Higher algebraic K-groups and $\mathcal{D}$-split sequences

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Abstract

In this paper, we use $\mathcal{D}$-split sequences and derived equivalences to provide formulas for calculation of higher algebraic K-groups (or mod-p K-groups) of certain matrix subrings which cover tiled orders, rings related to chains of Glaz-Vasconcelos ideals, and some other classes of rings. In our results, we do not assume any homological requirements on rings and ideals under investigation, and therefore extend sharply many existing results of this type in the algebraic K-theory literature to a more general context.

1 Introduction

One of the fundamental questions in the algebraic K-theory of rings is to understand and calculate higher algebraic K-groups K_n of rings, which were deeply developed in a very general context by Quillen in [19] for exact categories and by Waldhausen in [24] for Waldhausen categories. On the one hand, the usual methods for computing K_n may be the fundamental theorem, splitting morphisms, or certain long exact sequences of K_n-groups, namely, Mayer-Vietoris sequences, localization sequences or excision. In this direction there is a lot of literature (for example, see [8, 15, 18, 25, 26, 27], and others). On the other hand, we know that derived-equivalent rings share many common homological and numerical features, in particular, they have the isomorphic higher algebraic K_n-groups for all $n \geq 0$ (see [6]). This means that, in order to understand the higher K-groups K_n of a ring, one might refer to another ring which is derived-equivalent to the given one, and which may hopefully have a simple form so that its K_n-groups can be determined easily. This idea, however, seems not much to be benefited in the study of higher algebraic K-theory of rings, especially in dealing with calculation of K_n-groups.

In the present note, we shall use ring extensions and derived equivalences as reduction techniques to investigate the higher algebraic K_n-groups of certain matrix subrings which include many maximal orders, hereditary orders, tiled orders, endomorphism rings of chains of Glaz-Vasconcelos ideals, and other classes of rings. To produce such derived equivalences, we shall employ $\mathcal{D}$-split sequences defined in [10]. In this way, we reduce our calculation inductively to that of certain triangular matrix rings. The advantage of our method is: We not only drop all homological conditions on rings and ideals under investigation, but also extend many existing results (see [1, 8, 13]) of this type in the literature to a more general context. Our main results in this note can be stated as follows.

Theorem 1.1. *Suppose that $p \geq 2$, m and s are positive integers such that s is divisible by p. Let R be a $\mathbb{Z}/p^m\mathbb{Z}$-algebra with identity, and let I, I_i and I_{ij} be (not necessarily projective) ideals of R. We denote by $K_*(R)$ the $*$-th algebraic K-group of R with $* \in \mathbb{N}$.*

(1) *If $I_{ij} \subseteq I$ for all i, j, $I_{kj} \subseteq I_{ij}$ for $k \leq i$, $I_{ki} \subseteq I_{kj}$ for $j \leq i$ and $I_{ik}I_{kj} \subseteq I_{ij}$ for $i < k < j$, then*

$$S := \begin{pmatrix} R & I_{12} & I_{13} & \cdots & I_{1\,n} \\ I & R & I_{23} & \cdots & I_{2\,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ I & \cdots & I & R & I_{n-1\,n} \\ I & \cdots & I & I & R \end{pmatrix}$$

is a ring, and

$$K_*(S) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}] \simeq K_*(R) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}] \oplus \bigoplus_{j=2}^{n} K_*(R/I_{j-1\,j}) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}].$$

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(2) *For $2 \le i \le n$, suppose that R_i is a subalgebra of R with the same identity. If $I_{i+1} \subseteq I_i \subseteq R_i$ for all i, $I_j \subseteq I_{ij} \subseteq I$ for all i, j, and $I_{ik}I_{kj} \subseteq I_{ij}$ for $j < k < i$, then*

$$T := \begin{pmatrix} R & I_2 & I_3 & \cdots & I_n \\ I & R_2 & I_3 & \cdots & I_n \\ I & I_{32} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & R_{n-1} & I_n \\ I & I_{n2} & \cdots & I_{n\,n-1} & R_n \end{pmatrix}$$

is a ring, and

$$K_*(T) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}] \simeq K_*(R) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}] \oplus \bigoplus_{j=2}^{n} K_*(R_j/I_j) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}].$$

As pointed out in Section 6 below, Theorem 1.1 holds true for the mod-p K-groups $K_*(-, \mathbb{Z}/p\mathbb{Z})$ if we assume in Theorem 1.1 that R is a $\mathbb{Z}[\frac{1}{p}]$-algebra and $p \not\equiv 2$ (mod 4), that is, under these two assumptions, one can replace $K_*(-) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}]$ by $K_*(-, \mathbb{Z}/p\mathbb{Z})$ in Theorem 1.1.

The proof of the above result is based on the following observation. Note that the assumptions in Theorem 1.2(2) below is weaker than the ones in Theorem 1.1(2) above.

Theorem 1.2. *Let R be a ring with identity, and let I_{ij} be (not necessarily projective) ideals of R. We denote by $K_*(R)$ the $*$-th algebraic K-group of R with $* \in \mathbb{N}$.*

(1) *If $I_{kj} \subseteq I_{ij}$ for $k \le i$, $I_{ki} \subseteq I_{kj}$ for $j \le i$ and $I_{ik}I_{kj} \subseteq I_{ij}$ for $i < k < j$, then*

$$S := \begin{pmatrix} R & I_{12} & I_{13} & \cdots & I_{1\,n} \\ R & R & I_{23} & \cdots & I_{2\,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ R & \cdots & R & R & I_{n-1\,n} \\ R & \cdots & R & R & R \end{pmatrix}$$

is a ring, and

$$K_*(S) \simeq K_*(R) \oplus \bigoplus_{j=2}^{n} K_*(R/I_{j-1\,j}).$$

(2) *For $2 \le i \le n$, suppose that R_i is a subring of R with the same identity, that $I_i \subseteq R_i$ is a right ideal of R_i, and that I_i is a left ideal of R. If $I_{i+1} \subseteq I_i$ for all i, $I_j \subseteq I_{ij}$ for all i, j, $I_iI_{ij} \subseteq I_j$ for $j < i$, and $I_{ik}I_{kj} \subseteq I_{ij}$ for $j < k < i$, then*

$$T := \begin{pmatrix} R & I_2 & I_3 & \cdots & I_n \\ R & R_2 & I_3 & \cdots & I_n \\ R & I_{32} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & R_{n-1} & I_n \\ R & I_{n2} & \cdots & I_{n\,n-1} & R_n \end{pmatrix}$$

is a ring, and

$$K_*(T) \simeq K_*(R) \oplus \bigoplus_{j=2}^{n} K_*(R_j/I_j).$$

The strategy of our proofs of the theorems is first to use ring extensions, which are motivated from [28], and then to combine K-groups in Mayer-Vietoris sequences with K-groups of rings which are linked by derived equivalences produced from certain $\mathcal{D}$-split sequences.

This note is organized as follows. In Section 2, we recall some definitions and elementary facts on derived equivalences needed in the later proofs. In Section 3, we construct $\mathcal{D}$-split sequences by ring extensions and calculate the endomorphism rings of tilting modules related to these sequences. In Section 4, we prove the main results and state some of its consequences. Our proofs of the above results also give an explanation of the multiplicity factor $n-1$ in the isomorphisms of K_n-groups of the rings in [8] and [13]. In Section 5, we calculate K_0 and K_1 for some matrix subrings which are not covered by the main results. In fact, for K_0, we can remove some imposed conditions and say a little bit more, see Proposition 5.1 below. In Section 6, we show that the main result Theorem 1.1 holds for mod-p K-theory by outlining the key ingredients of its proof. In Section 7, we give some examples to show how our method can work, here GV-ideals in commutative rings enter into our play. These examples demonstrate also that the matrix rings studied in Section 3 really occur, as the endomorphism rings of chains of GV-ideals, in the field of commutative algebra.

2 Preliminaries

Let A be a ring with identity. By an A-module we mean a left A-module. Let A-Mod (respectively, A-mod) denote the category of all (respectively, finitely generated) left A-modules. Similarly, by A-Proj (respectively, A-proj) we denote the full subcategory of all (finitely generated) projective A-modules in R-Mod. For an A-module M, we denote by proj.dim$({}_AM)$ the projective dimension of M. Let $\mathscr{K}^b(A\text{-proj})$ be the bounded homotopy category of the additive category A-proj. The unbounded derived category of A-Mod is denoted by $\mathscr{D}(A)$, whereas the bounded derived category of A-Mod is denoted by $\mathscr{D}^{\mathrm{b}}(A)$. We say that two rings A and B are derived equivalent if $\mathscr{D}(A)$ and $\mathscr{D}(B)$ are equivalent as triangulated categories. It is well-known that if $\mathscr{D}^{\mathrm{b}}(A)$ and $\mathscr{D}^{\mathrm{b}}(B)$ are equivalent as triangulated categories then $\mathscr{D}(A)$ and $\mathscr{D}(B)$ are equivalent as triangulated categories.

Given an additive category $\mathcal{C}$ and an object X in $\mathcal{C}$, we denote by add(X) the full subcategory of $\mathcal{C}$ consisting of all objects which are direct summands of direct sums of finitely many copies of X.

For derived equivalences, Rickard's Morita theory [22] is very useful.

Theorem 2.1. [22] *For two rings A and B with identity, the following are equivalent:*
(a) $\mathscr{D}^{\mathrm{b}}(A)$ *and* $\mathscr{D}^{\mathrm{b}}(B)$ *are equivalent as triangulated categories.*
(b) $\mathscr{K}^b(A\text{-proj})$ *and* $\mathscr{K}^b(B\text{-proj})$ *are equivalent as triangulated categories.*
(c) $B \simeq \mathrm{End}_{\mathscr{K}^b(A\text{-proj})}(T^\bullet)$, *where* $T^\bullet$ *is a complex in* $\mathscr{K}^b(A\text{-proj})$ *satisfying*
(1) $\mathrm{Hom}(T^\bullet, T^\bullet[i]) = 0$ *for* $i \neq 0$, *and*
(2) add$(T^\bullet)$ *generates* $\mathscr{K}^b(A\text{-proj})$ *as a triangulated category.*

For derived equivalences, it is shown in [6] that the algebraic K-theory is an invariant. Recall that, for a ring A with identity, $K_n(A)$ denotes the n-th homotopy group of a certain space $K(A)$ produced by ones favorite K-theory defined for each $n \in \mathbb{N}$ (see [19], [24], [27]).

Theorem 2.2. [6] *If two rings A and B with identity are derived-equivalent, then their algebraic K-groups are isomorphic:* $K_*(A) \simeq K_*(B)$ *for all* $* \in \mathbb{N}$.

As is known, Morita equivalences are derived equivalences. Thus, if A and B are Morita equivalent, then their algebraic K-groups are isomorphic.

Another special class of derived equivalences can be constructed by tilting modules initialled from the representation theory of finite-dimensional algebras (for example, see [2]). Recall that a module T over a ring A is called a tilting module if the following three conditions are satisfied:

(1) T has a finite projective resolution $0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow T \longrightarrow 0$, where each P_i is a finitely generated projective A-module;

(2) $\mathrm{Ext}^i_A(T,T) = 0$ for all $i > 0$, and

(3) there is an exact sequence $0 \longrightarrow A \longrightarrow T_0 \longrightarrow \cdots \longrightarrow T_m \longrightarrow 0$ of A-modules with each T_i in add(T).

Note that, for a tilting module T, the projective resolution $P^\bullet$ of T satisfies (1) and (2) of Theorem 2.1(c). Thus, if ${}_AT$ is a tilting A-module then A and $\mathrm{End}_A(T)$ are derived-equivalent. To produce tilting modules, one may use the notion of $\mathcal{D}$-split sequences. Now let us recall the definition of $\mathcal{D}$-split sequences from [10].

Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ a full subcategory of $\mathcal{C}$. A sequence

$$X \xrightarrow{f} M \xrightarrow{g} Y$$

of morphisms between objects in $\mathcal{C}$ is called a $\mathcal{D}$-split sequence if

(1) $M \in \mathcal{D}$,

(2) f is a left $\mathcal{D}$-approximation of X, that is, $\mathrm{Hom}_{\mathcal{C}}(f,D'):\mathrm{Hom}_{\mathcal{C}}(M,D')\longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,D')$ is surjective for all $D'\in\mathcal{D}$, and g is a right $\mathcal{D}$-approximation of Y, that is, $\mathrm{Hom}_{\mathcal{C}}(D',g):\mathrm{Hom}_{\mathcal{C}}(D',M)\longrightarrow \mathrm{Hom}_{\mathcal{C}}(D',Y)$ is surjective for all $D'\in\mathcal{D}$, and

(3) f is a kernel of g, and g is a cokernel of f.

Examples of $\mathcal{D}$-split sequences include Auslander-Reiten sequences and short exact sequences of the form $0\to X\to P\to Y\to 0$ in A-Mod with P projective-injective. A non-example is the sequence $0\to\mathbb{Z}\to\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}\to 0$ of ableilan groups, that is, this sequence is not an $\mathrm{add}({}_{\mathbb{Z}}\mathbb{Q})$-split sequence. For more examples, one may find in [10], and also in the next section as well as in the last section of the present paper.

Given a $\mathcal{D}$-split sequence $X\to M'\to Y$, with $\mathcal{D}=\mathrm{add}(M)$ for M an object in $\mathcal{C}$, it is shown in [10] that there is a tilting module T over $\mathrm{End}_{\mathcal{C}}(X\oplus M)$ of projective dimension at most 1 such that $\mathrm{End}(T)$ is isomorphic to $\mathrm{End}_{\mathcal{C}}(M\oplus Y)$. Thus $\mathrm{End}_{\mathcal{C}}(X\oplus M)$ and $\mathrm{End}_{\mathcal{C}}(M\oplus Y)$ are derived-equivalent, and have the isomorphic algebraic K-theory by Theorem 2.2.

3 Ring extensions and derived equivalences

Ring extensions were used in [28] to study the finitistic dimensions of algebras. In this section, we shall use ring extensions to construct $\mathcal{D}$-split sequences which will be applied to calculation of the algebraic K-groups of rings in the next section.

We first establish the following general fact.

Lemma 3.1. *Let $B\subseteq A$ be an extension of rings with the same identity.*

(1) *If $\mathrm{Ext}^1_B({}_BA,{}_BB)=0$, then the sequence*

$$(*)\qquad 0\longrightarrow B\longrightarrow A\longrightarrow A/B\longrightarrow 0$$

is an $\mathrm{add}({}_BA)$-split sequence in B-Mod. Thus $\mathrm{End}_B({}_BB\oplus{}_BA)$ and $\mathrm{End}_B({}_BA\oplus A/B)$ are derived-equivalent.

(2) *If ${}_BA$ is projective, then the above sequence is an $\mathrm{add}({}_BA)$-split sequence.*

(3) *Suppose that $\mathrm{Ext}^1_B({}_BA,{}_BA)=0$. If ${}_BA$ is finitely presented with $\mathrm{proj.dim}({}_BA)\leq 1$ (for instance, ${}_BA$ is projective and finitely generated), then $A\oplus A/B$ is a tilting B-module of projective dimension at most 1. In particular, $\mathrm{End}_B(A\oplus A/B)$ is derived-equivalent to B.*

Proof. (1) We have the following exact sequence

$$0\to\mathrm{Hom}_B({}_BA,B)\longrightarrow\mathrm{Hom}_B(A,A)\longrightarrow\mathrm{Hom}_B(A,A/B)\longrightarrow\mathrm{Ext}^1_B(A,B)\longrightarrow\mathrm{Ext}^1_B(A,A).$$

The condition $\mathrm{Ext}^1_B(A,B)=0$ implies that the canonical surjection $A\to A/B$ is a right $\mathrm{add}({}_BA)$-approximation of A/B. To see that the inclusion $B\longrightarrow A$ is a left $\mathrm{add}({}_BA)$-approximation of B, we note that each homomorphism from ${}_BB$ to ${}_BA$ is given by an element a in A. Thus it can be extended to a homomorphism from ${}_BA$ to ${}_BA$ by the right multiplication of a. Clearly, one can check that this is also true for any homomorphism from ${}_BB$ to a direct summands of ${}_BA$. Thus we see that the inclusion map from B to A is a left $\mathrm{add}({}_BA)$-approximation of B. Thus $(*)$ is an $\mathrm{add}({}_BA)$-split sequence in B-Mod, and therefore $\mathrm{End}_B({}_BB\oplus{}_BA)$ and $\mathrm{End}_B({}_BA\oplus A/B)$ are derived-equivalent by [10, Theorem 1.1]. This finishes the proof of Lemma 3.1(1).

(2) is a special case of (1).

(3) Let $T:={}_BA\oplus\ A/B$. Since ${}_BA$ is finitely presented of projective dimension at most one, there is an exact sequence $0\to P_1\to P_0\to{}_BA\to 0$ such that P_i are finitely generated projective B-modules and the following diagram is commutative:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & P_1 \arrow[r, equal] \arrow[d] & P_1 \arrow[d] & & \\
0 \arrow[r] & P \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & A/B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A/B \arrow[r] & 0. \\
 & 0 & 0 & &
\end{tikzcd}$$

From this diagram we see that T is finitely presented of projective dimension at most one. Thus the conditions (1) and (3) in the definition of tilting modules are satisfied. It remains to show $\mathrm{Ext}^1_B(A\oplus A/B, A\oplus A/B)=0$. This is equivalent to that $\mathrm{Ext}^1_B(A,A/B)=0$, $\mathrm{Ext}^1_B(A/B,A/B)=0$ and $\mathrm{Ext}^1_B(A/B,A)=0$ since $\mathrm{Ext}^1_B(A,A)=0$ by assumption.

Indeed, we have seen that the inclusion map λ from B into A is always a left $\mathrm{add}({}_BA)$-approximation of ${}_BB$. Thus the induced map $\lambda^* := \mathrm{Hom}_B(\lambda, A)$ is surjective. Hence, by applying $\mathrm{Hom}_B(-,A)$ to the canonical exact sequence $(*)$, we get an exact sequence

$$0\to \mathrm{Hom}_B(A/B,A)\longrightarrow \mathrm{Hom}_B(A,A)\xrightarrow{\lambda^*}\mathrm{Hom}_B(B,A)\longrightarrow \mathrm{Ext}^1_B(A/B,A)\longrightarrow \mathrm{Ext}^1_B(A,A),$$

which shows $\mathrm{Ext}^1_B(A/B,A)=0$. If we apply $\mathrm{Hom}_B(A/B,-)$ to the canonical exact sequence, then we get an exact sequence:

$$\mathrm{Ext}^1_B(A/B,B)\longrightarrow \mathrm{Ext}^1_B(A/B,A)\longrightarrow \mathrm{Ext}^1_B(A/B,A/B)\longrightarrow 0$$

since the projective dimension of A/B is at most 1. This implies $\mathrm{Ext}^1_B(A/B,A/B)=0$. Similarly, applying $\mathrm{Hom}_B(A,-)$ to the canonical exact sequence $(*)$, we can deduce $\mathrm{Ext}^1_B(A,A/B)=0$. Thus we complete the proof of (3). $\square$

Remark. Sometimes the following observation is useful for getting $\mathcal{D}$-split sequences: Suppose that e and f are idempotent elements in a ring R and $a\in eRf$. Then the right multiplication map $Re\to Rf$, defined by $x\mapsto xa$ for $x\in Re$, is a left $\mathrm{add}(Rf)$-approximation of Re if and only if $eRf = afRf$. Thus, if the right multiplication map is injective, then $0\to Re\to Rf\to Rf/Rea\to 0$ is an $\mathrm{add}(Rf)$-split sequence if and only if $eRf=afRf$. For instance, the sequence $0\to\mathbb{Z}\xrightarrow{\cdot 2}\mathbb{Z}\to\mathbb{Z}/2\mathbb{Z}\to 0$ is not an $\mathrm{add}(\mathbb{Z})$-split sequence.

Let us mention an example of ring extensions which satisfy the conditions in Lemma 3.1. Recall that an extension $B\subseteq A$ of rings is called a quasi-Frobenius extension if ${}_BA$ is finitely generated and projective, and the bimodule ${}_AA_B$ is isomorphic to a direct summand of the direct sum of finitely many copies of ${}_A\mathrm{Hom}_B({}_BA_A, {}_BB)_B$. Thus each quasi-Frobenius extension $B\subseteq A$ provides an $\mathrm{add}({}_BA)$-split sequence

$$0\longrightarrow B\longrightarrow A\longrightarrow A/B\longrightarrow 0,$$

and a tilting B-module $A\oplus A/B$ by Lemma 3.1.

Now we consider some consequences of Lemma 3.1, which are needed in the next section.

Let R be a ring with identity and I_{ij} ideals in R with $1\le i<j\le n$, such that

(1) $I_{kj}\subseteq I_{ij}$ for $k\le i$,

(2) $I_{ki}\subseteq I_{kj}$ for $j\le i$, and

(3) $I_{ik}I_{kj}\subseteq I_{ij}$ for $i<k<j$. Then

$$B := \begin{pmatrix} R & I_{12} & I_{13} & \cdots & I_{1\,n}\\ R & R & I_{23} & \cdots & I_{2\,n}\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ R & R & \cdots & R & I_{n-1\,n}\\ R & R & \cdots & R & R\end{pmatrix}$$

is a ring. The rings of this form include tiled triangular orders and maximal orders [21]. They occur also as the endomorphism rings of chains of Glaz-Vasconcelos ideals of rings, see Section 7.

The following lemma shows that we may use derived equivalences to simplify the ring B.

Lemma 3.2. *Let B be the ring defined above. Then B is derived-equivalent to*

$$C := \begin{pmatrix} R & I_{12} & I_{13} & \cdots & I_{1\,n-1} & I_{1\,n-1}/I_{1n}\\ R & R & I_{23} & \cdots & I_{2n-1} & I_{2n-1}/I_{2n}\\ R & R & R & \ddots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \ddots & I_{n-2\,n-1} & I_{n-2\,n-1}/I_{n-2\,n}\\ R & R & R & \cdots & R & R/I_{n-1\,n}\\ 0 & 0 & 0 & \cdots & 0 & R/I_{n-1\,n}\end{pmatrix}.$$

Proof. We make the following conventions on notations. Let $S = M_n(R)$, the $n \times n$ matrix ring over R. Let e_i be the $n \times n$ matrix with 1_R in (i,i)-entry and zero in other entries. For convenience, we denote by $e_{i,j}(x)$ the matrix with x in (i,j)-position, and zero in other positions, and by B_{ij} the (i,j)-component of the matrix subring B of S, that is, the set of (i,j)-entries of all matrices in B. We define

$$A := \begin{pmatrix} R & I_{12} & \cdots & I_{1\,n-1} & I_{1\,n-1} \\ R & R & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & I_{n-2\,n-1} & I_{n-2\,n-1} \\ R & R & \cdots & R & R \\ R & R & \cdots & R & R \end{pmatrix}.$$

Note that the only difference between A and B is the last column. We can verify that A is a ring containing B as a subring.

Clearly, as a left B-module, ${}_BA \simeq Be_1 \oplus \cdots \oplus Be_{n-1} \oplus Be_{n-1}$. Thus ${}_BA$ is finitely generated and projective. Furthermore, it follows that B is Morita equivalent to $\mathrm{End}_B(B \oplus {}_BA)$ and that the latter is derived-equivalent to $\mathrm{End}_B({}_BA \oplus A/B)$ by Lemma 3.1. Thus B is derived-equivalent to $\mathrm{End}_B(Be_1 \oplus \cdots \oplus Be_{n-1} \oplus\ Ae_n/Be_n)$. For simplicity, we denote by Q the B-module Ae_n/Be_n. Note that $Ae_n \simeq Be_{n-1}$ as B-modules, and that we have a canonical exact sequence:

$$(*) \qquad 0 \longrightarrow Be_n \xrightarrow{\lambda} Be_{n-1} \xrightarrow{\pi} Q \longrightarrow 0,$$

where λ is the composition of the inclusion of Be_n into Ae_n with the right multiplication $\cdot e_{n,n-1}$, and π is the composite of the right multiplication by $\cdot e_{n-1,n}$ with the canonical surjective map $Ae_n \to Ae_n/Be_n$.

In the following, we shall prove that $\mathrm{End}_B(Be_1 \oplus \cdots \oplus Be_{n-1} \oplus Q)$ is isomorphic to C.

First, we define a map $\varphi : R \longrightarrow \mathrm{Hom}_B(Q,Q)$ as follows: For $b \in R$, let $\cdot e_nbe_n$ be the right multiplication map from Be_n to Be_n. This is well-defined by our assumptions. Also, let $\cdot e_{n-1}be_{n-1}$ be the right multiplication map from Be_{n-1} to itself. Then we see that $\lambda(\cdot e_{n-1}be_{n-1}) = (\cdot e_nbe_n)\lambda$. So, there is a unique $\alpha \in \mathrm{Hom}_B(Q,Q)$ making the following diagram commutative:

$$\begin{tikzcd} 0 \arrow[r] & Be_n \arrow[r, "\lambda"] \arrow[d, "\cdot e_nbe_n"'] & Be_{n-1} \arrow[r, "\pi"] \arrow[d, "\cdot e_{n-1}be_{n-1}"'] & Q \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & Be_n \arrow[r, "\lambda"] & Be_{n-1} \arrow[r, "\pi_j"] & Q \arrow[r] & 0. \end{tikzcd} \qquad (**)$$

Hence, we can define the image of b under φ is α. Clearly, if $b,b' \in R$, then $(b+b')\varphi = (b)\varphi + (b')\varphi$. Since $e_n(bb')e_n = e_nbe_nb'e_n$, we also have $(bb')\varphi = (b\varphi)(b'\varphi)$. Thus φ is a homomorphism of rings.

Now, we calculate the kernel of φ. Suppose $b \in R$ such that $\alpha = b\varphi = 0$. Then the map $\cdot e_{n-1}be_{n-1}$ factorizes through λ. This means that there is an element $r \in B_{n-1\,n}$ such that $\cdot e_{n-1}be_{n-1} = (\cdot e_{n-1}re_n)\lambda$ and $\cdot e_nbe_n = \lambda(\cdot e_{n-1}re_n)$. Hence $b = r \in B_{n-1\,n}$. Thus $\mathrm{Ker}(\varphi) \subseteq B_{nn} \cap B_{n-1\,n} = I_{n-1\,n}$. Since any map $\cdot e_nbe_n$ from Be_n to Be_n with $b \in B_{nn} \cap B_{n-1\,n}$ factorizes through λ, the corresponding α is zero. Hence $\mathrm{Ker}(\varphi)$ is $B_{n-1\,n}$.

Given an element $\alpha \in \mathrm{Hom}_B(Q,Q)$, we may form the following commutative diagram in B-Mod:

$$\begin{tikzcd} 0 \arrow[r] & Be_n \arrow[r, "\lambda"] \arrow[d, "b"'] & Be_{n-1} \arrow[r, "\pi"] \arrow[d, "a"'] & Q \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & Be_n \arrow[r, "\lambda"] & Be_{n-1} \arrow[r, "\pi"] & Q \arrow[r] & 0. \end{tikzcd}$$

Note that the homomorphism a exists and makes the right square of the above diagram commutative. Thus we have a homomorphism b making the left square commutative. We may identify a with an element in $B_{n-1\,n-1}$, say $a = \cdot e_{n-1,n-1}(r)$ with $r \in B_{n-1\,n-1}$, and identify b with an element in B_{nn}, say $b = \cdot e_{n,n}(s)$ with $s \in B_{nn}$. The commutativity of the left square means that $r = s \in B_{nn}$. This means that φ is surjective. Thus $\mathrm{End}_B(Q) \simeq R/I_{n-1\,n}$.

If we apply $\mathrm{Hom}_B(-,Be_j)$ to $(*)$ for $1 \le j \le n-1$ and use Lemma 3.1(3), we have the following exact commutative diagram with $e_{n,n-1}\cdot$ an isomorphism:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_B(Q,Be_j) \arrow[r, "(\pi)_*"] & \mathrm{Hom}_B(Be_{n-1},Be_j) \arrow[r, "(\lambda)_*"] \arrow[d, "\simeq"'] & \mathrm{Hom}_B(Be_n,Be_j) \arrow[r] \arrow[d, "\simeq"] & 0 \\ & & e_{n-1}Be_j \arrow[r, "e_{n,n-1}\cdot"] & e_nBe_j. & \end{tikzcd}$$

Thus $\mathrm{Hom}_B(Q,Be_j)=0$ for all $1\leq j\leq n-1$.

If we apply $\mathrm{Hom}_B(Be_j,-)$ to the exact sequence $(*)$ for $1\leq j\leq n-1$, we get an exact sequence

$$0\longrightarrow \mathrm{Hom}_B(Be_j,Be_n)\longrightarrow \mathrm{Hom}_B(Be_j,B_{n-1})\longrightarrow \mathrm{Hom}_B(Be_j,Q)\longrightarrow 0,$$

which shows that $\mathrm{Hom}_B(Be_j,Q)\simeq B_{jn-1}/B_{jn}=B_{jn-1}/I_{j\,n-1}$.

Now we identify $\mathrm{Hom}_B(Be_j,Be_i)$ with e_jBe_i for all $1\leq i,j\leq n-1$, and $\mathrm{Hom}_B(Be_j,Q)$ with $B_{jn-1}/B_{jn}=B_{jn-1}/I_{j\,n-1}$. Then we can see that $\mathrm{End}_B(Be_1\oplus\cdots\oplus Be_{n-1}\oplus Q)$ is isomorphic to C. This finishes the proof of Lemma 3.2. $\square$

A special case of Lemma 3.2 is the ring considered in [8] under certain homological assumptions and finiteness conditions. Here we start with a more general setting and remove all homological conditions on ideals as well as finiteness conditions on quotients.

Let R be a ring with identity, and I an arbitrary ideal in R. We consider the ring of the following form

$$B:=\begin{pmatrix} R & I^{t_{12}} & \cdots & I^{t_{1n}}\\ R & R & \ddots & \vdots\\ \vdots & \vdots & \ddots & I^{t_{n-1\,n}}\\ R & R & \cdots & R\end{pmatrix},$$

where t_{ij} are positive integers. Note that the conditions for B to be a ring are

(1) $t_{ij}\leq t_{i\,j+1}$, $t_{i+1\,j}\leq t_{ij}$ for $i<j$, and

(2) $t_{ij}\leq t_{ik}+t_{kj}$ for $i<k<j$.

The next result follows immediately from Lemma 3.2.

Lemma 3.3. *Assume that the above defined B is a ring. Then B is derived-equivalent to*

$$C:=\begin{pmatrix} R & I^{t_{12}} & \cdots & I^{t_{1\,n-1}} & I^{t_{1\,n-1}}/I^{t_{1\,n}}\\ R & R & \ddots & \vdots & \vdots\\ \vdots & \vdots & \ddots & I^{t_{n-2\,n-1}} & I^{t_{n-2\,n-1}}/I^{t_{n-2\,n}}\\ R & R & \cdots & R & R/I^{t_{n-1\,n}}\\ 0 & 0 & 0 & 0 & R/I^{t_{n-1\,n}}\end{pmatrix}.$$

Next, we consider a variation of the ring B in Lemma 3.2, which was considered in [5, 14] and cover some tiled orders in [21], and many other cases, for example, rings in [14], and some Auslander-regular, Cohen-Macaulay rings (not necessarily maximal orders, see [23]).

Let R be a ring with identity. Suppose that R_i is a subring of R with the same identity for $2\leq i\leq n$, that I_i is a left ideal of R for all $2\leq i\leq n$, and that I_{ij} is ideal of R, with $2\leq j<i\leq n$, which satisfies the following conditions:

(1) $I_i\subseteq R_i$ is a right ideal of R_i for all i,

(2) $I_n\subseteq I_{n-1}\subseteq\cdots\subseteq I_2$,

(3) $I_j\subseteq I_{ij}$ for all i,j,

(4) $I_iI_{ij}\subseteq I_j$ for $j<i$, and

(5) $I_{ik}I_{kj}\subseteq I_{ij}$ for $j<k<i$.

Here we do not assume that I_i is projective as a left R-module, nor that I_i is an ideal of R. Nevertheless one can check that

$$B:=\begin{pmatrix} R & I_2 & I_3 & \cdots & I_{n-1} & I_n\\ R & R_2 & I_3 & \cdots & I_{n-1} & I_n\\ R & I_{32} & R_3 & \ddots & \vdots & I_n\\ R & I_{42} & I_{43} & \ddots & I_{n-1} & \vdots\\ \vdots & \vdots & \vdots & \ddots & R_{n-1} & I_n\\ R & I_{n2} & I_{n3} & \cdots & I_{n\,n-1} & R_n\end{pmatrix},\qquad A:=\begin{pmatrix} R & R & I_3 & \cdots & I_{n-1} & I_n\\ R & R & I_3 & \cdots & I_{n-1} & I_n\\ R & R & R_3 & \ddots & \vdots & I_n\\ \vdots & \vdots & \vdots & \ddots & I_{n-1} & \vdots\\ R & R & I_{n-1\,3} & \cdots & R_{n-1} & I_n\\ R & R & I_{n3} & \cdots & I_{n\,n-1} & R_n\end{pmatrix},$$

$$C := \begin{pmatrix} R_2/I_2 & 0 & 0 & \cdots & 0 & 0 \\ R/I_2 & R & I_3 & I_4 & \cdots & I_n \\ R/I_{32} & R & R_3 & I_4 & \cdots & I_n \\ R/I_{42} & R & I_{43} & R_4 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & I_n \\ R/I_{n2} & R & I_{n3} & \cdots & I_{nn-1} & R_n \end{pmatrix}$$

with the usual matrix addition and multiplication form three rings with identity. Note that only the second column of A is different from the one of B.

We define a B-module Q as follows:

$$0 \longrightarrow Be_2 \xrightarrow{\lambda} Be_1 \xrightarrow{\pi} Q \longrightarrow 0,$$

where λ is a composition of the inclusion $Be_2 \to Ae_2$ with the isomorphism $Ae_2 \simeq Be_1$ as B-modules and where π is the cokernel of λ.

Now, we consider the endomorphism ring $\mathrm{End}_B(Q \oplus Be_1 \oplus Be_3 \oplus \cdots \oplus Be_n)$. By a proof similar to that of Lemma 3.2, one can show that the following lemma is true. We leave the details of its proof to the reader.

Lemma 3.4. *The above defined rings B and C are derived-equivalent.*

An alternative proof of Lemma 3.4 can be found in [5, Theorem 5.1.2], where A is replaced by the $n \times n$ matrix ring over R.

4 Higher algebraic K-theory of matrix subrings

In the algebraic K-theory of rings, the calculation of higher algebraic K-groups K_n seems to be one of the interesting and hard problems. In this section, we shall provide formulas for computation of the K_n-groups of certain rings by applying the results in the previous section. Our computation is based the philosophy that derived equivalences of rings preserve the K-theory and G-theory (see [6]), thus one can transfer the calculation of K_n of a ring to that of another ring which is derived-equivalent to and may be much more simpler than the original one. In the literature, there are many papers dealing with K_n-groups by exploiting excision, Mayer-Vietoris exact sequences or other related sequences (for example, see [8], [15], [18], [26], [27]). However, it seems that there are few papers using derived equivalences to calculate the higher algebraic K-groups. In the present section we shall show that sometimes our philosophy works powerfully though it may be difficult to find derived equivalences in general. For some new advances in constructing derived equivalences, we refer the reader to the recent papers [9, 11].

Let R be a ring with identity. We denote by $K_*(R)$ the series of algebraic K-groups of R with $* \in \{0,1,2,\cdots,\}$. The algebraic K-theory of matrix-like rings has been of interest since a long time. In [1], Berrick and Keating showed the following result.

Lemma 4.1. [1] *If R_i is a ring with identity for $i = 1,2$, and if M is an R_1-R_2-bimodule, then, for the triangular matrix ring*

$$S = \begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix},$$

there is an isomorphism of K-groups: $K_n(S) \simeq K_n(R_1) \oplus K_n(R_2)$ for all integers $n \in \mathbb{Z}$. Moreover, this isomorphism is induced from the canonical inclusion of $R_1 \oplus R_2$ into S.

For $n = 0$, this is classical. For $n = 1,2$, this was already shown by Dennis and Geller in 1976. We remark that Lemma 4.1 can be used to calculate the higher algebraic K-groups of algebras associated to finite EI-categories, or more generally, of "triangular" Artin algebras. Recall that an Artin algebra A over a commutative Artin ring is said to be triangular if the set of non-isomorphic indecomposable projective A-modules can be ordered as $P_1, P_2, \cdots, P_n$ such that $\mathrm{Hom}_A(P_j, P_i) = 0$ for all $j > i$. In this case, we have $K_*(A) \simeq \bigoplus_{j=1}^n K_*(\mathrm{End}_A(P_j))$ by Lemma 4.1. In particular, if A is a finite-dimensional hereditary algebra over an algebraically closed field k with n non-isomorphic simple modules, then $K_*(A) \simeq nK_*(k)$.

For a matrix ring of the form

$$T = \begin{pmatrix} R & I & \cdots & I \\ R & R & \ddots & \vdots \\ \vdots & \vdots & \ddots & I \\ R & R & \cdots & R \end{pmatrix}_{n\times n},$$

where R is a ring and I is an ideal in R such that the R-modules ${}_RI$ and I_R are projective, it was shown by Keating in [13] that there is an isomorphism of K-theory:

$$K_*(T) \simeq K_*(R) \oplus (n-1)K_*(R/I).$$

In [13], the author also considered the so-called trivial extension of a ring by a bimodule. It was shown that if T is the trivial extension of a ring R by an R-bimodule M, then $K_*(T) \simeq K_*(R)$ provided that M has finite projective dimension as a left T-module. Here the condition on M in this statement is necessary. See the counterexample $T := k[x]/(x^2)$ which is the trivial extension of k by k, where k is any field.

Recently, as a kind of generalization of the above result of Keating, the authors of [8] consider the following matrix ring: Let I be an ideal of a $\mathbb{Z}_p$-algebra R with identity, where $\mathbb{Z}_p$ is the p-adic integers (or, equivalently, $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$), and define

$$S = \begin{pmatrix} R & I^{t_{12}} & \cdots & I^{t_{1n}} \\ R & R & \ddots & \vdots \\ \vdots & \vdots & \ddots & I^{t_{n-1\,n}} \\ R & R & \cdots & R \end{pmatrix},$$

where t_{ij} are positive integers. Assume that S is a ring and that R/I^n is a finite ring for all $n \geq 1$. If both ${}_RI$ and I_R are projective, it is proved in [8] that the following isomorphism of the algebraic K-theory holds:

$$K_*(S)(1/s) \simeq K_*(R)(1/s) \oplus (n-1)K_*(R/I)(1/s),$$

where s is any rational integer such that p divides s, and where $G(1/s)$ denotes the group $G \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}]$ for an abelian group G.

We shall use our results in the previous section to extend all results on matrix rings mentioned above without any homological conditions on rings and ideals under investigation. Our proofs also explain the reason why the multiplicity $n-1$ appears in the above mentioned isomorphisms of the higher algebraic K-theory.

Lemma 4.2. *Let B be the matrix ring*

$$B := \begin{pmatrix} R & I_{12} & I_{13} & \cdots & I_{1\,n} \\ R & R & I_{23} & \cdots & I_{2\,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ R & R & \cdots & R & I_{n-1\,n} \\ R & R & \cdots & R & R \end{pmatrix}$$

defined in Lemma 3.2. Then

$$K_*(B) \simeq K_*(R) \oplus \bigoplus_{j=1}^{n-1} K_*(R/I_{j\,j+1}).$$

Proof. We show this lemma by induction on n. By Theorem 2.2 (see [6]), the algebraic K-theory and G-theory are invariant under derived equivalences. So, by Lemma 3.2, we have $K_*(B) \simeq K_*(C)$ (for notation see Section 3). Now it follows from Lemma 4.1 that $K_*(C) \simeq K_*(R/I_{n-1\,n}) \oplus K_*(B_{n-1})$, where B_{n-1} is the $(n-1)\times(n-1)$ left upper corner matrix subring of B. By induction, we have $K_*(B_{n-1}) \simeq K_*(R) \oplus K_*(R/I_{12}) \oplus \cdots \oplus K_*(R/I_{n-2\,n-1})$. Hence

$$K_*(B) \simeq K_*(R) \oplus K_*(R/I_{12}) \oplus \cdots \oplus K_*(R/I_{n-2\,n-1}) \oplus K_*(R/I_{n-1\,n}).$$

This finishes the proof of Lemma 4.2. □

In particular, as a consequence of Lemma 4.2, we can strengthen the result in [8] as the following corollary, here we drop all assumptions on rings and ideals.

Corollary 4.3. *Let R be an arbitrary ring with identity and I an arbitrary ideal in R. Then, for a ring of the following form*

$$S = \begin{pmatrix} R & I^{t_{12}} & \cdots & I^{t_{1n}} \\ R & R & \ddots & \vdots \\ \vdots & \vdots & \ddots & I^{t_{n-1\,n}} \\ R & R & \cdots & R \end{pmatrix},$$

where t_{ij} are positive integers, we have

$$K_*(S) \simeq K_*(R) \oplus \bigoplus_{j=2}^{n} K_*(R/I^{t_{j-1\,j}}).$$

As a special case of Corollary 4.3, we get the following result of [13] without the assumption that ${}_RI$ and I_R are projective.

Corollary 4.4. *Let R be a ring with identity and I an ideal in R. Suppose that t_j is a positive integers with $t_j \leq t_{j+1}$ for $j = 2, \cdots, n-1$. Let*

$$T = \begin{pmatrix} R & I^{t_2} & \cdots & I^{t_n} \\ R & R & \ddots & \vdots \\ \vdots & \vdots & \ddots & I^{t_n} \\ R & R & \cdots & R \end{pmatrix}.$$

Then T is a ring and

$$K_*(T) \simeq K_*(R) \oplus \bigoplus_{i=2}^{n} K_*(R/I^{t_i}).$$

Let us remark that if I is a nilpotent ideal in R with identity then $K_0(R) \simeq K_0(R/I)$. In general, this is not true for higher K-groups K_n with $n \geq 1$. Thus, for K_0, we may replace the direct summands $K_0(R/I^{t_j})$ by $K_0(R/I)$ in Corollary 4.4, and get $K_0(T) \simeq K_0(R) \oplus (n-1)K_0(R/I)$.

Similarly, we have the following result on the groups K_n of the ring defined in Lemma 3.4

Lemma 4.5. *Let B be the ring*

$$B := \begin{pmatrix} R & I_2 & I_3 & \cdots & I_{n-1} & I_n \\ R & R_2 & I_3 & \cdots & I_{n-1} & I_n \\ R & I_{32} & R_3 & \ddots & \vdots & I_n \\ R & I_{42} & I_{43} & \ddots & I_{n-1} & \vdots \\ \vdots & \vdots & \vdots & \ddots & R_{n-1} & I_n \\ R & I_{n2} & I_{n3} & \cdots & I_{n\,n-1} & R_n \end{pmatrix}$$

defined in Lemma 3.4. Then

$$K_*(B) \simeq K_*(R) \oplus \bigoplus_{j=2}^{n} K_*(R_j/I_j).$$

This result shows that the abelian group $K_n(B)$ of the ring B is independent of the choice of the ideals I_{ij} in R for all $n \geq 0$.

As a direct consequence of Lemma 4.5, we have the following corollary.

Corollary 4.6. *Let R be a ring with identity, and let I_j be an ideal of R with $2 \leq j \leq n$ such that $I_j \subseteq I_{j-1}$ for all j. Then, for the rings*

$$S := \begin{pmatrix} R & I_2 & I_3 & \cdots & I_n \\ R & R & I_3 & \cdots & I_n \\ R & I_2 & R & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & I_n \\ R & I_2 & \cdots & I_{n-1} & R \end{pmatrix}, \qquad T := \begin{pmatrix} R & I_2 & I_3 & \cdots & I_n \\ R & R & I_3 & \cdots & I_n \\ R & R & R & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & I_n \\ R & R & R & \dots & R \end{pmatrix},$$

we have

$$K_*(S) \simeq K_*(R) \oplus \bigoplus_{j=2}^{n} K_*(R/I_j) \simeq K_*(T).$$

Let us remark that we can also use our method in this section to calculate some corner rings eBe, though, in general, we cannot get an $\mathrm{add}({}_{eBe}eAe)$-split sequence

$$0 \longrightarrow eBe \longrightarrow eAe \longrightarrow eAe/eBe \longrightarrow 0,$$

with e an idempotent in B, from a given $\mathrm{add}({}_BA)$-split sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow A/B \longrightarrow 0.$$

For example, suppose that B is the ring defined in Lemma 4.2. If e is an idempotent in R, then, for the corner ring

$$B_1 := \begin{pmatrix} eRe & eI_{12}e & eI_{13}e & \cdots & eI_{1\,n}e \\ eRe & eRe & eI_{23}e & \cdots & eI_{2\,n}e \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ eRe & eRe & \cdots & eRe & eI_{n-1\,n}e \\ eRe & eRe & \cdots & eRe & eRe \end{pmatrix}$$

of B, we have

$$K_*(B_1) \simeq K_*(eRe) \oplus \bigoplus_{j=1}^{n-1} K_*(eRe/eI_{j\,j+1}e).$$

Also, we remark that, for any ring R, the functor $\mathrm{Hom}_R(-, {}_RR)$ is a duality between the category R-proj and the category R^{op}-proj, where R^{op} is the opposite ring of R. Thus, for each $n \geq 0$, we have $K_n(R) \simeq K_n(R^{op})$. From this fact, or from Lemma 3.1(3) for right modules, we can see that if S' is a ring of the form

$$S' := \begin{pmatrix} R & I_1 & I_1 & \cdots & I_1 \\ I_2 & R & I_2 & \cdots & I_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ I_{n-1} & \cdots & I_{n-1} & R & I_{n-1} \\ R & \cdots & R & R & R \end{pmatrix},$$

where R is a ring with identity and I_j is an ideal of R for each $1 \leq j < n$, then

$$K_*(S') \simeq K_*(R) \oplus \bigoplus_{j=1}^{n-1} K_*(R/I_j).$$

Note that S' is closely related to the ring S in Corollary 4.6.

Now, recall that a pullback diagram of rings:

$$(*) \qquad \begin{array}{ccc} R & \xrightarrow{f_1} & R_1 \\ {\scriptstyle h_2}\downarrow & & \downarrow{\scriptstyle h_1} \\ R_2 & \xrightarrow{f_2} & R_0 \end{array}$$

is called a Milnor square if one of f_2 and h_1 is surjective.

An example of Milnor squares is the following case: Let $R \subseteq S$ be an extension of rings with the same identity. If there is an ideal J of S such that $J \subseteq R$, then there is a canonical Milnor square

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R/J & \longrightarrow & S/J. \end{array}$$

Let R be the product $R_1 \times \cdots \times R_n$ of finitely many rings R_i with $1 \leq i \leq n$. A subdirect product of ring R is a subring $S \subseteq R$ for which each projection $S \to R_i$ carries S onto R_i for each i. In this case we say that the inclusion $S \subseteq R$ is an inclusion of a subdirect product.

The following lemma is useful and well-known for calculation of higher K-groups of rings.

Lemma 4.7. *For a given Milnor square* $(*)$*, the following are true:*

(1) (See [17, Theorem 3.3]) *There is a Mayer-Vietoris exact sequence:*

$$K_1(R) \xrightarrow{\left((f_1)_*,(h_2)_*\right)} K_1(R_1) \oplus K_1(R_2) \xrightarrow{\begin{pmatrix} (h_1)_* \\ -(f_2)_* \end{pmatrix}} K_1(R_0) \longrightarrow K_0(R) \xrightarrow{\left((f_1)_*,(h_2)_*\right)} K_0(R_1) \oplus K_0(R_2) \xrightarrow{\begin{pmatrix} (h_1)_* \\ -(f_2)_* \end{pmatrix}} K_0(R_0),$$

where f_* denotes the homomorphism induced by f.

(2) (See [4], [25, Theorem 5.5]) *Suppose that $(*)$ is a Milnor square of $\mathbb{Z}/p^m\mathbb{Z}$-algebras, where $p \geq 2$ and m are fixed positive integers. Let s be a non-zero integer such that p divides s. Then there is an exact sequence of K-groups, that is, the Mayer-Vietoris sequence:*

$$\begin{aligned} \cdots \longrightarrow \quad & K_{*+1}(R_1) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \oplus K_{*+1}(R_2) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \xrightarrow{\begin{pmatrix} (h_1)_* \\ -(f_2)_* \end{pmatrix}} K_{*+1}(R_0) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \longrightarrow K_*(R) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \\ & \xrightarrow{\left((f_1)_*,(h_2)_*\right)} K_*(R_1) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \oplus K_*(R_2) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \xrightarrow{\begin{pmatrix} (h_1)_* \\ -(f_2)_* \end{pmatrix}} K_*(R_0) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \longrightarrow \cdots . \end{aligned}$$

(3) *Suppose that $(*)$ is a Milnor square of $\mathbb{Z}/p^m\mathbb{Z}$-algebras, where $p \geq 2$ and m are fixed positive integers. Let s be a non-zero integer such that p divides s. If the induced homomorphism $(f_2)_*$ in* (2) *is an split epimorphism for all $* \in \mathbb{N}$, then there is an exact sequence for all $* \in \mathbb{N}$:*

$$0 \longrightarrow K_*(R) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \xrightarrow{\left((f_1)_*,(h_2)_*\right)} K_*(R_1) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \oplus K_*(R_2) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \xrightarrow{\begin{pmatrix} (h_1)_* \\ -(f_2)_* \end{pmatrix}} K_*(R_0) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \longrightarrow 0.$$

In particular, if the induced homomorphism $(f_2)_*$ in (2) *is an isomorphism for all $* \in \mathbb{N}$, then so is the induced homomorphism $(f_1)_*$.*

(4) (See [16, Theorem 13.33], [17]) *If both h_1 and f_1 are surjective, or if h_1 is surjective and f_1 is the inclusion of a subdirect product, then there is an exact sequence*

$$\begin{aligned} K_2(R) \quad & \xrightarrow{\left((f_1)_*,(h_2)_*\right)} K_2(R_1) \oplus K_2(R_2) \xrightarrow{\begin{pmatrix} (h_1)_* \\ -(f_2)_* \end{pmatrix}} K_2(R_0) \longrightarrow K_1(R) \xrightarrow{\left((f_1)_*,(h_2)_*\right)} \\ & K_1(R_1) \oplus K_1(R_2) \xrightarrow{\begin{pmatrix} (h_1)_* \\ -(f_2)_* \end{pmatrix}} K_1(R_0) \longrightarrow K_0(R) \xrightarrow{\left((f_1)_*,(h_2)_*\right)} K_0(R_1) \oplus K_0(R_2) \xrightarrow{\begin{pmatrix} (h_1)_* \\ -(f_2)_* \end{pmatrix}} K_0(R_0). \end{aligned}$$

Remark that there is a dual statement of (3) for $(f_1)_*$ being a split monomorphism for each $* \in \mathbb{N}$.

Now we turn to proving Theorem 1.1. Observe that the argument in our proof below is actually a combination of the previous results with Mayer-Vietoris exact sequences, and works also for many other cases. Here we prove only Theorem 1.1(1), and leave the details of the proof of Theorem 1.1(2) to the reader.

Proof of Theorem 1.1 (1): Let

$$J := \begin{pmatrix} I & I_{12} & \cdots & I_{1n} \\ I & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & I_{n-1\,n} \\ I & \cdots & I & I \end{pmatrix}, \quad B := \begin{pmatrix} R & I_{12} & \cdots & I_{1n} \\ I & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & I_{n-1\,n} \\ I & \cdots & I & R \end{pmatrix}, \quad A := \begin{pmatrix} R & I_{12} & \cdots & I_{1n} \\ R & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & I_{n-1\,n} \\ R & \cdots & R & R \end{pmatrix}.$$

By the assumptions in Theorem 1.1(1), we can verify that A and B are rings and that J is an ideal of A. Thus J is also an ideal of B. Note that B is a subalgebra of A. Let f be the inclusion of B into A. If we define

$$B' := B/J = \begin{pmatrix} R/I & 0 & \cdots & 0 \\ 0 & R/I & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & R/I \end{pmatrix}, \quad A' := A/J = \begin{pmatrix} R/I & 0 & \cdots & 0 \\ R/I & R/I & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ R/I & \cdots & R/I & R/I \end{pmatrix},$$

then we have a Milnor square

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ {\scriptstyle g'}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ B' & \xrightarrow{f'} & A', \end{array}$$

where g and g' are the canonical surjective maps, and where f' is the injective map induced from f. Since the map $f'_* : K_*(B') \to K_*(A')$ is an isomorphism for $* = 0, 1, 2, \cdots$, it follows that $f'_* \otimes \mathbb{Z}[\frac{1}{s}] : K_*(B') \otimes \mathbb{Z}[\frac{1}{s}] \to K_*(A') \otimes \mathbb{Z}[\frac{1}{s}]$ is an isomorphism. Thus we see from Lemma 4.7(3) that $f_* \otimes \mathbb{Z}[\frac{1}{s}]$ is also an isomorphism. It then follows from Corollary 4.2 that

$$K_*(B) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \simeq K_*(A) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \simeq K_*(R) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \oplus \bigoplus_{j=1}^{n-1} K_*(R/I_{j\,j+1}) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}].$$

This finishes the proof of Theorem 1.1(1).

If we define

$$J := \begin{pmatrix} I & I_2 & I_3 & \cdots & I_n \\ I & I_2 & I_3 & \cdots & \vdots \\ I & I_{32} & I_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & I_n \\ I & I_{n2} & \cdots & I_{n\,n-1} & I_n \end{pmatrix},$$

then the proof of Theorem 1.1(2) can be carried out similarly since we have Lemma 4.5. □

Now we mention the following corollary of Theorem 1.1. Here in its proof below we choose a suitable subring instead of an extension ring.

Corollary 4.8. *Suppose that $p \geq 2$ and m are positive integers. Let R be a $\mathbb{Z}/p^m\mathbb{Z}$-algebra with identity, and let I and J be two arbitrary ideals of R. Define*

$$S := \begin{pmatrix} R & I & \cdots & I \\ J & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & I \\ J & \cdots & J & R \end{pmatrix}_{n \times n}.$$

Then S is a ring, and we have

$$K_*(S) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \simeq K_*(R) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \oplus (n-1) K_*(R/(IJ + JI)) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}]$$

for all non-zero integer s such that s is divisible by p.

Proof. We define

$$B := \begin{pmatrix} R & IJ+JI & \cdots & IJ+JI \\ J & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & IJ+JI \\ J & \cdots & J & R \end{pmatrix}_{n\times n}, \qquad J' := \begin{pmatrix} J & IJ+JI & \cdots & IJ+JI \\ J & J & \ddots & \vdots \\ \vdots & \ddots & \ddots & IJ+JI \\ J & \cdots & J & J \end{pmatrix}_{n\times n}.$$

Then one can verify that B is a ring and $J' \subseteq B$ is an ideal in S. Note that B is a subring of S. Now, let $A := S$, $B' := B/J'$ and $A' := A/J'$. Then we may use the same argument as in the proof of Theorem 1.1 to show $K_*(B) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}] \simeq K_*(A) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}]$. But for the former, we have $K_*(B) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}] \simeq K_*(R) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}] \oplus (n-1)K_*(R/(IJ+JI)) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}]$ by Theorem 1.1(1). Thus Corollary 4.8 follows. □

Remark. In Corollary 4.8, if, in addition, $I^2 \subseteq J$ (for example, $I^2 = 0$, or $I \subseteq J$), we can show that

$$K_*(S) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \simeq K_*(R) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \oplus (n-1)K_*(R/I) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}]$$

for all non-zero integer s such that s is divisible by p. To see this, one just needs to consider $B := S$,

$$A := \begin{pmatrix} R & I & \cdots & I \\ I+J & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & I \\ I+J & \cdots & I+J & R \end{pmatrix}_{n\times n}, \quad \text{and} \quad J' := \begin{pmatrix} I & I & \cdots & I \\ J & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & I \\ J & \cdots & J & I \end{pmatrix}_{n\times n}.$$

Let us illustrate how the argument in the above proof of Theorem 1.1(1) can be applied to other cases.

Again, suppose that $p \geq 2$ and m are positive integers. Let R be a $\mathbb{Z}/p^m\mathbb{Z}$-algebra with identity and I an arbitrary ideal of R. For each finite partially ordered set P, we associate a ring $B := B(R,I,P)$ which is a subring of the matrix ring over R with indexing set P, it is defined as follows: Let $B = (B_{ij})_{i,j\in P}$ with $B_{ij} = R$ if $i \geq j$, and $B_{ij} = I$ otherwise. We may assume that $P = \{a_1, \cdots, a_n\}$ such that $a_i \leq a_j$ implies $i \leq j$. Under this assumption we see that $J' := M_n(I)$ is an ideal of B, which is also an ideal of

$$A := \begin{pmatrix} R & I & \cdots & I \\ R & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & I \\ R & \cdots & R & R \end{pmatrix}_{n\times n}.$$

Note that B is a subring of A. Let $B' := B/J'$ and $A' := A/J'$. We define C to be the diagonal matrix ring with the principal diagonal entries R/I. Then C is a subring of both B' and A'. Using this ring C, we can see that the inclusion f' of B' into A' induces an isomorphism $(f')_*$ for all $* \in \mathbb{N}$. Then we may use the same argument as the above to show that, for any s divisible by p,

$$K_*(B) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \simeq K_*(A) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \simeq K_*(R) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}] \oplus (n-1)K_*(R/I) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{s}].$$

We end this section by a couple of remarks concerning Theorem 1.1.

(1) In Theorem 1.1, if R is a $\mathbb{Z}_p$-algebra instead of a $\mathbb{Z}/p^m\mathbb{Z}$-algebra, and if R/I, R/I_i and R/I_{ij} are finite rings for all i, j, then Theorem 1.1 still holds true. Indeed, in this case we can use Charney's excision at the end of the paper [4] since $I \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}]$ has a unit. This is due to $\mathrm{Tor}_1^{\mathbb{Z}}(-, \mathbb{Z}[\frac{1}{s}]) = 0$ and to the fact that the quotient rings R/I, R/I_i and R/I_{ij} are $\mathbb{Z}/p^m\mathbb{Z}$-algebras for some $m > 0$. Indeed, we have an exact sequence

$$\mathrm{Tor}_1^{\mathbb{Z}}(R/I, \mathbb{Z}[\frac{1}{s}]) \longrightarrow I \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}] \longrightarrow R \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}] \longrightarrow (R/I) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}].$$

Clearly, the first and last terms vanish, this implies $I \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}] \simeq R \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{s}]$. So, the condition of Charney's result in [4] is satisfied. I thank X. J. Guo for explanation of this fact.

(2) A crucial fact of our proofs of the main results is: Given an extension $B \subseteq A$ of rings with the same identity such that ${}_BA$ is finitely generated and projective, we have $K_*(B) \simeq K_*(\mathrm{End}_B(A \oplus A/B))$ for all $* \in \mathbb{N}$. Moreover, we

may also compare the algebraic K-theory of B with that of A. For this purpose, we define Ω to be the kernel of the multiplication map $A\otimes_B A\to A$, it follows from the Additivity Theorem (see [19, Corollary 1, Section 3] that the exact sequence of the exact functors

$$0\longrightarrow \Omega\otimes_A - \longrightarrow A\otimes_B - \longrightarrow id \longrightarrow 0$$

on the category of finitely generated projective A-modules gives rise to three homomorphisms of abelian groups: $r_*: K_*(A)\to K_*(B)$, $t_*: K_*(B)\to K_*(A)$ and $\omega_*: K_*(A)\to K_*(A)$ such that $r_*t_* = 1_{K_*(A)}+\omega_*$. If, in addition, the n-fold tensor product of Ω over A vanishes for some natural number n, that is, $\Omega^{\otimes_A n}=0$ (for example, $\Omega=0$ in case the inclusion $B\subseteq A$ is an injective ring epimorphism), then the map t_* is split surjective, and $K_*(A)$ is a direct summand of $K_*(B)$. In general, neither t_* nor r_* is an isomorphism.

5 Lower K-theory for matrix subrings

In this section we consider the algebraic K-groups K_0 and K_1 for matrix subrings. Our results in this section are not covered by the main results in the previous sections.

We first consider the group K_0. In this case, we have the following result in which we do not assume that the rings considered are $\mathbb{Z}/p^m\mathbb{Z}$-algebras or $\mathbb{Z}[\frac{1}{p}]$-algebras.

Proposition 5.1. *Let R be an arbitrary ring with identity, and let I, J and I_{ij} be ideals in R.*

(1) *For the rings S and T defined in Theorem 1.1, we have*

$$K_0(S)\simeq K_0(R)\oplus\bigoplus_{j=2}^{n}K_0(R/I_{j-1\,j}),\quad K_0(T)\simeq K_0(R)\oplus\bigoplus_{j=2}^{n}K_0(R_j/I_j).$$

(2) *For the ring S defined in Corollary 4.8, we have*

$$K_0(S)\simeq K_0(R)\oplus(n-1)K_0(R/(IJ+JI)).$$

Moreover, if $I^2\subseteq J$, we have $K_0(S)\simeq K_0(R)\oplus(n-1)K_0(R/I)$.

The proof of this proposition is actually a combination of Corollary 4.6 and Lemma 4.7(1) and (3), and we leave the details of the proof to the interested reader.

Here arises an open question: We do not know, at moment, if Proposition 5.1 is true for higher algebraic K-groups K_n with $n\geq 1$. But, for K_1, we do have some partial answers. Before stating our result, we first prove the following lemma.

Lemma 5.2. *Let $B\subseteq A$ be an extension of rings with the same identity. Suppose that I is an idempotent ideal of A contained in B. If the inclusion $B\subseteq A$ induces an isomorphism $\gamma_i: K_i(B/I)\to K_i(A/I)$ for $i=1,2$, then $K_1(B)\simeq K_1(A)$.*

Proof. Let $K_i(B,I)$ denote the i-th relative K-group of the canonical surjective map $B\to B/I$. Then there is an exact sequence of K-groups (see [19]):

$$\cdots\longrightarrow K_n(R,I)\longrightarrow K_n(R)\longrightarrow K_n(R/I)\longrightarrow K_{n-1}(R,I)\longrightarrow K_{n-1}(R)\longrightarrow K_{n-1}(R/I)\longrightarrow\cdots,$$

and we may form the following commutative diagram of ableian groups with exact rows:

$$\begin{tikzcd}
K_2(B/I) \arrow[r] \arrow[d, "\gamma_2"] & K_1(B,I) \arrow[r] \arrow[d, "\gamma"] & K_1(B) \arrow[r] \arrow[d, "\beta"] & K_1(B/I) \arrow[r] \arrow[d, "\gamma_1"] & K_0(B,I) \arrow[d, "\simeq"] \\
K_2(A/I) \arrow[r] & K_1(A,I) \arrow[r] & K_1(A) \arrow[r] & K_1(A/I) \arrow[r] & K_0(A,I).
\end{tikzcd}$$

Here we use the fact that $K_0(B,I)$ is always independent of B. Now, by the Five Lemma in homological algebra, we know that the map β is isomorphic if γ is isomorphic. However, this follows from a result of Vaserstein (see [27, Chapter III, Section 2, Remark 2.2.1]), which states that if J is an ideal in a ring R with identity, then $K_1(R,J)$ is independent of R if and only if $J^2=J$. Thus γ is an isomorphism. $\square$

We should notice that, in general, $K_n(R,I)$ depends on R for $n\geq 1$. This is why the conclusions in Theorem 1.1 are localized.

So, with Lemma 5.2 in hand, we can prove the following proposition for K_1.

Proposition 5.3. *Let R be a ring with identity, and let $I \subseteq J$ be ideals in R. If I is an idempotent ideal of R, then, for the ring*

$$B := \begin{pmatrix} R & I & \cdots & I \\ J & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & I \\ J & \cdots & J & R \end{pmatrix}_{n\times n},$$

we have

$$K_1(B) \simeq K_1(R) \oplus (n-1)K_1(R/I).$$

Proof. Clearly, B is a subring of the ring

$$A := \begin{pmatrix} R & I & \cdots & I \\ R & R & \ddots & \vdots \\ \vdots & \ddots & \ddots & I \\ R & \cdots & R & R \end{pmatrix}_{n\times n},$$

and $J' := M_n(I)$, the $n\times n$ matrices over I, is an idempotent ideal of A and B, respectively. We know that $K_*(B/J')$ and $K_*(A/J')$ are isomorphic for all $* \in \mathbb{N}$. Hence Proposition 5.3 follows from Lemma 5.2 and Corollary 4.2 immediately. $\square$

Finally, we mention another type of matrix rings: Let R and S be rings with identity, and let ${}_RM_S$ and ${}_SN_R$ be bimodules. We define a ring

$$A := \begin{pmatrix} R & M \\ N & S \end{pmatrix}, \quad \begin{pmatrix} r & m \\ n & s \end{pmatrix} \cdot \begin{pmatrix} r' & m' \\ n' & s' \end{pmatrix} = \begin{pmatrix} rr' & rm'+ms' \\ nr'+sn' & ss' \end{pmatrix}$$

for $r,r' \in R, s,s' \in S, m,m' \in M$ and $n,n' \in N$. Note that $M' := \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ and $N' := \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix}$ are two ideals in A. Thus one has a Milnor diagram

$$\begin{array}{ccc} A & \longrightarrow & A/M' \\ \big\downarrow & & \big\downarrow \\ A/N' & \longrightarrow & A/(M'+N'). \end{array}$$

By Lemma 4.7(4), we can show that $K_i(A) \simeq K_i(R) \oplus K_i(S)$ for $i = 0,1$. This result can be used to reduce the calculation of lower K-groups of finite-dimensional algebras with radical-square-zero to local algebras.

6 Higher mod-p K-theory

In this section, we shall point out that our main result, Theorem 1.1, holds true for the mod-p K-theory $K_*(-,\mathbb{Z}/p\mathbb{Z})$ under the assumption that algebras considered are $\mathbb{Z}[\frac{1}{p}]$-algebras and $p \not\equiv 2 (\mathrm{mod}\, 4)$, where $p \geq 2$ is any positive integer.

Let R be a ring with identity. In [3], Browder developed K-theory with coefficients $\mathbb{Z}/p\mathbb{Z}$. This is the so-called mod-p K-theory $K_*(R,\mathbb{Z}/p\mathbb{Z})$ for $* \in \mathbb{Z}$. Note that $K_0(R,\mathbb{Z}/p\mathbb{Z}) = K_0(R)\otimes_{\mathbb{Z}}\mathbb{Z}/p\mathbb{Z}$, and $K_i(R,\mathbb{Z}/p\mathbb{Z}) = 0$ if $i < 0$ (see [3, p. 45]). Later, Weibel observed in [26] that excision holds and that Mayer-Vietoris sequences exist if the rings involved are $\mathbb{Z}[\frac{1}{p}]$-algebras. The mod-p K-theory is closely related to the usual K-theory in the following manner.

Lemma 6.1. Universal Coefficient Theorem (see [3] and [26]):

Let R be a $\mathbb{Z}[\frac{1}{p}]$-algebra with identity. For all $* \in \mathbb{N}$, there is a short exact sequence of abelian groups

$$0 \longrightarrow K_*(R)\otimes_{\mathbb{Z}}\mathbb{Z}/p\mathbb{Z} \longrightarrow K_*(R,\mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(K_{*-1}(R),\mathbb{Z}/p\mathbb{Z}) \longrightarrow 0.$$

If $p \not\equiv 2\,(\mathrm{mod}\ 4)$, then this sequence splits (not naturally), so that $K_*(R,\mathbb{Z}/p\mathbb{Z})$ is a $\mathbb{Z}/p\mathbb{Z}$-module. If $p \equiv 2\,(\mathrm{mod}\ 4)$, then $K_*(R,\mathbb{Z}/p\mathbb{Z})$ is a $\mathbb{Z}/2p\mathbb{Z}$-module.

Thus, if $p \not\equiv 2\,(\mathrm{mod}\ 4)$, we see that $K_*(R,\mathbb{Z}/p\mathbb{Z})$ is completely determined by the usual K-groups $K_*(R)$.

Another result which we need is a Mayer-Vietoris sequence for mod-p K-groups.

Lemma 6.2. [26, Corollary 1.3] *For a Milnor square* $(*)$ *of* $\mathbb{Z}[\frac{1}{p}]$*-algebras, there is a long exact sequence of abelian groups for all integers* $*$:

$$\cdots \longrightarrow \quad K_{*+1}(R_1,\mathbb{Z}/p\mathbb{Z})\oplus K_{*+1}(R_2,\mathbb{Z}/p\mathbb{Z}) \longrightarrow K_{*+1}(R_0,\mathbb{Z}/p\mathbb{Z}) \longrightarrow$$

$$K_*(R,\mathbb{Z}/p\mathbb{Z}) \longrightarrow K_*(R_1,\mathbb{Z}/p\mathbb{Z})\oplus K_*(R_2,\mathbb{Z}/p\mathbb{Z}) \longrightarrow K_*(R_0,\mathbb{Z}/p\mathbb{Z}) \longrightarrow \cdots .$$

Now it follows from the above two lemmas and Theorem 1.2 that Theorem 1.1 holds true for the mod-p K-groups $K_*(R,\mathbb{Z}/p\mathbb{Z})$ if R is a $\mathbb{Z}[\frac{1}{p}]$-algebra and if $p \not\equiv 2$ (mod 4), since the argument there in the proof of Theorem 1.1 works in our new situation.

If $p \equiv 2$ (mod 4), we do not know whether $K_*(R,\mathbb{Z}/p\mathbb{Z})$ can be fully controlled by the first and last terms in Lemma 6.1. In general, extensions of fixed abelian groups may not be isomorphic, for instance, the cyclic group of order 4 and the Klein group $(\mathbb{Z}/2\mathbb{Z})\times(\mathbb{Z}/2\mathbb{Z})$ both are extensions of the cyclic group of order 2 by itself, but they are not isomorphic.

7 Examples: GV-ideals

In this section we shall give some examples related to our results. The first one is constructed from a $\mathcal{D}$-split sequence which is induced by a surjective ring homomorphism.

Let B be a ring with identity and J an ideal of B. We define $A = B/J$. Then we have an exact sequence in B-Mod:

$$0 \to J \to B \xrightarrow{\pi} A \to 0,$$

where π is the canonical surjection.

For this sequence to be an $\mathrm{add}({}_BB)$-split sequence, we have to assume $\mathrm{Ext}^1_B(A,B) = 0$. This happens often in commutative algebra. For example, if B is a commutative noetherian ring, and J is an ideal of B such that J contains a regular sequence on B of length 2, then $\mathrm{Ext}^i_B(A,B) = 0$ for $i = 0,1$ (see [12, p. 101]). Another example is the so-called GV-ideals in integral domains. Here we will state the following general definition of GV-ideals.

Let R be an arbitrary ring with identity. Recall that an ideal I of R is called a GV-ideal (after the names Glaz and Vasconcelos, see [29, 7]) if the induced map $\mu_I : R \longrightarrow \mathrm{Hom}_R(I,R)$, given by $r \mapsto (x \mapsto xr)$ for $x \in I$, is an isomorphism of R-bimodules. This is equivalent to $\mathrm{Ext}^i_R(R/I,R) = 0$ for $i = 0,1$. Thus R is a GV-ideal of R. Note that $p\mathbb{Z}$ is not a GV-ideal of $\mathbb{Z}$ for any $p \in \mathbb{Z}$ with $|p| \neq 1$, even though we have $\mathbb{Z} \simeq \mathrm{Hom}_{\mathbb{Z}}(p\mathbb{Z},\mathbb{Z})$. We remark that the above definition of GV-ideals is more general than that in commutative rings where it is required that ${}_RI$ is finitely generated (see [29]).

Let $GV(R)$ be the set of all GV-ideals of R. For ideals I and J of R, we denote by $(I:J) := \{x \in R \mid Ix \subseteq J\}$. (This notation is different from what was usually used in ring theory, but soon we will see its convenience when elements compose). Clearly, $(I:R) = R, (R:I) = I$, and $(I:J)$ is an ideal of R.

The following lemma shows some properties of GV-ideals, which are of interest for our proofs.

Lemma 7.1. *Let B be a ring with identity, and let J be a GV-ideal in B. Then*

(1) *the sequence* $0 \to J \to B \xrightarrow{\pi} A \to 0$ *is an* $\mathrm{add}({}_BB)$*-split sequence in* B-Mod. *Thus* $\mathrm{End}_B(B\oplus J)$ *is derived-equivalent to* $\begin{pmatrix} B & B/J \\ 0 & B/J \end{pmatrix}$.

(2) $\mathrm{End}_B(J) \simeq B$ (*as rings and as B-bimodules*).

(3) *If I is an ideal in B, then* ${}_B\mathrm{Hom}_B(J,I)_B \simeq (J:I)$ *as B-bimodules. In particular, if $J \subseteq I$, then* $\mathrm{Hom}_B(J,I) \simeq B$.

(4) *If $x \in B$ such that $Jx = 0$, then $x = 0$.*

(5) *If I is an ideal in B with $J \subseteq I$, then $I \in GV(B)$.*

(6) *If $I,J \in GV(B)$, then $IJ \in GV(B)$.*

Proof. (1) is clear by [10, Theorem 1.1] and the definition of GV-ideals.

(2) By the definition of GV-ideals, the induced map $\mu_J : B \longrightarrow \mathrm{Hom}_B({}_BJ,B)$ is an isomorphism, this means that every homomorphism f from ${}_BJ$ to ${}_BB$ is given by the right multiplication of an element in B. Since J is an ideal in R, f is in fact an endomorphism of the module ${}_BJ$. Conversely, if $f \in \mathrm{End}_B(J)$, then f is a restriction of a right multiplication of an element of B. Hence $\mathrm{End}_B(J) \simeq B$.

(3) We define a map $\varphi : \mathrm{Hom}_B(J,I) \to (J : I)$ as follows: For $f \in \mathrm{Hom}_B(J,I)$, there is a unique element $b \in B$ such that the composition of f with the inclusion $\lambda : I \to B$ is the right multiplication map $\cdot b$ since $J \in GV(B)$. This means that $f\lambda = \cdot b$ and $b \in (J : I)$. So, we define $f \stackrel{\varphi}{\mapsto} b$. As $(J : I)$ is an ideal of B, it has a canonical bimodule structure. Now one can check that φ is an isomorphism of B-bimodules.

(4) This is a trivial consequence of the induced isomorphism $\mu_J : B \simeq \mathrm{Hom}_B(J,B)$.

(5)-(6) These statements were already proved in detail in [29] for commutative rings, the ideas of their proofs are as follows: It follows from (4) that $\mathrm{Hom}_B(I/J,B) = 0$. Further, by the isomorphism μ_J and the fact $\mu_J = \mu_I i_*$ where $i_* : \mathrm{Hom}_B(I,B) \to \mathrm{Hom}_B(J,B)$ is induced from the inclusion $i : J \to I$, one can check that μ_I is an isomorphism of B-bimodules. This proves (5).

Let $I,J \in GV(B)$. It follows from (4) that $\mu_{IJ} : B \to \mathrm{Hom}_B(IJ,B)$ is injective. We show that it is also surjective. In fact, since the composition of the maps $B \to \mathrm{Hom}_B(J,B) \to \mathrm{Hom}_B(J,\mathrm{Hom}_B(I,B)) \to \mathrm{Hom}_B(I \otimes_B J,R)$ is an isomorphism of B-B-bimodules, which is the composition of μ_{IJ} with the injective map $m_* : \mathrm{Hom}_B(IJ,B) \to \mathrm{Hom}_B(I \otimes_B J,B)$ induced from the surjective multiplication map $I \otimes_B J \to IJ$, we see that m_* is surjective, thus it is an isomorphism of B-B-bimodules. This implies that μ_{IJ} is surjective, and therefore (6) holds. □

From Lemma 7.1, we have the following

Proposition 7.2. *Let B be a ring with identity. Suppose that $I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_2 \subseteq I_1$ is a chain of ideals in B. If I_n is a GV-ideal in B, then*

(1) $\mathrm{End}_B(I_1 \oplus \cdots \oplus I_n)$ *is isomorphic to*

$$C := \begin{pmatrix} B & (I_1 : I_2) & (I_1 : I_3) & \cdots & (I_1 : I_n) \\ B & B & (I_2 : I_3) & \cdots & (I_2 : I_n) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ B & B & \cdots & B & (I_{n-1} : I_n) \\ B & B & \cdots & B & B \end{pmatrix}.$$

(2) $K_*\big(\mathrm{End}_B(\bigoplus_{j=1}^n I_j)\big) \simeq K_*(B) \oplus \bigoplus_{j=1}^{n-1} K_*\big(B/(I_j : I_{j+1})\big)$ *for all* $* \in \mathbb{N}$.

Proof. (1) Note that $\mathrm{End}_B(I_1 \oplus I_2 \oplus \cdots \oplus I_n)$ is the matrix ring with the entries $\mathrm{Hom}_B(I_i,I_j)$ for $1 \leq i,j \leq n$. Since I_n is a GV-ideal in B, every ideal I_j in the chain is a GV-ideal of B by Lemma 7.1(5). Now (1) follows from Lemma 7.1 immediately.

(2) This is a direct consequence of (1) and Lemma 4.2. □

As a consequence of Proposition 7.2 and Lemma 7.1(6), we have the following corollary.

Corollary 7.3. *If I is a GV-ideal in a ring B with identity, then, for any positive integer n,*

$$K_*\big(\mathrm{End}_B(\bigoplus_{j=1}^{n} I^j)\big) \simeq K_*(B) \oplus \bigoplus_{j=1}^{n-1} K_*\big(B/(I^j : I^{j+1})\big).$$

If we take $I_1 = B$, we have the following corollary.

Corollary 7.4. *Let B be a ring with identity. Suppose that $I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_2 \subseteq I_1 = B$ is a chain of GV-ideals in B. Then*

$$K_*\big(\mathrm{End}_B(B \oplus \bigoplus_{j=2}^{n} B/I_j)\big) \simeq K_*(B) \oplus K_*(B/I_2) \oplus \bigoplus_{j=2}^{n-1} K_*\big(B/(I_j : I_{j+1})\big).$$

Proof. For each j, we have an $\mathrm{add}({}_BB)$-split sequence by Lemma 7.1(1):

$$0 \longrightarrow I_j \longrightarrow {}_BB \longrightarrow B/I_j \longrightarrow 0.$$

This yields another $\mathrm{add}({}_BB)$-split sequence

$$0 \longrightarrow \bigoplus_{j=1}^{n} I_j \longrightarrow \bigoplus_{j=1}^{n} {}_BB \longrightarrow \bigoplus_{j=1}^{n} B/I_j \longrightarrow 0.$$

Hence $\mathrm{End}_B({}_BB\oplus\bigoplus_{j=2}^n I_j)$ and $\mathrm{End}_B({}_BB\oplus\bigoplus_{j=2}^n B/I_j)$ are derived-equivalent by [10, Theorem 1.1], and have the isomorphic algebraic K-groups K_*. By Proposition 7.2, we see that $K_*\big(\mathrm{End}_B({}_BB\oplus\bigoplus_{j=1}^n B/I_j)\big)\simeq K_*(B)\oplus\bigoplus_{j=1}^{n-1}K_*\big(B/(I_j:I_{j+1})\big)$ for all $*\in\mathbb{N}$. $\square$

As a concrete example, we consider the polynomial ring B:=$\mathbb{Z}[x]$ over $\mathbb{Z}$ in one variable x and its ideal $J:=(p,x)$ with p a prime number in $\mathbb{Z}$. It is known that J is a GV-ideal in B. Thus, for the ring $R:=\mathrm{End}_{\mathbb{Z}[x]}(\mathbb{Z}[x]\oplus J)$, by Proposition 7.2, we have

$$K_*(R)\simeq K_*(\mathbb{Z}[x])\oplus K_*(\mathbb{Z}/p\mathbb{Z}).$$

Since $\mathbb{Z}$ is a left noetherian ring of global dimension one, the Fundamental Theorem in algebraic K-theory says that the above isomorphism can be rewritten as

$$K_*(R)\simeq K_*(\mathbb{Z})\oplus K_*(\mathbb{Z}/p\mathbb{Z}).$$

By [20], we get

$$K_0(R)\simeq\mathbb{Z}\oplus\mathbb{Z},\qquad K_1(R)\simeq\mathbb{Z}/2\mathbb{Z}\oplus(\mathbb{Z}/p\mathbb{Z})^\times,$$
$$K_{2m}(R)=K_{2m}(\mathbb{Z})\quad\text{for } m\geq 1,\qquad K_{2m-1}(R)\simeq K_{2m-1}(\mathbb{Z})\oplus\mathbb{Z}/(p^m-1)\mathbb{Z}\quad\text{for } m\geq 2,$$

where $(\mathbb{Z}/p\mathbb{Z})^\times$ denotes the set of all non-zero elements of $\mathbb{Z}/p\mathbb{Z}$. Note that J is not a projective $\mathbb{Z}[x]$-module. In fact, we have a non-split exact sequence

$$0\longrightarrow\mathbb{Z}[x]\xrightarrow{\lambda}\mathbb{Z}[x]\oplus\mathbb{Z}[x]\xrightarrow{\pi}J\longrightarrow 0,$$

where λ sends $f(x)$ to $(xf(x),-pf(x))$, and π sends $(f(x),g(x))$ to $pf(x)+xg(x)$ for all $f(x),g(x)\in\mathbb{Z}[x]$. So, the result in [13] cannot be applied to R. However, the one in this note is applicable.

Finally, we mention the radical-full extensions in [28]. Recall that an extension $B\subseteq A$ of rings with the same identity is said to be left radical-full if $\mathrm{rad}(B)$ is a left ideal of A and $\mathrm{rad}(A)=\mathrm{rad}(B)A$, where $\mathrm{rad}(A)$ stands for the Jacobson radical of A. So, given a left radical-full extension $B\subseteq A$ of rings, we may form the ring $C:=\begin{pmatrix}A & \mathrm{rad}(B)\\ A & B\end{pmatrix}$. It follows from our results in this note that $K_n(C)\simeq K_n(A)\oplus K_n(B/\mathrm{rad}(B))$ for all $n\geq 0$ since for any ring extension $S\subseteq R$ and any ideal I in S, if I is a left ideal in R then the rings $\begin{pmatrix}R & I\\ R & S\end{pmatrix}$ and $\begin{pmatrix}S/I & 0\\ R/I & R\end{pmatrix}$ are derived-equivalent by Lemma 3.4.

Related to the last example, we have the following open question:

Question: Suppose that I and J are two arbitrary ideals in a ring R with identity. For the ring $S:=\begin{pmatrix}R & I\\ J & R\end{pmatrix}$ (or generally, the ring in Proposition 4.8), can one give a formula for $K_n(S)$ similar to the one in Theorem 1.2 for $n\geq 1$? (See also the question mentioned in Section 5).

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