section, we are interested in the bireflective subcategory $\mathscr{Y}$ in Lemma 3.5. This yields a recollement of triangulated categories:



where i_* and $j_!$ are inclusions. Recall that there is a ring homomorphism $\delta : R \to \Lambda$ with $\Lambda := \mathrm{End}_{\mathscr{D}(R)}(i^*(R))$.

Proposition 3.6. *If $H^0(i^*(R)) \in \mathscr{Y}$, then δ is the generalized localization of R at Σ. In particular, if $\mathscr{Y}$ is homological, then δ is the generalized localization of R at Σ.*

Proof. We first show that δ always has the property: For any Σ-exact ring homomorphism $\varphi : R \to S$, there exists a (not necessarily unique) ring homomorphism $\psi : \Lambda \to S$ such that $\varphi = \delta\psi$.

Let $\varphi : R \to S$ be a Σ-exact ring homomorphism. Since $S \otimes_R P^\bullet$ is exact in $\mathscr{C}(S)$ for $P^\bullet \in \Sigma$, we have $S \otimes^{\mathbb{L}}_R P^\bullet = S \otimes_R P^\bullet \simeq 0$ in $\mathscr{D}(S)$. Further, the functor $S \otimes^{\mathbb{L}}_R - : \mathscr{D}(R) \to \mathscr{D}(S)$ commutes with arbitrary direct sums, so $S \otimes^{\mathbb{L}}_R X^\bullet \simeq 0$ for each $X^\bullet \in \mathrm{Tria}(\Sigma)$.

Let $\mathscr{D}(R)/\mathrm{Tria}(\Sigma)$ be the Verdier quotient of $\mathscr{D}(R)$ by the full triangulated subcategory $\mathrm{Tria}(\Sigma)$. Then the functor i^* induces a triangle equivalence: $\mathscr{D}(R)/\mathrm{Tria}(\Sigma) \xrightarrow{\simeq} \mathscr{Y}$. Since $S \otimes^{\mathbb{L}}_R -$ sends $\mathrm{Tria}(\Sigma)$ to zero, there exists a triangle functor $F : \mathscr{Y} \to \mathscr{D}(S)$ together with a natural isomorphism of triangle functors:

$$\Phi : \; S \otimes^{\mathbb{L}}_R - \xrightarrow{\simeq} F\, i^* : \mathscr{D}(R) \longrightarrow \mathscr{D}(S).$$

This clearly induces the canonical ring homomorphisms:

$$\Lambda := \mathrm{End}_{\mathscr{D}(R)}(i^*(R)) \xrightarrow{F} \mathrm{End}_{\mathscr{D}(S)}\big(F(i^*(R))\big) \simeq \mathrm{End}_{\mathscr{D}(S)}(S \otimes^{\mathbb{L}}_R R) \simeq \mathrm{End}_{\mathscr{D}(S)}(S) \simeq S$$

where the first isomorphism is induced by the natural isomorphism $\Phi_R : S \otimes^{\mathbb{L}}_R R \to F(i^*(R))$ in $\mathscr{D}(S)$. Now, we define $\psi : \Lambda \to S$ to be the composition of the above homomorphisms of rings. Then $\varphi = \delta\psi$.

If $H^0(i^*(R)) \in \mathscr{Y}$, then the map δ is a ring epimorphism by Lemma 3.1(3). This implies that ψ is unique in $\varphi = \delta\psi$. Thus δ satisfies the condition (2) in Definition 3.3.

It remains to prove that δ is Σ-exact.

In fact, by Lemma 3.1(2), we have $\Lambda \simeq H^0(i^*(R))$ as R-modules and ${}_R\Lambda \in \mathscr{Y}$. Note that $\mathrm{Hom}_{\mathscr{D}(R)}(X^\bullet, Y^\bullet) = 0$ for $X^\bullet \in \mathrm{Tria}(\Sigma)$ and $Y^\bullet \in \mathscr{Y}$. In particular, $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, \Lambda[n]) = 0$ for any $P^\bullet \in \Sigma$ and $n \in \mathbb{Z}$. It follows that $H^n(\mathrm{Hom}_R(P^\bullet, \Lambda)) \simeq \mathrm{Hom}_{\mathscr{K}(R)}(P^\bullet, \Lambda[n]) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, \Lambda[n]) = 0$, and therefore the complex $\mathrm{Hom}_R(P^\bullet, \Lambda)$ is exact. Since $P^\bullet \in \mathscr{C}^b(R\text{-proj})$, we have $\mathrm{Hom}_R(P^\bullet, \Lambda) \in \mathscr{C}^b(\Lambda^{\mathrm{op}}\text{-proj})$. This implies that $\mathrm{Hom}_R(P^\bullet, \Lambda)$ is split exact, and therefore the complex $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Hom}_R(P^\bullet, \Lambda), \Lambda)$ over Λ is split exact. Now, we claim that the latter is isomorphic to the complex $\Lambda \otimes_R P^\bullet$ in $\mathscr{C}(\Lambda)$. Actually, this follows from the following general fact in homological algebra:

For any ring homomorphism $\delta : R \to \Lambda$ and $P \in R$-proj, there exists a natural isomorphism of Λ-modules:

$$\Lambda \otimes_R P \longrightarrow \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Hom}_R(P, \Lambda), \Lambda), \quad x \otimes p \mapsto [f \mapsto x\,(p)f]$$

for $x \in \Lambda$, $p \in P$ and $f \in \mathrm{Hom}_R(P, \Lambda)$.

Consequently, the complex $\Lambda \otimes_R P^\bullet$ is exact in $\mathscr{C}(\Lambda)$, and the homomorphism δ is Σ-exact. Hence δ is a generalized localization of R at Σ.

If $\mathscr{Y}$ is homological, then $H^0(i^*(R)) \in \mathscr{Y}$ by the equivalences of (1) and (4) in Lemma 3.2, and therefore, the second part of Proposition 3.6 follows. $\square$

Next, we show that, to judge whether $\mathscr{Y}$ is homological, one may check whether the bireflective subcategory defined by the dual of Σ, is homological. This will be used in the proof of Theorem 1.1.

Proposition 3.7. *Let* $\Sigma^* := \{\mathrm{Hom}_R(P^\bullet, R) \in \mathscr{C}^b(R^{\mathrm{op}}\text{-proj}) \mid P^\bullet \in \Sigma\}$ *and* $\mathscr{Y}' := \mathrm{Ker}(\mathrm{Hom}_{\mathscr{D}(R^{\mathrm{op}})}(\mathrm{Tria}(\Sigma^*), -))$. *Then* $\mathscr{Y}$ *is homological in* $\mathscr{D}(R)$ *if and only if so is* $\mathscr{Y}'$ *in* $\mathscr{D}(R^{\mathrm{op}})$.

Proof. We only prove the necessity of Proposition 3.7 because its sufficiency can be proved similarly.

Suppose that $\mathscr{Y}$ is homological in $\mathscr{D}(R)$. It follows from Lemma 3.2(4) and Proposition 3.6 that $\delta : R \to \Lambda$ is not only a homological ring epimorphism, but also the generalized localization of R at Σ. Moreover, by Lemma 3.4(3), δ is also the generalized localization of R at Σ^*.

Note that $\mathscr{Y}'$ is a bireflective subcategory of $\mathscr{D}(R^{\mathrm{op}})$ by Lemma 3.5. Now, let $\boldsymbol{L}$ be a left adjoint functor of the inclusion $\mathscr{Y}' \to \mathscr{D}(R^{\mathrm{op}})$. To show that $\mathscr{Y}'$ is homological in $\mathscr{D}(R^{\mathrm{op}})$, we employ the equivalences of (1) and (4) in Lemma 3.2, and prove that

(*a*) $H^0(\boldsymbol{L}(R)) \in \mathscr{Y}'$ and

(*b*) the ring homomorphism $\delta' : R \to \Lambda' := \mathrm{End}_{\mathscr{D}(R^{\mathrm{op}})}(\boldsymbol{L}(R))$ induced by $\boldsymbol{L}$ is homological.

Note that under the assumption (*a*), δ' is the generalized localization of R at Σ^* by Proposition 3.6. Since δ is a generalized localization of R at Σ^*, there exists a ring isomorphism $\rho : \Lambda' \to \Lambda$ such that $\delta = \delta'\rho$. Note that δ is a homological ring epimorphism. It follows that δ' is a homological ring epimorphism. So, it is enough to show (*a*).

In fact, by Lemma 3.5, we have

$$\mathscr{Y}' = \{Y^\bullet \in \mathscr{D}(R^{\mathrm{op}}) \mid \mathrm{Hom}_{\mathscr{D}(R^{\mathrm{op}})}\big(\mathrm{Hom}_R(P^\bullet, R), Y^\bullet[n]\big) = 0 \text{ for all } P^\bullet \in \Sigma \text{ and } n \in \mathbb{Z}\}.$$

Let $P^\bullet \in \Sigma$ and set $P^{\bullet *} := \mathrm{Hom}_R(P^\bullet, R)$. Then

$$\mathrm{Hom}_{\mathscr{D}(R^{\mathrm{op}})}\big(P^{\bullet *}, Y^\bullet[n]\big) \simeq H^n\big(\mathbb{R}\mathrm{Hom}_R(P^{\bullet *}, Y^\bullet)\big) \simeq H^n\big(Y^\bullet \otimes^{\mathbb{L}}_R P^\bullet\big) \simeq H^n\big(Y^\bullet \otimes^\bullet_R P^\bullet\big).$$

Thus

$$\mathscr{Y}' = \{Y^\bullet \in \mathscr{D}(R^{\mathrm{op}}) \mid H^n\big(Y^\bullet \otimes^\bullet_R P^\bullet\big) = 0 \text{ for all } P^\bullet \in \Sigma \text{ and } n \in \mathbb{Z}\}.$$

Since $\delta : R \to \Lambda$ is the generalized localization of R at Σ by Proposition 3.6, $H^n(\Lambda \otimes_R P^\bullet) = 0$ for any $P^\bullet \in \Sigma$ and $n \in \mathbb{Z}$. This shows $\Lambda_R \in \mathscr{Y}'$. Note that $\delta : R^{\mathrm{op}} \to \Lambda^{\mathrm{op}}$ is a homological ring epimorphism. Hence $\mathscr{D}(\Lambda^{\mathrm{op}})$ can be regarded as a full triangulated subcategory of $\mathscr{D}(R^{\mathrm{op}})$. Moreover, since $\mathscr{D}(\Lambda^{\mathrm{op}}) = \mathrm{Tria}(\Lambda_\Lambda)$ and $\mathscr{Y}'$ is closed under direct sums in $\mathscr{D}(R^{\mathrm{op}})$, we have $\mathscr{D}(\Lambda^{\mathrm{op}}) \subseteq \mathscr{Y}'$. Now, we claim that $H^n(\boldsymbol{L}(R)) \in \Lambda^{\mathrm{op}}\text{-Mod}$ for all n, and therefore $H^n(\boldsymbol{L}(R)) \in \mathscr{Y}'$ for all n.

Actually, since $\boldsymbol{L}(R) \in \mathscr{Y}'$, we have $H^n(\boldsymbol{L}(R) \otimes^\bullet_R P^\bullet) = 0$ for all $P^\bullet \in \Sigma$ and $n \in \mathbb{Z}$. Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ to the complex $\boldsymbol{L}(R) \otimes^\bullet_R P^\bullet$ of $\mathbb{Z}$-modules, we see that, for all n,

$$\begin{aligned}\mathrm{Hom}_{\mathscr{D}(R)}\big(P^\bullet, \mathrm{Hom}_{\mathbb{Z}}(\boldsymbol{L}(R), \mathbb{Q}/\mathbb{Z})[n]\big) &\simeq H^n\big(\mathrm{Hom}^\bullet_R(P^\bullet, \mathrm{Hom}_{\mathbb{Z}}(\boldsymbol{L}(R), \mathbb{Q}/\mathbb{Z}))\big)\\ &\simeq H^n\big(\mathrm{Hom}^\bullet_{\mathbb{Z}}(\boldsymbol{L}(R)\otimes^\bullet_R P^\bullet, \mathbb{Q}/\mathbb{Z})\big)\\ &\simeq \mathrm{Hom}_{\mathbb{Z}}\big(H^{-n}(\boldsymbol{L}(R)\otimes^\bullet_R P^\bullet), \mathbb{Q}/\mathbb{Z}\big)\\ &= 0.\end{aligned}$$

Define $W^\bullet := \mathrm{Hom}_{\mathbb{Z}}(\boldsymbol{L}(R), \mathbb{Q}/\mathbb{Z})$. Then $W^\bullet \in \mathscr{Y}$ by Lemma 3.5. However, since $\mathscr{Y}$ is homological in $\mathscr{D}(R)$ by assumption, $\mathscr{Y} = \mathscr{D}(\Lambda)$ by Lemma 3.2. Thus $W^\bullet \in \mathscr{D}(\Lambda)$ and $H^{-n}(W^\bullet) \in \Lambda$-Mod. Since $H^{-n}(W^\bullet) \simeq \mathrm{Hom}_{\mathbb{Z}}(H^n(\boldsymbol{L}(R)), \mathbb{Q}/\mathbb{Z})$, we infer that $H^n(\boldsymbol{L}(R)) \in \Lambda^{\mathrm{op}}$-Mod by the general result:

Let $R \to S$ be a ring epimorphism and N be an R^{op}-module. If $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}) \in S$-Mod, then $N \in S^{\mathrm{op}}$-Mod.

For a proof of this result, one may use [**33**, Exercise 3.2.4] to show that the natural map $N \to N \otimes_R S$ is an isomorphism of R^{op}-modules. □

4. Weak tilting modules and recollements.

This section is devoted to preparations for proofs of our main results in this paper. First, we introduce a special class of modules, called weak tilting modules, which can be constructed from both good tilting and cotilting modules, and then discuss bireflective subcategories (of derived module categories) arising from weak tilting modules. Finally, we shall describe when these subcategories are homological. In particular, we shall establish a key proposition, Proposition 4.4, which will be applied in later sections.

Throughout this section, let R be an arbitrary ring, M an R-module and S the endomorphism ring of ${}_RM$. Then M becomes naturally an R-S-bimodule. Further, let n be a natural number.

Definition 4.1. The R-module M is called an *n-weak tilting module* if the following conditions are fulfilled:

$(R1)$ There exists an exact sequence of R-modules: $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$, such that $P_i \in \mathrm{add}({}_RR)$ for all $0 \le i \le n$,

$(R2)$ $\mathrm{Ext}^j_R(M, M) = 0$ for all $j \ge 1$, and

$(R3)$ there exists an exact sequence of R-modules

$$0 \longrightarrow {}_RR \longrightarrow M_0 \xrightarrow{\nu} M_1 \longrightarrow \cdots \longrightarrow M_n \longrightarrow 0$$

such that $M_i \in \mathrm{Prod}({}_RM)$ for all $0 \le i \le n$, and

$(R4)$ the right S-module M is strongly S-Mittag-Leffler (see Definition 2.3).

Classical tilting modules are weak tilting modules. If a weak tilting R-module M satisfies $\mathrm{Prod}({}_RM) = \mathrm{Add}({}_RM)$ (for example, M_S is of finite length), then ${}_RM$ is a classical tilting module. Moreover, if S is right noetherian, then any right S-module is S-Mittag-Leffler (see Section 2.3), and thus $(R4)$ is always satisfied.

Let

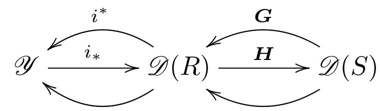
$$\boldsymbol{G} := {}_RM \otimes^{\mathbb{L}}_S - : \mathscr{D}(S) \longrightarrow \mathscr{D}(R), \quad \boldsymbol{H} := \mathbb{R}\mathrm{Hom}_R(M, -) : \mathscr{D}(R) \longrightarrow \mathscr{D}(S) \quad \text{and}$$

$$\mathscr{Y} := \{Y^\bullet \in \mathscr{D}(R) \mid \mathrm{Hom}_{\mathscr{D}(R)}(M,\, Y^\bullet[m]) = 0 \text{ for all } m \in \mathbb{Z}\}.$$

Then $\mathscr{Y} = \mathrm{Ker}(\boldsymbol{H})$. Moreover, if M satisfies $(R1)$, then $\mathscr{Y}$ is a bireflective subcategory of $\mathscr{D}(R)$ by Lemma 3.5.

If M satisfies both $(R1)$ and $(R2)$, then the pair $(\boldsymbol{G}, \boldsymbol{H})$ induces a triangle equivalence: $\mathscr{D}(S) \xrightarrow{\simeq} \mathrm{Tria}({}_RM)$ (see [**24**, Chapter 5, Corollary 8.4, Theorem 8.5]). Thus, by Lemma 3.5, Proposition 3.6 and Lemma 3.2, we have the following useful result for constructing recollements of derived module categories.

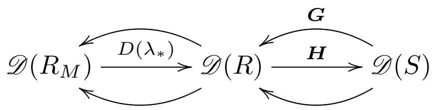
Lemma 4.2. *Suppose that the R-module M satisfies $(R1)$ and $(R2)$. Then there exists a recollement*:



$(*)$

with i_* being the inclusion.

If $\mathscr{Y}$ is homological in $\mathscr{D}(R)$, then the generalized localization $\lambda : R \to R_M$ of R at M exists and is homological, which induces a recollement of derived module categories:



In the following, we shall consider when $\mathscr{Y}$ is homological. In general, this category may not be homological since the category

$$\mathcal{E} := \mathscr{Y} \cap R\text{-Mod} = \{Y \in R\text{-Mod} \mid \mathrm{Ext}^m_R(M, Y) = 0 \text{ for all } m \geq 0\}$$

may not be an abelian subcategory of R-Mod. By Lemma 3.2, whether $\mathscr{Y}$ is homological is completely determined by the cohomology groups of $i_*i^*(R)$. So, to calculate these cohomology groups efficiently, we shall use weak tilting modules.

From now on, we assume that ${}_RM$ *is a n-weak tilting module*, and define $M^\bullet$ to be the complex

$$\cdots \longrightarrow 0 \longrightarrow M_0 \xrightarrow{\mu} M_1 \longrightarrow \cdots \longrightarrow M_n \longrightarrow 0 \longrightarrow \cdots$$

arising from $(R3)$ in Definition 4.1, where M_i is in degree i for $0 \leq i \leq n$.

For each R-module X, let $\theta_X : M \otimes_S \mathrm{Hom}_R(M, X) \to X$ be the evaluation map. Then $\mu : M_0 \to M_1$ induces another homomorphism $\widetilde{\mu} : \mathrm{Coker}(\theta_{M_0}) \to \mathrm{Coker}(\theta_{M_1})$ of R-modules. The kernel of $\widetilde{\mu}$ will determine when $\mathscr{Y}$ is homological.

Lemma 4.3. (1) *If $X \in \mathrm{Prod}({}_RM)$, then θ_X is injective and* $\mathrm{Coker}(\theta_X) \in \mathcal{E}$.

(2) *There are isomorphisms in $\mathscr{D}(R)$*:

$$\boldsymbol{GH}(R) \simeq {}_RM \otimes^{\mathbb{L}}_S \mathrm{Hom}_R(M,\, M^\bullet) \simeq {}_RM \otimes_S \mathrm{Hom}_R(M,\, M^\bullet).$$

Moreover,

$$H^j(i_*i^*(R)) \simeq \begin{cases} 0 & \text{if } j < 0, \\ H^{j+1}\big({}_RM \otimes_S \mathrm{Hom}_R(M,\, M^\bullet)\big) & \text{if } j > 0. \end{cases}$$

(3) *For $n = 0$, the complex $i_*i^*(R)$ is isomorphic in $\mathscr{D}(R)$ to the stalk complex* $\mathrm{Coker}(\theta_{M_0})$. *For $n \geq 1$, the complex $i_*i^*(R)$ is isomorphic in $\mathscr{D}(R)$ to a complex of the form*: $0 \to E^0 \to E^1 \to \cdots \to E^{n-1} \to 0$, *such that $E^m \in \mathcal{E}$ for $0 \leq m \leq n-1$.*

Proof. M is an R-S-bimodule with $S = \mathrm{End}_R(M)$. So we have an adjoint pair $(\boldsymbol{G}, \boldsymbol{H})$ of functors. Let

$$\theta:\ M \otimes_S \mathrm{Hom}_R(M, -) \longrightarrow \mathrm{Id}_{R\text{-Mod}} \quad \text{and} \quad \varepsilon : \boldsymbol{GH} \longrightarrow \mathrm{Id}_{\mathscr{D}(R)}$$

be the counit adjunctions with respect to the adjoint pairs $(M \otimes_S -, \mathrm{Hom}_R(M, -))$ and $(\boldsymbol{G}, \boldsymbol{H})$, respectively.

For each $X^\bullet \in \mathscr{D}(R)$, it follows from the recollement $(*)$ in Lemma 4.2 that there exists a canonical distinguished triangle in $\mathscr{D}(R)$:

$$\boldsymbol{GH}(X^\bullet) \xrightarrow{\varepsilon_{X^\bullet}} X^\bullet \longrightarrow i_*i^*(X^\bullet) \longrightarrow \boldsymbol{GH}(X^\bullet)[1].$$

(1) Suppose $X \in \mathrm{Prod}({}_RM)$. To verify that θ_X is injective, it is sufficient to show that $\theta_{M^I} : M \otimes_S \mathrm{Hom}_R(M, M^I) \to M^I$ is injective for any nonempty set I. Since $\mathrm{Hom}_R(M, M^I) \simeq \mathrm{Hom}_R(M, M)^I$, the injection of θ_{M^I} is equivalent to saying that the canonical map $\rho_I :\ M \otimes_S S^I \to M^I$ in Definition 2.3 is injective. This holds exactly if M is S-Mittag-Leffler. Thus θ_X is injective by $(R4)$.

We prove $\mathrm{Coker}(\theta_X) \in \mathcal{E}$ by showing the existence of the commutative diagram in $\mathscr{D}(R)$:

$$\begin{array}{ccccccc}
\boldsymbol{GH}(X) & \xrightarrow{\ \varepsilon_X\ } & X & \longrightarrow & i_*i^*(X) & \longrightarrow & \boldsymbol{GH}(X)[1] \\
\simeq\downarrow & & \Vert & & \downarrow\simeq & & \downarrow\simeq \\
M \otimes_S \mathrm{Hom}_R(M, X) & \xrightarrow{\ \theta_X\ } & X & \longrightarrow & \mathrm{Coker}(\theta_X) & \longrightarrow & M \otimes_S \mathrm{Hom}_R(M, X)[1]
\end{array}$$

To check the first square in the above diagram, we define $F := {}_RM \otimes_S -$ and $G' := \mathrm{Hom}_R(M, -)$. According to Corollary 2.2, it suffices to prove $X \in \mathcal{R}_{G'}$ and $G'(X) \in \mathcal{L}_F$, where the categories $\mathcal{R}_{G'}$ and $\mathcal{L}_F$ are introduced in Lemma 2.1. Note that $X \in \mathcal{R}_{G'}$ is due to the axiom $(R2)$, while $G'(X) \in \mathcal{L}_F$ if and only if $\mathrm{Tor}_j^S(M, S^I) = 0$ for any $j > 0$ and any set I. However, since M is a weak tilting module, the right S-module M is strongly S-Mittag-Leffler by $(R4)$, and therefore $\mathrm{Tor}_j^S(M, S^I) = 0$ by Lemma 2.4(3). This implies $G'(X) \in \mathcal{L}_F$.

With the help of the above diagram and the recollement $(*)$ in Lemma 4.2, we have $i_*i^*(X) \in \mathscr{Y}$, and therefore

$$i_*i^*(X) \simeq \mathrm{Coker}(\theta_X) \in \mathscr{Y} \cap R\text{-Mod} = \mathcal{E}.$$

This finishes the proof of (1).

(2) By $(R3)$ in Definition 4.1, $M^\bullet$ is a bounded complex such that each of its terms

belongs to $\mathrm{Prod}({}_RM)$. Notice that $\theta_{M^\bullet}$ is injective in $\mathscr{C}(R)$ since θ_{M_i} is injective for each $0 \le i \le n$ due to (1). This clearly induces a complex $\mathrm{Coker}(\theta_{M^\bullet})$ of the form:

$$0 \to \mathrm{Coker}(\theta_{M_0}) \overset{\partial_0}{\to} \mathrm{Coker}(\theta_{M_1}) \overset{\partial_1}{\to} \cdots \to \mathrm{Coker}(\theta_{M_{n-1}}) \overset{\partial_{n-1}}{\to} \mathrm{Coker}(\theta_{M_n}) \to 0 \ \text{ in } \ \mathscr{C}(R).$$

It follows from $(R3)$ that there is a quasi-isomorphism $R \to M^\bullet$ in $\mathscr{K}(R)$. Consequently, one can easily construct the commutative diagram in $\mathscr{D}(R)$:

$$\begin{array}{ccccccc}
\boldsymbol{GH}(R) & \xrightarrow{\varepsilon_R} & R & \longrightarrow & i_*i^*(R) & \longrightarrow & \boldsymbol{GH}(R)[1] \\
\simeq\downarrow & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\
M \otimes_S \mathrm{Hom}_R(M, M^\bullet) & \xrightarrow{\theta_{M^\bullet}} & M^\bullet & \longrightarrow & \mathrm{Coker}(\theta_{M^\bullet}) & \longrightarrow & M \otimes_S \mathrm{Hom}_R(M, M^\bullet)[1]
\end{array}$$

In particular,

$$i_*i^*(R) \simeq \mathrm{Coker}(\theta_{M^\bullet}) \tag{$*$}$$

in $\mathscr{D}(R)$, and therefore $H^j(i_*i^*(R)) \simeq H^j(\mathrm{Coker}(\theta_{M^\bullet}))$ for any $j \in \mathbb{Z}$. This shows $H^j(i_*i^*(R)) = 0$ for $j < 0$ or $j > n$. Now, it follows from $R \simeq M^\bullet$ in $\mathscr{D}(R)$ that there is a triangle in $\mathscr{D}(R)$:

$$M \otimes_S \mathrm{Hom}_R(M, M^\bullet) \longrightarrow R \longrightarrow \mathrm{Coker}\big(\theta_{M^\bullet}\big) \longrightarrow M \otimes_S \mathrm{Hom}_R(M, M^\bullet)[1].$$

Applying the cohomology functor H^j to this triangle, one gets

$$H^j(i_*i^*(R)) \simeq H^j\big(\mathrm{Coker}(\theta_{M^\bullet})\big) \simeq H^{j+1}(M \otimes_S \mathrm{Hom}_R(M, M^\bullet)) \text{ for any } j > 0. \tag{$**$}$$

Thus (2) follows.

(3) For $n = 0$, the conclusion follows from $i_*i^*(R) \simeq \mathrm{Coker}(\theta_{M^\bullet})$ trivially. So we may assume $n \ge 1$. Since the $(n+1)$-term of the complex $M \otimes_S \mathrm{Hom}_R(M, M^\bullet)$ is zero, we see from $(*)$ and $(**)$ that $H^n(\mathrm{Coker}(\theta_{M^\bullet})) = 0$. This implies that the $(n-1)$-th differential ∂_{n-1} of the complex $\mathrm{Coker}(\theta_{M^\bullet})$ is surjective. It follows that $\mathrm{Coker}(\theta_{M^\bullet})$ is isomorphic in $\mathscr{D}(R)$ to the complex:

$$0 \longrightarrow \mathrm{Coker}(\theta_{M_0}) \xrightarrow{\partial_0} \mathrm{Coker}(\theta_{M_1}) \xrightarrow{\partial_1} \cdots \longrightarrow \mathrm{Coker}(\theta_{M_{n-2}}) \xrightarrow{\partial_{n-2}} \mathrm{Ker}(\partial_{n-1}) \longrightarrow 0. \tag{$\dagger$}$$

Since $M_m \in \mathrm{Prod}({}_RM)$ for $0 \le m \le n$ by $(R3)$, it follows from (1) that $\mathrm{Coker}(\theta_{M_m}) \in \mathcal{E}$. As $\mathcal{E}$ is always closed under kernels of surjective homomorphisms in R-Mod, $\mathrm{Ker}(\partial_{n-1}) \in \mathcal{E}$. This means that $(\dagger)$ is a bounded complex with all of its terms in $\mathcal{E}$.

Consequently, the complex $i_*i^*(R)$ is isomorphic in $\mathscr{D}(R)$ to the complex $(\dagger)$ with the required form in Lemma 4.3(3). This finishes the proof. □

Remark. By Lemma 4.3(2), up to isomorphism, the cohomology groups $H^j({}_RM \otimes_S \mathrm{Hom}_R(M, M^\bullet))$ for $j \in \mathbb{Z}$ are independent of the choice of the complex $M^\bullet$ in $(R3)$ of Definition 4.1.

Proposition 4.4. *The following statements are equivalent*:

(1) *The full triangulated subcategory* $\mathscr{Y}$ *of* $\mathscr{D}(R)$ *is homological.*
(2) *The category* $\mathcal{E}$ *is an abelian subcategory of* R-Mod.
(3) $H^j({}_RM \otimes_S \operatorname{Hom}_R(M, M^\bullet)) = 0$ *for any* $j \geq 2$.
(4) *The kernel of the map* $\widetilde{\mu} : \operatorname{Coker}(\theta_{M_0}) \to \operatorname{Coker}(\theta_{M_1})$ *belongs to* $\mathcal{E}$.

Proof. The equivalence of (1) and (2) follows from the one of (1) and (6) in Lemma 3.2 together with Lemma 4.3(3), while the equivalence of (1) and (3) follows from the one of (1) and (2) in Lemma 3.2 together with Lemma 4.3(2). Now we prove that (1) and (4) are equivalent. By Lemma 4.3(2) and the equivalence of (1) and (3) in Lemma 3.2, (1) is equivalent to $H^0(i_*i^*(R)) \in \mathscr{Y}$. By the proof of Lemma 4.3(3), $H^0(i_*i^*(R)) \simeq H^0(\operatorname{Coker}(\theta_{M^\bullet})) \simeq \operatorname{Ker}(\partial_0) = \operatorname{Ker}(\widetilde{\mu})$, where $\partial_0 = \widetilde{\mu}$. Thus (1) is equivalent to $\operatorname{Ker}(\widetilde{\mu}) \in \mathscr{Y} \cap R\text{-Mod} = \mathcal{E}$. □

As a consequence of Proposition 4.4, we have characterizations for $\mathscr{Y}$ to be homological.

Corollary 4.5. (1) *If the map* $\widetilde{\mu} : \operatorname{Coker}(\theta_{M_0}) \to \operatorname{Coker}(\theta_{M_1})$ *is surjective, then* $\mathscr{Y}$ *is homological.*
(2) *If* $n = 2$, *then* $\mathscr{Y}$ *is homological if and only if* $M \otimes_S \operatorname{Ext}^2_R(M, R) = 0$.

Proof. (1) $\mathcal{E}$ is closed under kernels of surjective homomorphisms in R-Mod, and both $\operatorname{Coker}(\theta_{M_0})$ and $\operatorname{Coker}(\theta_{M_1})$ belong to $\mathcal{E}$ by Lemma 4.3(1). So, if $\widetilde{\mu}$ is surjective, then $\operatorname{Ker}(\widetilde{\mu}) \in \mathcal{E}$, and therefore $\mathscr{Y}$ is homological by Proposition 4.4(4).

(2) By Proposition 4.4(3), we have to check $H^j({}_RM \otimes_S \operatorname{Hom}_R(M, M^\bullet)) = 0$ for $j \geq 2$. Note that $H^j(M \otimes_S \operatorname{Hom}_R(M, M^\bullet)) = 0$ for all $j > n$.

By $(R2)$, $\operatorname{Ext}^j_R(M, M) = 0$ for all $j \geq 1$. It follows that $\operatorname{Ext}^j_R(M, M^I) \simeq \operatorname{Ext}^j_R(M, M)^I = 0$ for any nonempty set I, and therefore $\operatorname{Ext}^j_R(M, X) = 0$ for any $X \in \operatorname{Prod}(M)$. By $(R3)$, there exists an exact sequence in R-Mod:

$$0 \longrightarrow R \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_n \longrightarrow 0$$

with $M_i \in \operatorname{Prod}(M)$ for $0 \leq i \leq n$. Since $\operatorname{Ext}^j_R(M, X) = 0$ for any $X \in \operatorname{Prod}(M)$ and $j \geq 1$, $H^k(\operatorname{Hom}_R(M, M^\bullet)) \simeq \operatorname{Ext}^k_R(M, R)$ for $k \geq 1$.

If $n = 2$, then we consider the complex $M \otimes_S \operatorname{Hom}_R(M, M^\bullet)$:

$$0 \longrightarrow M \otimes_S \operatorname{Hom}_R(M, M_0) \longrightarrow M \otimes_S \operatorname{Hom}_R(M, M_1) \longrightarrow M \otimes_S \operatorname{Hom}_R(M, M_2) \longrightarrow 0.$$

Since ${}_RM \otimes_S - : S\text{-Mod} \to R\text{-Mod}$ is right exact, $H^2(M \otimes_S \operatorname{Hom}_R(M, M^\bullet)) \simeq M \otimes_S H^2(\operatorname{Hom}_R(M, M^\bullet)) \simeq M \otimes_S \operatorname{Ext}^2_R(M, R)$. Now, (2) follows from Proposition 4.4(3). □

An application of Corollary 4.5 is the result.

Corollary 4.6. *Suppose that the complex* $M^\bullet$ *decomposes into a direct sum of* $U^\bullet$ *and* $V^\bullet$ *as complexes of* R-*modules*

$$U^\bullet : \quad \cdots \longrightarrow 0 \longrightarrow U_0 \xrightarrow{s} U_1 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \cdots ,$$

$$V^\bullet: \qquad \cdots \longrightarrow 0 \longrightarrow V_0 \xrightarrow{t} V_1 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0 \longrightarrow \cdots$$

such that $V_1 \in \mathrm{Add}({}_RM)$. *Then* $\mathscr{Y}$ *is homological.*

Proof. By definition, $M_0 = U_0 \oplus V_0$, $M_1 = U_1 \oplus V_1$ and $\mu = \left(\begin{smallmatrix} s & 0 \\ 0 & t \end{smallmatrix}\right)$. According to Corollary 4.5(1), it suffices to show that $\widetilde{s} : \mathrm{Coker}(\theta_{U_0}) \to \mathrm{Coker}(\theta_{U_1})$ and $\widetilde{t} : \mathrm{Coker}(\theta_{V_0}) \to \mathrm{Coker}(\theta_{V_1})$ are surjective.

Since $H^j(M^\bullet) = 0 = U_i$ for $j \geq 1$ and $2 \leq i \leq n$, the map s is surjective, and therefore $\widetilde{s}$ is surjective. As ${}_RM$ is finitely generated by $(R1)$, the functor $\mathrm{Hom}_R(M, -) : R\text{-Mod} \to S\text{-Mod}$ commutes with direct sums. So, if $X \in \mathrm{Add}({}_RM)$, then $\theta_X : M \otimes_S \mathrm{Hom}_R(M, X) \to X$ is an isomorphism. It follows that $\mathrm{Coker}(\theta_{V_1}) = 0$ due to $V_1 \in \mathrm{Add}({}_RM)$. Thus $\widetilde{t}$ is surjective. Now, Corollary 4.6 follows from Corollary 4.5(1). □

As another consequence of Proposition 4.4, we mention the result.

Corollary 4.7. (1) *If* $M_0 \in \mathrm{Add}({}_RM)$, *then* ${}_RM$ *is a classical tilting module.*
(2) *If* $n \leq 1$ *or* $M_1 \in \mathrm{Add}({}_RM)$, *then* $\mathscr{Y}$ *is homological in* $\mathscr{D}(R)$.

Proof. (1) Suppose that $M_0 \in \mathrm{Add}({}_RM)$. Then $\theta_{M_0} : M \otimes_S \mathrm{Hom}_R(M, M_0) \to M_0$ is an isomorphism, and therefore $\mathrm{Coker}(\theta_{M_0}) = 0$. By the proof of Proposition 4.4, we have $H^0(i_*i^*(R)) \simeq \mathrm{Ker}(\widetilde{\mu}) = 0$. Note that $\mathrm{End}_{\mathscr{D}(R)}(i^*(R)) \simeq H^0(i^*(R)) = H^0(i_*i^*(R))$ as R-modules by Lemma 3.1(2). This implies $\mathrm{End}_{\mathscr{D}(R)}(i^*(R)) = 0$ and so $\mathscr{Y} = 0$ by Lemma 3.1(1). Now, it follows from Lemma 4.2 that $\mathbb{R}\mathrm{Hom}_R(M, -) : \mathscr{D}(R) \to \mathscr{D}(S)$ is a triangle equivalence. Consequently, ${}_RM$ is a classical tilting module (see [**24**, Theorem 4.1]).

(2) follows from Corollary 4.6. □

5. Proofs of Theorem 1.1 and Corollary 1.2.

In this section, we first develop several properties of (good) tilting modules, and then give a method to construct weak tilting modules. With these preparations in hand, we will prove Theorem 1.1 and Corollary 1.2.

Throughout this section, A denotes a ring and n is a natural number. In addition, we assume that T is a *good* n-tilting A-module with $(T1), (T2)$ and $(T3)'$ (see Introduction for notation). Let $B := \mathrm{End}_A(T)$.

First of all, we collect some basic properties of good tilting modules. For proofs, we refer to [**6**, Proposition 1.4 and Lemma 1.5].

Lemma 5.1. *The following hold true for the tilting module* ${}_AT$.

(1) *The torsion class* $T^\perp := \{X \in A\text{-Mod} \mid \mathrm{Ext}^i_A(T, X) = 0$ *for all* $i \geq 1\}$ *in* A-Mod *is closed under arbitrary direct sums in* A-Mod.

(2) *The right* B-*module* T *has a finitely generated projective resolution of length at most* n:

$$0 \longrightarrow \mathrm{Hom}_A(T_n, T) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(T_1, T) \longrightarrow \mathrm{Hom}_A(T_0, T) \longrightarrow T_B \longrightarrow 0$$

with $T_i \in \mathrm{add}({}_AT)$ *for all* $0 \leq i \leq n$.

(3) *The map* $A^{\mathrm{op}} \to \mathrm{End}_{B^{\mathrm{op}}}(T)$*, defined by* $a \mapsto [t \mapsto at]$ *for* $a \in A$ *and* $t \in T$*, is an isomorphism of rings. Moreover,* $\mathrm{Ext}^i_{B^{\mathrm{op}}}(T,T)=0$ *for all* $i \geq 1$.

(4) *If* $T_n = 0$ *in* $(T3)'$*, then* ${}_AT$ *is an* $(n-1)$*-tilting module.*

Throughout this section, we define

$$G := {}_AT \otimes^{\mathbb{L}}_B - : \mathscr{D}(B) \longrightarrow \mathscr{D}(A), \quad H := \mathbb{R}\mathrm{Hom}_A(T,-) : \mathscr{D}(A) \longrightarrow \mathscr{D}(B),$$

$$Q^\bullet := \cdots \longrightarrow 0 \longrightarrow \mathrm{Hom}_A(T,T_0) \longrightarrow \mathrm{Hom}_A(T,T_1) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(T,T_n) \longrightarrow 0 \longrightarrow \cdots$$

where $\mathrm{Hom}_A(T,T_i)$ is of degree i for $0 \leq i \leq n$, and $Q^{\bullet *} := \mathrm{Hom}_B(Q^\bullet, B) \in \mathscr{C}(B^{\mathrm{op}}\text{-proj})$. Note that $Q^{\bullet *}$ is isomorphic in $\mathscr{C}^b(B^{\mathrm{op}}\text{-proj})$ to the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathrm{Hom}_A(T_n,T) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(T_1,T) \longrightarrow \mathrm{Hom}_A(T_0,T) \longrightarrow 0 \longrightarrow \cdots$$

The following lemma is taken from [**6**, Theorem 2.2], which says that $\mathscr{D}(A)$ is not equivalent to $\mathscr{D}(B)$ in general.

Lemma 5.2. *The functor* $H : \mathscr{D}(A) \to \mathscr{D}(B)$ *is fully faithful, and* $\mathrm{Im}(H) = \mathrm{Ker}(\mathrm{Hom}_{\mathscr{D}(B)}(\mathrm{Ker}(G),-))$.

The next result supplies a way to understand T by some special objects or by subcategories of derived module categories. In particular, the category $\mathrm{Ker}(G)$ is a bireflective subcategory of $\mathscr{D}(B)$.

Lemma 5.3. *For the tilting* A*-module* T*, we have*

(1) $H(A) \simeq Q^\bullet$ *in* $\mathscr{D}(B)$ *and* $\mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, Q^\bullet[m]) = 0$ *for any* $m \neq 0$.

(2) $\mathrm{Ker}(G) = \{Y^\bullet \in \mathscr{D}(B) \mid \mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, Y^\bullet[i]) = 0$ *for all* $i \in \mathbb{Z}\}$.

(3) *Let* $j_! : \mathrm{Tria}(Q^\bullet) \to \mathscr{D}(B)$ *and* $i_* : \mathrm{Ker}(G) \to \mathscr{D}(B)$ *be the inclusions. Then there exists a recollement of triangulated categories together with a triangle equivalence*:

$$\mathrm{Ker}(G) \underset{\longleftarrow}{\overset{\overset{i^*}{\longleftarrow}}{\xrightarrow{\;i_*\;}}} \mathscr{D}(B) \underset{\underset{j_*}{\longleftarrow}}{\overset{\overset{j_!}{\longleftarrow}}{\xrightarrow{\;j^!\;}}} \mathrm{Tria}(Q^\bullet) \xrightarrow[\simeq]{G\,j_*} \mathscr{D}(A) \qquad (\star)$$

such that $G\,j_*\,j^!$ *is naturally isomorphic to* G.

Proof. Lemma 5.3 is implied in [**6**]. For the convenience of the reader, we include a proof here.

(1) By $(T3)'$, the stalk complex A is quasi-isomorphic in $\mathscr{C}(A)$ to the complex $T^\bullet$ of the form:

$$\cdots \longrightarrow 0 \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_n \longrightarrow 0 \longrightarrow \cdots$$

where $T_i \in \mathrm{add}(T)$ is in degree i for $0 \leq i \leq n$. Further, by $(T2)$, we have $T_i \in T^\perp := \{X \in A\text{-Mod} \mid \mathrm{Ext}^i_A(T,X) = 0 \text{ for all } i \geq 1\}$. It follows from Lemma 2.1(1) that

$H(A) \simeq H(T^\bullet) \simeq \mathrm{Hom}_A(T, T^\bullet) = Q^\bullet$ in $\mathscr{D}(B)$. Since the functor H is fully faithful by Lemma 5.2,

$$\begin{aligned}\mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, Q^\bullet[m]) &\simeq \mathrm{Hom}_{\mathscr{D}(B)}(H(A), H(A)[m]) \\ &\simeq \mathrm{Hom}_{\mathscr{D}(A)}(A, A[m]) \simeq \mathrm{Ext}_A^m(A, A) = 0\end{aligned}$$

for any $m \neq 0$. This shows (1).

(2) Since $Q^\bullet \in \mathscr{C}^b(B\text{-proj})$ and $Q^{\bullet *}$ is quasi-isomorphic to T_B by Lemma 5.1(2), there are natural isomorphisms of triangle functors:

$$\mathbb{R}\mathrm{Hom}_B(Q^\bullet, -) \xrightarrow{\simeq} Q^{\bullet *} \otimes_B^{\mathbb{L}} - \xrightarrow{\simeq} {}_{\mathbb{Z}}T \otimes_B^{\mathbb{L}} - : \mathscr{D}(B) \longrightarrow \mathscr{D}(\mathbb{Z}).$$

Note that $H^m(\mathbb{R}\mathrm{Hom}_B(Q^\bullet, Y^\bullet)) \simeq \mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, Y^\bullet[m])$ for $m \in \mathbb{Z}$ and $Y^\bullet \in \mathscr{D}(B)$. This shows (2).

(3) Since $Q^\bullet \in \mathscr{C}^b(B\text{-proj})$, we know from (2) and Lemma 3.5 that there exists a recollement of triangulated categories:

$$\mathrm{Ker}(G) \underset{i^!}{\overset{i^*}{\underset{i_*}{\rightleftarrows}}} \mathscr{D}(B) \underset{j_*}{\overset{j_!}{\underset{j^!}{\rightleftarrows}}} \mathrm{Tria}(Q^\bullet) \qquad (\star\star)$$

On the one hand, by the correspondence between recollements and TTF (torsion, torsion-free) triples (see, for example, [**11**, Section 2.3]), we infer from $(\star\star)$ that $\mathrm{Im}(j_*) = \mathrm{Ker}(\mathrm{Hom}_{\mathscr{D}(B)}(\mathrm{Ker}(G), -))$ and that the functor $j_* : \mathrm{Tria}(Q^\bullet) \to \mathrm{Im}(j_*)$ is a triangle equivalence with the restriction of $j^!$ to $\mathrm{Im}(j_*)$ as its quasi-inverse. On the other hand, it follows from Lemma 5.2 that $\mathrm{Im}(H) = \mathrm{Ker}(\mathrm{Hom}_{\mathscr{D}(B)}(\mathrm{Ker}(G), -))$ and the functor $H : \mathscr{D}(A) \to \mathrm{Im}(H)$ is a triangle equivalence with the restriction of G to $\mathrm{Im}(H)$ as its quasi-inverse. Consequently, $\mathrm{Im}(j_*) = \mathrm{Im}(H)$ and the composition $G j_* : \mathrm{Tria}(Q^\bullet) \to \mathscr{D}(A)$ of j_* with G is a triangle equivalence.

For any $X^\bullet \in \mathscr{D}(B)$, by the recollement $(\star\star)$, there exists a canonical triangle in $\mathscr{D}(B)$:

$$i_*i^!(X^\bullet) \longrightarrow X^\bullet \longrightarrow j_*j^!(X^\bullet) \longrightarrow i_*i^!(X^\bullet)[1].$$

Since $\mathrm{Im}(i_*i^!) = \mathrm{Im}(i_*) = \mathrm{Ker}(G)$, $G(X^\bullet) \xrightarrow{\simeq} Gj_*j^!(X^\bullet)$ in $\mathscr{D}(B)$. This proves (3). □

Next, we shall investigate when the subcategory $\mathrm{Ker}(G)$ of $\mathscr{D}(B)$ is homological.

Lemma 5.4. $\mathrm{Ker}(G)$ *is a homological subcategory of* $\mathscr{D}(B)$ *if and only if* $\mathrm{Ker}(\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(T, -))$ *is a homological subcategory of* $\mathscr{D}(B^{\mathrm{op}})$.

Proof. In Proposition 3.7, we take $R := B$ and $\Sigma := \{Q^\bullet\}$. Then $\Sigma^* = \{Q^{\bullet *}\}$. Since $Q^{\bullet *}$ is quasi-isomorphic to T_B by Lemma 5.1(2), there is a natural isomorphism of triangle functors:

$$\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(T, -) \xrightarrow{\simeq} \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(Q^{\bullet *}, -) : \mathscr{D}(B^{\mathrm{op}}) \longrightarrow \mathscr{D}(\mathbb{Z}).$$

This implies

$$\begin{aligned}\operatorname{Ker}\big(\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(T,\,-)\big) &= \operatorname{Ker}\big(\mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(Q^{\bullet *},\,-)\big)\\ &= \{Y^\bullet \mid \mathrm{Hom}_{\mathscr{D}(B^{\mathrm{op}})}(Q^{\bullet *},\,Y^\bullet[m]) = 0 \text{ for } m\in\mathbb{Z}\}.\end{aligned}$$

Thus Lemma 5.4 follows from Lemma 3.5 and Proposition 3.7. □

Now we point out that each good tilting module can produce a weak tilting module over its endomorphism ring. This guarantees that we can apply Proposition 4.4 to show Theorem 1.1.

Lemma 5.5. *The right B-module T_B is an n-weak tilting module.*

Proof. Clearly, $\mathrm{proj.dim}(T_B) \le n$. Moreover, $(R1)$ and $(R2)$ hold for T_B by Lemmas 5.1(2) and 5.1(3), respectively. Now, we check $(R3)$ for T_B.

In fact, according to $(T1)$, the module ${}_AT$ admits a projective resolution of A-modules:

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\ \pi\ } T \longrightarrow 0$$

with $P_i \in \mathrm{Add}({}_AA)$ for $0 \le i \le n$. Since $\mathrm{Ext}^j_A(T,T) = 0$ for all $j \ge 1$ by $(T2)$, the sequence

$$0 \longrightarrow B \longrightarrow \mathrm{Hom}_A(P_0,T) \longrightarrow \mathrm{Hom}_A(P_1,T) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(P_n,T) \longrightarrow 0$$

of right B-modules is exact. Note that $\mathrm{Hom}_A(P_i,T) \in \mathrm{Prod}(T_B)$ due to $P_i \in \mathrm{Add}({}_AA)$. Thus T_B satisfies $(R3)$.

It remains to prove that T_B satisfies $(R4)$.

Actually, by Lemma 5.1(3), the map $A^{\mathrm{op}} \to \mathrm{End}_{B^{\mathrm{op}}}(T)$, defined by $a \mapsto [t \mapsto at]$ for $a \in A$ and $t \in T$, is an isomorphism of rings. Further, it follows from Lemma 2.5 that the right A^{op}-module T is strongly A^{op}-Mittag-Leffler. Hence the right $\mathrm{End}_{B^{\mathrm{op}}}(T)$-module T is strongly $\mathrm{End}_{B^{\mathrm{op}}}(T)$-Mittag-Leffler. Thus, by definition, the B^{op}-module T is an n-weak tilting module. □

Proof of Theorem 1.1. Recall that the complex $P^\bullet$ is the deleted projective resolution of ${}_AT$:

$$\cdots \longrightarrow 0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \longrightarrow \cdots$$

appearing in $(T1)$. Here P_i is in degree $-i$ for $0 \le i \le n$.

By Lemma 5.5, T is an n-weak tilting B^{op}-module and the exact sequence in $(R3)$ can be chosen as

$$0 \longrightarrow B_B \longrightarrow \mathrm{Hom}_A(P_0,T) \longrightarrow \mathrm{Hom}_A(P_1,T) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(P_n,T) \longrightarrow 0.$$

In particular, the complex $M^\bullet$ in Proposition 4.4 can be chosen as

$$\mathrm{Hom}_A(P^\bullet,T): \quad \cdots \longrightarrow 0 \longrightarrow \mathrm{Hom}_A(P_0,T) \longrightarrow \mathrm{Hom}_A(P_1,T) \longrightarrow$$

$$\cdots \longrightarrow \mathrm{Hom}_A(P_n, T) \longrightarrow 0 \longrightarrow \cdots$$

Let

$$\boldsymbol{H} = \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(T, -) : \mathscr{D}(B^{\mathrm{op}}) \longrightarrow \mathscr{D}(A^{\mathrm{op}}).$$

It follows from Lemma 5.4 that $\mathrm{Ker}(G)$ is homological in $\mathscr{D}(B)$ if and only if so is $\mathrm{Ker}(\boldsymbol{H})$ in $\mathscr{D}(B^{\mathrm{op}})$. That is, ${}_AT$ is homological if and only if

(a) $\mathrm{Ker}(\boldsymbol{H})$ is a homological subcategory of $\mathscr{D}(B^{\mathrm{op}})$.

Now, in Proposition 4.4, we take $R := B^{\mathrm{op}}$, $S := A^{\mathrm{op}}$ and $M := {}_RT_S$. By Proposition 4.4, (a) is equivalent to

(b) $H^j(\mathrm{Hom}_{B^{\mathrm{op}}}(T, M^\bullet) \otimes_A T) = 0$ for all $j \geq 2$, where $\mathrm{Hom}_{B^{\mathrm{op}}}(T, M^\bullet) := \mathrm{Hom}_{B^{\mathrm{op}}}(T, \mathrm{Hom}_A(P^\bullet, T))$ is the complex of the form:

$$0 \longrightarrow \mathrm{Hom}_{B^{\mathrm{op}}}(T, \mathrm{Hom}_A(P_0, T)) \longrightarrow \mathrm{Hom}_{B^{\mathrm{op}}}(T, \mathrm{Hom}_A(P_1, T)) \longrightarrow$$
$$\cdots \longrightarrow \mathrm{Hom}_{B^{\mathrm{op}}}(T, \mathrm{Hom}_A(P_n, T)) \longrightarrow 0,$$

with $\mathrm{Hom}_{B^{\mathrm{op}}}(T, \mathrm{Hom}_A(P_i, T))$ in degree i for $0 \leq i \leq n$.

In the following we show $\mathrm{Hom}_A(P^\bullet, A) \simeq \mathrm{Hom}_{B^{\mathrm{op}}}(T, \mathrm{Hom}_A(P^\bullet, T))$ as complexes over A^{op}.

In fact, since T is a good tilting A-module, it follows from the axiom $(T3)'$ that there exists an exact sequence $0 \to A \to T_0 \to T_1$ with $T_i \in \mathrm{add}(T)$ for $i = 0, 1$. Applying the functor $\Phi := \mathrm{Hom}_A(-, {}_AT_B)$ to this sequence, we obtain another exact sequence $\Phi(T_1) \to \Phi(T_0) \to \Phi(A) \to 0$ of B^{op}-modules by Lemma 5.1(2), and the exact commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_A(X, A) & \longrightarrow & \mathrm{Hom}_A(X, T_0) & \longrightarrow & \mathrm{Hom}_A(X, T_1) \\ & \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} \\ 0 \longrightarrow & \mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(A), \Phi(X)) & \longrightarrow & \mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(T_0), \Phi(X)) & \longrightarrow & \mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(T_1), \Phi(X)) \end{array}$$

where the isomorphisms in the second and third columns are due to $T_0 \in \mathrm{add}(T)$ and $T_1 \in \mathrm{add}(T)$, respectively. Consequently, $\Phi : \mathrm{Hom}_A(X, A) \to \mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(A), \Phi(X))$ in the first column is an isomorphism. This implies

$$\mathrm{Hom}_A(-, A) \xrightarrow{\simeq} \mathrm{Hom}_{B^{\mathrm{op}}}(\Phi(A), \Phi(-)) \xrightarrow{\simeq}$$
$$\mathrm{Hom}_{B^{\mathrm{op}}}(T, \mathrm{Hom}_A(-, T)) : A\text{-Mod} \to A^{\mathrm{op}}\text{-Mod}.$$

Thus $\mathrm{Hom}_A(P^\bullet, A) \simeq \mathrm{Hom}_{B^{\mathrm{op}}}(T, \mathrm{Hom}_A(P^\bullet, T))$ as complexes over A^{op}. This completes the proof of the first part of Theorem 1.1, while the second part of Theorem 1.1 follows from Lemma 4.2. □

REMARK 5.6. (1) Up to isomorphism, the cohomologies $H^m(\mathrm{Hom}_A(P^\bullet, A) \otimes_A T_B)$ in Theorem 1.1 are independent of the choice of the projective resolutions of ${}_AT$. Moreover, by Lemma 4.3(2) and the proof of Theorem 1.1, there are isomorphisms in $\mathscr{D}(B^{\mathrm{op}})$:

$$\mathrm{Hom}_A(P^\bullet, A)\otimes_A T_B \simeq \mathrm{Hom}_A(P^\bullet, A)\otimes^{\mathbb{L}}_A T_B \simeq \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(T,\, B)\otimes^{\mathbb{L}}_A T_B.$$

(2) In Theorem 1.1, if $n=2$, then ${}_AT$ is homological if and only if $\mathrm{Ext}^2_A(T,\,A)\otimes_A T=0$. This follows from $H^m(\mathrm{Hom}_A(P^\bullet, A)\otimes_A T_B)=0$ for $m\geq 3$ and $H^2(\mathrm{Hom}_A(P^\bullet, A)\otimes_A T_B)\simeq H^2(\mathrm{Hom}_A(P^\bullet, A))\otimes_A T\simeq \mathrm{Ext}^2_A(T, A)\otimes_A T$.

To prove Corollary 1.2, we first establish a lemma.

Lemma 5.7. *The complex* $\mathrm{Hom}_A(P^\bullet, A)$ *is isomorphic in* $\mathscr{D}(\mathbb{Z})$ *to the complex*:

$$\begin{aligned}\mathrm{Hom}_A(T, T^\bullet):\ \ &\cdots\longrightarrow 0\longrightarrow \mathrm{Hom}_A(T,T_0)\longrightarrow \mathrm{Hom}_A(T,T_1)\longrightarrow\\ &\cdots\longrightarrow \mathrm{Hom}_A(T,T_n)\longrightarrow 0\longrightarrow\cdots\end{aligned}$$

In particular, if A is commutative, then $\mathrm{Hom}_A(P^\bullet, A)\otimes_A T_B\simeq \mathrm{Hom}_A(T,T^\bullet)\otimes^{\mathbb{L}}_A T_B$ *in* $\mathscr{D}(B^{\mathrm{op}})$*, where $T^\bullet$ denotes the complex in $(T3)'$ without the term A.*

Proof. The maps π and ω in $(T1)$ and $(T3)'$ induce two canonical quasi-isomorphisms $\widetilde{\pi}: P^\bullet\to T$ and $\widetilde{\omega}: A\to T^\bullet$ in $\mathscr{C}(A)$, respectively. Consequently, both $\widetilde{\pi}$ and $\widetilde{\omega}$ are isomorphisms in $\mathscr{D}(A)$. Since $\widetilde{\pi}$ and $\widetilde{\omega}$ are chain maps in $\mathscr{C}(A)$, we obtain two chain maps in $\mathscr{C}(\mathbb{Z})$:

$$\mathrm{Hom}_A(P^\bullet, A)\xrightarrow{(\widetilde{\omega})^*}\mathrm{Hom}^\bullet_A(P^\bullet, T^\bullet)\xleftarrow{(\widetilde{\pi})_*}\mathrm{Hom}_A(T, T^\bullet).$$

Now, we claim that both chain maps are quasi-isomorphisms.

Indeed, applying $H^i(-)$ to these chain maps for $i\in\mathbb{Z}$, we construct the commutative diagram:

$$\begin{array}{ccccc}
H^i(\mathrm{Hom}_A(P^\bullet, A)) & \xrightarrow{H^i((\widetilde{\omega})^*)} & H^i(\mathrm{Hom}^\bullet_A(P^\bullet, T^\bullet)) & \xleftarrow{H^i((\widetilde{\pi})_*)} & H^i(\mathrm{Hom}_A(T,\,T^\bullet))\\
\downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq\\
\mathrm{Hom}_{\mathscr{K}(A)}(P^\bullet, A[i]) & \xrightarrow{(\widetilde{\omega})^*} & \mathrm{Hom}_{\mathscr{K}(A)}(P^\bullet, T^\bullet[i]) & \xleftarrow{(\widetilde{\pi})_*} & \mathrm{Hom}_{\mathscr{K}(A)}(T, T^\bullet[i])\\
\downarrow q_1 & & \downarrow q_2 & & \downarrow q_3\\
\mathrm{Hom}_{\mathscr{D}(A)}(P^\bullet, A[i]) & \xrightarrow[\simeq]{(\widetilde{\omega})^*} & \mathrm{Hom}_{\mathscr{D}(A)}(P^\bullet, T^\bullet[i]) & \xleftarrow[\simeq]{(\widetilde{\pi})_*} & \mathrm{Hom}_{\mathscr{D}(A)}(T, T^\bullet[i])
\end{array}$$

where the maps q_j, $1\leq j\leq 3$, are induced by the localization functor $q:\mathscr{K}(A)\to\mathscr{D}(A)$, and where the isomorphisms in the third row are due to the isomorphisms $\widetilde{\omega}$ and $\widetilde{\pi}$ in $\mathscr{D}(A)$.

Since $P^\bullet$ is a bounded complex of projective A-modules, both q_1 and q_2 are bijective. This implies that $H^i((\widetilde{\omega})^*)$ is also bijective, and therefore $(\widetilde{\omega})^*$ is a quasi-isomorphism.

Note that $(\widetilde{\pi})_*$ is a quasi-isomorphism if and only if $H^i((\widetilde{\pi})_*)$ is bijective for each $i\in\mathbb{Z}$. This is also equivalent to saying that q_3 is bijective in the above diagram. Actually, to prove the bijection of q_3, it is enough to show that, for $X\in\mathrm{add}({}_AT)$ and $i\in\mathbb{Z}$, the canonical map $\mathrm{Hom}_{\mathscr{K}(A)}(T, X[i])\to\mathrm{Hom}_{\mathscr{D}(A)}(T, X[i])$ induced by q is bijective since $T^\bullet$ is a bounded complex with each term in $\mathrm{add}({}_AT)$. However, this follows directly from

$(T2)$. Thus $(\widetilde{\pi})_*$ is a quasi-isomorphism.

Consequently, the complexes $\mathrm{Hom}_A(P^\bullet, A)$ and $\mathrm{Hom}_A(T, T^\bullet)$ are isomorphic in $\mathscr{D}(\mathbb{Z})$.

If A is commutative, then each A-module can be naturally regarded as a right A-module and even as an A-A-bimodule. Particularly, $T^\bullet$ can be regarded as a complex of A-A-bimodules. In this sense, both $\widetilde{\pi} : P^\bullet \to T$ and $\widetilde{\omega} : A \to T^\bullet$ are quasi-isomorphisms of complexes of A-A-bimodules. Moreover, the chain maps $(\widetilde{\omega})^*$ and $(\widetilde{\pi})_*$ are quasi-isomorphisms in $\mathscr{C}(A^{\mathrm{op}})$. Thus $\mathrm{Hom}_A(P^\bullet, A) \simeq \mathrm{Hom}_A(T, T^\bullet)$ in $\mathscr{D}(A^{\mathrm{op}})$. Note that $\mathrm{Hom}_A(P^\bullet, A) \otimes_A T_B \simeq \mathrm{Hom}_A(P^\bullet, A) \otimes^{\mathbb{L}}_A T_B$ in $\mathscr{D}(B^{\mathrm{op}})$. As a result, $\mathrm{Hom}_A(P^\bullet, A) \otimes_A T_B \simeq \mathrm{Hom}_A(T, T^\bullet) \otimes^{\mathbb{L}}_A T_B$ in $\mathscr{D}(B^{\mathrm{op}})$. □

Proof of Corollary 1.2. (1) Let $P^\bullet_M := (P^{-i}_M)_{0\leq i\leq n}$ and $P^\bullet_N := (P^{-i}_N)_{0\leq i\leq n}$ denote deleted projective resolutions of M and N, respectively. Then $P^\bullet = P^\bullet_M \oplus P^\bullet_N$. Suppose $m \in \mathbb{N}$ with $m \geq 2$. Then

$$H^m\big(\mathrm{Hom}_A(P^\bullet, A) \otimes_A T_B\big) \simeq H^m\big(\mathrm{Hom}_A(P^\bullet_M, A) \otimes_A T\big) \oplus H^m\big(\mathrm{Hom}_A(P^\bullet_N, A) \otimes_A T\big).$$

Due to $\mathrm{proj.dim}({}_AM) \leq 1$, we have $P^{-i}_M = 0$ for all $2 \leq i \leq n$. This implies $H^m(\mathrm{Hom}_A(P^\bullet_M, A) \otimes_A T) = 0$. Note that ${}_AT = M \oplus N$ is strongly A-Mittag-Leffler by Lemma 2.5, and therefore so is the first syzygy $\Omega_A(N)$ of ${}_AN$. Since $\Omega_A(N)$ is finitely generated, it has a finitely generated projective resolution by Lemma 2.4(4). Hence we can assume $P^{-j}_N \in \mathrm{add}({}_AA)$ for all $1 \leq j \leq n$. Now, we consider the natural transformation $\zeta : \mathrm{Hom}_A(-, A) \otimes_A T \to \mathrm{Hom}_A(-, T)$ from A-Mod to B^{op}-Mod. If $X \in \mathrm{add}({}_AA)$, then ζ_X is an isomorphism. In particular, $\mathrm{Hom}_A(P^{-j}_N, A) \otimes_A T \xrightarrow{\simeq} \mathrm{Hom}_A(P^{-j}_N, T)$ for all $1 \leq j \leq n$. Since $m \geq 2$ and ${}_AT$ is a tilting module,

$$H^m\big(\mathrm{Hom}_A(P^\bullet_N, A) \otimes_A T\big) \simeq H^m\big(\mathrm{Hom}_A(P^\bullet_N, T)\big) = \mathrm{Ext}^m_A(N, T) = 0.$$

Consequently, $H^m(\mathrm{Hom}_A(P^\bullet, A) \otimes_A T_B) = 0$ and (1) follows from Theorem 1.1.

(2) Suppose that ${}_AT$ is homological. By Theorem 1.1, $H^m(\mathrm{Hom}_A(P^\bullet, A)\otimes_A T_B) = 0$ for all $m \geq 2$. Furthermore, we shall show $T_n = 0$ if $H^n(\mathrm{Hom}_A(P^\bullet, A) \otimes_A T_B) = 0$.

In fact, since A is a commutative ring, every one-sided A-module is automatically an A-bimodule. It follows from the proof of Lemma 5.7 that $\mathrm{Hom}_A(P^\bullet, A) \simeq \mathrm{Hom}_A(T, T^\bullet)$ in $\mathscr{D}(A^{\mathrm{op}})$. Note that the tensor functor $- \otimes_A T_B : A^{\mathrm{op}}\text{-Mod} \to B^{\mathrm{op}}\text{-Mod}$ is right exact. This means

$$0 = H^n\big(\mathrm{Hom}_A(P^\bullet, A) \otimes_A T_B\big) \simeq H^n\big(\mathrm{Hom}_A(P^\bullet, A)\big) \otimes_A T \simeq H^n\big(\mathrm{Hom}_A(T, T^\bullet)\big) \otimes_A T.$$

In particular, $H^n(\mathrm{Hom}_A(T_n, T^\bullet)) \otimes_A T_n = 0$, due to $T_n \in \mathrm{add}({}_AT)$.

Recall that the complex $\mathrm{Hom}_A(T_n, T^\bullet)$ is of the form

$$\cdots \longrightarrow 0 \longrightarrow \mathrm{Hom}_A(T_n, T_0) \longrightarrow$$
$$\cdots \longrightarrow \mathrm{Hom}_A(T_n, T_{n-1}) \longrightarrow \mathrm{Hom}_A(T_n, T_n) \longrightarrow 0 \longrightarrow \cdots$$

As $\mathrm{Hom}_A(T_n, T_{n-1}) = 0$ by our assumption in Corollary 1.2(2), we obtain $H^n(\mathrm{Hom}_A(T_n, T^\bullet)) = \mathrm{Hom}_A(T_n, T_n)$. Thus $\mathrm{End}_A(T_n) \otimes_A T_n = 0$. It follows from the surjective map

$$\mathrm{End}_A(T_n) \otimes_A T_n \longrightarrow T_n,\ f \otimes x \mapsto (x)f \quad \text{for } f \in \mathrm{End}_A(T_n) \ \text{ and } \ x \in T_n$$

that $T_n = 0$.

Now, by our assumption, $\mathrm{Hom}_A(T_{i+1}, T_i) = 0$ for $1 \leq i \leq n-1$. Thus, by induction, we can show $T_j = 0$ for $2 \leq j \leq n$. It then follows from Lemma 5.1(4) that T is a 1-tilting A-module.

The sufficiency of Corollary 1.2(2) follows from Theorem 1.1, see also [**11**, Theorem 1.1(1)]. □

6. Applications to cotilting modules: Proof of Theorem 1.3.

In this section, we shall apply the results in Section 4 to deal with cotilting modules. First, we construct weak tilting modules from good cotilting modules, and then use Proposition 4.4 to show Corollary 6.3 and give a proof of Theorem 1.3. In the course of our discussions, we also develop some criterions for bireflective subcategories induced from cotilting modules to be homological.

Suppose that A is a ring and W is a fixed injective cogenerator for A-Mod. Recall that an A-module W is called a *cogenerator* for A-Mod if, for any A-module Y, there exists an injective homomorphism $Y \to W^I$ in A-Mod with I a set.

Definition 6.1. An A-module U is called an *n-cotilting module* if the following three conditions are satisfied:

$(C1)$ $\mathrm{inj.dim}({}_AU) \leq n$;

$(C2)$ $\mathrm{Ext}^j_A(U^I, U) = 0$ for each $j \geq 1$ and for every nonempty set I; and

$(C3)$ there exists an exact sequence of A-modules

$$0 \longrightarrow U_n \longrightarrow \cdots \longrightarrow U_1 \longrightarrow U_0 \longrightarrow W \longrightarrow 0$$

such that $U_i \in \mathrm{Prod}({}_AU)$ for all $0 \leq i \leq n$.

An n-cotilting A-module U is said to be *good* if it satisfies $(C1)$, $(C2)$ and

$(C3)'$ there is an exact sequence of A-modules

$$0 \longrightarrow U_n \longrightarrow \cdots \longrightarrow U_1 \longrightarrow U_0 \longrightarrow W \longrightarrow 0$$

such that $U_i \in \mathrm{add}({}_AU)$ for all $0 \leq i \leq n$.

We say that U is a (good) cotilting A-module if ${}_AU$ is (good) n-cotilting for some $n \in \mathbb{N}$.

As in the case of tilting modules, for a given n-cotilting A-module U with $(C1)$–$(C3)$, the A-module $U' := \bigoplus_{i=0}^n U_i$ is a good n-cotilting module which is equivalent to the given one in the sense that $\mathrm{Prod}(U) = \mathrm{Prod}(U')$.

From now on, we assume that U is a *good* n-cotilting A-module with $(C_1), (C_2)$ and $(C_3)'$, and call U a *good n-cotilting A-module with respect to W*. Let $R := \mathrm{End}_A(U)$, $M := \mathrm{Hom}_A(U, W)$ and $\Lambda := \mathrm{End}_A(W)$. Then M is an R-Λ-bimodule.

LEMMA 6.2. (1) *The R-module M has a finitely generated projective resolution of length at most n:*

$$0 \longrightarrow \mathrm{Hom}_A(U,U_n) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(U,U_1) \longrightarrow \mathrm{Hom}_A(U,U_0) \longrightarrow M \longrightarrow 0$$

with $U_m \in \mathrm{add}({}_AU)$ for all $0 \le m \le n$.

(2) *The Hom-functor $\mathrm{Hom}_A(U,-) : A\text{-Mod} \to R\text{-Mod}$ induces an isomorphism of rings: $\Lambda \simeq \mathrm{End}_R(M)$, and $\mathrm{Ext}^i_R(M,M) = 0$ for all $i \ge 1$.*

(3) *The module ${}_RM$ satisfies $(R1)$–$(R3)$.*

PROOF. (1) Applying $\mathrm{Hom}_A(U,-)$ to $(C_3)'$, we obtain the sequence in (1) with all $\mathrm{Hom}_A(U,U_i) \in \mathrm{add}({}_RR)$. Its exactness follows directly from $(C2)$. Thus $\mathrm{proj.dim}({}_RM) \le n$.

(2) Let Ψ be the Hom-functor $\mathrm{Hom}_A(U,-) : A\text{-Mod} \to R\text{-Mod}$. Then $\Psi(U) = R$, $\Psi(W) = M$ and $\mathrm{Hom}_A(X,W) \xrightarrow{\simeq} \mathrm{Hom}_R(\Psi(X),\Psi(W))$ for any $X \in \mathrm{add}({}_AU)$.

If $n = 0$, then $W = U_0$ and $M = \mathrm{Hom}_A(U,U_0)$ as R-modules. In this case, one can easily check (2).

Suppose $n \ge 1$. By (1), the R-module $M = \Psi(W)$ has a finitely generated projective resolution

$$0 \longrightarrow \Psi(U_n) \longrightarrow \cdots \longrightarrow \Psi(U_1) \longrightarrow \Psi(U_0) \longrightarrow \Psi(W) \longrightarrow 0$$

with $U_m \in \mathrm{add}(U)$ for $0 \le m \le n$. Applying $\mathrm{Hom}_A(-,W)$ to the resolution of W in $(C3)'$, we can construct the commutative diagram:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathrm{Hom}_A(W,W) & \longrightarrow & \mathrm{Hom}_A(U_0,W) & \longrightarrow & \mathrm{Hom}_A(U_1,W) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Hom}_A(U_n,W) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} & & & & \downarrow{\scriptstyle\simeq} & & \\
0 & \longrightarrow & \mathrm{Hom}_R(\Psi(W),\Psi(W)) & \longrightarrow & \mathrm{Hom}_R(\Psi(U_0),\Psi(W)) & \longrightarrow & \mathrm{Hom}_R(\Psi(U_1),\Psi(W)) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Hom}_R(\Psi(U_n),\Psi(W)) & \longrightarrow & 0.
\end{array}$$

Since the module ${}_AW$ is injective, the first row in the diagram is exact. This implies that Ψ is an isomorphism of rings and that the entire sequence of the second row in the diagram is exact. Thus $\mathrm{Ext}^i_R(M,M) = \mathrm{Ext}^i_R(\Psi(W),\Psi(W)) = 0$ for all $i \ge 1$ by definition.

(3) Clearly, $(R1)$ and $(R2)$ follow from (1) and (2), respectively. It remains to show $(R3)$ for M. In fact, by $(C1)$, there exists an exact sequence of A-modules: $0 \to U \to I_0 \to I_1 \to \cdots \to I_n \to 0$ where I_i is an injective module for $0 \le i \le n$. Since W is an injective cogenerator for A-Mod, $I_i \in \mathrm{Prod}({}_AW)$. Moreover, due to $(C2)$, the sequence

$$0 \longrightarrow R \longrightarrow \mathrm{Hom}_A(U,I_0) \longrightarrow \mathrm{Hom}_A(U,I_1) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(U,I_n) \longrightarrow 0$$

is exact. Since $\mathrm{Hom}_A(U,-)$ commutes with arbitrary direct products, it follows from $I_i \in \mathrm{Prod}({}_AW)$ that $\mathrm{Hom}_A(U,I_i) \in \mathrm{Prod}({}_R\mathrm{Hom}_A(U,W)) = \mathrm{Prod}({}_RM)$ and that ${}_RM$ satisfies $(R3)$. □

By Lemma 6.2(2), the ring $\mathrm{End}_R(M)$ can be identified naturally with Λ (up to isomorphism of rings). Now, we define

$$\boldsymbol{G} := {}_RM \otimes^{\mathbb{L}}_{\Lambda} - : \mathscr{D}(\Lambda) \longrightarrow \mathscr{D}(R) \quad \text{and} \quad \boldsymbol{H} := \mathbb{R}\mathrm{Hom}_R(M,-) : \mathscr{D}(R) \longrightarrow \mathscr{D}(\Lambda).$$

Since ${}_RM$ satisfies both $(R1)$ and $(R2)$ in Definition 4.1, it follows from Lemma 4.2 that there exists a recollement of triangulated categories:

$$\xymatrix{\mathrm{Ker}(\boldsymbol{H}) \ar[r]^{i_*} & \mathscr{D}(R) \ar@/_1pc/[l]_{i^*} \ar@/^1pc/[l] \ar[r]^{\boldsymbol{H}} & \mathscr{D}(\Lambda) \ar@/_1pc/[l]_{\boldsymbol{G}} \ar@/^1pc/[l]}$$

where (i^*, i_*) is an adjoint pair of functors with i_* the inclusion. In this situation, we are interested in the following

PROBLEM. When is $\mathrm{Ker}(\boldsymbol{H})$ homological in $\mathscr{D}(R)$?

We do not know whether ${}_RM$ satisfies $(R4)$ and cannot directly apply Proposition 4.4 to ${}_RM$. However, the following holds true.

COROLLARY 6.3. *Suppose that A is a ring together with an injective cogenerator W for A-*Mod. *Let U be a good n-cotilting A-module with respect to W. Suppose that $\Lambda := \mathrm{End}_A(W)$ is a right noetherian ring.*

(1) *The following assertions are equivalent*:

(a) $\mathrm{Ker}(\boldsymbol{H})$ *is homological in $\mathscr{D}(R)$.*

(b) $H^m({}_R\mathrm{Hom}_A(U, W) \otimes_\Lambda \mathrm{Hom}_A(W, I^\bullet)) = 0$ *for $m \geq 2$, where $I^\bullet$ is a deleted injective coresolution of ${}_AU$*:

$$\cdots \longrightarrow 0 \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_n \longrightarrow 0 \longrightarrow \cdots$$

with I_i in degree i for all $0 \leq i \leq n$.

(2) *If one of the above assertions in* (1) *holds, then the generalized localization $\lambda : R \to R_M$ of R at M exists and induces a recollement of derived module categories*:

$$\xymatrix{\mathscr{D}(R_M) \ar[r]^{D(\lambda_*)} & \mathscr{D}(R) \ar@/_1pc/[l] \ar@/^1pc/[l] \ar[r]^{\boldsymbol{H}} & \mathscr{D}(\Lambda) \ar@/_1pc/[l]_{\boldsymbol{G}} \ar@/^1pc/[l]}$$

PROOF. (1) By the proof of Lemma 6.2(3), the sequence in $(R3)$ can be chosen as follows:

$$0 \longrightarrow R \longrightarrow \mathrm{Hom}_A(U, I_0) \longrightarrow \mathrm{Hom}_A(U, I_1) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(U, I_n) \longrightarrow 0.$$

In this case, the complex $M^\bullet$ can be defined as the complex:

$$\mathrm{Hom}_A(U, I^\bullet):\ 0 \longrightarrow \mathrm{Hom}_A(U, I_0) \longrightarrow \mathrm{Hom}_A(U, I_1) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(U, I_n) \longrightarrow 0.$$

Since Λ is right noetherian, M is a weak tilting R-module. It follows from Proposition 4.4 that (a) is equivalent to

(b') $H^j({}_RM \otimes_\Lambda \mathrm{Hom}_R(M, M^\bullet)) = 0$ for any $j \geq 2$, where $M^\bullet := \mathrm{Hom}_A(U, I^\bullet)$.

To prove that (a) and (b) in Corollary 6.3 are equivalent, it is sufficient to show that (b') and (b) are equivalent. For this purpose, we shall show $\mathrm{Hom}_R(M, M^\bullet) \simeq \mathrm{Hom}_A(W, I^\bullet)$ as complexes over Λ.

Let $\Psi = \mathrm{Hom}_A(U, -) : A\text{-Mod} \to R\text{-Mod}$. Then $\Psi(W) = M$ and $M^\bullet = \Psi(I^\bullet)$. The functor Ψ induces a natural transformation of functors

$$\mathrm{Hom}_A(W, -) \longrightarrow \mathrm{Hom}_R(\Psi(W), \Psi(-)) : A\text{-Mod} \longrightarrow \Lambda\text{-Mod}.$$

This yields a chain map $\mathrm{Hom}_A(W, I^\bullet) \to \mathrm{Hom}_R(\Psi(W), \Psi(I^\bullet)) = \mathrm{Hom}_R(M, M^\bullet)$ in $\mathscr{C}(\Lambda)$. Note that all terms I_i of $I^\bullet$ are injective A-modules. To verify that the chain map is an isomorphism of complexes, it is enough to show that, for any injective A-module X, the functor Ψ induces an isomorphism of Λ-modules:

$$\mathrm{Hom}_A(W, X) \xrightarrow{\simeq} \mathrm{Hom}_R(\Psi(W), \Psi(X)).$$

However, this follows from $(C3)'$ and Lemma 6.2(1) even for an arbitrary A-module X.

Consequently, $\mathrm{Hom}_A(W, I^\bullet) \simeq \mathrm{Hom}_R(M, M^\bullet)$ as complexes over Λ. Thus (b') and (b), and therefore, also (a) and (b), are equivalent.

(2) follows from Lemma 4.2. □

As a consequence of Corollary 6.3, we have the following result.

COROLLARY 6.4. *Let U be a good n-cotilting A-module with respect to an injective cogenerator ${}_AW$. Suppose that* $\mathrm{End}_A(W)$ *is a right Noether ring (for example, A is an Artin algebra and W is the dual of A). If $n \leq 1$, then* $\mathrm{Ker}(\boldsymbol{H})$ *is homological in* $\mathscr{D}(R)$.

PROOF OF THEOREM 1.3. Recall that ${}_AW$ is the injective cogenerator $D(A)$ over the Artin algebra A. Then $\mathrm{End}_A(W)$ is isomorphic to A, and therefore right noetherian. Since ${}_AU$ is a good 1-cotilting module with respect to W, the category $\mathrm{Ker}(\boldsymbol{H})$ is homological by Corollary 6.4. Now, Theorem 1.3 follows from Corollary 6.3. □

7. Homological tilting modules over Gorenstein rings.

In this section, we shall apply Corollary 1.2 to construct two classes of infinitely generated, good tilting modules over Gorenstein rings such that one is homological, and the other is not.

A ring A is called *Gorenstein* (or *Iwanaga–Gorenstein*) if A is left and right noetherian and if both $\mathrm{inj.dim}({}_AA)$ and $\mathrm{inj.dim}(A_A)$ are finite. In this case, $\mathrm{inj.dim}({}_AA) = \mathrm{inj.dim}(A_A)$. The ring A is called *n-Gorenstein* if it is Gorenstein with $\mathrm{inj.dim}({}_AA) = n$.

Let A be an n-Gorenstein ring. It is known that, for an A-module M, $\mathrm{proj.dim}({}_AM) < \infty$ if and only if $\mathrm{inj.dim}({}_AM) < \infty$. Moreover, these two dimensions are at most n (for example, see [**17**, Theorem 9.1.10]). In particular, each injective A-module has projective dimension at most n and each projective A-module has injective dimension at most n.

Let

$$0 \longrightarrow {}_AA \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_n \longrightarrow 0$$

be a minimal injective resolution of ${}_AA$. Then $T := \bigoplus_{i=0}^{n} I_i$ is an n-tilting A-module (see [**2, 32**]). This module is good and has projective dimension exactly n. In the following, T is called the *canonical n-tilting module* associated with A. Moreover, $\mathrm{Add}({}_AT)$ coincides with the full subcategory of A-Mod consisting of all injective A-modules (see [**26**, Corollaries 4.6 and 4.7]).

From now on, A is a *commutative n-Gorenstein ring.* Let $\mathfrak{h}_i$ be the set of all prime ideals of A with height i. Then $I_i = \bigoplus_{\mathfrak{p}\in\mathfrak{h}_i} E(A/\mathfrak{p})$ for all $0 \le i \le n$ (see [**5**, Section 1]).

For an A-module M, we denote by $E(M)$ its injective envelope.

Now we construct infinitely generated, homological tilting modules over noncommutative Gorenstein rings by one-point extensions of rings.

Suppose that $\mathfrak{m}$ is a fixed maximal ideal of A. Define $\Lambda := \begin{pmatrix} A/\mathfrak{m} & 0 \\ A/\mathfrak{m} & A \end{pmatrix}$, where $A/\mathfrak{m}$ is regarded as an A-module via the canonical surjection $A \to A/\mathfrak{m}$. Then Λ is a left and right noetherian ring.

Each Λ-module can be written as a triple (X, Y, f) with X an $(A/\mathfrak{m})$-module, Y an A-module and $f : X \to Y$ a homomorphism in A-Mod. Since there is a canonical surjection from Λ to A, each A-module can be regarded as a Λ-module. In the following, for an A-module Y, the Λ-module $(0, Y, 0)$ will be denoted by ${}_\Lambda Y$ for simplicity. Evidently, $\mathrm{proj.dim}({}_\Lambda Y) = \mathrm{proj.dim}({}_A Y)$.

Let

$$
{}_\Lambda N := (A/\mathfrak{m}, 0, 0), \quad {}_\Lambda V := (A/\mathfrak{m}, E(A/\mathfrak{m}), \mu_\mathfrak{m}) \quad \text{and} \quad {}_\Lambda W := N \oplus V \oplus \bigoplus_{\mathfrak{p}\neq\mathfrak{m}} E(A/\mathfrak{p}),
$$

where $\mu_\mathfrak{m} : A/\mathfrak{m} \to E(A/\mathfrak{m})$ is the canonical inclusion and $\mathfrak{p}$ runs through all prime ideals of A.

PROPOSITION 7.1. *Suppose that the localization $A_\mathfrak{m}$ of A at $\mathfrak{m}$ has global dimension $n \ge 1$. Then*

(1) *Λ is an $(n+1)$-Gorenstein ring and W is an infinitely generated, good $(n+1)$-tilting Λ-module.*

(2) *If $n = 1$ (for example, $A_\mathfrak{m}$ is a Dedekind domain), then W is a homological 2-tilting Λ-module.*

PROOF. (1) First, we show that $\mathrm{Add}(W)$ is exactly the full category of Λ-Mod consisting of all injective Λ-modules.

To see this, let $e := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \Lambda$. Then $e\Lambda e \simeq A$, $V \simeq \mathrm{Hom}_A(e\Lambda, E(A/\mathfrak{m}))$ and $E(A/\mathfrak{p}) \simeq \mathrm{Hom}_A(e\Lambda, E(A/\mathfrak{p}))$ as Λ-modules for $\mathfrak{p} \neq \mathfrak{m}$. Since ${}_Ae\Lambda$ is finitely generated, the functor $\mathrm{Hom}_A(e\Lambda, -) : A\text{-Mod} \to \Lambda\text{-Mod}$ commutes with direct sums. Then ${}_\Lambda W \simeq N \oplus \mathrm{Hom}_A(e\Lambda, T)$. As the modules ${}_AT$ and ${}_\Lambda N$ are injective, the Λ-module W is also injective. On the other hand, Λ is a left noetherian ring, this means that direct sums of injective Λ-modules are again injective. Thus $\mathrm{Add}(W)$ consists of injective Λ-modules. Further, each Λ-module (X, Y, f) has a submodule ${}_\Lambda Y$ which can be embedded into a module in $\mathrm{Add}(\mathrm{Hom}_A(e\Lambda, T))$, and has a quotient module $(X, 0, 0)$ which can be embedded into a module in $\mathrm{Add}(N)$. Thus $\mathrm{Add}(W)$ coincides with the category of all injective Λ-modules.

Second, we show $\mathrm{proj.dim}(W) = n + 1$.

Since the localization $A \to A_{\mathfrak{m}}$ of A at $\mathfrak{m}$ is flat, it is homological. Thanks to $A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}} \simeq A/\mathfrak{m}$, we have $\text{proj.dim}({}_A A/\mathfrak{m}) = \text{proj.dim}({}_{A_{\mathfrak{m}}} A/\mathfrak{m})$, while the latter is equal to the global dimension of $A_{\mathfrak{m}}$. Thus $\text{proj.dim}({}_A A/\mathfrak{m}) = n$. It follows from $\Omega_\Lambda(N) = A/\mathfrak{m}$ that $\text{proj.dim}({}_\Lambda N) = n+1$. Note that there is a short exact sequence of Λ-modules

$$0 \longrightarrow (A/\mathfrak{m}, A/\mathfrak{m}, \text{Id}) \longrightarrow V \longrightarrow E(A/\mathfrak{m})/(A/\mathfrak{m}) \longrightarrow 0,$$

where Id stands for the identity map of $A/\mathfrak{m}$. In this sequence, the module $(A/\mathfrak{m}, A/\mathfrak{m}, \text{Id})$ is projective, but V is not projective due to $n \geq 1$. This implies $\text{proj.dim}({}_\Lambda V) = \text{proj.dim}({}_A E(A/\mathfrak{m})/(A/\mathfrak{m}))$. Since A is Gorenstein and $\text{proj.dim}({}_A A/\mathfrak{m}) < \infty$, $\text{proj.dim}({}_A E(A/\mathfrak{m})/(A/\mathfrak{m})) \leq n$, and therefore $\text{proj.dim}({}_\Lambda V) \leq n$. Moreover, $\text{proj.dim}({}_\Lambda E(A/\mathfrak{p})) = \text{proj.dim}({}_A E(A/\mathfrak{p})) \leq n$. Thus $\text{proj.dim}(W) = n+1$.

Third, we prove $\text{inj.dim}({}_\Lambda \Lambda) = n+1$.

Note that $\text{inj.dim}({}_\Lambda E(A/\mathfrak{m})) = 1$ because there is a short exact sequence of Λ-modules:

$$0 \longrightarrow {}_\Lambda E(A/\mathfrak{m}) \longrightarrow V \longrightarrow N \longrightarrow 0, \qquad (\blacklozenge)$$

where both V and N are indecomposable and injective. Since $A_{\mathfrak{m}}$ has global dimension n, it is regular and $\mathfrak{m}$ is of height n, that is, $\mathfrak{m} \in \mathfrak{h}_n$. Based on the form of the minimal injective coresolution of ${}_A A$, one can describe a minimal injective coresolution of ${}_\Lambda A$ as follows:

$$0 \longrightarrow {}_\Lambda A \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} \longrightarrow V \oplus \bigoplus_{\mathfrak{p} \in \mathfrak{h}_n \setminus \{\mathfrak{m}\}} E(A/\mathfrak{p}) \longrightarrow N \longrightarrow 0.$$

This shows $\text{inj.dim}({}_\Lambda A) = n+1$. Since $A_{\mathfrak{m}}$ is regular, the dual of the Koszul complex determined by a regular sequence of $\mathfrak{m}$ provides an injective resolution of ${}_A A/\mathfrak{m}$:

$$0 \longrightarrow {}_A A/\mathfrak{m} \longrightarrow E(A/\mathfrak{m}) \longrightarrow E(A/\mathfrak{m})^{s_1} \longrightarrow E(A/\mathfrak{m})^{s_2} \longrightarrow \cdots \longrightarrow E(A/\mathfrak{m})^{s_n} \longrightarrow 0$$

with $s_i \in \mathbb{N}$ for all $1 \leq i \leq n$. So there is a long exact sequence of Λ-modules:

$$0 \longrightarrow (A/\mathfrak{m}, A/\mathfrak{m}, \text{Id}) \longrightarrow V \longrightarrow E(A/\mathfrak{m})^{s_1} \longrightarrow E(A/\mathfrak{m})^{s_2} \longrightarrow \cdots \longrightarrow E(A/\mathfrak{m})^{s_n} \longrightarrow 0.$$

This sequence together with $(\blacklozenge)$ gives rise to an injective resolution of $(A/\mathfrak{m}, A/\mathfrak{m}, \text{Id})$ as follows:

$$0 \longrightarrow (A/\mathfrak{m}, A/\mathfrak{m}, \text{Id}) \longrightarrow V \longrightarrow W_1 \longrightarrow W_2 \longrightarrow \cdots \longrightarrow W_n \longrightarrow W_{n+1} \longrightarrow 0,$$

where $W_j \in \text{add}(V \oplus N)$ for all $1 \leq j \leq n+1$. This forces $\text{inj.dim}((A/\mathfrak{m}, A/\mathfrak{m}, \text{Id})) \leq n+1$. Consequently, $\text{inj.dim}({}_\Lambda \Lambda) = n+1$ since ${}_\Lambda \Lambda = A \oplus (A/\mathfrak{m}, A/\mathfrak{m}, \text{Id})$.

Note that Λ is left and right noetherian and that $\text{Add}(W)$ is the category of all injective Λ-modules. Since $\text{inj.dim}({}_\Lambda \Lambda) = n+1 = \text{proj.dim}(W)$, it follows from [**17**, Proposition 9.1.6] that Λ is $(n+1)$-Gorenstein. From the injective resolution of ${}_\Lambda \Lambda$ constructed above, it follows that W is a good $(n+1)$-tilting Λ-module. Clearly, it is infinitely generated since V is infinitely generated.

(2) Let ${}_\Lambda M := V \oplus \bigoplus_{\mathfrak{p}\neq\mathfrak{m}} E(A/\mathfrak{p})$. Then $W = M \oplus N$. Moreover, $\Omega_\Lambda(N) = {}_\Lambda A/\mathfrak{m}$, which is finitely generated.

If $n = 1$, then $\mathrm{proj.dim}({}_\Lambda N) = 2$ and $\mathrm{proj.dim}({}_\Lambda M) \leq 1$. In this case, the 2-tilting Λ-module W does satisfy the assumptions of Corollary 1.2(1). Thus W is homological. □

Under the assumptions of Proposition 7.1, let U be the canonical $(n+1)$-tilting Λ-module associated with the Gorenstein ring Λ. Then $\mathrm{add}(U) = \mathrm{add}(W)$. Thus U is homological if and only if W is homological. By Proposition 7.1(2), if $n = 1$, then U is homological. This means that, over non-commutative 2-Gorenstein rings, the canonical tilting modules may be homological.

The following result, however, shows that, over commutative n-Gorenstein rings with $n \geq 2$, the canonical tilting modules are never homological.

Proposition 7.2. *If $n \geq 2$, then ${}_AT$ is not homological.*

Proof. If $\mathfrak{p}$ and $\mathfrak{q}$ are two prime ideals of A, then $\mathrm{Hom}_A(E(A/\mathfrak{p}), E(A/\mathfrak{q})) \neq 0$ if and only if $\mathfrak{p} \subseteq \mathfrak{q}$ (see [**17**, Theorem 3.3.8]). This implies that if $0 \leq i < j \leq n$, then $\mathrm{Hom}_A(E(A/\mathfrak{p}), E(A/\mathfrak{q})) = 0$ for $\mathfrak{p} \in \mathfrak{h}_j$ and $\mathfrak{q} \in \mathfrak{h}_i$, and therefore $\mathrm{Hom}_A(I_j, I_i) = 0$. If we take $T_i = I_i$ for $0 \leq i \leq n$, then the tilting A-module T satisfies the assumptions in Corollary 1.2(2). Thus, by Corollary 1.2(2), ${}_AT$ is not homological since $\mathrm{proj.dim}({}_AT) = n \geq 2$. □

In Proposition 7.2, the subcategory $\mathrm{Ker}({}_AT \otimes^{\mathbb{L}}_B -)$ cannot be realized as the derived module category $\mathscr{D}(C)$ of a ring C with a homological ring epimorphism $B \to C$. Thus tilting modules of higher projective dimension do not have to be homological in general.

For general constructions of homological tilting modules over arbitrary rings, we shall discuss them in a forthcoming paper.

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Hongxing Chen
School of Mathematical Sciences
Capital Normal University
100048 Beijing, China
E-mail: chx19830818@163.com

Changchang Xi
School of Mathematical Sciences
Capital Normal University
100048 Beijing, China
and
School of Mathematics and
Information Science
Henan Normal University
Henan, China
E-mail: xicc@cnu.edu.cn