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Erratum

Erratum to "On the finitistic dimension conjecture I: related to representation-finite algebras" [J. Pure Appl. Algebra 193 (2004) 287–305]☆

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The purpose of this note is to correct a mistake in the proofs of the results, Theorems 3.1 and 4.5, in the above-mentioned paper [1]. The mistake arises due to the statement "the first syzygy of a B-module is an A-module". All errors caused by this mistake can be corrected by the following lemmas. Thus all results in [1] remain true.

In the present note, we keep the original notation used in [1].

Lemma 0.1. *Let B be a subalgebra of an artin algebra A with the same identity such that the Jacobson radical $rad(B)$ of B is a left ideal in A. If X is a B-module, then $rad(\Omega_B(X))$ and $\Omega^2_B(X)$ are A-modules, where $\Omega_B(X)$ stands for the first syzygy of the B-module X.*

Proof. Let X be a B-module, and let $f : P_B(X) \longrightarrow X$ be a projective cover of X. Thus the top of X and the top of $P_B(X)$ are isomorphic, and the kernel of f is contained in the radical of $P_B(X)$. We denote the radical of the B-module X by $\mathrm{rad}(X)$. Note that any surjection $g : X \longrightarrow Y$ between B-modules X and Y induces a surjection $g' : \mathrm{rad}(X) \longrightarrow \mathrm{rad}(Y)$ with kernel $\mathrm{Ker}(g') = \mathrm{Ker}(g) \cap \mathrm{rad}(X)$. Thus we get the following exact sequence in B-mod:

$$0 \longrightarrow \Omega_B(X) \xrightarrow{g'} \mathrm{rad}(P_B(X)) \xrightarrow{f'} \mathrm{rad}({}_BX) \longrightarrow 0.$$

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Let us denote the multiplication map $\mathrm{rad}(B)\otimes_B X \longrightarrow \mathrm{rad}({}_BX) = \mathrm{rad}(B)X$ by μ_X, and the inclusion $\mathrm{rad}({}_BX) \to X$ by q_X. Then we have the following commutative diagram in B-mod:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1^B(\mathrm{rad}(B), X) \arrow[r, "\delta"] & \mathrm{rad}(B)\otimes_B \Omega_B(X) \arrow[r, "1\otimes g"] \arrow[d, "\mu_{\Omega_B(X)} q_{\Omega_B(X)}"'] & \mathrm{rad}(B)\otimes_B (P_B(X)) \arrow[r, "1\otimes f"] \arrow[d, "\mu_{P_B(X)}"'] & \mathrm{rad}(B)\otimes_B X \arrow[r] \arrow[d, "\mu_X"'] & 0 \\
 & 0 \arrow[r] & \Omega_B(X) \arrow[r, "g'"] & \mathrm{rad}(P_B(X)) \arrow[r, "f'"] & \mathrm{rad}({}_BX) \arrow[r] & 0
\end{tikzcd}$$

Since $\mathrm{rad}({}_BX) = \mathrm{rad}(B)X$ and since the map $\mu_{P_B(X)}$ is an isomorphism, we get an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^B(\mathrm{rad}(B), X) \xrightarrow{\delta} \mathrm{rad}(B)\otimes_B \Omega_B(X) \xrightarrow{\mu} \mathrm{rad}(B)\Omega_B(X) {=} \mathrm{rad}(\Omega_B(X)) \longrightarrow 0.$$

Since $\mathrm{rad}(B)$ is an A-B-bimodule, we see that the morphism $\mathrm{Tor}_1^B(\mathrm{rad}(B), X) \longrightarrow \mathrm{rad}(B)\otimes_B \Omega_B(X)$ is an A-module homomorphism. Thus $\mathrm{rad}(\Omega_B(X))$, as the quotient of the A-module homomorphism δ, is an A-module. Note that the A-module structure of $\mathrm{rad}(\Omega_B(X))$ is induced from $\mathrm{rad}(B)\otimes_B \Omega_B(X)$, that is, the A-module structure on $\mathrm{rad}(B)\Omega_B(X)$ is given by $a \cdot (bx) = (ab)x$ for all $a \in A$, $b \in \mathrm{rad}(B)$ and $x \in \Omega_B(X)$. Now it follows from the surjection $P_B(\Omega_B(X)) \longrightarrow \Omega_B(X)$ that we have another exact sequence in B-mod:

$$(*) \qquad 0 \longrightarrow \Omega_B^2(X) \longrightarrow \mathrm{rad}(P_B(\Omega_B(X)) \longrightarrow \mathrm{rad}(\Omega_B(X)) \longrightarrow 0.$$

Since the A-module structures on $\mathrm{rad}(P_B(\Omega_B(X)))$ and $\mathrm{rad}(\Omega_B(X))$ are given by the left multiplication of elements in A, the map $\mathrm{rad}(P_B(\Omega_B(X))) \longrightarrow \mathrm{rad}(\Omega_B(X))$ is an A-homomorphism. Thus its kernel $\Omega_B^2(X)$ is an A-module. $\square$

We stress that $\mathrm{rad}({}_BX)$ might not be an A-module in general, this can be seen by the following example which is given by R. Farnsteiner, and informed to me by C.M. Ringel. However, if X is a projective B-module, then $\mathrm{rad}(X) \simeq \mathrm{rad}(B)\otimes_B X$ is an A-module.

Example. Let A be the 2 by 2 matrix algebra over the k-algebra $k[x]/(x^2)$. If we take B to be the subalgebra of A generated by $\mathrm{rad}(A)$ and the identity of A. Then $\mathrm{rad}(A) = \mathrm{rad}(B)$. Note that B is a local algebra and has a 2-dimensional uniserial module X. The radical of X is one-dimensional and cannot be an A-module because a simple A-module must be 2-dimensional. This shows that $\mathrm{rad}({}_BX)$ may not have an A-module structure even under the assumption "$\mathrm{rad}(B) = \mathrm{rad}(A)$".

Lemma 0.2. *Suppose B is a subalgebra of A such that $\mathrm{rad}(B)$ is a left ideal in A. For any B-module X and integer $i \geqslant 2$, there is a projective A-module Q and an A-module Z such that $\Omega_B^i(X) \simeq \Omega_A(Z) \oplus Q$ as A-modules.*

If $\mathrm{rad}(B)$ is an ideal in A, then there is an exact sequence of A-modules:

$$0 \longrightarrow \Omega_B^i(X) \longrightarrow \Omega_A^2(Y) \oplus P \longrightarrow S \longrightarrow 0,$$

where P is projective, and S is an A-module such that ${}_BS$ is semisimple. In particular, if $\mathrm{rad}(B) = \mathrm{rad}(A)$, the module S is even a semisimple A-module.

Proof. We take a minimal projective resolution of the B-module ${}_BX$:

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} {}_BX \longrightarrow 0.$$

This gives an exact sequence of A-modules:

$$(*) \qquad 0 \longrightarrow \Omega^i_B(X) \longrightarrow \mathrm{rad}_B(P_{i-1}) \xrightarrow{d_{i-1}} \mathrm{rad}_B(P_{i-2}).$$

Since $\mathrm{rad}_B(P_i)$ is an A-module by the left multiplication of elements in A, we know that the map d_{i-1} in $(*)$ is an A-homomorphism with the image $\mathrm{rad}(\Omega^{i-1}_B(X))$. Now we have the following sequence

$$A\otimes_B P_{i-1} \xrightarrow{g} A\otimes_B P_{i-2} \longrightarrow Y \longrightarrow 0,$$

where Y is the cokernel of the map $g := id_A\otimes_B d_{i-1}$. Then one has embeddings $\Omega^i_B(X) \hookrightarrow \mathrm{rad}({}_BP_{i-1}) \simeq \mathrm{rad}(B)\otimes_B P_{i-1} \hookrightarrow A\otimes_B P_{i-1}$, with the last inclusion following from the projectivity of P_{i-1}.

This implies that $\Omega^i_B(X)$ can be embedded in the projective A-module $A\otimes_B P_{i-1}$. If we denote the cokernel of this embedding by Z, then we know that $\Omega^i_B(X) \simeq \Omega_A(Z)\oplus Q$ with Q a projective A-module. Note that the modules Z and Q depend upon X. This finishes the first part of the lemma.

Now suppose that the minimal projective presentation of the A-module Y is given by

$$0 \longrightarrow \Omega^2_A(Y) \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow Y \longrightarrow 0,$$

with Q_i projective. Then an elementary homological calculation shows that there is a projective A-module Q' such that $\mathrm{Im}(g) \simeq \Omega_A(Y)\oplus Q'$. Since $Q_1\oplus Q' \longrightarrow \mathrm{Im}(g)$ is a projective cover of the A-module $\mathrm{Im}(g)$ with the kernel $\Omega^2_A(Y)$, we know that there is a projective A-module P such that $\mathrm{Ker}(g) \simeq \Omega^2_A(Y)\oplus P$. Thus we have the following exact commutative diagram in A-mod:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega^i_B(X) & \longrightarrow & \mathrm{rad}(P_{i-1}) & \longrightarrow & \mathrm{rad}(\Omega^{i-1}_B(X)) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega^2_A(Y)\oplus P & \longrightarrow & A\otimes_B P_{i-1} & \longrightarrow & A\otimes_B P_{i-2} & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & S_1 & \longrightarrow & S_2 & \longrightarrow & S_3 & \longrightarrow & Y & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}$$

Since S_2 is isomorphic to $(A/\mathrm{rad}(B))\otimes_B P_{i-1}$, we know that S_2 is a semisimple B-module because $\mathrm{rad}(B)$ is an ideal in A and $\mathrm{rad}(B)(A/\mathrm{rad}(B)) = 0$. Note that if $\mathrm{rad}(B) = \mathrm{rad}(A)$, then S_2 itself is a semisimple A-module. Thus, as a submodule of S_2, the module S_1 is also semisimple as a B-module. This gives the second part of the lemma. $\square$

By Lemma 0.1, we can give a new proof of Theorem 3.1: one has to replace $\Omega_B(X)$ by $\Omega_B^2(X)$ and raise the given bound by 1 in the original proof. Theorem 4.5 in [1] can be proved as follows:

Theorem 0.3. *Let A, B and C be three artin algebras with the same identity such that* (i) $C \subseteq B \subseteq A$, *and* (ii) *the Jacobson radical of C is a left ideal of B, and the Jacobson radical of B is a left ideal of A. If A is representation-finite, then C has finite finitistic dimension.*

Proof. Let ${}_CX$ be a C-module of finite projective dimension. It follows from Lemma 0.1 that $\Omega_C^2(X)$ is a B-module. So we may consider the following exact sequence of B-modules:

$$0 \longrightarrow \Omega_B\Omega_C^2(X) \longrightarrow P \longrightarrow \Omega_C^2(X) \longrightarrow 0,$$

where P is a projective B-module. By Lemma 0.2, there is a B-module Y and a projective B-module Q' such that $\Omega_C^2(X) = \Omega_B(Y) \oplus Q'$. Thus the above exact sequence can be rewritten as:

$$0 \longrightarrow \Omega_B^2(Y) \longrightarrow P \longrightarrow \Omega_C^2(X) \longrightarrow 0.$$

Again by Lemma 0.2, there is an A-module Z and a projective A-module Q such that $\Omega_B^2(Y) = \Omega_A(Z) \oplus Q$. So we have the following exact sequence:

$$0 \longrightarrow \Omega_A(Z) \oplus Q \longrightarrow P \longrightarrow \Omega_C^2(X) \longrightarrow 0.$$

Now, if we consider this sequence as a sequence in C-mod, then we may use the idea in [1] to finish the proof of this theorem. $\square$

As a consequence of the two lemmas, all proofs in [1] remain unchanged, but must take into account the second syzygy of a B-module.

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References

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