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Recollements of derived categories I: Construction from exact contexts

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A new method is established to construct recollements of derived categories of rings from exact contexts and noncommutative tensor products. This method is applicable to a large variety of situations, including Milnor squares, localizations, ring extensions and ring epimorphisms.

Contents

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1. Introduction

In 1982 Beilinson, Bernstein and Deligne introduced recollements of triangulated categories in contexts of the derived categories of perverse sheaves over singular spaces, which provide a derived version of Grothendieck's six functors on abelian categories (see [13,4]). Since then, they have been used in algebraic geometry and topology and recently in representation theory. For instance, Happel used recollements of bounded derived module categories to establish reduction techniques for homological conjectures (see [14]). More recently, recollements have become of interest in studying tilting theory of infinitely generated tilting modules and algebraic K-groups of rings (see [1,3,6,8–10]).

To produce recollements of triangulated categories, there are many methods available. For instance, one may take stable categories of Frobenius categories or quotient categories by smashing subcategories. But, to construct recollements of derived module categories of rings seems to be a difficult question, and little is known about it. A known example is the Cline-Parshall-Scott construction: For a stratifying ideal ReR of a ring R generated by an idempotent element e, there is a recollement $(\mathscr{D}(R/ReR), \mathscr{D}(R), \mathscr{D}(eRe))$ of derived module categories of rings $R/ReR, R$ and eRe (see [11]), where $\mathscr{D}(R)$ denotes the (unbounded) derived module category of R. Another example is the recollement from an infinitely generated tilting module of projective dimension at most 1 (see [8]).

The purpose of the present paper is to establish a general method for constructing recollements of derived module categories of rings by homological exact contexts. The advantage of such a construction is that all rings in the recollements can be described explicitly from data of given exact contexts. Consequently, we construct recollements of derived module categories from Milnor squares, localizations, ring extensions and ring epimorphisms.

To state our results more precisely, we introduce a few terminology.

Recall from [5] that an *exact context* is a quadruple (λ, μ, M, m) consisting of two ring homomorphisms $\lambda : R \to S$ and $\mu : R \to T$, an S-T-bimodule M and a fixed element $m \in M$, such that the sequence

$$(*) \quad 0 \longrightarrow R \xrightarrow{(\lambda,\mu)} S \oplus T \xrightarrow{\binom{\cdot m}{-m\cdot}} M \longrightarrow 0$$

is an exact sequence of abelian groups, where $\cdot m$ and $m\cdot$ denote the right and left multiplication maps by m, respectively. An exact context (λ, μ, M, m) is said to be *homological* if $\mathrm{Tor}_i^R(T, S) = 0$ for all $i \geq 1$.

Given an exact context (λ, μ, M, m), we associate it with an unitary (associative) ring $T \boxtimes_R S$, called the *noncommutative tensor product* of (λ, μ, M, m) (see [5] for details), and a triangular matrix ring $\Lambda := \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$. Our main result reads as follows.

Theorem 1.1. *Let (λ, μ, M, m) be a homological exact context with $T \boxtimes_R S$ as its noncommutative tensor product. Then there exists a recollement of the derived module categories of rings:*

$$\mathscr{D}(T \boxtimes_R S) \longrightarrow \mathscr{D}(\Lambda) \longrightarrow \mathscr{D}(R).$$

If the projective dimensions of ${}_RS$ and T_R are finite, then this recollement can be restricted to a recollement of bounded derived module categories:

$$\mathscr{D}^b(T \boxtimes_R S) \longrightarrow \mathscr{D}^b(\Lambda) \longrightarrow \mathscr{D}^b(R).$$

Theorem 1.1 enables us to study categorical and homological properties of R by those of S and T through the related rings Λ and $T \boxtimes_R S$, and vice versa. For instance, this is used in [7] to investigate upper bounds for the global and finitistic dimensions of R by those of S and T.

As a consequence of Theorem 1.1, we obtain recollements of derived module categories from localizations of commutative rings.

Corollary 1.2. *Let R be a commutative ring, Φ a multiplicative subset of R, and $\lambda : R \to S$ the localization of R at Φ. If λ is injective (for example, if R is an integral domain), then there exists a recollement of derived module categories*

$$\xymatrix{\mathscr{D}(\Psi^{-1}S') \ar[r] & \mathscr{D}\big(\mathrm{End}_R(S \oplus S/R)\big) \ar[r] \ar@/_1pc/[l] \ar@/^1pc/[l] & \mathscr{D}(R) \ar@/_1pc/[l] \ar@/^1pc/[l]}$$

where $S' := \mathrm{End}_R(S/R)$ is commutative, Ψ is the image of Φ under the induced map $R \to S'$ by right multiplication and $\Psi^{-1}S'$ is the localization of S' at Ψ.

Recollements in Corollary 1.2 (see also Corollary 4.3) can be used to study infinitely generated tilting modules of projective dimension at most 1 (see [8]).

As another consequence of Theorem 1.1, we construct recollements of derived module categories from ring epimorphisms.

Let $\lambda : R \to S$ be a ring homomorphism. We consider λ as a complex $Q^\bullet$ of left R-modules with R and S in degrees -1 and 0, respectively. Let $S' := \mathrm{End}_{\mathscr{D}(R)}(Q^\bullet)$. Then we have the corollary.

Corollary 1.3. *Suppose that $\lambda : R \to S$ is a homological ring epimorphism with $\mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big) = 0$. If the projective dimension of ${}_RS$ is at most 1, then there exists a recollement of derived module categories:*

$$\xymatrix{\mathscr{D}(\mathrm{End}_{S'}(S' \otimes_R S)) \ar[r] & \mathscr{D}(\mathrm{End}_{\mathscr{D}(R)}(S \oplus Q^\bullet)) \ar[r] \ar@/_1pc/[l] \ar@/^1pc/[l] & \mathscr{D}(R). \ar@/_1pc/[l] \ar@/^1pc/[l]}$$

Corollary 1.3 not only generalizes [8, Corollary 6.6] where λ is required to be injective, but also provides derived equivalences of rings if $S' \otimes_R S$ vanishes (see Corollary 4.9).

Finally, we apply Theorem 1.1 to trivially twisted extensions of finite-dimensional algebras (see [21]).

Let A_1 and A_2 be finite-dimensional algebras over a field k given by the quivers $\Gamma = (\Gamma_0, \Gamma_1)$ with relations $\{\sigma_i \mid i \in I_0\}$, and $\Delta = (\Delta_0 := \Gamma_0, \Delta_1)$ with relations $\{\tau_j \mid j \in J_0\}$, respectively. Then the *trivially twisted extension* A of A_1 and A_2 over $A_0 := k\Gamma_0$ is given by the quiver $Q = (Q_0 := \Gamma_0, Q_1 := \Gamma_1 \dot\cup \Delta_1)$, with the relations $\{\sigma_i \mid i \in I_0\} \cup \{\tau_j | j \in J_0\} \cup \{\alpha\beta \mid \alpha \in \Gamma_1, \beta \in \Delta_1\}$, where $\alpha\beta$ means that α comes first and then β follows.

Let $\lambda : A_0 \to A_1$ and $\mu : A_0 \to A_2$ be the canonical inclusions, and let M be the quotient of A by the ideal generated by $\{\beta\alpha \mid \alpha \in \Gamma_1, \beta \in \Delta_1\}$. Then M is an A_1-A_2-bimodule, $(\lambda, \mu, M, 1)$ is an exact context and its noncommutative tensor product is isomorphic to A (see [5, Example 4.4]). Let $B := \begin{pmatrix} A_1 & M \\ 0 & A_2 \end{pmatrix}$ be the triangular matrix ring. Since A_0 is semisimple, $\mathrm{Tor}_i^{A_0}(A_2, A_1) = 0$ for all $i \geq 1$, and therefore we get the following result for free.

Corollary 1.4. *Let A be the trivially twisted extension of k-algebras A_1 and A_2 over a semisimple k-algebra A_0. Then there exists a recollement of derived module categories:*

$$\xymatrix{\mathscr{D}(A) \ar[r] & \mathscr{D}(B) \ar[r] \ar@/_1pc/[l] \ar@/^1pc/[l] & \mathscr{D}(A_0) \ar@/_1pc/[l] \ar@/^1pc/[l]}$$

which can be restricted to a recollement of bounded derived module categories.

Corollary 1.4 can be used to construct finite-dimensional algebras which have two different derived stratifications with all of their derived simple factors being finite-dimensional algebras (compared with examples in [2,8,10]). For instance, we take A_1 and A_2 to be finite-dimensional local algebras. Moreover, the recollement in Corollary 1.4 can be applied to describe the algebraic K-groups of A in terms of the ones of A_0, A_1 and A_2 (see [6]).

The contents of this paper are outlined as follows. In Section 2, we fix notation and recall a few basic definitions often used in proofs. In Section 3, we prove Theorem 1.1. In Section 4, we construct recollements of derived module categories from homotopy pullback squares of rings, ring extensions and ring epimorphisms.

2. Preliminaries

In this section, we shall fix some notation on derived categories and recall the definitions of recollements and homological ring epimorphisms.

Let $\mathcal{C}$ be an additive category.

Throughout the paper, a full subcategory $\mathcal{B}$ of $\mathcal{C}$ is always assumed to be closed under isomorphisms, that is, if $X \in \mathcal{B}$ and $Y \in \mathcal{C}$ with $Y \simeq X$, then $Y \in \mathcal{B}$.

Given two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, we denote the composite of f and g by fg which is a morphism from X to Z. The induced morphisms $\mathrm{Hom}_{\mathcal{C}}(Z, f) : \mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Z, Y)$ and $\mathrm{Hom}_{\mathcal{C}}(f, Z) : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ are denoted by f^* and f_*, respectively.

We denote the composition of a functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ with a functor $G : \mathcal{D} \to \mathcal{E}$ between categories $\mathcal{D}$ and $\mathcal{E}$ by GF which is a functor from $\mathcal{C}$ to $\mathcal{E}$. The kernel and the image of the functor F are denoted by $\mathrm{Ker}(F)$ and $\mathrm{Im}(F)$, respectively.

Let $\mathscr{C}(\mathcal{C})$ be the category of all complexes over $\mathcal{C}$ with chain maps, and $\mathscr{K}(\mathcal{C})$ the homotopy category of $\mathscr{C}(\mathcal{C})$. When $\mathcal{C}$ is abelian, the derived category of $\mathcal{C}$ is denoted by $\mathscr{D}(\mathcal{C})$, which is the localization of $\mathscr{K}(\mathcal{C})$ at all quasi-isomorphisms. It is well known that both $\mathscr{K}(\mathcal{C})$ and $\mathscr{D}(\mathcal{C})$ are triangulated categories. For a triangulated category, its shift functor is denoted by [1] universally.

In this paper, all rings are associative and with identity, and all ring homomorphisms preserve identity. Unless stated otherwise, all modules are referred to left modules.

Let R be a ring. We denote by R-Mod the category of all unitary R-modules. By our convention of the composite of two morphisms, if $f : M \to N$ is a homomorphism of R-modules, then the image of $x \in M$ under f is denoted by $(x)f$ instead of $f(x)$.

As usual, we shall simply write $\mathscr{C}(R)$, $\mathscr{K}(R)$ and $\mathscr{D}(R)$ for $\mathscr{C}(R\text{-Mod})$, $\mathscr{K}(R\text{-Mod})$ and $\mathscr{D}(R\text{-Mod})$, respectively, and identify R-Mod with the subcategory of $\mathscr{D}(R)$ consisting of all stalk complexes concentrated in degree zero. Further, we denote by $\mathscr{D}^b(R)$ the full subcategory of $\mathscr{D}(R)$ consisting of all complexes which are isomorphic in $\mathscr{D}(R)$ to bounded complexes of R-modules.

Let $(X^\bullet, d_{X^\bullet})$ and $(Y^\bullet, d_{Y^\bullet})$ be two chain complexes over R-Mod. The *mapping cone* of a chain map $h^\bullet : X^\bullet \to Y^\bullet$ is usually denoted by $\mathrm{Con}(h^\bullet)$. The triangle of the form $X^\bullet \xrightarrow{h^\bullet} Y^\bullet \to \mathrm{Con}(h^\bullet) \to X^\bullet[1]$ in $\mathscr{K}(R)$ is called a *distinguished triangle*.

For each $n \in \mathbb{Z}$, the n-th cohomology functor from $\mathscr{D}(R)$ to R-Mod is denoted by $H^n(-)$.

Let $M^\bullet$ be a complex of R-S-bimodules. Then the tensor functor $M^\bullet \otimes_S^\bullet - : \mathscr{K}(S) \to \mathscr{K}(R)$ and the Hom-functor $\mathrm{Hom}_R^\bullet(M^\bullet, -) : \mathscr{K}(R) \to \mathscr{K}(S)$ form a pair of adjoint triangle functors. Denote by $M^\bullet \otimes_S^{\mathbb{L}} -$ the total left-derived functor of $M^\bullet \otimes_S^\bullet -$, and by $\mathbb{R}\mathrm{Hom}_R(M^\bullet, -)$ the total right-derived functor of $\mathrm{Hom}_R^\bullet(M^\bullet, -)$. Note that $\big(M^\bullet \otimes_S^{\mathbb{L}} -, \mathbb{R}\mathrm{Hom}_R(M^\bullet, -)\big)$ is an adjoint pair of triangle functors.

If a triangle functor $H : \mathscr{K}(R) \to \mathscr{K}(S)$ preserves acyclicity, that is, $H(X^\bullet)$ is acyclic whenever $X^\bullet$ is acyclic, then H induces a triangle functor $D(H) : \mathscr{D}(R) \to \mathscr{D}(S)$ defined by $X^\bullet \mapsto H(X^\bullet)$. In this case, $D(H)$ is called the *derived functor* of H.

Now, we recall the notion of recollements of triangulated categories, which was first defined in [4] to study "exact sequences" of derived categories of perverse sheaves over geometric objects.

Definition 2.1. Let $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories. We say that $\mathcal{D}$ is a *recollement* of $\mathcal{D}'$ and $\mathcal{D}''$ (or there is a recollement $(\mathcal{D}'', \mathcal{D}, \mathcal{D}')$) if there are six triangle functors among the three categories:

$$\mathcal{D}'' \;\underset{\overset{\longleftarrow}{i^!}}{\overset{\overset{i^*}{\longleftarrow}}{\xrightarrow{\,i_*=i_!\,}}}\; \mathcal{D} \;\underset{\overset{\longleftarrow}{j_*}}{\overset{\overset{j_!}{\longleftarrow}}{\xrightarrow{\,j^!=j^*\,}}}\; \mathcal{D}'$$

such that

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs,

(2) i_*, j_* and $j_!$ are fully faithful functors,

(3) $i^! j_* = 0$ (and thus also $j^! i_! = 0$ and $i^* j_! = 0$), and

(4) for each object $X \in \mathcal{D}$, there are triangles $i_! i^!(X) \to X \to j_* j^*(X) \to i_! i^!(X)[1]$ and $j_! j^!(X) \to X \to i_* i^*(X) \to j_! j^!(X)[1]$ in $\mathcal{D}$.

A prominent example of recollements of triangulated categories is given by homological ring epimorphisms. Let $\lambda : R \to S$ be a homomorphism of rings. Then the restriction functor $\lambda_* : S\text{-Mod} \to R\text{-Mod}$ induced by λ gives rise to a triangle functor $D(\lambda_*) : \mathscr{D}(S) \to \mathscr{D}(R)$. If λ is a *homological* ring epimorphism, that is, a ring epimorphism with $\mathrm{Tor}_n^R(S, S) = 0$ for all $n \geq 1$ (see [12]), then $D(\lambda_*)$ is fully faithful and there exists a recollement $\big(\mathscr{D}(S), \mathscr{D}(R), \mathrm{Tria}({}_RQ^\bullet)\big)$ of triangulated categories (see [16, Section 4]), where $Q^\bullet$ is the mapping cone of λ and $\mathrm{Tria}({}_RQ^\bullet)$ stands for the smallest full triangulated subcategory of $\mathscr{D}(R)$ containing $Q^\bullet$ and being closed under coproducts. In general, $\mathrm{Tria}({}_RQ^\bullet)$ may not be equivalent to the derived module category of a ring.

Equivalences of triangulated (or derived) categories are extreme cases of recollements, that is, either $\mathcal{D}' = 0$ or $\mathcal{D}'' = 0$. Recall from [18] that a bounded complex $U^\bullet$ of finitely generated projective R-modules is called a *tilting complex* over R if $U^\bullet$ is self-orthogonal (that is, $\mathrm{Hom}_{\mathscr{D}(R)}(U^\bullet, U^\bullet[n]) = 0$ for any $n \neq 0$) and $\mathrm{Tria}(U^\bullet) = \mathscr{D}(R)$. For a tilting complex $U^\bullet$ over R, $\mathscr{D}(R)$ is equivalent to $\mathscr{D}(\mathrm{End}_{\mathscr{D}(R)}(U^\bullet))$ as triangulated categories (see [18, Theorem 6.4]). In this case, R and $\mathrm{End}_{\mathscr{D}(R)}(U^\bullet)$ are said to be *derived equivalent*. For advances in constructing derived equivalences of algebras, we refer the reader to [22].

3. Exact contexts and recollements of derived module categories

This section is devoted to the proof of Theorem 1.1.

We assume that $(\lambda : R \to S, \mu : R \to T, M, m)$ is an exact context. Recall that its noncommutative tensor product $T \boxtimes_R S$ has $T \otimes_R S$ as the underlying abelian group, and its multiplication $\circ$ is defined: for any $(t_i, s_i) \in T \times S$ with $i = 1, 2$,

$$(t_1 \otimes s_1) \circ (t_2 \otimes s_2) := t_1\big(1 \otimes x + y \otimes 1\big)s_2,$$

where $(x, y) \in S \oplus T$ is chosen such that $s_1 m t_2 = xm + my$. For more details, we refer to [5].

An exact context (λ, μ, M, m) is called an *exact pair* if $M = S \otimes_R T$ and $m = 1 \otimes 1$. The following result is shown in [5, Corollary 5.4 and Lemma 3.7].

Lemma 3.1. (1) *If λ is a ring epimorphism, then (λ, μ) is an exact pair and $T \boxtimes_R S \simeq \mathrm{End}_T(T \otimes_R S)$ as rings.*

(2) *Let R be a commutative ring, and let S and T be R-algebras via λ and μ, respectively. If (λ, μ) is an exact pair, then $T \boxtimes_R S$ coincides with the usual tensor product $T \otimes_R S$ of R-algebras T and S.*

Let $\rho = \mu \otimes S : \; S \to T \otimes_R S,\; s \mapsto 1 \otimes s$ and $\phi = T \otimes \lambda : \; T \to T \otimes_R S,\; t \mapsto t \otimes 1$ for $s \in S$ and $t \in T$, and let

$$\beta : M \longrightarrow T \otimes_R S,\ x \mapsto 1 \otimes s_x + t_x \otimes 1,$$

where $x \in M$ and $(s_x, t_x) \in S \oplus T$ is chosen such that $x = s_x m + m t_x$. We define

$$\Lambda := \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}, \quad \Gamma := \begin{pmatrix} T \boxtimes_R S & T \boxtimes_R S \\ T \boxtimes_R S & T \boxtimes_R S \end{pmatrix}, \quad \theta := \begin{pmatrix} \rho & \beta \\ 0 & \phi \end{pmatrix} : \ \Lambda \longrightarrow \Gamma.$$

Then θ is a ring homomorphism and $\lambda\rho = \mu\phi$. Furthermore, let

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \Lambda \quad \text{and} \quad \varphi : \Lambda e_1 \longrightarrow \Lambda e_2,\ \begin{pmatrix} s \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} sm \\ 0 \end{pmatrix} \quad \text{for} \ \ s \in S.$$

Then φ is a homomorphism of finitely generated projective Λ-modules. Clearly, Λe_1 and Λe_2 are right R-modules via λ and μ, respectively, and the map $\cdot m : S \to M$ is a homomorphism of S-R-bimodules. Thus φ is a homomorphism of Λ-R-bimodules.

From now on, let $P^\bullet$ be the complex $0 \to \Lambda e_1 \xrightarrow{\varphi} \Lambda e_2 \to 0$ in $\mathscr{C}(\Lambda)$ with Λe_1 and Λe_2 in degrees -1 and 0, respectively, that is $P^\bullet = \mathrm{Con}(\varphi)$. Then $P^{\bullet *} := \mathrm{Hom}^\bullet_\Lambda(P^\bullet, \Lambda)$ is given by $0 \to e_2\Lambda \xrightarrow{\varphi_*} e_1\Lambda \to 0$ in $\mathscr{C}(\Lambda^{op})$ with $e_2\Lambda$ and $e_1\Lambda$ in degrees 0 and 1, respectively. Moreover, $P^\bullet$ is a bounded complex over $\Lambda \otimes_{\mathbb{Z}} R^{\mathrm{op}}$, and there is a distinguished triangle $\Lambda e_1 \xrightarrow{\varphi} \Lambda e_2 \to P^\bullet \to \Lambda e_1[1]$ in $\mathscr{K}(\Lambda \otimes_{\mathbb{Z}} R^{\mathrm{op}})$.

According to [5, Theorem 1.1(1)], $\theta : \Lambda \to \Gamma$ is a universal localization, and thus a ring epimorphism. This implies that $\theta_* : \Gamma\text{-Mod} \to \Lambda\text{-Mod}$ is fully faithful. Now, we regard Γ-Mod as a full subcategory of Λ-Mod and define a full subcategory of $\mathscr{D}(\Lambda)$,

$$\mathscr{D}(\Lambda)_\Gamma := \{X^\bullet \in \mathscr{D}(\Lambda) \mid H^n(X^\bullet) \in \Gamma\text{-Mod for all } n \in \mathbb{Z}\}.$$

Lemma 3.2. (1) *There exists a recollement of triangulated categories:*

$$(\star) \qquad \mathscr{D}(\Lambda)_\Gamma \;\underset{\overset{i^*}{\longleftarrow}\;\underset{i^!}{\longleftarrow}}{\xrightarrow{\;i_*\;}}\; \mathscr{D}(\Lambda) \;\underset{\overset{j_!}{\longleftarrow}\;\underset{j_*}{\longleftarrow}}{\xrightarrow{\;j^!\;}}\; \mathscr{D}(R)$$

where i_* is the embedding, $j_! := {}_\Lambda P^\bullet \otimes^{\mathbb{L}}_R -,\ \ j^! := \mathrm{Hom}^\bullet_\Lambda(P^\bullet, -) \simeq {}_R P^{\bullet *} \otimes^\bullet_\Lambda -$, and $j_* := \mathbb{R}\mathrm{Hom}_R(P^{\bullet *}, -)$.

(2) *The map $\theta : \Lambda \to \Gamma$ is homological if and only if $\mathrm{Tor}^R_i(T, S) = 0$ for all $i \geq 1$. In this case, $\mathscr{D}(\Gamma)$ is equivalent to $\mathscr{D}(\Lambda)_\Gamma$ as triangulated categories.*

(3) *If λ is a homological ring epimorphism, then $\mathrm{Tor}^R_i(S, T) = 0$ for all $i \geq 1$ and there are isomorphisms $i_* i^*(\Lambda e_1) \simeq i_* i^*(\Lambda e_2) \simeq \Lambda e_2 \otimes^{\mathbb{L}}_R S$ in $\mathscr{D}(\Lambda)$.*

Proof. (1) Since $P^\bullet$ is a bounded complex of finitely generated projective Λ-modules, the functor $\mathrm{Hom}^\bullet_\Lambda(P^\bullet, -) : \mathscr{K}(\Lambda) \to \mathscr{K}(R)$ preserves acyclicity. Then $\mathbb{R}\mathrm{Hom}_\Lambda(P^\bullet, -) : \mathscr{D}(\Lambda) \to \mathscr{D}(R)$ can be replaced by $\mathrm{Hom}^\bullet_\Lambda(P^\bullet, -)$ up to natural isomorphism. Moreover, there is a natural isomorphism $P^{\bullet *} \otimes^\bullet_\Lambda - \xrightarrow{\simeq} \mathrm{Hom}^\bullet_\Lambda(P^\bullet, -) : \mathscr{C}(\Lambda) \to \mathscr{C}(R)$. This implies that $P^{\bullet *} \otimes^\bullet_\Lambda -$ also preserves acyclicity. Consequently, the derived functors $P^{\bullet *} \otimes^\bullet_\Lambda -$ and $P^{\bullet *} \otimes^{\mathbb{L}}_\Lambda -$ from $\mathscr{D}(\Lambda)$ to $\mathscr{D}(R)$ are naturally isomorphic. Thus both $(j_!, j^!)$ and $(j^!, j_*)$ are adjoint pairs. Now, the recollement $(\star)$ follows from [5, Lemma 5.5(3)].

(2) The first statement is [5, Theorem 1.1(2)], while the equivalence follows from [8, Proposition 3.6(b)].

(3) Since λ is homological, it follows from [12, Theorem 4.4] that λ induces an isomorphism $S \otimes^{\mathbb{L}}_R \lambda : S \xrightarrow{\simeq} S \otimes^{\mathbb{L}}_R S$ in $\mathscr{D}(S)$. This implies $S \otimes^{\mathbb{L}}_R \mathrm{Con}(\lambda) = 0$ in $\mathscr{D}(S)$. By the exact sequence $(*)$ in the introduction, $\mathrm{Con}(\lambda) \simeq \mathrm{Con}(m\cdot)$ in $\mathscr{D}(R)$. Consequently, $S \otimes^{\mathbb{L}}_R \mathrm{Con}(m\cdot) = 0$ in $\mathscr{D}(S)$. Now, we apply $S \otimes^{\mathbb{L}}_R - : \mathscr{D}(R) \to \mathscr{D}(S)$ to the triangle $T \xrightarrow{m\cdot} M \to \mathrm{Con}(m\cdot) \to T[1]$ and obtain an isomorphism $S \otimes^{\mathbb{L}}_R T \simeq S \otimes^{\mathbb{L}}_R M$ in $\mathscr{D}(S)$.

Since $M \in S$-Mod and λ is homological, there is a canonical isomorphism $S \otimes_R^{\mathbb{L}} M \simeq M$ in $\mathscr{D}(S)$. Thus $S \otimes_R^{\mathbb{L}} T \simeq M$ in $\mathscr{D}(S)$. This shows $\mathrm{Tor}_i^R(S,T) = 0$ for all $i \geq 1$.

The isomorphism $i_*i^*(\Lambda e_1) \simeq i_*i^*(\Lambda e_2)$ in $\mathscr{D}(\Lambda)$ follows from $i^*(P^\bullet) \simeq i^*j_!(R) = 0$. Moreover, since λ is homological, the multiplication map $S \otimes_R S \to S$ induces an isomorphism $S \otimes_R^{\mathbb{L}} S \xrightarrow{\simeq} S$ in $\mathscr{D}(S)$. This implies that $\Lambda e_1[1] \otimes_R^{\mathbb{L}} \mathrm{Hom}_\Lambda(P^\bullet, \Lambda e_1) \simeq \Lambda e_1$ in $\mathscr{D}(\Lambda)$ since $\Lambda e_1 = S$ and $\mathrm{Hom}_\Lambda(P^\bullet, \Lambda e_1) \simeq S[-1]$. Now, the isomorphism $i_*i^*(\Lambda e_1) \simeq \Lambda e_2 \otimes_R^{\mathbb{L}} S$ in $\mathscr{D}(\Lambda)$ can be concluded from the proof of [5, Lemma 5.8]. $\square$

Corollary 3.3. *Let (λ, μ, M, m) be a homological exact context. Then there exists a recollement of derived module categories*

$$(\lozenge) \qquad \mathscr{D}(\Gamma) \underset{i^!}{\overset{i^*}{\leftleftarrows}} \xrightarrow{i_*} \mathscr{D}(\Lambda) \underset{j_*}{\overset{j_!}{\leftleftarrows}} \xrightarrow{j^!} \mathscr{D}(R)$$

with six triangle functors given by

$$i^* := \Gamma \otimes_\Lambda^{\mathbb{L}} -, \quad i_* := D(\theta_*), \quad i^! = \mathbb{R}\mathrm{Hom}_\Lambda(\Gamma, -),$$
$$j_! := {}_\Lambda P^\bullet \otimes_R^{\mathbb{L}} -, \quad j^! := \mathrm{Hom}_\Lambda^\bullet(P^\bullet, -) \simeq {}_R P^{\bullet *} \otimes_\Lambda^\bullet -, \quad j_* := \mathbb{R}\mathrm{Hom}_R(P^{\bullet *}, -).$$

To prove Theorem 1.1, we describe relations among projective dimensions of modules over rings related by exact contexts. For an R-module X, the projective dimension of X is denoted by $\mathrm{pdim}({}_RX)$.

Lemma 3.4. *Let (λ, μ, M, m) be a homological exact context. Then*

(1) $\mathrm{pdim}({}_RS) \leq \max\{1, \mathrm{pdim}({}_\Lambda\Gamma)\}$ *and* $\mathrm{pdim}({}_\Lambda\Gamma) \leq \max\{2, \mathrm{pdim}({}_RS) + 1\}$. *In particular,* $\mathrm{pdim}({}_RS) < \infty$ *if and only if* $\mathrm{pdim}({}_\Lambda\Gamma) < \infty$.

(2) $\mathrm{pdim}(T_R) \leq \max\{1, \mathrm{pdim}(\Gamma_\Lambda)\}$ *and* $\mathrm{pdim}(\Gamma_\Lambda) \leq \max\{2, \mathrm{pdim}(T_R) + 1\}$. *In particular,* $\mathrm{pdim}(T_R) < \infty$ *if and only if* $\mathrm{pdim}(\Gamma_\Lambda) < \infty$.

Proof. Note that the ring homomorphisms $\mu^{\mathrm{op}} : R^{\mathrm{op}} \to T^{\mathrm{op}}$ and $\lambda^{\mathrm{op}} : R^{\mathrm{op}} \to S^{\mathrm{op}}$, together with (M, m) form an exact context. So it is sufficient to show (1) because (2) can be shown similarly.

We first show $\mathrm{pdim}({}_RS) \leq \max\{1, \mathrm{pdim}({}_\Lambda\Gamma)\}$.

For this aim, we use the recollement given in Corollary 3.3. Clearly, there is a triangle $P^\bullet \otimes_R^{\mathbb{L}} P^{\bullet *} \to \Lambda \xrightarrow{\theta} \Gamma \to P^\bullet \otimes_R^{\mathbb{L}} P^{\bullet *}[1]$ in $\mathscr{D}(\Lambda)$. This implies that $\mathrm{Con}(\theta)$ is isomorphic in $\mathscr{D}(\Lambda)$ to the complex $P^\bullet \otimes_R^{\mathbb{L}} P^{\bullet *}[1]$. By $(*)$ in the introduction, we have $\mathrm{Con}(\lambda) \simeq \mathrm{Con}(m\cdot)$ in $\mathscr{D}(R)$. Consequently, $P^{\bullet *}[1] \simeq S \oplus \mathrm{Con}(m\cdot) \simeq S \oplus \mathrm{Con}(\lambda)$ in $\mathscr{D}(R)$ and $\mathrm{Con}(\theta) \simeq P^\bullet \otimes_R^{\mathbb{L}} S \oplus P \otimes_R^{\mathbb{L}} \mathrm{Con}(\lambda)$ in $\mathscr{D}(\Lambda)$. Since $P^\bullet \otimes_R^{\mathbb{L}} - : \mathscr{D}(R) \to \mathscr{D}(\Lambda)$ is a fully faithful functor, $\mathrm{Hom}_{\mathscr{D}(R)}(S, Y[n]) \simeq \mathrm{Hom}_{\mathscr{D}(\Lambda)}(P^\bullet \otimes_R^{\mathbb{L}} S, P^\bullet \otimes_R^{\mathbb{L}} Y[n])$ for every $Y \in R$-Mod and $n \in \mathbb{N}$.

Suppose $\mathrm{pdim}({}_\Lambda\Gamma) < \infty$ and define $s := \max\{1, \mathrm{pdim}({}_\Lambda\Gamma)\}$. To show $\mathrm{pdim}({}_RS) \leq s$, we shall prove $\mathrm{Hom}_{\mathscr{D}(R)}(S, Y[n]) = 0$ for any $n > s$ and $Y \in R$-Mod. Since $P^\bullet \otimes_R^{\mathbb{L}} S$ is a direct summand of $\mathrm{Con}(\theta)$ in $\mathscr{D}(\Lambda)$, it is enough to show $\mathrm{Hom}_{\mathscr{D}(\Lambda)}(\mathrm{Con}(\theta), P^\bullet \otimes_R^{\mathbb{L}} Y[n]) = 0$ for any $n > s$.

Recall that $\mathrm{Con}(\theta)$ is the complex $0 \to \Lambda \xrightarrow{\theta} \Gamma \to 0$ with Λ and Γ in degrees -1 and 0, respectively. Then $\mathrm{Con}(\theta)$ is isomorphic in $\mathscr{D}(\Lambda)$ to a bounded complex

$$X^\bullet : \ 0 \longrightarrow X^{-s} \longrightarrow X^{1-s} \longrightarrow \cdots \longrightarrow X^{-1} \longrightarrow X^0 \longrightarrow 0$$

such that X^i are projective Λ-modules for all $0 \leq i \leq s$. Let ${}_pY$ be a deleted projective resolution of ${}_RY$. Then

$$\begin{aligned}\mathrm{Hom}_{\mathscr{D}(\Lambda)}(\mathrm{Con}(\theta), P^\bullet \otimes^{\mathbb{L}}_R Y[n]) &\simeq \mathrm{Hom}_{\mathscr{D}(\Lambda)}(X^\bullet, P^\bullet \otimes^{\mathbb{L}}_R Y[n]) \\ &= \mathrm{Hom}_{\mathscr{D}(\Lambda)}(X^\bullet, P^\bullet \otimes^\bullet_R ({}_pY)[n]) \\ &\simeq \mathrm{Hom}_{\mathscr{K}(\Lambda)}(X^\bullet, P^\bullet \otimes^\bullet_R ({}_pY)[n]) \\ &= 0\end{aligned}$$

for any $n > s$, where the last equality follows from the fact that all terms in positive degrees of the complex $P^\bullet \otimes^\bullet_R ({}_pY)$ are 0. This shows $\mathrm{pdim}({}_RS) \le s$.

Next, we show $\mathrm{pdim}({}_\Lambda\Gamma) \le \max\{2, \mathrm{pdim}({}_RS)+1\}$.

Suppose $\mathrm{pdim}({}_RS) = m < \infty$, and let

$$M^\bullet : 0 \longrightarrow M^{-m} \longrightarrow M^{1-m} \longrightarrow \cdots \longrightarrow M^{-1} \longrightarrow M^0 \longrightarrow 0$$

be a deleted projective resolution of ${}_RS$, where M^i are projective R-modules for all $-m \le i \le 0$. Then $P^\bullet \otimes^{\mathbb{L}}_R S = P^\bullet \otimes^\bullet_R M^\bullet$ in $\mathscr{D}(\Lambda)$. Note that $\mathrm{Con}(\lambda)$ is isomorphic in $\mathscr{D}(R)$ to a complex of the form:

$$\widetilde{M^\bullet} : 0 \longrightarrow M^{-m} \longrightarrow M^{1-m} \longrightarrow \cdots \longrightarrow M^{-1} \oplus R \longrightarrow M^0 \longrightarrow 0.$$

In particular, $\widetilde{M^i} = 0$ for $i > 0$ or $i < -\max\{1, m\}$. Then $P^\bullet \otimes^{\mathbb{L}}_R \mathrm{Con}(\lambda) = P^\bullet \otimes^\bullet_R \widetilde{M^\bullet}$ in $\mathscr{D}(\Lambda)$. Thus $\mathrm{Con}(\theta) \simeq (P^\bullet \otimes^\bullet_R M^\bullet) \oplus (P^\bullet \otimes^\bullet_R \widetilde{M^\bullet})$ in $\mathscr{D}(\Lambda)$. Recall that $P^\bullet$ is the two-term complex $0 \to \Lambda e_1 \xrightarrow{\varphi} \Lambda e_2 \to 0$ with Λe_1 and Λe_2 in degrees -1 and 0, respectively. This implies that $\mathrm{Con}(\theta)$ is isomorphic in $\mathscr{D}(\Lambda)$ to a complex of the form:

$$N^\bullet : 0 \longrightarrow N^{-t} \longrightarrow N^{1-t} \longrightarrow \cdots \longrightarrow N^{-1} \longrightarrow N^0 \longrightarrow 0,$$

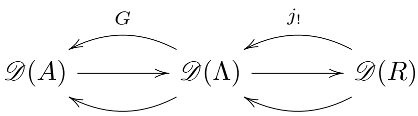
where $t := \max\{2, m+1\}$ and N^i is a projective Λ-module for all $-t \le i \le 0$. Now, let $X \in \Lambda\text{-Mod}$. Then

$$\mathrm{Hom}_{\mathscr{D}(\Lambda)}(\mathrm{Con}(\theta), X[n]) \simeq \mathrm{Hom}_{\mathscr{D}(\Lambda)}(N^\bullet, X[n]) \simeq \mathrm{Hom}_{\mathscr{K}(\Lambda)}(N^\bullet, X[n]) = 0$$

for any $n > t$. Applying $\mathrm{Hom}_{\mathscr{D}(\Lambda)}(-, X[n])$ to the triangle $\mathrm{Con}(\theta)[-1] \to \Lambda \xrightarrow{\theta} \Gamma \to \mathrm{Con}(\theta)$ in $\mathscr{D}(\Lambda)$, we have $\mathrm{Ext}^n_\Lambda(\Gamma, X) \simeq \mathrm{Hom}_{\mathscr{D}(\Lambda)}(\Gamma, X[n]) = 0$ for any $n > t$. This shows $\mathrm{pdim}({}_\Lambda\Gamma) \le t$. □

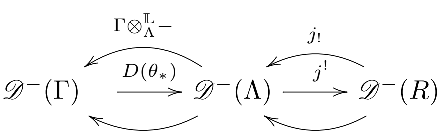
Proof of Theorem 1.1. Let $A := T \boxtimes_R S$. Then $\Gamma = M_2(A)$ and the A-Γ-bimodule (A, A) induces a Morita equivalence $(A, A) \otimes_\Gamma - : \Gamma\text{-Mod} \to A\text{-Mod}$, which gives a triangle equivalence $\mathscr{D}(\Gamma) \xrightarrow{\simeq} \mathscr{D}(A)$.

Since (λ, μ, M, m) is a homological exact context, it follows from Corollary 3.3 that there exists a recollement of derived module categories:



where $G := (A, A) \otimes^{\mathbb{L}}_\Lambda -$ and $j_! = {}_\Lambda P^\bullet \otimes^{\mathbb{L}}_R -$. This shows the first part of Theorem 1.1.

By [17, Theorem 3], the recollement in Corollary 3.3 restricts to a recollement at $\mathscr{D}^-$-level



if and only if ${}_\Lambda\Gamma$ is isomorphic to a bounded complex of projective Λ-modules, that is $\mathrm{pdim}({}_\Lambda\Gamma) < \infty$. Moreover, this $\mathscr{D}^-$-level recollement restricts to a recollement $\big(\mathscr{D}^b(\Gamma), \mathscr{D}^b(\Lambda), \mathscr{D}^b(R)\big)$ of bounded derived categories if $\mathrm{pdim}(\Gamma_\Lambda) < \infty$. However, by Corollary 3.4, $\mathrm{pdim}({}_\Lambda\Gamma) < \infty$ if and only if $\mathrm{pdim}({}_R S) < \infty$, while $\mathrm{pdim}(\Gamma_\Lambda) < \infty$ if and only if $\mathrm{pdim}(T_R) < \infty$. Identifying $\mathscr{D}^b(\Gamma)$ with $\mathscr{D}^b(A)$ up to equivalence, we have shown the second part of Theorem 1.1. □

4. Constructions of recollements of derived module categories

As consequences of Theorem 1.1, we establish concrete constructions of recollements of derived categories for algebras and rings.

4.1. From ring constructions

In this subsection, we apply Theorem 1.1 to construct recollements of derived categories of rings from pullbacks of rings and strictly pure extensions. In the course, we also prove Corollary 1.2.

A commutative diagram of ring homomorphisms

$$(\natural): \quad \begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ R_2 & \xrightarrow{j_2} & R' \end{array}$$

is called a *homotopy pullback square* of rings if the sequence $0 \to R \xrightarrow{(i_1,\, i_2)} R_1 \oplus R_2 \xrightarrow{\left(\begin{smallmatrix} j_1 \\ -j_2 \end{smallmatrix}\right)} R' \to 0$ is exact as abelian groups. If j_1 (or equivalently, i_2) is surjective, then $(\natural)$ is called a *Milnor square*. In this case, Milnor showed in [15] that there is a Mayer-Vietoris sequence of algebraic K-groups, that is, a 6-term exact sequence of algebraic K-groups starting from K_1-groups.

Given the square $(\natural)$, the ring R' is an R_1-R_2-bimodule and the quadruple $(i_1, i_2, R', 1)$ is an exact context, where 1 denotes the identity of R'. Let $\Lambda := \begin{pmatrix} R_1 & R' \\ 0 & R_2 \end{pmatrix}$.

Corollary 4.1. *Suppose that* $\mathrm{Coker}(i_2) \otimes_R \mathrm{Coker}(i_1) = 0$ *and* $\mathrm{Tor}_i^R(R_2, R_1) = 0$ *for all* $i \geq 1$. *Then there exists a recollement* $\big(\mathscr{D}(R'), \mathscr{D}(\Lambda), \mathscr{D}(R)\big)$ *of derived module categories.*

Proof. By Theorem 1.1, it suffices to show that $\mathrm{Coker}(i_2) \otimes_R \mathrm{Coker}(i_1) = 0$ implies $R_2 \boxtimes_R R_1 \simeq R'$ as rings.

Since $(i_1, i_2, R', 1)$ is an exact context, it follows from [5, Proposition 5.3] that its noncommutative tensor product $A := R_2 \boxtimes_R R_1$ satisfies the universal property:

Let (B, f_1, f_2, f') be a quadruple with B a ring, $f_1 : R_1 \to B$ and $f_2 : R_2 \to B$ ring homomorphisms and $f' : R' \to B$ a homomorphism of abelian groups such that $i_1 f_1 = i_2 f_2$ and $((r_1)j_1(r_2)j_2)f' = (r_1)f_1(r_2)f_2$ for $r_1 \in R_1$ and $r_2 \in R_2$. Then there is a unique ring homomorphism $\sigma : A \to B$ such that $(f_1, f_2, f') = (\rho, \phi, \beta)\sigma$, where $\rho : R_1 \to A, \phi : R_2 \to A$ and $\beta : R' \to A$ are defined by the exact context $(i_1, i_2, R', 1)$ (see Section 3).

Now, we consider the quadruple $(R', j_1, j_2, \mathrm{id}_{R'})$, where $\mathrm{id}_{R'}$ is the identity map of R', and obtain a unique ring homomorphism $\sigma : A \to R'$ such that $(j_1, j_2, \mathrm{id}_{R'}) = (\rho, \phi, \beta)\sigma$. Clearly, $(i_2, i_1, R', 1)$ forms an exact context. Since $\mathrm{Coker}(i_2) \otimes_R \mathrm{Coker}(i_1) = 0$, we see from [5, Lemma 3.8] that (i_2, i_1) is an exact pair, that is, $(i_2, i_1, R_2 \otimes_R R_1, 1 \otimes 1)$ is an exact context. Consequently, the equality $(j_2, j_1) = (\phi, \rho)\sigma$ implies that σ is an isomorphism. Thus $A \simeq R'$ as rings. □

The following result provides a sufficient condition for Corollary 4.1 to hold true.

Lemma 4.2. *If $i_1 : R \to R_1$ in ($\natural$) is a homological ring epimorphism, then* $\mathrm{Coker}(i_2) \otimes_R \mathrm{Coker}(i_1) = 0$ *and* $\mathrm{Tor}_i^R(R_2, R_1) = 0$ *for all* $i \geq 1$.

Proof. Suppose that i_1 is homological. In particular, i_1 is a ring epimorphism. Since $(i_2, i_1, R', 1)$ is an exact context, we see from [5, Corollary 3.10] that (i_2, i_1) is an exact pair. Further, $\mathrm{Coker}(i_2) \otimes_R \mathrm{Coker}(i_1) = 0$ by [5, Lemma 3.8]. Since λ is homological, $i_1 \otimes^{\mathbb{L}}_R R_1 : R \otimes^{\mathbb{L}}_R R_1 \to R_1 \otimes^{\mathbb{L}}_R R_1$ is an isomorphism. It follows from the square ($\natural$) that $j_2 \otimes^{\mathbb{L}}_R R_1 : R_2 \otimes^{\mathbb{L}}_R R_1 \to R' \otimes^{\mathbb{L}}_R R_1$ is an isomorphism. Since R' can be regarded as a right R_1-module via j_1, the multiplication map $R' \otimes_R R_1 \to R'$ induces an isomorphism $R' \otimes^{\mathbb{L}}_R R_1 \simeq R'$ in $\mathscr{D}(R_1^{\mathrm{op}})$. Thus $R_2 \otimes^{\mathbb{L}}_R R_1 \simeq R'$. This implies $\mathrm{Tor}_n^R(R_2, R_1) = 0$ for all $n \geq 1$. $\square$

Combining Corollary 4.1 with Lemma 4.2, we have the corollary.

Corollary 4.3. *Let $\lambda : R \subseteq S$ be an extension of rings. Define $T := \mathrm{End}_R(S/R)$ and $\Lambda := \mathrm{End}_R(S \oplus S/R)$. Suppose that R is commutative and λ is a homological ring epimorphism. Then there exists a recollement $\big(\mathscr{D}(T \otimes_R S), \mathscr{D}(\Lambda), \mathscr{D}(R)\big)$ of derived module categories, where $T \otimes_R S$ is the usual tensor product of R-algebras.*

Proof. Let $\mu : R \to T$ be the right multiplication map defined by $r \mapsto (x \mapsto xr)$ for any $r \in R$ and $x \in S/R$. Since R is commutative and λ is a ring epimorphism, S is commutative. Further, since λ is injective and $\mathrm{Tor}_1^R(S, S) = 0$, it follows from [8, Lemma 6.5 and Lemma 6.4(2)] that the pair (λ, μ) is exact, T is commutative and $\Lambda \simeq \begin{pmatrix} S & S \otimes_R T \\ 0 & T \end{pmatrix}$ as rings. In particular, both S and T are R-algebras via λ and μ, respectively. By Lemma 3.1(2), $T \boxtimes_R S \simeq T \otimes_R S \simeq S \otimes_R T$ as rings. Thus the quadruple $(\lambda, \mu, \rho, \phi)$ forms a homotopy pullback square of commutative rings. Since λ is homological, the existence of the recollement is a consequence of Lemma 4.2 and Corollary 4.1. $\square$

Proof of Corollary 1.2. Since S is the localization of R at Φ, the module ${}_R S$ is always flat. Thus λ is a homological ring epimorphism. By Corollary 4.3, it suffices to show that $S' \otimes_R S$ is isomorphic to $\Psi^{-1} S'$.

We consider the following well-defined map

$$f : \; S' \otimes_R S \longrightarrow \Psi^{-1} S', \;\; y \otimes \frac{r}{x} \mapsto \frac{(r)\lambda' \, y}{(x)\lambda'}$$

where $y \in S'$, $r \in R$ and $x \in \Phi$. Clearly, this map preserves multiplication and is surjective. It is injective because the map

$$g : \; \Psi^{-1} S' \longrightarrow S' \otimes_R S, \;\; \frac{y}{(x)\lambda'} \mapsto y \otimes \frac{1}{x}$$

for $y \in S'$ and $x \in \Phi$, is well defined and satisfies the equality $fg = 1$. Thus f is an isomorphism of rings. This implies $S' \otimes_R S \simeq \Psi^{-1} S'$. $\square$

Finally, we apply Theorem 1.1 to strictly pure extensions of rings. For the convenience of the reader, we first recall some definitions from [20,5] about pure extensions.

An extension $R \subseteq C$ of rings is said to be *pure* if there exists a splitting $C = R \oplus X$ of R-R-bimodules. A pure extension $R \subseteq C$ is said to be *strictly pure* if the R-R-bimodule X is even a C-C-bimodule, that is, X is an ideal of C and the composite of the inclusion $R \to C$ with the canonical surjection $C \to C/X$ is the identity map.

Let $\alpha : R \to C$ and $\beta : R \to D$ be strictly pure extensions with split decompositions $C = R \oplus X$ and $D = R \oplus Y$ as R-R-bimodules. Then (α, β) can be completed into an exact context in the following way:

Let $M = R \oplus X \oplus Y$, the direct sum of abelian groups. We can endow M with a ring structure in an obvious way, such that C and D are subrings of M: $X \cdot Y = Y \cdot X = 0$ in M. Then $(\alpha, \beta, M, 1)$ is an exact context and its noncommutative tensor product is the ring $D \boxtimes_R C = R \oplus X \oplus Y \oplus Y \otimes_R X$, up to isomorphism of R-R-bimodules, with the multiplication:

$$X \circ Y = X \circ (Y \otimes_R X) = (Y \otimes_R X) \circ Y = (Y \otimes_R X) \circ (Y \otimes_R X) = 0;$$
$$y \circ x = y \otimes x, \quad y' \circ (y \otimes x) = y'y \otimes x, \quad (y \otimes x) \circ x' = y \otimes xx' \in Y \otimes_R X$$

for any $x, x' \in X$ and $y, y' \in Y$. Observe that M is the quotient ring of $T \boxtimes_R S$ modulo the ideal $Y \otimes_R X$ and that $(\alpha, \beta, M, 1)$ is homological if and only if $\mathrm{Tor}_i^R(Y, X) = 0$ for all $i \geq 1$. Let $\Lambda := \begin{pmatrix} C & M \\ 0 & D \end{pmatrix}$. Theorem 1.1 implies the corollary.

Corollary 4.4. *Let $\alpha : R \to C$ and $\beta : R \to D$ be strictly pure extensions of rings. If $\mathrm{Tor}_i^R(Y, X) = 0$ for all $i \geq 1$, then there exists a recollement $\big(\mathscr{D}(D \boxtimes_R C), \mathscr{D}(\Lambda), \mathscr{D}(R)\big)$ of derived module categories.*

4.2. From ring epimorphisms

In this section, we focus on constructing recollements of derived module categories from exact contexts induced from ring epimorphisms. In particular, we prove Corollary 1.3.

Let $\lambda : R \to S$ be a ring homomorphism. We consider λ as a complex $Q^\bullet$ of left R-modules with R and S in degrees -1 and 0, respectively. Then there exists a canonical distinguished triangle in $\mathscr{K}(R)$:

$$(**) \quad R \xrightarrow{\lambda} S \xrightarrow{\pi} Q^\bullet \xrightarrow{\nu} R[1]$$

Now, we set $S' := \mathrm{End}_{\mathscr{D}(R)}(Q^\bullet)$ and define $\lambda' : R \to S'$ by $r \mapsto f^\bullet$ for $r \in R$, where $f^\bullet$ is a chain map with $f^{-1} := \cdot r$, $f^0 := \cdot(r)\lambda$ and $f^i = 0$ for $i \neq 0, -1$. Here, $\cdot r$ stands for the right multiplication map by r. These data can be recorded in the commutative diagram:

$$\begin{array}{ccccccc}
R & \xrightarrow{\lambda} & S & \xrightarrow{\pi} & Q^\bullet & \xrightarrow{\nu} & R[1] \\
\downarrow{\scriptstyle \cdot r} & & \downarrow{\scriptstyle \cdot(r)\lambda} & & \downarrow{\scriptstyle f^\bullet} & & \downarrow{\scriptstyle (\cdot r)[1]} \\
R & \xrightarrow{\lambda} & S & \xrightarrow{\pi} & Q^\bullet & \xrightarrow{\nu} & R[1]
\end{array}$$

The map λ' is called the ring homomorphism *associated to* λ. If λ is injective, then we shall identify $Q^\bullet$ with S/R in $\mathscr{D}(R)$, and λ' with the map $R \to \mathrm{End}_R(S/R)$ induced by right multiplication.

Let $\Lambda := \mathrm{End}_{\mathscr{D}(R)}\big(S \oplus Q^\bullet\big)$. We consider the quadruple $\big(\lambda, \lambda', \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet), \pi\big)$. If λ is an injective ring epimorphism with $\mathrm{Tor}_1^R(S, S) = 0$, then (λ, λ') is an exact pair (see [8, Lemma 6.5(3)]). This fact can be generalized to the following case.

Lemma 4.5. *Suppose* $\mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big) = 0$. *Then*

(1) $\big(\lambda, \lambda', \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet), \pi\big)$ *is an exact context.*

(2) *If λ is a ring epimorphism, then*

  (*i*) (λ, λ') *is an exact pair, and*

  (*ii*) $\Lambda \simeq \begin{pmatrix} S & \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet) \\ 0 & S' \end{pmatrix}$ *as rings.*

Proof. (1) Applying $\mathrm{Hom}_{\mathscr{D}(R)}(-, Q^\bullet)$ to the triangle $(**)$, we have the long exact sequence

$$\mathrm{Hom}_{\mathscr{D}(R)}(S[1], Q^\bullet) \xrightarrow{(\lambda[1])_*} \mathrm{Hom}_{\mathscr{D}(R)}(R[1], Q^\bullet) \xrightarrow{\nu_*} \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, Q^\bullet) \xrightarrow{\pi_*} \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet) \xrightarrow{\lambda_*} \mathrm{Hom}_{\mathscr{D}(R)}(R, Q^\bullet).$$

Since $\mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big) = 0$, the map $\mathrm{Hom}_R(S, \lambda)$ is injective. As $\mathrm{Hom}_{\mathscr{D}(R)}(S[1], S) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(S, S[-1]) \simeq \mathrm{Ext}_R^{-1}(S, S) = 0$, we obtain $\mathrm{Hom}_{\mathscr{D}(R)}(S[1], Q^\bullet) = 0$ by applying $\mathrm{Hom}_{\mathscr{D}(R)}(S[1], -)$ to the triangle $(**)$. Thus ν_* is injective. Since $\mathrm{Hom}_R(\lambda, S) : \mathrm{Hom}_R(S, S) \to \mathrm{Hom}_R(R, S)$ is surjective, the map $\mathrm{Hom}_{\mathscr{K}(R)}(\lambda, Q^\bullet) : \mathrm{Hom}_{\mathscr{K}(R)}(S, Q^\bullet) \to \mathrm{Hom}_{\mathscr{K}(R)}(R, Q^\bullet)$ is surjective. Note that $\mathrm{Hom}_{\mathscr{K}(R)}(R, Q^\bullet) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(R, Q^\bullet)$. Thus $\lambda_* : \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet) \to \mathrm{Hom}_{\mathscr{D}(R)}(R, Q^\bullet)$ is surjective.

Let $\cdot\pi : S \to \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet)$ stand for the right multiplication map by π. Then we have the exact commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{Ker}(\lambda) & \longrightarrow & R & \xrightarrow{\lambda} & S & \longrightarrow & \mathrm{Coker}(\lambda) & \longrightarrow & 0 \\
 & & \downarrow{\simeq} & & \downarrow{\lambda'} & & \downarrow{\cdot\pi} & & \downarrow{\simeq} & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathscr{D}(R)}(R[1], Q^\bullet) & \xrightarrow{\nu_*} & \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, Q^\bullet) & \xrightarrow{\pi_*} & \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet) & \xrightarrow{\lambda_*} & \mathrm{Hom}_{\mathscr{D}(R)}(R, Q^\bullet) & \longrightarrow & 0
\end{array}$$

where the isomorphisms follow from $\mathrm{Hom}_{\mathscr{D}(R)}(R[n], Q^\bullet) \simeq H^{-n}(Q^\bullet)$ for each $n \in \mathbb{Z}$. Consequently, the chain map $\big(\lambda',\ \cdot\pi\big) : \mathrm{Con}(\lambda) \to \mathrm{Con}(\pi_*)$ is a quasi-isomorphism. Observe that π_* is the left multiplication map by π. Thus the middle square in the above diagram is a pull-back and push-out diagram, and therefore $\big(\lambda, \lambda', \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet), \pi\big)$ is an exact context.

(2) Suppose that λ is a ring epimorphism. Then (λ, λ') is an exact pair, due to Lemma 3.1(1). Observe that $\mathrm{End}_S(S) = \mathrm{End}_R(S)$ and the homomorphism $\mathrm{Hom}_R(\lambda, S) : \mathrm{Hom}_R(S, S) \to \mathrm{Hom}_R(R, S)$ is an isomorphism. This implies $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, S) = 0$. Now (2) follows from $\mathrm{End}_{\mathscr{D}(R)}(S) \simeq \mathrm{End}_R(S) \simeq S$. $\square$

Corollary 4.6. *Let $\lambda : R \to S$ be a ring epimorphism with $\mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big) = 0$. If $\mathrm{Tor}_i^R(S', S) = 0$ for any $i \geq 1$, then there exists a recollement $\big(\mathscr{D}(\mathrm{End}_{S'}(S' \otimes_R S)),\ \mathscr{D}(\Lambda),\ \mathscr{D}(R)\big)$ of derived module categories.*

Proof. Since $\mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big) = 0$, the quadruple $\big(\lambda, \lambda', \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet), \pi\big)$ is an exact context by Lemma 4.5(1). Since λ is a ring epimorphism, the pair (λ, λ') is exact by Lemma 4.5(2), and $S \boxtimes_R S' \simeq \mathrm{End}_{S'}(S' \otimes_R S)$ as rings by Lemma 3.1(1). Now, Corollary 4.6 follows from Theorem 1.1. $\square$

Note that Corollary 4.6 generalizes [8, Corollary 6.6] where λ is required to be injective and homological, and where ${}_R S$ is assumed to have projective dimension at most 1.

To show Corollary 1.3, we first prove the following lemma in which ${U^\bullet}^{(I)}$ stands for the direct sum of I copies of $U^\bullet$ in $\mathscr{D}(R)$.

Lemma 4.7. *Let $\lambda : R \to S$ be a ring homomorphism, and let I be a nonempty set. Define $U^\bullet := S \oplus Q^\bullet$. Then $\mathrm{Hom}_{\mathscr{D}(R)}(U^\bullet, {U^\bullet}^{(I)}[n]) = 0$ for any $0 \neq n \in \mathbb{Z}$ if and only if*

(1) $\mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big) = 0$ *and*

(2) $\mathrm{Ext}_R^i(S, S^{(I)}) = 0 = \mathrm{Ext}_R^{i+1}(S, R^{(I)})$ *for any $i \geq 1$.*

Proof. First, we notice two general facts:

(a) Applying $\mathrm{Hom}_{\mathscr{D}(R)}(-, S^{(I)})$ to $(**)$, one can prove

$$\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, S^{(I)}[i]) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(S, S^{(I)}[i]) \ \text{ for } \ i \in \mathbb{Z} \setminus \{0\} \ \text{ and}$$

$$\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, S^{(I)}) \simeq \mathrm{Ker}\big(\mathrm{Hom}_R(\lambda, S^{(I)})\big).$$

(b) Applying $\mathrm{Hom}_{\mathscr{D}(R)}(-, R^{(I)})$ to $(**)$, one can show

$$\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, R^{(I)}[j]) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(S, R^{(I)}[j]) \;\text{ for }\; j \in \mathbb{Z} \setminus \{0, 1\}.$$

Next, we show the necessity of the first part of Lemma 4.7.

Suppose $\mathrm{Hom}_{\mathscr{D}(R)}(U^\bullet, U^{\bullet\,(I)}[n]) = 0$ for any $n \neq 0$. Then $\mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet[-1]) = 0$ and $\mathrm{Ext}^i_R(S, S^{(I)}) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, S^{(I)}[i]) = 0$ for any $i \geq 1$. Consequently, $\mathrm{Hom}_R(S, \lambda) : \mathrm{Hom}_R(S, R) \to \mathrm{Hom}_R(S, S)$ is injective. This means that the condition (1) holds. Further, applying $\mathrm{Hom}_{\mathscr{D}(R)}(S, -)$ to the triangle

$$(\dagger\dagger) \qquad R^{(I)} \xrightarrow{\lambda^{(I)}} S^{(I)} \xrightarrow{\pi^{(I)}} Q^{\bullet (I)} \to R^{(I)}[1],$$

we get $\mathrm{Ext}^{i+1}_R(S, R^{(I)}) \simeq \mathrm{Hom}_{\mathscr{D}(R)}\big(S, Q^{\bullet(I)}[i]\big) = 0$. Thus (1) and (2) are satisfied.

Finally, we show the sufficiency of the conditions. Assume (1) and (2). Then it follows from (a) and (b) that $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, S^{(I)}[n]) = 0 = \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, R^{(I)}[m+1])$ for $n \in \mathbb{Z} \setminus \{0\}$ and $m \in \mathbb{Z} \setminus \{-1, 0\}$. Applying $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet -)$ to the triangle $(\dagger\dagger)$, one can show $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, Q^{\bullet\,(I)}[m]) = 0$ for $m \in \mathbb{Z} \setminus \{-1, 0\}$. Furthermore, we shall show that $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, Q^{\bullet\,(I)}[-1]) = 0$ if $\mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big) = 0$:

In fact, $\mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)^I\big) \simeq \mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big)^I = 0$, where $\mathrm{Ker}(\lambda)^I$ stands for the direct product of I copies of $\mathrm{Ker}(\lambda)$. Since $\mathrm{Ker}(\lambda)^I$ contains $\mathrm{Ker}(\lambda)^{(I)}$ as a submodule, we have $\mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)^{(I)}\big) = 0$ and $\mathrm{Ker}\big(\mathrm{Hom}_R(S, \lambda^{(I)})\big) \simeq \mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)^{(I)}\big) = 0$. Now, it follows from the following exact commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, Q^{\bullet\,(I)}[-1]) & \xrightarrow{(\nu[-1])^*} & \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, R^{(I)}) & \xrightarrow{(\lambda^{(I)})^*} & \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, S^{(I)}) \\
& \downarrow & & \downarrow \pi_* & & \downarrow \pi_* \\
0 \longrightarrow & \mathrm{Ker}\big(\mathrm{Hom}_R(S, \lambda^{(I)})\big) & \longrightarrow & \mathrm{Hom}_R(S, R^{(I)}) & \xrightarrow{(\lambda^{(I)})^*} & \mathrm{Hom}_R(S, S^{(I)})
\end{array}$$

that $\mathrm{Ker}\big(\mathrm{Hom}_R(S, \lambda^{(I)})\big) \simeq \mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)^{(I)}\big) = 0$ and $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, Q^{\bullet\,(I)}[-1]) = 0$. Thus $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, Q^{\bullet\,(I)}[n]) = 0$ for $n \neq 0$.

It remains to prove $\mathrm{Hom}_{\mathscr{D}(R)}(S, Q^{\bullet\,(I)}[n]) = 0$ for $n \neq 0$. Actually, applying $\mathrm{Hom}_{\mathscr{D}(R)}(S, -)$ to the triangle $(\dagger\dagger)$, we have the long exact sequence:

$$\cdots \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(S, S^{(I)}[j]) \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^{\bullet(I)}[j]) \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(S, R^{(I)}[j+1]) \xrightarrow{(\lambda^{(I)})^*} \mathrm{Hom}_{\mathscr{D}(R)}(S, S^{(I)}[j+1]) \longrightarrow \cdots$$

for $j \in \mathbb{Z}$. Since $\mathrm{Hom}_{\mathscr{D}(R)}(S, S^{(I)}[r]) = 0$ for $0 \neq r \in \mathbb{Z}$ and $\mathrm{Hom}_{\mathscr{D}(R)}(S, R^{(I)}[t]) = 0$ for $t \in \mathbb{Z} \setminus \{0, 1\}$, we have $\mathrm{Hom}_{\mathscr{D}(R)}(S, Q^{\bullet(I)}[j]) = 0$ for $j \in \mathbb{Z} \setminus \{-1, 0\}$ and $\mathrm{Hom}_{\mathscr{D}(R)}(S, Q^{\bullet(I)}[-1]) \simeq \mathrm{Ker}\big(\mathrm{Hom}_R(S, \lambda^{(I)})\big) = 0$. It then follows that $\mathrm{Hom}_{\mathscr{D}(R)}(S, Q^{\bullet\,(I)}[n]) = 0$ and $\mathrm{Hom}_{\mathscr{D}(R)}(U^\bullet, U^{\bullet\,(I)}[n]) = 0$ for any $n \neq 0$. This finishes the proof of the sufficiency. □

Proof of Corollary 1.3. Since λ is a ring epimorphism, it follows from Lemma 4.5(2) that (λ, λ') is an exact pair and Λ can be identified with the ring $\begin{pmatrix} S & \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet) \\ 0 & S' \end{pmatrix}$. In the following, we focus on the exact context (λ, μ, M, m) defined by $\mu := \lambda'$, $M := \mathrm{Hom}_{\mathscr{D}(R)}(S, Q^\bullet)$ and $m := \pi$, and use the recollement $(\star)$ given by the six triangle functors in Lemma 3.2(1). By Theorem 1.1, to prove Corollary 1.3, it suffices to show $\mathrm{Tor}^R_i(S', S) = 0$ for all $i \geq 1$.

Suppose $\mathrm{pdim}(_RS) \leq 1$. Then $\mathrm{Tor}^R_i(S', S) = 0$ for all $i \geq 2$. It remains to show $\mathrm{Tor}^R_1(S', S) = 0$. Since $\Lambda e_2 = S' \oplus S \otimes_R S'$ as right R-modules, it is enough to show $\mathrm{Tor}^R_1(\Lambda e_2, S) = 0$.

Let $\eta : Id_{\mathscr{D}(\Lambda)} \to j_*j^!$ be the unit adjunction with respect to the adjoint pair $(j^!, j_*)$. Then, for any $X^\bullet \in \mathscr{D}(\Lambda)$, there exists a canonical triangle in $\mathscr{D}(\Lambda)$:

$$i_*i^!(X^\bullet) \to X^\bullet \xrightarrow{\eta_{X^\bullet}} j_*j^!(X^\bullet) \to i_*i^!(X^\bullet)[1],$$

where $j_*j^!(X^\bullet) = \mathbb{R}\mathrm{Hom}_R(P^{\bullet *}, \mathrm{Hom}^\bullet_\Lambda(P^\bullet, X^\bullet))$.

Let $0 \to P^{-1} \xrightarrow{\delta} P^0 \to {}_RS \to 0$ be a projective resolution of ${}_RS$ with all P^j projective R-modules. This exact sequence gives rise to a triangle $P^{-1} \to P^0 \to S \to P^{-1}[1]$ in $\mathscr{D}(R)$. Then, by the recollement $(\star)$, there is the following exact commutative diagram:

$$\begin{array}{ccccccc}
i_*i^!(\Lambda e_2 \otimes_R P^{-1}) & \longrightarrow & \Lambda e_2 \otimes_R P^{-1} & \xrightarrow{\eta_{\Lambda e_2 \otimes_R P^{-1}}} & j_*j^!(\Lambda e_2 \otimes_R P^{-1}) & \longrightarrow & i_*i^!(\Lambda e_2 \otimes_R P^{-1})[1] \\
\downarrow & & \downarrow {\scriptstyle 1\otimes\delta} & & \downarrow {\scriptstyle j_*j^!(1\otimes\delta)} & & \downarrow \\
i_*i^!(\Lambda e_2 \otimes_R P^{0}) & \longrightarrow & \Lambda e_2 \otimes_R P^{0} & \xrightarrow{\eta_{\Lambda e_2 \otimes_R P^{0}}} & j_*j^!(\Lambda e_2 \otimes_R P^{0}) & \longrightarrow & i_*i^!(\Lambda e_2 \otimes_R P^{0})[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
i_*i^!(\Lambda e_2 \otimes^{\mathbb{L}}_R S) & \longrightarrow & \Lambda e_2 \otimes^{\mathbb{L}}_R S & \xrightarrow{\eta_{\Lambda e_2 \otimes^{\mathbb{L}}_R S}} & j_*j^!(\Lambda e_2 \otimes^{\mathbb{L}}_R S) & \longrightarrow & i_*i^!(\Lambda e_2 \otimes^{\mathbb{L}}_R S)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
i_*i^!(\Lambda e_2 \otimes_R P^{-1})[1] & \longrightarrow & \Lambda e_2 \otimes_R P^{-1}[1] & \longrightarrow & j_*j^!(\Lambda e_2 \otimes_R P^{-1})[1] & \longrightarrow & i_*i^!(\Lambda e_2 \otimes_R P^{-1})[2]
\end{array}$$

Since $i_*i^*(\Lambda e_1) \simeq \Lambda e_2 \otimes^{\mathbb{L}}_R S$ in $\mathscr{D}(\Lambda)$ by Lemma 3.2(3) and $j^!i_* = 0$ by $(\star)$, it follows that $j_*j^!(\Lambda e_2 \otimes^{\mathbb{L}}_R S) \simeq j_*j^!i_*i^*(\Lambda e_1) = 0$. Then $j_*j^!(1 \otimes \delta)$ is an isomorphism, and therefore so is $H^0(j_*j^!(1 \otimes \delta))$.

Suppose that $H^0(\eta_P) : P \to H^0\big(j_*j^!(P)\big)$ is an injective map for any projective Λ-module P. Then $H^0(\eta_{\Lambda e_2 \otimes_R P^{-1}})$ is injective since ${}_RP^{-1}$ is projective. It follows from the isomorphism $H^0(j_*j^!(1 \otimes \delta))$ that the map $1 \otimes \delta : \Lambda e_2 \otimes_R P^{-1} \to \Lambda e_2 \otimes_R P^0$ is injective. This implies $\mathrm{Tor}^R_1(\Lambda e_2, S) = 0$, as desired.

Now, we shall show that $H^0(\eta_P) : P \to H^0\big(j_*j^!(P)\big)$ is indeed an injective map for any projective Λ-module P.

First, we observe that $H^0(\eta_P)$ is injective if and only if $\mathrm{Hom}_{\mathscr{D}(\Lambda)}(\Lambda, P) \xrightarrow{j^!} \mathrm{Hom}_{\mathscr{D}(R)}\big(j^!(\Lambda), j^!(P)\big)$ is injective. To see this point, we consider the composite of maps:

$$\omega^n_{X^\bullet} : \mathrm{Hom}_{\mathscr{D}(\Lambda)}(\Lambda, X^\bullet[n]) \xrightarrow{j^!} \mathrm{Hom}_{\mathscr{D}(R)}\big(j^!(\Lambda), j^!(X^\bullet)[n]\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{D}(\Lambda)}(\Lambda, j_*j^!(X^\bullet)[n])$$

for $n \in \mathbb{Z}$, where the isomorphism is induced by the adjoint pair $(j^!, j_*)$. Then $\omega^n_{X^\bullet} = \mathrm{Hom}_{\mathscr{D}(\Lambda)}(\Lambda, \eta_{X^\bullet[n]})$. It is known that the n-th cohomology functor $H^n(-) : \mathscr{D}(\Lambda) \to \Lambda\text{-Mod}$ is naturally isomorphic to the Hom-functor $\mathrm{Hom}_{\mathscr{D}(\Lambda)}(\Lambda, -[n])$. So $\omega^n_{X^\bullet}$ coincides with $H^n(\eta_{X^\bullet}) : H^n(X^\bullet) \to H^n(j_*j^!(X^\bullet))$. It follows that the map $H^0(\eta_P)$ is injective if and only if so is the map $\mathrm{Hom}_{\mathscr{D}(\Lambda)}(\Lambda, P) \xrightarrow{j^!} \mathrm{Hom}_{\mathscr{D}(R)}\big(j^!(\Lambda), j^!(P)\big)$.

Second, if $\mathrm{Hom}_{\mathscr{D}(\Lambda)}(i_*i^*(\Lambda), P) = 0$, then $\mathrm{Hom}_{\mathscr{D}(\Lambda)}(\Lambda, P) \xrightarrow{j^!} \mathrm{Hom}_{\mathscr{D}(R)}\big(j^!(\Lambda), j^!(P)\big)$ is injective.

Indeed, let $\varepsilon : j_!j^! \to Id_{\mathscr{D}(\Lambda)}$ be the counit adjunction with respect to the adjoint pair $(j_!, j^!)$. Then, for each $X^\bullet \in \mathscr{D}(\Lambda)$, there is a canonical triangle $j_!j^!(X^\bullet) \xrightarrow{\varepsilon_{X^\bullet}} X^\bullet \to i_*i^*(X^\bullet) \to j_!j^!(X^\bullet)[1]$ in $\mathscr{D}(\Lambda)$. Now, we consider the morphisms:

$$\mathrm{Hom}_{\mathscr{D}(\Lambda)}(\Lambda, X^\bullet[m]) \xrightarrow{j^!} \mathrm{Hom}_{\mathscr{D}(R)}\big(j^!(\Lambda), j^!(X^\bullet)[m]\big) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{D}(\Lambda)}\big(j_!j^!(\Lambda), X^\bullet[m]\big)$$

for any $m \in \mathbb{Z}$, where the last map is an isomorphism given by the adjoint pair $(j_!, j^!)$. One can check that the composite of the above two morphisms coincides with $\mathrm{Hom}_{\mathscr{D}(\Lambda)}(\varepsilon_\Lambda, X^\bullet[m])$. Thus, to show that $j^!$:

$\mathrm{Hom}_{\mathscr{D}(\Lambda)}(\Lambda, P) \to \mathrm{Hom}_{\mathscr{D}(R)}\big(j^!(\Lambda), j^!(P)\big)$ is injective, it suffices to show that $\mathrm{Hom}_{\mathscr{D}(\Lambda)}(\varepsilon_\Lambda, P)$ is injective. For this aim, we apply $\mathrm{Hom}_{\mathscr{D}(\Lambda)}(-, P)$ to the triangle $j_!j^!(\Lambda) \xrightarrow{\varepsilon_\Lambda} \Lambda \to i_*i^*(\Lambda) \to j_!j^!(\Lambda)[1]$ and get the exact sequence of abelian groups

$$\mathrm{Hom}_{\mathscr{D}(\Lambda)}(i_*i^*(\Lambda), P) \longrightarrow \mathrm{Hom}_{\mathscr{D}(\Lambda)}(\Lambda, P) \longrightarrow \mathrm{Hom}_{\mathscr{D}(\Lambda)}(j_!j^!(\Lambda), P).$$

If $\mathrm{Hom}_{\mathscr{D}(\Lambda)}(i_*i^*(\Lambda), P) = 0$, then $\mathrm{Hom}_{\mathscr{D}(\Lambda)}(\varepsilon_\Lambda, P)$ is injective, and thus the map $j^! : \mathrm{Hom}_{\mathscr{D}(\Lambda)}(\Lambda, P) \to \mathrm{Hom}_{\mathscr{D}(R)}\big(j^!(\Lambda), j^!(P)\big)$ is injective, as desired.

Third, we show that if $\mathrm{Hom}_R(S, S') = 0$, then $\mathrm{Hom}_{\mathscr{D}(\Lambda)}(i_*i^*(\Lambda), P) = 0$ for arbitrary projective Λ-module P.

By Lemma 3.2(3), $i_*i^*(\Lambda e_2) \simeq i_*i^*(\Lambda e_1) \simeq \Lambda e_2 \otimes^{\mathbb{L}}_R S$ in $\mathscr{D}(\Lambda)$. So, $\mathrm{Hom}_{\mathscr{D}(\Lambda)}(i_*i^*(\Lambda), P) = 0$ if and only if $\mathrm{Hom}_{\mathscr{D}(\Lambda)}(\Lambda e_2 \otimes^{\mathbb{L}}_R S, P) = 0$. Further, we consider the following isomorphisms

$$\mathrm{Hom}_{\mathscr{D}(\Lambda)}(\Lambda e_2 \otimes^{\mathbb{L}}_R S, P) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(S, \mathbb{R}\mathrm{Hom}_\Lambda(\Lambda e_2, P)) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(S, e_2P) \simeq \mathrm{Hom}_R(S, e_2P).$$

Since $e_2\Lambda \simeq S'$ as R-modules, $\mathrm{Hom}_R(S, e_2\Lambda) \simeq \mathrm{Hom}_R(S, S') = 0$. As ${}_\Lambda P$ is a projective module, there is an index set I such that e_2P is a direct summand of $(S')^{(I)}$. Observe that $(S')^{(I)}$ is a submodule of the direct product $(S')^I$ of I copies of ${}_RS'$. Consequently, $\mathrm{Hom}_R\big(S, (S')^{(I)}\big)$ is a subgroup of $\mathrm{Hom}_R\big(S, (S')^I\big)$ which is isomorphic to $\mathrm{Hom}_R(S, S')^I$. Thus $\mathrm{Hom}_R(S, (S')^{(I)}) = 0 = \mathrm{Hom}_R(S, e_2P)$. This implies $\mathrm{Hom}_{\mathscr{D}(\Lambda)}(i_*i^*(\Lambda), P) = 0$.

Now, we show $\mathrm{Hom}_R(S, S') = 0$. Actually, we shall prove a stronger statement, namely $\mathrm{Hom}_{\mathscr{D}(R)}(S, S'[n]) = 0$ for any $n \in \mathbb{Z}$.

Since λ is a ring epimorphism such that $\mathrm{Tor}_1^R(S, S) = 0$, it follows from [19, Theorem 4.8] that $\mathrm{Ext}^1_R(S, S^{(I)}) \simeq \mathrm{Ext}^1_S(S, S^{(I)}) = 0$ for any set I. Thanks to $\mathrm{pdim}({}_RS) \le 1$, we can apply Lemma 4.7 to $U^\bullet := S \oplus Q^\bullet$ and obtain $\mathrm{Hom}_{\mathscr{D}(R)}(U^\bullet, U^\bullet[m]) = 0$ for $m \neq 0$. This implies $\mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, Q^\bullet[m]) = 0$ for $m \neq 0$ and

$$H^m(\mathbb{R}\mathrm{Hom}_R(Q^\bullet, Q^\bullet)) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, Q^\bullet[m]) = \begin{cases} 0 & \text{if } m \neq 0, \\ S' & \text{if } m = 0. \end{cases}$$

Thus the complex $\mathbb{R}\mathrm{Hom}_R(Q^\bullet, Q^\bullet)$ is isomorphic in $\mathscr{D}(R)$ to the stalk complex S'. On the one hand, due to the adjoint pair $\big(Q^\bullet \otimes^{\mathbb{L}}_R -, \mathbb{R}\mathrm{Hom}_R(Q^\bullet, -)\big)$ of the triangle functors, we have

$$\begin{aligned} \mathrm{Hom}_{\mathscr{D}(R)}(S, S'[n]) &\simeq \mathrm{Hom}_{\mathscr{D}(R)}\big(S, \mathbb{R}\mathrm{Hom}_R(Q^\bullet, Q^\bullet)[n]\big) \\ &\simeq \mathrm{Hom}_{\mathscr{D}(R)}\big(S, \mathbb{R}\mathrm{Hom}_R(Q^\bullet, Q^\bullet[n])\big) \\ &\simeq \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet \otimes^{\mathbb{L}}_R S, Q^\bullet[n]) \end{aligned}$$

for any $n \in \mathbb{Z}$. On the other hand, since λ is homological by assumption, the morphism $\lambda \otimes^{\mathbb{L}}_R S : R \otimes^{\mathbb{L}}_R S \to S \otimes^{\mathbb{L}}_R S$ is an isomorphism in $\mathscr{D}(R)$. It then follows from the triangle

$$R \otimes^{\mathbb{L}}_R S \xrightarrow{\lambda \otimes^{\mathbb{L}}_R S} S \otimes^{\mathbb{L}}_R S \longrightarrow Q^\bullet \otimes^{\mathbb{L}}_R S \longrightarrow R \otimes^{\mathbb{L}}_R S[1]$$

that $Q^\bullet \otimes^{\mathbb{L}}_R S = 0$. Hence $\mathrm{Hom}_{\mathscr{D}(R)}(S, S'[n]) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet \otimes^{\mathbb{L}}_R S, Q^\bullet[n]) = 0$ for any $n \in \mathbb{Z}$. Thus, for any projective Λ-module P, the map $H^0(\eta_P) : P \to H^0(j_*j^!(P))$ is injective. This shows Corollary 1.3. □

In the following we consider when ring homomorphisms give rise to derived equivalences.

Corollary 4.8. *Let $\lambda : R \to S$ be a ring homomorphism. Then $S \oplus Q^\bullet$ is a tilting complex in $\mathscr{D}(R)$ if and only if $\mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big) = 0$, $\mathrm{Ext}^1_R(S, S) = 0$ and there is an exact sequence $0 \to P_1 \to P_0 \to S \to 0$ of R-modules with all P_i finitely generated projective R-modules.*

Proof. Let $U^\bullet := S \oplus Q^\bullet$. Clearly, $U^\bullet$ is a generator of $\mathscr{D}(R)$, that is, $\mathrm{Tria}(U^\bullet) = \mathscr{D}(R)$, since $R \in \mathrm{Tria}(U^\bullet)$ by $(**)$. Thus $U^\bullet$ is a tilting complex in $\mathscr{D}(R)$ if and only if it is self-orthogonal and ${}_RS$ has a projective resolution of finite length by finitely generated projective R-modules. Furthermore, if $\mathrm{pdim}({}_RS) < \infty$ and $\mathrm{Ext}^{i+1}_R(S, R^{(I)}) = 0$ for any $i \geq 1$, then $\mathrm{pdim}({}_RS) \leq 1$. Now Corollary 4.8 is a consequence of Lemma 4.7. $\square$

Observe that if $\lambda : R \to S$ is injective and ${}_RS$ is finitely generated and projective, then all conditions in the corollary are fulfilled.

Corollary 4.9. *Let $\lambda : R \to S$ be a homological ring epimorphism with $\mathrm{Hom}_R\big(S, \mathrm{Ker}(\lambda)\big) = 0$. Then $S' \otimes_R S = 0$ if and only if there is an exact sequence $0 \to P_1 \to P_0 \to {}_RS \to 0$ of R-modules such that P_0 and P_1 are finitely generated and projective. In this case, the rings Λ and R are derived equivalent.*

Proof. If $S' \otimes_R S = 0$, then $S \boxtimes_R S' = 0$ and $j^! : \mathscr{D}(\Lambda) \to \mathscr{D}(R)$ is a triangle equivalence by Lemma 3.2(1). Thus $j^!(\Lambda)$ is a tilting complex over R. Since $j^!(\Lambda) \simeq (S \oplus \mathrm{Con}(m\cdot))[-1] \simeq (S \oplus \mathrm{Con}(\lambda))[-1] = (S \oplus Q^\bullet)[-1]$ in $\mathscr{D}(R)$, we see that $S \oplus Q^\bullet$ is also a tilting complex over R. Clearly, the necessity of Corollary 4.9 is a consequence of Corollary 4.8.

It remains to show the sufficiency of Corollary 4.9. Since ${}_RS$ has a finite projective resolution by finitely generated projective R-modules, we have $\mathbb{R}\mathrm{Hom}_R(Q^\bullet, Q^\bullet) \otimes^{\mathbb{L}}_R S \simeq \mathbb{R}\mathrm{Hom}_R(Q^\bullet, Q^\bullet \otimes^{\mathbb{L}}_R S)$ in $\mathscr{D}(\mathbb{Z})$, where $Q^\bullet$ is regarded as a complex of R-R-bimodules. As λ is homological and $\mathrm{pdim}({}_RS) \leq 1$, we deduce from the proof of Corollary 1.3 that $\mathbb{R}\mathrm{Hom}_R(Q^\bullet, Q^\bullet) \simeq S'$ in $\mathscr{D}(R^{\mathrm{op}})$ and $Q^\bullet \otimes^{\mathbb{L}}_R S = 0$ in $\mathscr{D}(R)$. Thus $S' \otimes^{\mathbb{L}}_R S = 0$ in $\mathscr{D}(\mathbb{Z})$, which implies $S' \otimes_R S = 0$. $\square$

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