ON THE SEMI-SIMPLICITY OF CYCLOTOMIC TEMPERLEY-LIEB ALGEBRAS

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Abstract. In [7], a class of associative algebras called cyclotomic Temperley-Lieb algebras over a commutative ring was introduced. In this note, we provide a necessary and sufficient condition for a cyclotomic Temperley-Lieb algebra to be semi-simple.

1. Introduction

The Temperley-Lieb algebras were first introduced in [9] in order to study the single bond transfer matrices for the Ising model and for the Potts model. Jones [4] defined a trace function on a Temperley-Lieb algebra so that he could construct Jones polynomial of a link when the trace is non-degenerate. It is known that the trace is non-degenerate if the Temperley-Lieb algebra is semi-simple. So it is an interesting question to provide a criterium for a Temperley-Lieb algebra to be semi-simple. In [10, §5], Westbury computed explicitly the determinants of Gram matrices associated to all "cell modules" via Tchebychev polynomials. This implies that a Temperley-Lieb algebra is semi-simple if and only if such polynomials do not take values zero for the papramaters.

As a generalization of a Temperley-Lieb algebra, the cyclotomic Temperley-Lieb algebra $TL_{m,n}(\boldsymbol{\delta})$ of type $G(m,1,n)$ was introduced in [7]. It is proved in [7] that $TL_{m,n}(\boldsymbol{\delta})$ is a cellular algebra in the sense of [2]. Thus $TL_{m,n}(\boldsymbol{\delta})$ is semi-simple if and only if all of its "cell modules" are pairwise non-isomorphic irreducible. In order to describe a cell module to be irreducible, Rui and Xi computed the determinants of Gram matrices of certain cell modules [7, 8.1]. In general, it is hard to compute the determinants for all cell modules.

In this note, we shall consider the semi-simplicity of cyclotomic Temperley-Lieb algebras, this is an analog question considered in [8] (see [1] for the case $m = 1$). Following [5], we study two functors F and G between certain categories in section 3. Via these functors and [7, 8.1], we can show Theorem 4.6, the main result of this paper, which says that the semi-simplicity of a cyclotomic Temperley-Lieb algebra can be determined by generalized Tchebychev polynomials and the parameters $\bar{\delta}_i, 1 \le i \le m$.

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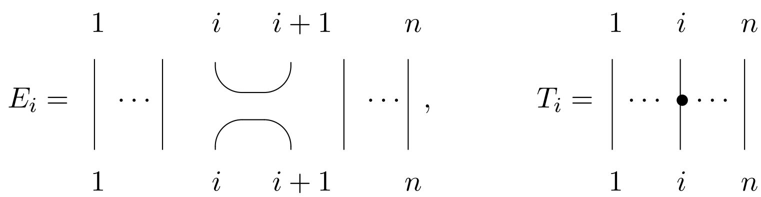
2. Cyclotomic Temperley-Lieb algebras

In this section, we recall some of results on the cyclotomic Temperley-Lieb algebras in [7]. Throughout the paper, we fix two natural numbers m and n.

A labelled Temperley-Lieb diagram (or labelled TL-diagram) D of type $G(m,1,n)$ is a Temperley-Lieb diagram with $2n$ vertices and n arcs. Each arc is labelled by an element in $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$, which will be considered as the number of dots on it. The following are two special labelled TL-diagrams.



An arc in a labelled TL-diagram D is horizontal if both of its endpoints are in the same row of D. Otherwise, it is vertical. A dot will be replaced by $m-1$ dots if it moves from one endpoint of a horizontal arc to another. A dot in a vertical arc can move freely from one endpoint to another.

Given a horizontal arc $\{i,j\}$ of D with $i<j$. We say i (resp. j) the left (resp. right) endpoint of the arc. For a horizontal (resp. vertical) arc, we always assume that the dots on this arc concentrate on the left endpoint (resp. the endpoint on the top row of the labelled TL-diagram D).

Suppose an arc l_1 joins another arc l_2 with a common endpoint j. A dot can move from l_1 to l_2. We always assume that a dot on the endpoint $j \in l_1$ can be replaced by a dot on $j \in l_2$.

Given two labelled TL-diagrams D_1 and D_2 of type $G(m,1,n)$. Following [7], we define a new labelled TL-diagram $D_1 \circ D_2$ as follows: First, compose D_1 and D_2 in the same way as was done for the Temperley-Lieb algebra to get a new diagram P. Second, applying the rule for the movement of dots to relabel each arc of P. We get a new labelled TL-diagram, and this is defined to be $D_1 \circ D_2$. Let $n(\bar{i}, D_1, D_2)$ be the number of the relabelled closed cycles on which there are $\bar{i}$ dots.

Definition 2.1. [7, 3.3] Let R be a commutative ring containing 1 and $\delta_0, \dots, \delta_{m-1}$. Put $\boldsymbol{\delta} = (\delta_0, \dots, \delta_{m-1})$. A **cyclotomic Temperley-Lieb algebra** $TL_{m,n}(\boldsymbol{\delta})$ is an associative algebra over R with a basis consisting of all labelled TL-diagrams of type $G(m,1,n)$, and the multiplication is given by $D_1 \cdot D_2 = \prod_{i=0}^{m-1} \delta_i^{n(\bar{i}, D_1, D_2)} D_1 \circ D_2$.

It was shown in [7] that $TL_{m,n}(\boldsymbol{\delta})$ can be defined by generators and relations. For the details we refer to [7, 2.1].

In the remaining part of this section, we recall some results on the representations of $TL_{m,n}(\boldsymbol{\delta})$. First, we give the notion of a cellular algebra in [2], which depends on the existence of certain basis. There is also a basis-free definition of cellular algebras, for this we refer to [6].

Definition 2.2. [2, 1.1] An associative R–algebra A is called a **cellular algebra** with cell datum (I, M, C, i) if the following conditions are satisfied:

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has an R–basis $C^{\lambda}_{S,T}$ where (S, T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is an R–linear anti–automorphism of A with $i^2 = id$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^{\lambda}_{S,T}$ can be written as

$$aC^{\lambda}_{S,T} = \sum_{U \in M(\lambda)} r_a(U, S) C^{\lambda}_{U,T} + r',$$

where r' belongs to $A^{<\lambda}$ consisting of all R-linear combination of basis elements with upper index μ strictly smaller than λ, and the coefficients $r_a(U, S) \in R$ do not depend on T.

For each $\lambda \in I$, one can define a cell module $\Delta(\lambda)$ and a symmetric, associative bilinear form $\Phi_\lambda : \Delta(\lambda) \otimes_R \Delta(\lambda) \to R$ in the following way (see [2, §2]): As an R-module, $\Delta(\lambda)$ has an R-basis $\{C^{\lambda}_S \mid S \in M(\lambda)\}$, the module structure is given by

$$aC^{\lambda}_S = \sum_{U \in M(\lambda)} r_a(U, S) C^{\lambda}_U. \tag{2.1}$$

The bilinear form Φ_λ is defined by

$$\Phi_\lambda(C^{\lambda}_S, C^{\lambda}_T) C^{\lambda}_{U,V} \equiv C^{\lambda}_{U,S} C^{\lambda}_{T,V} \pmod{A^{<\lambda}},$$

where U and V are arbitrary elements in $M(\lambda)$.

Let $\mathrm{rad}\Delta(\lambda) = \{c \in \Delta(\lambda) \mid \Phi_\lambda(c, c') = 0 \text{ for all } c' \in \Delta(\lambda)\}$. Then $\mathrm{rad}\Delta(\lambda)$ is a submodule of $\Delta(\lambda)$. Put $L(\lambda) = \Delta(\lambda)/\mathrm{rad}\Delta(\lambda)$. Then either $L(\lambda) = 0$ or $L(\lambda)$ is irreducible [2, 3.2]. We will need the following result next section.

Lemma 2.3. *$rad\Delta(\lambda)$ is annihilated by $A^{\leq\lambda}$.*

Proof. Let $a = C^{\mu}_{S_1,T_1} \in A^{\leq\lambda}$ and $C^{\lambda}_S \in rad\Delta(\lambda)$. If $\mu < \lambda$, then $aC^{\lambda}_S = 0$ in $\Delta(\lambda)$. If $\mu = \lambda$, then we still have $aC^{\lambda}_S = 0$ since $r_a(S_1, S) = \Phi_\lambda(C^{\lambda}_{T_1}, C^{\lambda}_S)$ and $C^{\lambda}_S \in rad\Delta(\lambda)$. □

From now on, we assume that R **is a splitting field of** $x^m - 1$. Then $x^m - 1 = \prod_{i=1}^m (x - u_i)$ for some $u_i \in R, 1 \leq i \leq m$. Let $G_{m,n}$ be the R-subalgebra of $TL_{m,n}(\boldsymbol{\delta})$ generated by $T_1, T_2, \cdots, T_n$. Let $\Lambda(m, n) = \{(i_1, i_2, \cdots, i_n) \mid 1 \leq i_j \leq m\}$. Define $\mathbf{i} \leq \mathbf{j}$ if $i_k \geq j_k$ for all $1 \leq k \leq n$. Then $(\Lambda(m, n), \leq)$ is a poset. For any $\mathbf{i} \in \Lambda(m, n)$, set $C^{\mathbf{i}}_{1,1} = \prod_{j=1}^n \prod_{l=i_j+1}^m (t_j - u_l)$.

Lemma 2.4. *The set* $\{C^{\mathbf{i}}_{1,1} \mid \mathbf{i} \in \Lambda(m,n)\}$ *is a cellular basis of* $G_{m,n}$.

The cell module for $\mathbf{i} \in \Lambda(m,n)$ with respect to the above cellular basis will be denoted by $\Delta(\mathbf{i})$.

An (n,k)-*labelled parenthesis graph* is a graph consisting of n vertices $\{1,2,...,n\}$ and k horizontal arcs (hence $2k \leqslant n$ and there are $n-2k$ free vertices which do not belong to any arc) such that

(1) there are at most $m-1$ dots on each arc,

(2) there are no arcs $\{i,j\}$ and $\{q,l\}$ satisfying $i<q<j<l$

(3) there is no arc $\{i,j\}$ and free vertex q such that $i<q<j$.

Let $P(n,k)$ be the set of all (n,k)-labelled parenthesis graphs. A labelled TL-diagram D with k horizontal arcs can be determined by a triple pair (v_1,v_2,x), $x \in G_{m,n-2k}$ and $v_1,v_2 \in P(n,k)$ (see [7, §5]) and vice versa. Such a D will be denoted by $v_1 \otimes v_2 \otimes x$. In this case, we define $\mathrm{top}(D)=v_1$ and $\mathrm{bot}(D)=v_2$.

Let $\Lambda_{m,n} = \{(k,\mathbf{i}) \mid 0 \le k \le [n/2],, \mathbf{i} \in \Lambda(m,n-2k)\}$. For any $(k,\mathbf{i}),(l,\mathbf{j}) \in \Lambda_{m,n}$, say $(k,\mathbf{i}) \le (l,\mathbf{j})$ if either $k>l$ or $k=l$ and $\mathbf{i} \le \mathbf{j}$. Then $(\Lambda_{m,n},\le)$ is a poset. For $v_1,v_2 \in P(n,k)$ and $\mathbf{i} \in \Lambda(m,n-2k)$, define $C^{(k,\mathbf{i})}_{v_1,v_2} = v_1 \otimes v_2 \otimes C^{\mathbf{i}}_{1,1}$.

Proposition 2.5. [7, 5.3] *Let* R *be a splitting field of* x^m-1. *The set* $\{C^{(k,\mathbf{i})}_{v_1,v_2} \mid (k,\mathbf{i}) \in \Lambda_{n,m}, v_1,v_2 \in P(n,k)\}$ *is a cellular basis of* $TL_{m,n}(\boldsymbol{\delta})$.

Let $\Delta(k,\mathbf{i})$ be the cell module with respect to the cellular basis given in Proposition 2.5. Then

$$\Delta(k,\mathbf{i}) \cong V(n,k) \otimes_R v_0 \otimes_R \Delta(\mathbf{i}) \tag{2.2}$$

where $V(n,k)$ is the free R-module generated by $P(n,k)$ and v_0 is a fix element in $P(n,k)$. The following theorem is known as branching rule for the cell module $\Delta(k,\mathbf{i})$.

Proposition 2.6. [7, 7.1] *Suppose* $chR \nmid m$. *For* $\mathbf{i}=(i_1,i_2,\cdots,i_{n-2k}) \in \Lambda(m,n-2k)$, *define* $\mathbf{i}_0=(i_1,i_2,\cdots,i_{n-2k-1}) \in \Lambda(m,n-2k-1)$ *and* $\mathbf{i}\cup j=(i_1,i_2,\cdots,i_{n-2k},j) \in \Lambda(m,n-2k+1)$. *Then there is a short exact sequence*

$$0 \longrightarrow \Delta(k,\mathbf{i}_0) \longrightarrow \Delta(k,\mathbf{i})\downarrow \longrightarrow \bigoplus_{j=1}^{m} \Delta(k-1,\mathbf{i}\cup j) \longrightarrow 0, \tag{2.3}$$

where we denote by $M\downarrow$ *the restriction of a* $TL_{m,n}(\boldsymbol{\delta})$*-module* M *to a* $TL_{m,n-1}(\boldsymbol{\delta})$*-module.*

Proof. It is proved in [7, 7.1] that

$$0 \longrightarrow \Delta(k,\mathbf{i}_0) \longrightarrow \Delta(k,\mathbf{i})\downarrow \longrightarrow V(n-1,k-1) \otimes_R v_0 \otimes_R \Delta(\mathbf{i}) \otimes_R R\langle t_{n-2k+1}\rangle \longrightarrow 0.$$

Since $chR \nmid m$, $R\langle t_{n-2k+1}\rangle$ is semi-simple. Therefore, $R\langle t_{n-2k+1}\rangle \cong \oplus_{j=1}^m \Delta(j)$, where $\Delta(j)$ is the cell module of $R\langle t_{n-2k+1}\rangle$ with respect to the cellular basis given in Lemma 2.4 (the case

$m = 1$). By direct computation, we have

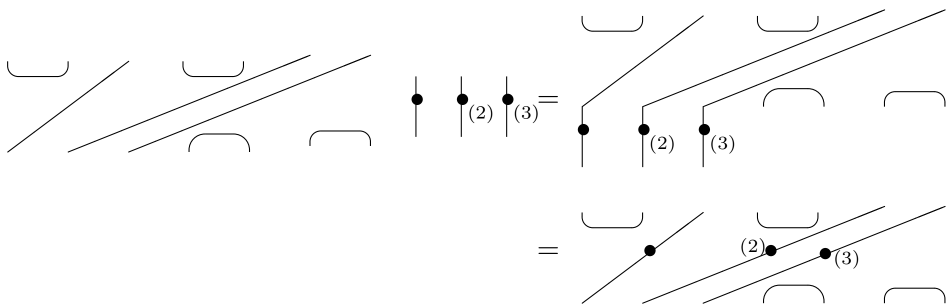
$$\Delta(\mathbf{i}) \otimes_R \Delta(j) \cong \Delta(\mathbf{i} \cup j).$$

By (2.2), we have (2.3). $\square$

As $G_{m,n}$-module, $\Delta(0, \mathbf{i}) \cong \Delta(\mathbf{i})$. Note that a cellular algebra is semi-simple if and only if all of its cell modules are pairwise non-isomorphic irreducible [2]. Therefore, that $TL_{m,n}(\boldsymbol{\delta})$ is semi-simple implies all $\Delta(\mathbf{i})$ are pairwise non-isomorphic irreducible. So, $G_{m,n}$ is semi-simple which is equivalent to the fact $chR \nmid m$. Moreover, $u_i \neq u_j$ for any $i \neq j$, $1 \leq i, j \leq m$.

In the sub-sequel, we assume $chR \nmid m$, $u_i = \xi^i$, $1 \leq i \leq m$ where ξ is a primitive m-th root of unity. The reason is that the semi-simplicity of $G_{m,n}$ is necessary for $TL_{m,n}(\boldsymbol{\delta})$ to be semi-simple.

For the latter use, we need another construction of the cell modules as follows. Let $J^{\geq k}_{m,n}$ (resp. $J^{>k}_{m,n}$) be the free R-submodule of $TL_{m,n}$ generated by labelled TL-diagrams with l horizontal arcs such that $l \geq k$ (resp. $l > k$). Let $I^k_{m,n}(\boldsymbol{\delta})$ be the submodule of $J^{\geq k}_{m,n}/J^{>k}_{m,n}$ generated by the coset of $v \otimes v_0 \otimes x$, with $v \in P(n,k)$, $x \in G_{m,n-2k}$, and $v_0 = \mathrm{top}(E_{n-2k+1} \cdots E_{n-1}) \in P(n,k)$. Then $I^k_{m,n}(\boldsymbol{\delta})$ is a right $G_{m,n-2k}$-module in which $x \in G_{m,n-2k}$ acts on the free vertices of $\mathrm{bot}(D)$ of, $D \in I^k_{m,n}(\boldsymbol{\delta})$. In the following we give an example to illustrate the action.



By the construction of cell modules, we have

$$\Delta(k, \mathbf{i}) \cong I^k_{m,n}(\boldsymbol{\delta}) \otimes_{G_{m,n-2k}} \Delta(\mathbf{i}) \tag{2.4}$$

Moreover, $\{v \otimes v_0 \otimes_{G_{m,n-2k}} C^{\mathbf{i}}_{11} \mid v \in P(n,k)\}$ is a free R-basis of $\Delta(k, \mathbf{i})$.

3. Restriction and induction

In this section, we assume that there is at least one non-zero parameter, say δ_i. Otherwise $\bar{\delta}_j = 0$ for $1 \leq j \leq m$ (see (4.1) for the definition of $\bar{\delta}_j$). By [7, 8.1], $TL_{m,n}(\boldsymbol{\delta})$ is not semi-simple.

Lemma 3.1. *Suppose $\delta_i \neq 0$. Let $e = \delta_i^{-1} T_n^i E_{n-1} \in TL_{m,n}(\boldsymbol{\delta})$. Then $e^2 = e$, and $eTL_{m,n}(\boldsymbol{\delta})e \cong TL_{m,n-2}(\boldsymbol{\delta})$.*

Proof. Each element in $eTL_{m,n}(\delta_{\mathbf{i}})e$ is a linear combination of the labelled TL-diagrams D in which top(D) (resp. bot(D)) contains a horizontal arc $\{n-1,n\}$ where there are i (resp. 0) dots. Let D^0 be the labelled TL-diagram obtained from D by removing the horizontal arc $\{n-1,n\}$ on top(D) and bot(D). By the definition of the product of two labelled TL-diagrams in Definition 2.1, one can verify easily that the R-linear isomorphism $\phi : eTL_{m,n}(\delta_{\mathbf{i}})e \to TL_{m,n-2}$ with $\phi(D) = \delta_i D^0$, is an algebraic isomorphism. □

Now we may use the idempotent e to define two functors F and G as follows.

Definition 3.2. Let $F : TL_{m,n}(\boldsymbol{\delta})$-mod $\longrightarrow TL_{m,n-2}(\boldsymbol{\delta})$-mod with $F(M) = eM$ and $G : TL_{m,n-2}(\boldsymbol{\delta})$-mod $\longrightarrow TL_{m,n}(\boldsymbol{\delta})$-mod with $G(M) = TL_{m,n}(\boldsymbol{\delta})e \otimes_{TL_{m,n-2}(\boldsymbol{\delta})} M$.

Proposition 3.3. *Assume* $\mathbf{i} \in \Lambda(m, n-2k)$.

a) If φ *is a non-zero* $TL_{m,n-2}(\boldsymbol{\delta})$*-homomorphism, then* $G(\varphi) \neq 0$.

b) FG *is an identity functor.*

c) $G(\Delta(k-1,\mathbf{i})) = \Delta(k,\mathbf{i}), \quad G(\Delta(k-1,\mathbf{i})\downarrow) = \Delta(k,\mathbf{i})\downarrow$*;*

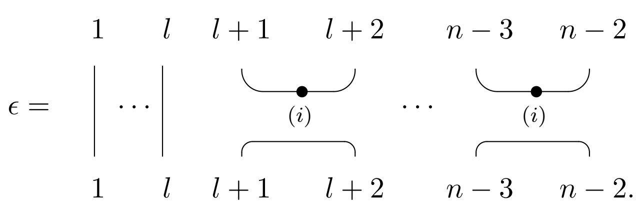
d) $F(\Delta(k,\mathbf{i})) = \Delta(k-1,\mathbf{i}), \quad F(\Delta(k,\mathbf{i})\downarrow) = \Delta(k-1,\mathbf{i})\downarrow$.

Proof. (a) and (b) follows from a general result in [3, 6.2]. (d) follows from (c) and (b) by applying the functor F on both side of (c).

Let $v_0 = \mathrm{top}(E_{n-2k+1}E_{n-2k+3}\cdots E_{n-1}) \in P(n,k)$. We claim, as $TL_{m,n}(\boldsymbol{\delta})$-modules,

$$I^k_{m,n}(\boldsymbol{\delta}) \cong TL_{m,n}(\boldsymbol{\delta})e \otimes_{TL_{m,n-2}(\boldsymbol{\delta})} I^{k-1}_{m,n-2}(\boldsymbol{\delta}). \tag{3.1}$$

In fact, let $l = n-2k$. Then $\epsilon = T^i_{l+1}T^i_{l+3}\cdots T^i_{n-3}E_{l+1}E_{l+3}\cdots E_{n-3} \in I^{k-1}_{m,n-2}(\boldsymbol{\delta})$, that is,



Suppose $D_1e \otimes D_2 \in TL_{m,n}(\boldsymbol{\delta})e \otimes_{TL_{m,n-2}(\boldsymbol{\delta})} I^{k-1}_{m,n-2}(\boldsymbol{\delta})$. Then $D_2 \cdot \epsilon = \delta_i^{k-1} D_2$, $eD_2 = D_2e$ and

$$D_1e \otimes D_2 = \delta_i^{1-k} D_1e \otimes D_2\epsilon = \delta_i^{-k} D_1 D_2^0 e \otimes \epsilon.$$

where D_2^0 can be obtained from D_2 by adding two horizontal arcs $\{n-1,n\}$ on the top and bottom row of D_2. Obviously, $D_1D_2^0 \in I^k_{m,n}(\delta_{\mathbf{i}})$. Therefore, any element in $TL_{m,n}(\boldsymbol{\delta})e \otimes_{TL_{m,n-2}(\boldsymbol{\delta})} I^{k-1}_{m,n-2}(\boldsymbol{\delta})$ can be expressed as a linear combination of the element $D_3e \otimes \epsilon$ with $D_3 = D_1D_2^0$. Define the R-linear map $\alpha : TL_{m,n}(\boldsymbol{\delta})e \otimes_{TL_{m,n-2}(\boldsymbol{\delta})} I^{k-1}_{m,n-2}(\boldsymbol{\delta}) \to I^k_{m,n}(\boldsymbol{\delta})$ with $\alpha(D_3e \otimes \epsilon) = D_3$.

Then α is an epimorphism. If $D_3 = 0$, then either $0 = D_3 \in TL_{m,n}(\boldsymbol{\delta})$ or $\mathrm{bot}(D_3)$ contains at least one extra arc, say $(i', i'+1)$, $i' \le n-2k-1$, in which there are s dots. So,

$$D_3 e \otimes \epsilon = \delta_i^{-1} D_3 T_{i'}^{i-s} E_{i'} T_{i'}^s e \otimes \epsilon = \delta_i^{-1} D_3 e \otimes T_{i'}^{i-s} E_{i'} T_{i'}^s \epsilon = \delta_i^{-1} D_3 e \otimes 0 = 0.$$

Therefore, α is injective. By (3.1) and (2.2),

$$\begin{aligned} G(\Delta(k-1,\mathbf{i})) &= TL_{m,n}(\boldsymbol{\delta})e \otimes_{TL_{m,n-2}(\boldsymbol{\delta})} \left(I_{m,n-2}^{k-1}(\boldsymbol{\delta}) \otimes_{G_{m,n-2k}} \Delta(\mathbf{i})\right) \\ &\cong \left(TL_{m,n}(\boldsymbol{\delta})e \otimes_{TL_{m,n-2}(\boldsymbol{\delta})} I_{m,n-2}^{k-1}(\boldsymbol{\delta})\right) \otimes_{G_{m,n-2k}} \Delta(\mathbf{i}) \\ &\cong I_{m,n}^{k}(\boldsymbol{\delta}) \otimes_{G_{m,n-2k}} \Delta(\mathbf{i}) \\ &= \Delta(k,\mathbf{i}). \end{aligned}$$

This completes the proof of the first isomorphism given in (c). The second isomorphism can be proved similarly. □

Definition 3.4. For any $TL_{m,n}(\boldsymbol{\delta})$-modules M and N, define

$$\left\langle M, N \right\rangle_n = \left\langle M, N \right\rangle_{TL_{m,n}(\boldsymbol{\delta})} = \dim_R \mathrm{Hom}_{TL_{m,n}(\boldsymbol{\delta})}(M,N).$$

Proposition 3.5. *Suppose* $\mathbf{i} \in \Lambda(m,n), \mathbf{j} \in \Lambda(m,n-2k)$ *and* $k_0 \in \mathbb{N}$. *Then* $\left\langle \Delta(k_0,\mathbf{i}), \Delta(k+k_0,\mathbf{j}) \right\rangle_{n+2k_0} \neq 0$ *if and only if* $\left\langle \Delta(0,\mathbf{i}), \Delta(k,\mathbf{j}) \right\rangle_n \neq 0$.

Proof. " $\Leftarrow$" follows from Proposition 3.3(a) and (c) by applying G repeatedly.

"$\Rightarrow$" Suppose $0 \neq \varphi \in \mathrm{Hom}_{TL_{m,n+2k_0}(\boldsymbol{\delta})}(\Delta(k_0,\mathbf{i}), \Delta(k+k_0,\mathbf{j}))$ and $W = \varphi(\Delta(k_0,\mathbf{i}))$. Let $e = \delta_i^{-1} T_{n+2k_0-1}^i E_{n+2k_0-1}$. We claim

$$eW \neq 0. \tag{3.2}$$

Otherwise, we have $eW = 0$. Let $v_i = \mathrm{top}(E_i) = \mathrm{bot}(E_i)$. Then

$$E_1 = \delta_i^{-2}(v_1 \otimes v_{n+2k_0-1} \otimes id) \cdot T_{n+2k_0-1}^i E_{n+2k_0-1} T_{n+2k_0-1}^i \cdot (v_{n+2k_0-1} \otimes v_1 \otimes id).$$

So, $E_1 W = 0$ which implies $EW = 0$ with $E = E_1 E_3 \cdots E_{2k_0-1}$. On the other hand, Let $U_0 = rad\Delta(k_0,\mathbf{i})$. Then either $\Delta(k_0,\mathbf{i}) = U_0$ or $\Delta(k_0,\mathbf{i})/U_0$ is irreducible [2, 3.2]. Let $\mathbf{m} = (m,m,\cdots,m) \in \Lambda(m,n)$. Since $E \in TL_{m,n+2k_0}^{(k_0,\mathbf{m})} \subset TL_{m,n+2k_0}^{\le(k_0,\mathbf{i})}$, Lemma 2.3 shows $EU_0 = 0$. We have $W = \varphi(\Delta(k_0,\mathbf{i})) \cong \Delta(k_0,\mathbf{i})/U$. We claim $U \subset U_0$. Otherwise, $U + U_0 = \Delta(k_0,\mathbf{i})$ and hence $U/(U_0 \cap U) \cong \Delta(k_0,\mathbf{i})/U_0$ is irreducible. So, there is a composition series of $\Delta(k_0,\mathbf{i})$ such that the multiplicity of $L(k_0,\mathbf{i})$ is greater than 2, a contradiction.

Let $y = \mathrm{top}(T_1^i T_3^i \cdots T_{2k_0-1}^i E)$. Then $v = y \otimes v_0 \otimes C_{1,1}^{\mathbf{i}} \in \Delta(k_0,\mathbf{i})$ is a non-zero element, where v_0 is a fixed element in $P(n+2k_0,k_0)$. Since $\delta_i \neq 0$, $T_1^i T_3^i \cdots T_{2k_0-1}^i E \cdot v = (\delta_i)^{k_0} v \neq 0$, which implies $v \notin U$. Therefore, $T_1^i T_3^i \cdots T_{2k_0-1}^i E(v+U) = \delta_i^{k_0}(v+U) \not\equiv 0 \bmod U$, which contradicts to the fact $eW = 0$. This completes the proof of (3.2).

If $eW \neq 0$, then $F(\varphi) \neq 0$. Now, the result follows from induction and (3.2). □

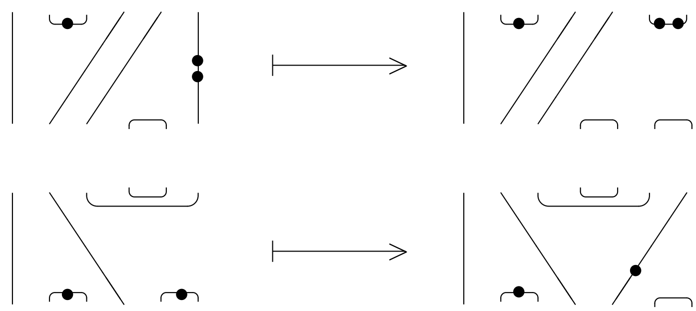
Proposition 3.6. *Suppose M is a $TL_{m,n}(\boldsymbol{\delta})$-module. Then $M\uparrow\cong G(M)\downarrow$, where $M\uparrow$ is the induced module of a $TL_{m,n}(\boldsymbol{\delta})$-module M to $TL_{m,n+1}(\boldsymbol{\delta})$. In particular, for any $\mathbf{i}\in\Lambda(m,n-2k)$, $\Delta(k,\mathbf{i})\uparrow\cong\Delta(k+1,\mathbf{i})\downarrow$.*

Proof. Suppose $x\in TL_{m,n+1}(\boldsymbol{\delta})$. Add $(n+2)$-th vertex on top(x) and bot(x) to get a new labelled TL-diagram D in which

(1) the $(n+2)$-th vertex of top(D) joins the vertex j if $\{j,n+1\}$ is an arc in x. Here $n+1$ is the $(n+1)$-th vertex in bot(x). Moreover, if there are s dots on the arc $\{j,n+1\}$, so is the new arc $\{j,n+2\}$

(2) $\{n+1,n+2\}$ is a horizontal arc in bot(D) in which there is no dot.

We give two examples to illustrate the above definition.



Define an R-linear map $\alpha: TL_{m,n+1}(\boldsymbol{\delta})\to TL_{m,n+2}(\boldsymbol{\delta})e$ with $\alpha(x)=D$. Obviously, α is an R-linear isomorphism. By the definition of the product of two labelled TL-diagrams, α is a left $TL_{m,n+1}(\boldsymbol{\delta})$-module and right $TL_{m,n}(\boldsymbol{\delta})$-module isomorphism. That is,

$$TL_{m,n+1}(\boldsymbol{\delta})\cong TL_{m,n+2}(\boldsymbol{\delta})e. \tag{3.3}$$

For any $TL_{m,n}(\boldsymbol{\delta})$-module M,

$$\begin{aligned} M\uparrow &\cong TL_{m,n+1}(\boldsymbol{\delta})\otimes_{TL_{m,n}(\boldsymbol{\delta})} M\\ &\cong TL_{m,n+2}(\boldsymbol{\delta})e\otimes_{TL_{m,n}(\boldsymbol{\delta})} M \text{ by (3.3)}\\ &\cong G(M)\downarrow. \end{aligned}$$

□

Corollary 3.7. *Suppose $chR\nmid m$. Assume $\mathbf{i}=(i_1,i_2,\dots,i_n)\in\Lambda(m,n)$. If $\mathbf{j}=(i_1,i_2,\dots,i_n,j)\in\Lambda(m,n+1)$, then $\left\langle \Delta(0,\mathbf{i})\uparrow,\Delta(0,\mathbf{j})\right\rangle_{n+1}\neq 0$.*

Proof. By Proposition 3.6, $\left\langle \Delta(0,\mathbf{i})\uparrow,\Delta(0,\mathbf{j})\right\rangle_{n+1}=\left\langle \Delta(1,\mathbf{i})\downarrow,\Delta(0,\mathbf{j})\right\rangle_{n+1}$. Now Proposition 2.6 implies that, for all $\mathbf{j}=(i_1,i_2,\dots,i_n,j), 1\le j\le m$, $\left\langle \Delta(1,\mathbf{i})\downarrow,\Delta(0,\mathbf{j})\right\rangle_{n+1}\neq 0$. □

Proposition 3.8. *Suppose $chR\nmid m$ and $\left\langle \Delta(0,\mathbf{i}),\Delta(k,\mathbf{j})\right\rangle_n\neq 0$ for $\mathbf{i}\in\Lambda(m,n)$ and $\mathbf{j}\in\Lambda(m,n-2k)$.*

(a) If $\mathbf{i}^0 = (i_1, i_2, \ldots, i_{n-1}) \in \Lambda(m, n-1)$*, then* $\left\langle \Delta(0, \mathbf{i}^0), \Delta(k, \mathbf{j}) \downarrow \right\rangle_{n-1} \neq 0$.

(b) Let $\mathbf{j}^0 = (j_1, j_2, \ldots, j_{n-2k-1})$ *and* $\mathbf{j}^1 = (j_1, j_2, \ldots, j_{n-2k}, j_0), 1 \leq j_0 \leq m$*. Then either* $\left\langle \Delta(0, \mathbf{i}^0), \Delta(k, \mathbf{j}^0) \right\rangle_{n-1} \neq 0$ *or* $\left\langle \Delta(0, \mathbf{i}^0), \Delta(k-1, \mathbf{j}^1) \right\rangle_{n-1} \neq 0$.

Proof. Since $\mathbf{i}^0 \in \Lambda(m, n-1)$, Corollary 3.7 implies $\left\langle \Delta(0, \mathbf{i}^0) \uparrow, \Delta(0, \mathbf{i}) \right\rangle_n \neq 0$. Since $chR \nmid m$, $\Delta(\mathbf{i})$ is a simple $G_{m,n}$-module, forcing $\Delta(0, \mathbf{i})$ to be an irreducible $TL_{m,n}(\boldsymbol{\delta})$-module. So, $\left\langle \Delta(0, \mathbf{i}^0) \uparrow, \Delta(k, \mathbf{j}) \right\rangle_n \neq 0$. Using Frobenius reciprocity, we get (a).

Let $V = \Delta(k, \mathbf{j}) \downarrow$. By Proposition 2.6, there is a submodule $W \subset V$ such that $W \cong \Delta(k, \mathbf{j}^0)$. $\mathbf{j}^0 = (j_1, j_2, \ldots, j_{n-2k-1})$.

Let $0 \neq S$ be the image of $\Delta(0, \mathbf{i}^0)$ in V. Since $\Delta(0, \mathbf{i}^0)$ is irreducible, $S \cong \Delta(0, \mathbf{i}^0)$. If $S \subset W$, $\left\langle \Delta(0, \mathbf{i}^0), \Delta(k, \mathbf{j}^0) \right\rangle_{n-1} \neq 0$.

If $S \not\subset W$, then $S \cap W = 0$. Thus, $(S \oplus W)/W \cong S/(W \cap S) = S$ is an irreducible submodule of V/W. By Proposition 2.6,

$$V/W \cong \bigoplus_{j=1}^{m} \Delta(k-1, \mathbf{j} \cup j).$$

Hence there is a $\mathbf{j}^1 = (j_1, j_2, \ldots, j_{n-2k}, j_0) \in \Lambda(m, n-2k+1)$ such that $(S \oplus W)/W \subset \Delta(k-1, \mathbf{j}^1)$, forcing $\left\langle \Delta(0, \mathbf{i}^0), \Delta(k-1, \mathbf{j}^1) \right\rangle_{n-1} \neq 0$. □

4. Semi-simplicity of the cyclotomic Temperley-Lieb algebras

In this section, we shall give the necessary and sufficient conditions on the semi-simplicity of $TL_{m,n}(\boldsymbol{\delta})$. The key is [7, 8.1]. First, let us recall some of results in [7].

Let $u_i = \xi^i$ where ξ is a primitive m-th root of unity. For any $\mathbf{i} = (i_1, i_2, \cdots, i_{n-2}) \in \Lambda(m, n-2)$, let

$$\Psi_{\mathbf{i}}(n, 1) = \begin{pmatrix} A & B_1 & & & & \\ B_1^T & A & B_2 & & & \\ & B_2^T & A & B_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & A & B_{n-2} \\ & & & & B_{n-2}^T & A \end{pmatrix},$$

where $B_j = (b_{st})$ with $b_{st} = u_{i_j}^{s-t}$ for $1 \leq s, t \leq m$, and B_i^T stands for the transpose of B_i, and

$$A = \begin{pmatrix} \delta_0 & \delta_1 & \cdots & \delta_{m-1} \\ \delta_1 & \delta_2 & \cdots & \delta_0 \\ \vdots & \vdots & \cdots & \vdots \\ \delta_{m-1} & \delta_0 & \cdots & \delta_{m-2} \end{pmatrix}.$$

Let $p(x) = \delta_0 x^{m-1} + \delta_1 x^{m-2} + \cdots + \delta_{m-1}$. Write

$$\frac{p(x)}{x^m - 1} = \frac{\bar{\delta}_1}{x - u_1} + \frac{\bar{\delta}_2}{x - u_2} + \cdots + \frac{\bar{\delta}_m}{x - u_m}. \tag{4.1}$$

Then

$$\bar{\delta}_j = p(u_j) / \prod_{i \neq j} (u_j - u_i). \tag{4.2}$$

Following [7], we partition $\mathbf{i} = (i_1, i_2, ..., i_{n-2})$ into $(i_{1,1}, i_{1,2}, ..., i_{1,j_1}, i_{2,1}, i_{2,2}, ..., i_{2,j_2}, ..., i_{r,j_r})$ with $j_1 + j_2 + ... + j_r = n - 2$ such that m divides $i_{p,q} + i_{p,q+1}$ for all p with $1 \leqslant q < j_p$ and that m does not divide $i_{p,j_p} + i_{p+1,1}$ for all $1 \leqslant p < r$. Let

$$P_n(x_1, ..., x_n) = \det \begin{pmatrix} x_1 & 1 & & & \\ 1 & x_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & x_{n-1} & 1 \\ & & & 1 & x_n \end{pmatrix}.$$

We call $P_n(x_1, x_2, \cdots, x_n)$ the n-th generalized Tchebychev polynomial. The following result was proved in [7, §8].

Proposition 4.1. *Keep the setup. Then*

$$\det \Psi_{\mathbf{i}}(n,1) = (-1)^{\frac{1}{2}m(m-1)(n-1)} m^{m(n-1)} \frac{(\bar{\delta}_1 \bar{\delta}_2 ... \bar{\delta}_m)^{n-1}}{\prod_{p=1}^{r} (\bar{\delta}_{m - i_{p,j_p}} \prod_{q=1}^{j_p} \bar{\delta}_{i_{p,q}})} \prod_{p=1}^{r} P_{j_p}(\bar{\delta}_{i_{p,1}}, \bar{\delta}_{i_{p,2}}, ..., \bar{\delta}_{i_{p,j_p}}).$$

Proposition 4.2. *Suppose* $\mathbf{i} \in \Lambda(m,n), \mathbf{j} \in \Lambda(m, n-2)$. *If* $\langle \Delta(0,\mathbf{i}), \Delta(1,\mathbf{j}) \rangle_n \neq 0$, *then* $\det \Psi_{\mathbf{j}}(n,1) = 0$.

Proof. Since $\langle \Delta(0,\mathbf{i}), \Delta(1,\mathbf{j}) \rangle_n \neq 0$, there is a $\varphi \in \mathrm{Hom}_{TL_{m,n}(\boldsymbol{\delta})}(\Delta(0,\mathbf{i}), \Delta(1,\mathbf{j}))$ such that $\varphi(v) \neq 0$ for some $v \in \Delta(0,\mathbf{i})$. Consider an element

$$T = \sum_{i=1}^{n-1} \sum_{s=0}^{m-1} T_i^s E_i T_i^s \in TL_{m,n}(\boldsymbol{\delta})$$

We have $T\varphi(v) = \varphi(Tv) = \varphi(0) = 0$. Write

$$\varphi(v) = \sum_{i=1}^{n-1} \sum_{s=0}^{m-1} a_{i,s} v_i^{(s)} \otimes v_0 \otimes C_{1,1}^{\mathbf{j}},$$

where $v_i^{(s)} = \mathrm{top}(T_i^s E_i)$ and v_0 is a fixed element in $P(n,1)$. Since

$$(v_1 \otimes v_1 \otimes C_{1,1}^{\mathbf{j}})(v_2 \otimes v_2 \otimes C_{1,1}^{\mathbf{j}}) \equiv v_1 \otimes v_2 \otimes \phi_{v_1,v_2}^{(n,1)}(t_1, t_2, ..., t_{n-2})(C_{1,1}^{\mathbf{j}})^2 \pmod{TL_{n,m}^{<(1,\mathbf{j})}},$$

for some elements $\phi^{(n,1)}_{v_1,v_2}(t_1,t_2,.....,t_{n-2})$ in $G_{m,n-2}$. By a direct computation, we have

$$0 = T\varphi(v) = \sum_{1\le i,j\le n-1}\sum_{0\le s,t\le m-1} \phi^{(n,1)}_{v_i^{(s)},v_j^{(t)}}(u_{j_1},u_{j_2},\dots,u_{j_{n-2}})a_{j,t}v_i^{(s)}\otimes v_0\otimes C^{\mathbf{j}}_{1,1}$$

Therefore, for all i,s, we have $\sum_{1\le j\le n-1}\sum_{0\le t\le m-1}\phi^{(n,1)}_{v_i^{(s)},v_j^{(t)}}(u_{j_1},u_{j_2},\dots,u_{j_{n-2}})a_{j,t} = 0$.

Since $\varphi(v)\ne 0$, there is at least one of $a_{i,t}\ne 0$, which implies $\det\Psi_{\mathbf{j}}(n,1)=0$. □

Proposition 4.3. *Suppose R is a splitting field of x^m-1 with $chR\nmid m$. If $\det\Psi_{\mathbf{i}}(l,1)\ne 0$ for all $2\le l\le n$ and $\mathbf{i}\in\Lambda(m,l-2)$, then $TL_{m,n}(\boldsymbol{\delta})$ is semi-simple.*

Proof. It is proved in [7] that $TL_{m,n}(\boldsymbol{\delta})$ is a cellular algebra. Note that a cellular algebra is semi-simple if and only if all of its cell modules are pairwise non-isomorphic irreducible (see [2]). So, $TL_{m,n}(\boldsymbol{\delta})$ is not semi-simple if there is a cell module, say $\Delta(k_1,\mathbf{i})$, which is not irreducible. Thus, the length of $\Delta(k_1,\mathbf{i})$ is strictly greater than 1, and there is an irreducible proper submodule D of $\Delta(k_1,\mathbf{i})$. Note that any simple module of a cellular algebra is the simple head of a cell module. Therefore, D is the simple quotient of a cell module, say $\Delta(k_2,\mathbf{j})$. Since D is a composition factor of $\Delta(k_1,\mathbf{i})$, it follows from Definition 2.2 and (2.1) that $(k_1,\mathbf{i})\le(k_2,\mathbf{j})$. Moreover, $(k_1,\mathbf{i})\ne(k_2,\mathbf{j})$. Otherwise, $\Delta(k_1,\mathbf{i})$ would have a simple head D. So, the multiplicity of D in $\Delta(k_1,\mathbf{i})$ is at least two, a contradiction. We have $\left\langle \Delta(k_2,\mathbf{j}),\Delta(k_1,\mathbf{i})\right\rangle_n\ne 0$. Moreover, either $k_1>k_2$ or $k_1=k_2$ and $\mathbf{i}<\mathbf{j}$.

Suppose $k_1>k_2$. Using Proposition 3.5, we can assume $\mathbf{j}\in\Lambda(m,l), l=n-2k_2$. Let $k=k_1-k_2$. Then $\left\langle \Delta(0,\mathbf{j}),\Delta(k,\mathbf{i})\right\rangle_l\ne 0$. Applying Proposition 3.8 repeatedly, we can assume $k=1$. By Proposition 4.2, $\det\Psi_{\mathbf{i}}(l,1)=0$, a contradiction.

Suppose $k_1=k_2$ and $\mathbf{i}<\mathbf{j}$. By Proposition 3.5, $\left\langle \Delta(0,\mathbf{j}),\Delta(0,\mathbf{i})\right\rangle_{n-2k_1}\ne 0$, a contradiction since $\Delta(0,\mathbf{j})\not\cong\Delta(0,\mathbf{i})$ and both of them are irreducible.

Thus we have shown that under our assumption all cell modules are irreducible. It is clear that they are also pairwise non-isomorphic. Hence $TL_{m,n}(\boldsymbol{\delta})$ is semi-simple. □

Lemma 4.4. *Suppose $\det\Psi_{\mathbf{i}}(n,1)\ne 0$ for all $\mathbf{i}\in\Lambda(m,n-2)$ with $m\ge 2$. Then $\bar\delta_i\ne 0$ for any $i, 1\le i\le m$.*

Proof. Take $\mathbf{i}=(m,m,\cdots,m)\in\Lambda(m,n-2)$. Then $\mathbf{i}$ can be divided into one part with $j_1=n-2$. By Proposition 4.1, $\bar\delta_i\ne 0$, $1\le i\le m-1$ since they are the factors of $\det\Psi_{\mathbf{i}}(n,1)$. Take $\mathbf{i}=(1,1,\cdots,1)\in\Lambda(m,n-2)$. Then $\mathbf{i}$ can be divided into either one part if $m=2$ or $n-2$ parts if $m>2$. By Proposition 4.1, $\bar\delta_m\ne 0$ since it is a factor of $\det\Psi_{\mathbf{i}}(n,1)$ in any case. □

It is proved in [7, 8.1] that $\det\Psi_{\mathbf{i}}(n,1)\ne 0$ for all $\mathbf{i}\in\Lambda(m,n-2)$ and $chR\nmid m$ if $TL_{m,n}(\boldsymbol{\delta})$ is semi-simple. The following is the inverse of this result.

Proposition 4.5. *Suppose R is a splitting field of x^m-1 with $chR\nmid m$ and $m\ge 2$. If $\det\Psi_{\mathbf{i}}(n,1)\ne 0$ for all $\mathbf{i}\in\Lambda(m,n-2)$, then $TL_{m,n}(\boldsymbol{\delta})$ is semi-simple.*

Proof. By Proposition 4.3, we need prove $\det \Psi_{\mathbf{i}}(l,1) \neq 0$ for all $2 \leq l \leq n, \mathbf{i} \in \Lambda(m, l-2)$ under our assumption. If $\det \Psi_{\mathbf{i}}(l,1) = 0$ for some $l, l \neq n$ and $\mathbf{i} \in \Lambda(m, l-2)$, then $P_{j_p}(\bar{\delta}_{i_{p,1}}, \bar{\delta}_{i_{p,2}}, ..., \bar{\delta}_{i_{p,j_p}}) = 0$ for some $p, 1 \leq p \leq r$ by Proposition 4.1 and Lemma 4.4.

On the other hand, take $\mathbf{i}_0 = (i_1, i_2, \cdots, i_{l-2}, a, a, \cdots, a) \in \Lambda(m, n-2)$ with $m \nmid (i_{l-2} + a)$. By Proposition 4.1, $P_{j_p}(\bar{\delta}_{i_{p,1}}, \bar{\delta}_{i_{p,2}}, ..., \bar{\delta}_{i_{p,j_p}})$ must be a factor of $\det \Psi_{\mathbf{i}_0}(n,1)$ and hence $\det \Psi_{\mathbf{i}_0}(n,1) = 0$, a contradiction. □

Remark. The reason we assume $m \geq 2$ is that we need the fact that i_{l-2} and a cannot be in the same part. When $m = 1$, we cannot use the above argument. However, one can get a necessary and sufficient condition for $TL_{n,1}$ to be semi-simple [10, §5].

Together with [7, 8.1] and Proposition 4.5, we have the main result of this paper as follows. Note that Theorem 4.6 is not true if $m = 1$.

Theorem 4.6. *Suppose $m \geq 2$. Let R be a splitting field of $x^m - 1$, containing $1, \delta_0, \cdots, \delta_{m-1}$. Then the following conditions are equivalent.*

(a) $TL_{m,n}(\boldsymbol{\delta})$ is semi-simple.

(b) $TL_{m,n}(\boldsymbol{\delta})$ is split semi-simple.

(c) $chR \nmid m$ and $\det \Psi_{\mathbf{i}}(n,1) \neq 0$ for all $\mathbf{i} \in \Lambda(m, n-2)$.

(d) All cell modules $\Delta(k, \mathbf{i})$ with $(k, \mathbf{i}) \in \Lambda_{n,m}$ are pairwise non-isomorphic irreducible.

(e) All cell modules $\Delta(k, \mathbf{i})$ with $(k, \mathbf{i}) \in \Lambda_{n,m}, k \in \{0, 1\}$ are pairwise non-isomorphic irreducible.

Proof. Since $TL_{m,n}(\boldsymbol{\delta})$ is a cellular algebra, (a), (b) and (d) are equivalent. By [7, 8.1], (c) and (e) are equivalent. By Proposition 4.5 and [7, 8.1], (a) and (c) are equivalent. □

The following Corollary follows immediately from [7, 8.1] and Proposition 4.5.

Corollary 4.7. *Keep the setup. Then $TL_{m,n}(\boldsymbol{\delta})$ is semi-simple if and only if*

(a) $chR \nmid m$

(b) $P_1(\bar{\delta}_i) = \bar{\delta}_i \neq 0, 1 \leq i \leq m$.

(c) For any $(i_1, i_2, \cdots, i_l) \in \Lambda(m, l)$ with $m \mid (i_j + i_{j+1}), 1 \leq j \leq l-1$, $P_l(\bar{\delta}_{i_1}, \bar{\delta}_{i_2}, \cdots, \bar{\delta}_{i_l}) \neq 0$, $2 \leq l \leq n$.

Remark. When $m = 1$, $\Lambda(m, n)$ contains only one element $(1, 1, \cdots, 1)$ which can be partitioned into one part. In this case, Corollary 4.7 is Westbury's Theorem given in [10, §5].

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