



# Cellularity of twisted semigroup algebras

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## ABSTRACT

In this paper, the cellularity of twisted semigroup algebras over an integral domain is investigated by introducing the concept of cellular twisted semigroup algebras of type  $\mathcal{JH}$ . Partition algebras, Brauer algebras and Temperley–Lieb algebras all are examples of cellular twisted semigroup algebras of type  $\mathcal{JH}$ . Our main result shows that the twisted semigroup algebra of a regular semigroup is cellular of type  $\mathcal{JH}$  with respect to an involution on the twisted semigroup algebra if and only if the twisted group algebras of certain maximal subgroups are cellular algebras. Here we do not assume that the involution of the twisted semigroup algebra induces an involution of the semigroup itself. Moreover, for a twisted semigroup algebra, we do not require that the twisting decomposes essentially into a constant part and an invertible part, or takes values in the group of units in the ground ring. Note that trivially twisted semigroup algebras are the usual semigroup algebras. So, our results extend not only a recent result of East, but also some results of Wilcox.

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## 1. Introduction

Motivated by a multiplicative property of the famous Kazhdan–Lusztig canonical basis for Hecke algebras of type A, the notion of cellular algebras was initially introduced by Graham and Lehrer in [4]. The theory of cellular algebras provides a systematical method for understanding the representation theory of many important algebras in mathematics and physics. Roughly speaking, a cellular algebra is an associative algebra with an involution and a basis such that the multiplication of basis elements can be expressed by a “straightening formula” (see Definition 2.3). An advantage of a cellular algebra is that many questions in representation theory are reduced to relatively easy ones in linear algebra; for instance, the parametrization of non-isomorphic irreducible representations and the semisimplicity of a finite-dimensional cellular algebra over a field are reduced to calculations of non-zero bilinear forms [4]. Moreover, homological properties like global dimension and quasi-heredity for cellular algebras can be characterized by Cartan determinants (see [9,16]). Examples of cellular algebras include all Hecke algebras of finite types,  $q$ -Schur algebras, Brauer algebras, Temperley–Lieb algebras, partition algebras, Birman–Wenzl algebras and many other diagram algebras (see [3–5,11,13,14]). The theory of cellular algebras also opens a way for a characteristic-free investigation of these algebras (for example, see [10] for Brauer algebras). On the one hand, it is worth noticing that all these algebras just mentioned give us cellular semigroup algebras by specializing their defining parameters. On the other hand, it is not difficult to see that for some semigroups, their semigroup algebras over a field cannot be cellular, for example, the upper triangular  $2 \times 2$  matrix algebra over a field can be considered as a semigroup algebra, this algebra is not cellular by [8]. So, a natural question is: When is a semigroup algebra cellular?

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Another question is: how to get different cellular structures by deforming semigroup algebras? In [2], East investigated the cellularity of inverse semigroup algebras, he gave a sufficient condition for a finite inverse semigroup algebra to be cellular. Wilcox [12] extended the result in [2] and considered the cellularity of twisted finite regular semigroup algebras, he gave a sufficient condition for a twisted finite regular semigroup algebra over a commutative ring  $R$  to be cellular with respect to an involution which is determined by an involution on the semigroup itself and an involution on the ground ring  $R$ .

In the present paper, we continue to investigate the cellularity of twisted regular semigroup algebras. Since involution is one of the substantial ingredients for a cellular algebra, we first analyze involutions of twisted semigroup algebras and introduce the notion of  $\mathcal{H}$ -type and  $\mathcal{J}$ -type for involutions as well as the notion of  $\mathcal{JH}$ -type for twisted semigroup algebras. To answer our questions mentioned above, we show in Theorem 5.3 that the twisted semigroup algebra of a regular semigroup is cellular of type  $\mathcal{JH}$  with respect to an involution on the twisted semigroup algebra if and only if the twisted group algebras of certain maximal subgroups contained in the semigroup are cellular. For groups, it is known that the group algebras of symmetric groups, dihedral groups, and the finite Coxeter groups are cellular (see [4,3]). We have mentioned that the case of an inverse semigroup was considered by East in [2], and the case of a regular semigroup with the assumption of existence of an involution on the semigroup is considered by Wilcox in [12]. Thus, comparing with the result of Wilcox in [12], our result is obtained without the assumption that an involution of the twisted semigroup algebra of a regular semigroup must be an involution of the semigroup itself. As is known, requiring a semigroup to have an involution is a strong condition on the semigroup itself. Moreover, in our consideration we only assume the ground ring to be an integral domain, and drop the restriction that a twisting takes values in the group of units of the ground ring. Thus, our result also extends the main result of Wilcox in [12].

It is well-known in [4] that, for a cellular algebra with cell datum  $(I, M, C, \delta)$  (see Definition 2.3), the irreducible representations can be indexed by a subset of the poset  $I$ . Moreover, it is shown in [9] that a cellular algebra has finite global dimension if and only if its Cartan determinant is 1. Thus we may parameterize the irreducible representations of a cellular semigroup algebra, and determine when a cellular semigroup algebra is of finite global dimension, or quasi-hereditary by applying the general methods of cellular algebras (see [4,9,16,17]).

The paper is organized as follows: In Section 2, we recall some definitions and basic facts needed in later proofs, and introduce the  $\mathcal{J}$ -type and the  $\mathcal{JH}$ -type for cellular twisted semigroup algebras. In Section 3, we discuss some properties of a twisting on a semigroup. Section 4 is devoted to dealing with involutions on twisted semigroup algebras. Here we shall introduce the notion of type  $\mathcal{J}$  and type  $\mathcal{H}$  for involutions, and give a characterization of such involutions. The main results, Theorems 5.2 and 5.3, are stated and proved in Section 5. In Section 6, we apply the general theory of cellular algebras to twisted semigroup algebras and give a criterion for a cellular twisted semigroup algebra to be semisimple.

## 2. Preliminaries

In this section, we shall first recall some basic definitions and facts on semigroups. For further information on semigroups we refer to any standard text books, for example, the book by Howie [7]. After this, we recall the concept of a cellular algebra in [4], and the notion of a twisting in [12]. Also, we shall introduce types for cellular twisted semigroup algebras.

### 2.1. Definitions and basic facts on semigroups

Let  $\mathfrak{S}$  be a semigroup, and  $\mathfrak{S}^1$  the semigroup obtained from  $\mathfrak{S}$  by adding an identity if  $\mathfrak{S}$  has no identity, otherwise we put  $\mathfrak{S}^1 = \mathfrak{S}$ . In the theory of semigroups, the Green's relations  $\mathcal{L}$ ,  $\mathcal{R}$ ,  $\mathcal{J}$ ,  $\mathcal{H}$  and  $\mathcal{D}$  on  $\mathfrak{S}$  are of fundamental importance [6]. They are defined in the following way: for  $x, y \in \mathfrak{S}$ ,

$$\begin{aligned} x\mathcal{L}y &\Leftrightarrow \mathfrak{S}^1x = \mathfrak{S}^1y; \\ x\mathcal{R}y &\Leftrightarrow x\mathfrak{S}^1 = y\mathfrak{S}^1; \\ x\mathcal{J}y &\Leftrightarrow \mathfrak{S}^1x\mathfrak{S}^1 = \mathfrak{S}^1y\mathfrak{S}^1; \\ \mathcal{H} &= \mathcal{L} \cap \mathcal{R}; \\ \mathcal{D} &= \mathcal{L} \vee \mathcal{R} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}. \end{aligned}$$

If  $\mathcal{K}$  is one of the Green's relations and  $a \in \mathfrak{S}$ , we denote by  $K_a$  the  $\mathcal{K}$ -class of  $\mathfrak{S}$  containing  $a$ , and by  $\mathfrak{S}/\mathcal{K}$  the set of all  $\mathcal{K}$ -classes of  $\mathfrak{S}$ . Further, we define

$$\begin{aligned} L_a &\leq L_b \quad \text{if } \mathfrak{S}^1a \subseteq \mathfrak{S}^1b, \\ R_a &\leq R_b \quad \text{if } a\mathfrak{S}^1 \subseteq b\mathfrak{S}^1, \\ J_a &\leq J_b \quad \text{if } \mathfrak{S}^1a\mathfrak{S}^1 \subseteq \mathfrak{S}^1b\mathfrak{S}^1. \end{aligned}$$

Then we have a partial order on each of the sets  $\mathfrak{S}/\mathcal{L}$ ,  $\mathfrak{S}/\mathcal{R}$  and  $\mathfrak{S}/\mathcal{J}$ . It is well known that  $\mathcal{D} = \mathcal{J}$  if  $\mathfrak{S}$  is a finite semigroup. In this case, the set  $\mathfrak{S}/\mathcal{D}$  of  $\mathcal{D}$ -classes of  $\mathfrak{S}$  inherits a partial order defined by

$$D_s \leq D_t \Leftrightarrow s \in \mathfrak{S}^1t\mathfrak{S}^1, \quad \text{for } s, t \in \mathfrak{S}.$$

Clearly,  $D_{xy} \leq D_y$  for all  $x, y \in \mathfrak{S}$ .

We say that a semigroup  $\mathfrak{S}$  satisfies the condition  $\min_L$  (respectively,  $\min_R$  or  $\min_J$ ) if the partially ordered set  $\mathfrak{S}/\mathcal{L}$  (respectively,  $\mathfrak{S}/\mathcal{R}$  or  $\mathfrak{S}/\mathcal{J}$ ) satisfies the descending chain condition.

By a 0-simple semigroup, we mean a semigroup with zero satisfying the condition that  $\mathfrak{S}a\mathfrak{S} = \mathfrak{S}$  for every  $a \in \mathfrak{S} \setminus \{0\}$ . Note that 0-simple semigroups occur very often. The following is a method to get 0-simple semigroups:

Let  $\mathfrak{S}$  be a semigroup and  $a \in \mathfrak{S}$ . Then  $I(a) := \{b \in \mathfrak{S}^1 a \mathfrak{S}^1 \mid \mathfrak{S}^1 b \mathfrak{S}^1 \subset \mathfrak{S}^1 a \mathfrak{S}^1\}$  is an ideal of  $\mathfrak{S}^1 a \mathfrak{S}^1$ . It is clear that  $\mathfrak{S}^1 a \mathfrak{S}^1 = J_a \cup I(a)$ . Now we consider the set  $\mathfrak{S}_a := J_a \cup \{0\}$ , where 0 stands for a single element, and is not in  $J_a$ . We define an operation  $\circ$  on  $\mathfrak{S}_a$  as follows: for all  $x, y \in J_a$

$$x \circ y = \begin{cases} xy & \text{if } xy \in J_a, \\ 0 & \text{otherwise,} \end{cases}$$

where  $xy$  is the product of  $x$  and  $y$  in  $\mathfrak{S}$ , and  $x \circ 0 = 0 \circ x = 0 \circ 0 = 0$ . It is easy to check that  $(\mathfrak{S}_a, \circ)$  is a semigroup with zero element 0. This semigroup  $\mathfrak{S}_a$  is called the *principal factor* of  $\mathfrak{S}$  determined by  $a$ . It is well known that each principal factor of  $\mathfrak{S}$  is either a 0-simple semigroup or a null semigroup (that is, a semigroup  $\mathfrak{S}$  with zero and  $\mathfrak{S}^2 = \{0\}$ ) (see [7] for a detailed discussion).

An element  $a$  of a semigroup  $\mathfrak{S}$  is called *regular* if there exists  $b \in \mathfrak{S}$  such that  $aba = a$ . Equivalently,  $a \in \mathfrak{S}$  is regular if and only if the  $\mathcal{L}$ -class  $L_a$  (respectively, the  $\mathcal{R}$ -class  $R_a$ , or the  $\mathcal{D}$ -class  $D_a$ ) contains an idempotent element of  $\mathfrak{S}$ . A semigroup  $\mathfrak{S}$  is *regular* if each element of  $\mathfrak{S}$  is regular. If  $a$  and  $b$  are two elements in a regular semigroup  $\mathfrak{S}$  and if  $\mathcal{K}$  is a Green relation on  $\mathfrak{S}$ , then, for all  $x, y \in J_a$ , we have  $a\mathcal{K}b$  in  $\mathfrak{S}$  if and only if  $x\mathcal{K}y$  in  $\mathfrak{S}_a$ .

A regular semigroup  $\mathfrak{S}$  is called an *inverse semigroup* if each  $\mathcal{L}$ -class and each  $\mathcal{R}$ -class of  $\mathfrak{S}$  contains precisely one idempotent element; and is called a *completely 0-simple semigroup* if  $\mathfrak{S}$  is a 0-simple semigroup satisfies the conditions  $\min_L$  and  $\min_R$ .

Now let  $I$  and  $\Lambda$  be non-empty sets and  $G$  a group, and let  $P = (p_{\lambda k})$  be a  $\Lambda \times I$ -matrix with entries in the 0-group  $G^0 := G \cup \{0\}$ . Suppose  $P$  is *regular* in the sense that no row and no column consist entirely of zero elements. More precisely, for any  $k \in I$ , there exists a  $\lambda \in \Lambda$  such that  $p_{\lambda k} \neq 0$ , and for any  $\lambda \in \Lambda$ , there exists  $l \in I$  such that  $p_{\lambda l} \neq 0$ . Let  $\mathfrak{S} = (G \times I \times \Lambda) \cup \{0\}$ . We define a composition on  $\mathfrak{S}$  by

$$(a, k, \lambda)(b, l, \mu) = \begin{cases} (ap_{\lambda l}b, k, \mu) & \text{if } p_{\lambda l} \neq 0; \\ 0 & \text{if } p_{\lambda l} = 0 \end{cases}$$

and

$$(a, k, \lambda)0 = 0(a, k, \lambda) = 00 = 0,$$

where  $a, b \in G, k, l \in I$ , and  $\lambda, \mu \in \Lambda$ . One can verify that  $\mathfrak{S}$  becomes a completely 0-simple semigroup with respect to the above composition. It is called the  *$I \times \Lambda$  Rees matrix semigroup* over the 0-group  $G^0$  with the regular sandwich matrix  $P$ , and is denoted by  $\mathcal{M}^0(G, I, \Lambda; P)$ . The Rees theorem says that each completely 0-simple semigroup is isomorphic to some  $\mathcal{M}^0(G, I, \Lambda; P)$ , and vice versa.

Similarly, in the above construction, we may replace  $G$  by an  $R$ -algebra  $A$  over a commutative ring  $R$  with identity and use the above multiplication to define an  $R$ -algebra structure on  $\bigoplus_{(\lambda, k) \in I \times \Lambda} A$  with 0 as zero element, this algebra is called a Munn algebra, and is denoted by  $[A, I, \Lambda; P]$ .

**Lemma 2.1** ([7, p. 62]).

- (1) The following statements are equivalent for  $(a, k, \lambda), (b, l, \mu) \in \mathcal{M}^0(G, I, \Lambda; P)$ :
  - (1)  $(a, k, \lambda)\mathcal{L}(b, l, \mu)$  if and only if  $\lambda = \mu$ .
  - (2)  $(a, k, \lambda)\mathcal{R}(b, l, \mu)$  if and only if  $k = l$ .
  - (3)  $(a, k, \lambda)\mathcal{H}(b, l, \mu)$  if and only if  $k = l$  and  $\lambda = \mu$ .
- (2) Any non-zero elements of a completely 0-simple semigroup are in the same  $\mathcal{D}$ -class, and any two non-zero maximal subgroups contained in a completely 0-simple semigroup are isomorphic.

By [7, Theorem III.2.8, p.66], we may assume in  $\mathcal{M}^0(G, I, \Lambda; P)$  that  $I \cap \Lambda = \{0\}$ ,  $G = (G, 0, 0)$  and  $p_{0,0} = e$ , where  $e$  is the identity of  $G$ . In addition, for any non-zero maximal subgroup  $G'$  of  $\mathcal{M}^0(G, I, \Lambda; P)$ , we have that  $\mathcal{M}^0(G, I, \Lambda; P) \cong \mathcal{M}^0(G', I, \Lambda; P)$  as semigroups. In what follows, we always suppose that  $\mathcal{M}^0(G, I, \Lambda; P)$  satisfies the above assumptions.

Suppose that  $\mathfrak{S}$  is a finite regular semigroup. It follows from the regularity that the principal factor  $\mathfrak{S}_a$  is a 0-simple semigroup for every  $a \in \mathfrak{S}$ . Observe that a finite semigroup satisfies the conditions  $\min_L$  and  $\min_R$ . So the semigroup  $\mathfrak{S}_a$  is a completely 0-simple semigroup. Thus each principal factor of a finite regular semigroup is a completely 0-simple semigroup.

Finally, let us mention the following facts on a semigroup, which will be used later in the proofs.

**Lemma 2.2** ([7] p. 45, p. 59). Let  $\mathfrak{S}$  be an arbitrary semigroup.

- (1) If  $a, x \in \mathfrak{S}$ , then either  $xa \in J_a$  or  $xa \in I(a)$ .
- (2) Suppose  $a, e = e^2 \in \mathfrak{S}$ . If  $a\mathcal{L}e$ , then  $a = ae$ . Similarly, if  $a\mathcal{R}e$ , then  $a = ea$ .

## 2.2. Cellular algebras and twisting maps of semigroups

Throughout this paper,  $R$  is a commutative ring with identity.

First, let us recall the original definition of cellular algebras introduced by Graham and Lehrer, which is given by means of multiplicative properties of a basis.

**Definition 2.3** ([4]). An associative  $R$ -algebra  $A$  (possibly without identity) is called a *cellular algebra* with cell datum  $(I, M, C, \delta)$  if the following conditions are satisfied:

- (C1)  $I$  is a partially ordered set. Associated with each  $\lambda \in I$  there is a set  $M(\lambda)$ . The algebra  $A$  has an  $R$ -basis  $C_{S,T}^\lambda$ , where  $\lambda$  runs through  $I$ , and where  $(S, T)$  runs through  $M(\lambda) \times M(\lambda)$ .
- (C2)  $\delta$  is an  $R$ -linear anti-automorphism of  $A$  of order 2, and sends  $C_{S,T}^\lambda$  to  $C_{T,S}^\lambda$ .
- (C3) If  $\lambda \in I$  and  $S, T \in M(\lambda)$ , then, for each  $a \in A$ ,

$$aC_{S,T}^\lambda = \sum_{U \in M(\lambda)} r_a(U, S)C_{U,T}^\lambda + r',$$

where the coefficients  $r_a(U, S) \in R$  do not depend on  $T$ , and where  $r'$  is a linear combination of basis elements  $C_{X,Y}^\mu$  with upper index  $\mu$  strictly smaller than  $\lambda$ .

Note that in the above definition we do not require that  $I$  and  $M(\lambda)$  are finite sets. If they are finite sets and if the  $R$ -algebra  $A$  has an identity, then the above definition of an cellular algebra coincides with the one in [4]. In the present paper, a cellular  $R$ -algebra  $A$  with an identity is called a **unitary** cellular algebra. Examples of (unitary) cellular algebras include Temperley–Lieb algebras, Brauer algebras, Hecke algebras of finite type, partition algebras and certain Birman–Wenzl algebras (see [3,4,14,15]). The significance of cellular algebras is that the irreducible representations can be determined by methods in linear algebra. For more details and further information on cellular algebras we refer to [4,9] and the references therein.

In the following, an  $R$ -linear anti-automorphism  $\delta$  of  $A$  with  $\delta^2 = id$  is called an  *$R$ -involution*.

Next, let us introduce some notations related to semigroup algebras.

Let  $\mathfrak{S}$  be a semigroup. We denote by  $R[\mathfrak{S}]$  the semigroup algebra of  $\mathfrak{S}$  over  $R$ . In general, if  $I$  is a subset of  $\mathfrak{S}$ , then  $R[I]$  denotes the set of  $R$ -linear combinations of elements in  $I$ , that is,  $R[I]$  is a free  $R$ -module with  $I$  as a basis. So each element of  $R[I]$  is a finite summation of the form  $\sum_{x \in I} r_x x$ ,  $r_x \in R$ ,  $x \in I$ . In particular, if  $I_1$  and  $I_2$  are subsets of  $\mathfrak{S}$ , then  $R[I_1 \cap I_2] = R[I_1] \cap R[I_2]$ . If  $\mathfrak{S}$  is a semigroup with zero  $\theta$ , then  $R[\theta]$  is an ideal of  $R[\mathfrak{S}]$ , and we define  $R_0[\mathfrak{S}] = R[\mathfrak{S}]/R[\theta]$ . This  $R$ -algebra  $R_0[\mathfrak{S}]$  is called the *contracted semigroup algebra* of  $\mathfrak{S}$  over  $R$ . If  $\mathfrak{S}$  has no zero, then we define  $R_0[\mathfrak{S}] = R[\mathfrak{S}]$ . Clearly, an element  $a$  of  $R_0[\mathfrak{S}]$  is a finite linear combination  $a = \sum r_s s$  of elements  $s \in \mathfrak{S} \setminus \{\theta\}$ . The support of  $a \in R[\mathfrak{S}]$ , denoted by  $\text{supp}(a)$ , is the set  $\{s \in \mathfrak{S} \setminus \{\theta\} \mid r_s \neq 0\}$ .

Finally, we recall the definition of a twisting on a semigroup  $\mathfrak{S}$  in [12].

**Definition 2.4.** (1) A *twisting* of  $\mathfrak{S}$  into  $R$  is a map

$$\pi : \mathfrak{S} \times \mathfrak{S} \longrightarrow R, \quad (x, y) \mapsto \pi(x, y) \quad \text{for } x, y \in \mathfrak{S},$$

which satisfies

$$(TW) \quad \pi(x, y)\pi(xy, z) = \pi(x, yz)\pi(y, z)$$

for  $x, y, z \in \mathfrak{S}$ .

- (2) A twisting  $\pi$  of  $\mathfrak{S}$  into  $R$  is called an  $\mathcal{LR}$ -twisting of  $\mathfrak{S}$  if it satisfies the following two properties for all  $x, y, z \in \mathfrak{S}$ :
  - (LR1) If  $x\mathcal{L}y$ , then  $\pi(x, z) = \pi(y, z)$ .
  - (LR2) If  $y\mathcal{R}z$ , then  $\pi(x, y) = \pi(x, z)$ .
- (3) Let  $\pi$  be a twisting of  $\mathfrak{S}$  into  $R$ . The *twisted semigroup algebra*  $R^\pi[\mathfrak{S}]$  of  $\mathfrak{S}$  over  $R$  with respect to the twisting  $\pi$ , is defined to be an  $R$ -algebra with the  $R$ -basis  $\mathfrak{S}$  in which the multiplication  $\bullet$  is defined by

$$x \bullet y = \pi(x, y)(xy) \quad \text{for all } x, y \in \mathfrak{S},$$

and is extended by linearity.

- (4) In case  $\mathfrak{S}$  has a zero element  $\theta$ , the *twisted contracted semigroup algebra* of  $\mathfrak{S}$  over  $R$  with respect to a twisting  $\pi$  is defined as  $R^\pi[\mathfrak{S}]/R[\theta]$ , denoted by  $R_0^\pi[\mathfrak{S}]$ .

It is shown in [12] that Brauer algebras, Temperley–Lieb algebras and partition algebras are examples of twisted semigroup algebras.

We end this section by introducing the notion of  $\mathcal{J}$ -type and  $\mathcal{JH}$ -type for twisted semigroup algebras. These types reflect a relationship between a cellular basis and a Green relation on a semigroup.

**Definition 2.5.** Let  $R$  be a commutative ring with identity, and let  $\mathfrak{S}$  be a semigroup (with zero). A twisted (or contracted) semigroup algebra  $R^\pi[\mathfrak{S}]$  (or  $R_0^\pi[\mathfrak{S}]$ ) is called a *cellular algebra of type  $\mathcal{J}$*  with cell datum  $(I, M, C, \delta)$  if the conditions (C1)–(C3) in Definition 2.3 and the following additional condition are satisfied:

- (C4) For every  $\lambda \in I$ , there exists a  $\mathcal{J}$ -class  $J$  of  $\mathfrak{S}$  such that  $\text{supp}(C_{S,T}^\lambda) \subseteq J$  for all  $S, T \in M(\lambda)$ . Note that the  $J$  may depend on the given  $\lambda$ .

**Definition 2.6.** A cellular twisted (contracted) semigroup algebra  $R^\pi[\mathfrak{S}](R_0^\pi[\mathfrak{S}])$  with datum  $(I, M, C, \delta)$  is of type  $\mathcal{JH}$  if it is of type  $\mathcal{J}$ , and for each  $\lambda \in I$  and  $S, T \in M(\lambda)$ , there exists an  $\mathcal{H}$ -class  $H$  of  $\mathfrak{S}$  such that  $\text{supp}(C_{S,T}^\lambda) \subseteq H$ .

We remark that  $R[\mathfrak{S}]$  and  $R^\pi[\mathfrak{S}]$  are equal as  $R$ -modules. So the notion of support in  $R^\pi[\mathfrak{S}]$  makes sense with respect to the canonical basis  $\mathfrak{S}$ . Also, we may speak of the support of an element in  $R_0^\pi[\mathfrak{S}]$ .

For a basis-free definition of a cellular algebra we refer to [8]. For properties and more examples of cellular algebras we refer to [3,4,14,15] and the references therein. For further information about semigroup algebras, we refer the reader to [1].

### 3. Properties of a twisting of a semigroup

In this section, we investigate the structure of a semigroup algebra with a twisting.

Let  $\pi$  be a twisting of a semigroup  $\mathfrak{S}$  into  $R$ , and  $a \in \mathfrak{S}$ . We define a multiplication  $\odot$  on  $R[J_a]$  as follows: for  $x, y \in J_a$ ,

$$x \odot y := \begin{cases} \pi(x, y)xy & \text{if } xy \in J_a; \\ 0 & \text{otherwise} \end{cases}$$

and  $x \odot 0 = 0 = 0 \odot x$ , where  $xy$  is the product of  $x$  and  $y$  in  $\mathfrak{S}$  and  $0$  is the zero element of the  $R$ -module  $R[J_a]$ . Then we extend this multiplication  $R$ -linearly to the whole  $R$ -module  $R[J_a]$ . We claim that  $R[J_a]$  is an  $R$ -algebra with respect to the multiplication  $\odot$  (possibly without identity). In fact, it is sufficient to show that  $(R[J_a], \odot)$  is a twisted contracted semigroup algebra of the semigroup  $\mathfrak{S}_a := J_a \cup \{0\}$  with multiplication  $\circ$ .

We define

$$\pi_a : \mathfrak{S}_a \times \mathfrak{S}_a \longrightarrow R, \quad (x, y) \mapsto \pi_a(x, y),$$

where

$$\pi_a(x, y) = \begin{cases} \pi(x, y) & \text{if } xy \in J_a, \\ 0 & \text{if } xy \notin J_a. \end{cases}$$

Now, we verify that  $\pi_a$  is a twisting of  $\mathfrak{S}_a$  into  $R$ . By Definition 2.4, it suffices to verify that  $\pi_a$  satisfies (TW). Suppose  $x, y, z \in J_a$ . We consider the following two cases:

(1) If  $xyz \in J_a$ , then  $\mathfrak{S}^1 a \mathfrak{S}^1 = \mathfrak{S}^1 xyz \mathfrak{S}^1 \subseteq \mathfrak{S}^1 xy \mathfrak{S}^1 \subseteq \mathfrak{S}^1 x \mathfrak{S}^1 = \mathfrak{S}^1 a \mathfrak{S}^1$ . This means that  $xy \in J_a$ . Similarly,  $yz \in J_a$ . Hence  $x \circ y = xy, y \circ z = yz$ . Thus

$$\begin{aligned} \pi_a(x, y)\pi_a(x \circ y, z) &= \pi(x, y)\pi(xy, z) = \pi(x, yz)\pi(y, z) \\ &= \pi_a(x, yz)\pi_a(y, z) = \pi_a(x, y \circ z)\pi_a(y, z). \end{aligned}$$

(2) If  $xyz \notin J_a$ , then  $x \circ y \circ z = 0 = x \circ y \circ z$ . Thus we have to consider the following two cases:

(1) If  $xy \notin J_a$ , then  $\pi_a(x, y) = 0$  and  $\pi_a(x, y)\pi_a(x \circ y, z) = 0$ . On the other hand, we have

(i) If  $yz \in J_a$ , then  $\pi_a(x, y \circ z) = \pi_a(x, yz) = 0$ . Clearly,  $\pi_a(x, y \circ z)\pi_a(y, z) = 0$ .

(ii) If  $yz \notin J_a$ , then  $\pi_a(y, z) = 0$  and  $\pi_a(x, y \circ z)\pi_a(y, z) = 0$ .

(2) If  $xy \in J_a$ , then  $\pi_a(x \circ y, z) = 0$  and  $\pi_a(x, y)\pi_a(x \circ y, z) = 0$ . Further, we have

(i) If  $yz \in J_a$ , then  $\pi_a(x, y \circ z) = 0$  and  $\pi_a(x, y \circ z)\pi_a(y, z) = 0$ . Thus  $\pi_a(x, y)\pi_a(x \circ y, z) = \pi_a(x, y \circ z)\pi_a(y, z)$ .

(ii) If  $yz \notin J_a$ , then  $\pi_a(y, z) = 0$  and  $\pi_a(x, y \circ z)\pi_a(y, z) = 0$ . Thus  $\pi_a(x, y)\pi_a(x \circ y, z) = \pi_a(x, y \circ z)\pi_a(y, z)$ .

Hence, in any cases, the condition (TW) holds true. Thus  $\pi_a$  is a twisting of the semigroup  $(\mathfrak{S}_a, \circ)$ . By definition, we see that the  $R$ -algebra  $(R[J_a], \odot)$  is the twisted contracted semigroup algebra of the semigroup  $(\mathfrak{S}_a, \circ)$  over  $R$  with respect to the twisting  $\pi_a$ . This  $R$ -algebra  $(R[J_a], \odot)$  is denote by  $R^\pi[J_a]$ . Thus we have proved the following proposition.

**Proposition 3.1.** Let  $R$  be a commutative ring with identity and  $\mathfrak{S}$  a semigroup. If  $\pi$  is a twisting of  $\mathfrak{S}$  into  $R$ , then, for every  $a \in \mathfrak{S}$ , the map  $\pi_a$  is a twisting of the semigroup  $(\mathfrak{S}_a := J_a \cup \{0\}, \circ)$  into  $R$ .

Note that if  $\mathcal{K}$  is a Green relation on a regular semigroup  $\mathfrak{S}$  and if  $a \in \mathfrak{S}$  and  $x, y \in J_a$ , then  $x\mathcal{K}y$  in  $\mathfrak{S}$  if and only if  $x\mathcal{K}y$  in  $(J_a \cup \{0\}, \circ)$ . The following is straightforward.

**Proposition 3.2.** Let  $R$  be a commutative ring with identity and  $\mathfrak{S}$  a semigroup. Suppose  $\pi$  is an  $\mathcal{LR}$ -twisting on  $\mathfrak{S}$  into  $R$ . Then, for every  $a \in \mathfrak{S}$ , the  $\pi_a$  is an  $\mathcal{LR}$ -twisting of  $\mathfrak{S}_a$  into  $R$ .

**Proposition 3.3.** Let  $\pi$  be a twisting of  $\mathfrak{S} = \mathcal{M}^0(G, I, \Lambda; P)$  into  $R$ . Then  $\pi$  is an  $\mathcal{LR}$ -twisting if and only if, for  $(g, i, \lambda), (h, j, \mu) \in \mathfrak{S}$  with  $g, h \in G, i, j \in I$ , and  $\lambda, \mu \in \Lambda$ , we have  $\pi((g, i, \lambda), (h, j, \mu)) = \pi((e, 0, \lambda), (e, j, 0))$ , where  $e$  is the identity of  $G$ .

**Proof.** By Lemma 2.1, we have  $(g, i, \lambda)\mathcal{L}(e, 0, \lambda)$  and  $(h, j, \mu)\mathcal{R}(e, j, 0)$ . Thus  $\pi$  is an  $\mathcal{LR}$ -twisting of  $\mathfrak{S}$  into  $R$  if and only if

$$\pi((g, i, \lambda), (h, j, \mu)) = \pi((e, 0, \lambda), (h, j, \mu)) = \pi((e, 0, \lambda), (e, j, 0)). \quad \square$$

**Proposition 3.4.** Let  $R$  be a commutative ring with identity and  $\mathfrak{S} = \mathcal{M}^0(G, I, \Lambda; P)$ , and let  $\pi$  be a twisting of  $\mathfrak{S}$  into  $R$ . Then  $R_0^\pi[\mathfrak{S}]$  is isomorphic to the Munn algebra  $[R[G], I, \Lambda; \bar{P}]$ , where  $\bar{P} = (\pi_{\lambda i} p_{\lambda i})$  with  $\pi_{\lambda i} = \pi((e, 0, \lambda), (e, i, 0))$ , where  $e$  is the identity of  $G$ .

**Proof.** For an element  $a \in R[G]$ , let  $(a)_{i\lambda}$  be the  $I \times \Lambda$  matrix over  $R[G]$  such that the  $(i, \lambda)$ -entry is  $a$ , and all other entries are zero. We define a map  $\varphi$  by

$$\varphi : \mathfrak{S} \longrightarrow [R[G], I, \Lambda; \bar{P}], \quad (a, i, \lambda) \mapsto (a)_{i\lambda} \quad \text{for } a \in G, i \in I, \lambda \in \Lambda,$$

and extend this map linearly to  $R_0^\pi[\mathfrak{S}]$ . By Proposition 3.3, it is a routine calculation that  $\varphi$  is an algebra isomorphism of  $R_0^\pi[\mathfrak{S}]$  onto  $[R[G], I, \Lambda; \bar{P}]$ .  $\square$

#### 4. Involutions on twisted semigroup algebras

The aim of this section is to characterize involutions on the twisted semigroup algebra  $R^\pi[\mathfrak{S}]$  of a semigroup  $\mathfrak{S}$ , where  $\pi$  is a twisting of  $\mathfrak{S}$ . First, we introduce the following types for involutions.

**Definition 4.1.** Let  $\delta$  be an  $R$ -involution on the (contracted) twisted semigroup algebra  $R^\pi[\mathfrak{S}]$ .

- (1)  $\delta$  is called an involution of type  $\mathcal{J}$  if, for every  $s \in \mathfrak{S}$ , we have  $\text{supp}(\delta(s)) \subseteq J_s$ .
- (2)  $\delta$  is called an involution of type  $\mathcal{H}$  if, for every  $s \in \mathfrak{S}$ , there exists an  $\mathcal{H}$ -class  $H$  of  $\mathfrak{S}$  such that  $\text{supp}(\delta(s)) \subseteq H$  for every  $s \in H_s$ .
- (3)  $\delta$  is called an involution of type  $\mathcal{HL}$  if  $\delta$  is of type  $\mathcal{H}$ , and if, for each  $x \in \mathfrak{S}$ , there exists at least one idempotent element  $e_x \in J_x$  such that  $\delta(e_x) = e_x$ .

The following proposition establishes a relationship between a cellular twisted semigroup algebra of type  $\mathcal{J}$  and an involution of type  $\mathcal{J}$ .

**Proposition 4.2.** Let  $R$  be a commutative ring with identity and  $\mathfrak{S}$  a semigroup. If  $R^\pi[\mathfrak{S}]$  is a cellular twisted semigroup algebra of type  $\mathcal{J}$  with cell datum  $(I, M, C, \delta)$ , then  $\delta$  is an involution of type  $\mathcal{J}$  on  $R^\pi[\mathfrak{S}]$ .

**Proof.** For  $s \in \mathfrak{S}$ , we write  $s$  as an  $R$ -linear combination of the basis elements  $C_{S,T}^\lambda$  with  $\lambda \in \Lambda (\subseteq I)$  and  $S, T \in M(\lambda)$ . Put

$$\Lambda' = \left\{ \lambda \in \Lambda \mid J_s \cap \text{supp}(C_{S,T}^\lambda) \neq \emptyset \text{ for a basis element } C_{S,T}^\lambda \right\}.$$

By Definition 2.5(C4), the support of each  $C_{S,T}^\lambda$  belongs to a  $\mathcal{J}$ -class. Since different  $\mathcal{J}$ -classes in  $\mathfrak{S}$  are disjoint, we know that  $s$  is in fact a linear combination of  $C_{S,T}^\lambda$  with  $\lambda \in \Lambda'$ , say  $s = \sum_{\lambda \in \Lambda'} \sum_{S,T} r_{S,T}^\lambda C_{S,T}^\lambda$  with  $r_{S,T}^\lambda \in R$ . Clearly, there must be at least one basis element  $C_{S,T}^\lambda$  in the expression such that  $s \in \text{supp}(C_{S,T}^\lambda) \subseteq J$ , where  $J$  is a  $\mathcal{J}$ -class in  $\mathfrak{S}$ . This implies that  $s \in J$  and  $J = J_s$ . By Definition 2.5(C4), all  $C_{S,T}^\lambda$  in the expression of  $s$  have support in  $J_s$ . Now, by applying the involution, we infer that  $\delta(s) = \sum_{\lambda \in \Lambda'} \sum_{S,T} r_{S,T}^\lambda C_{T,S}^\lambda$ . Thus  $\delta(s)$  has support in  $J_s$ . This is what we want to prove.  $\square$

The following proposition gives a characterization of involutions of type  $\mathcal{J}$  on twisted semigroup algebras.

**Proposition 4.3.** Let  $\mathfrak{S}$  be a semigroup, and let  $\pi$  be a twisting of  $\mathfrak{S}$  into  $R$ . If there is an  $R$ -involution  $\delta_a$  on  $R_0^\pi[J_a]$  for every  $a \in \mathfrak{S}$  such that

- (1)  $\delta_x = \delta_a$  for every  $x \in J_a$ ; and
  - (2) for  $x, y \in \mathfrak{S}$ , we have  $\delta_{xy}(x \bullet y) = \delta_y(y) \bullet \delta_x(x)$ , where  $xy$  is the product in  $\mathfrak{S}$ , and where  $\delta_y(y) \bullet \delta_x(x)$  are defined in  $R^\pi[\mathfrak{S}]$ ,
- then the map  $\delta$  defined by

$$\delta : R[\mathfrak{S}] \longrightarrow R[\mathfrak{S}], \quad \sum_{x \in \mathfrak{S}} r_x x \mapsto \sum_{x \in \mathfrak{S}} r_x \delta_x(x)$$

is a unique  $R$ -involution of type  $\mathcal{J}$  on  $R^\pi[\mathfrak{S}]$  such that, as  $R$ -linear maps,  $\delta(x) = \delta_s(x)$  for every  $x \in J_s$  with  $s \in \mathfrak{S}$ .

Conversely, any involution of type  $\mathcal{J}$  on  $R^\pi[\mathfrak{S}]$  can be obtained in this way.

**Proof.** Suppose that the conditions of Proposition 4.3 are satisfied. Then  $\delta$  is well-defined and  $R$ -linear. To verify that  $\delta$  is an involution on  $R^\pi[\mathfrak{S}]$ , it suffices to show that  $\delta^2 = \text{id}$  and  $\delta$  is an anti-automorphism of the algebra  $R^\pi[\mathfrak{S}]$ . Suppose  $a = \sum_{s \in \Lambda} k_s s$  and  $b = \sum_{t \in \Gamma} l_t t$  are two elements of  $R^\pi[\mathfrak{S}]$ , where  $k_s, l_t \in R$ , and where  $\Lambda$  and  $\Gamma$  are finite subsets of  $\mathfrak{S}$ . Then

$$\begin{aligned} \delta(a \bullet b) &= \delta \left( \sum_{s \in \Lambda} \sum_{t \in \Gamma} k_s l_t (s \bullet t) \right) = \sum_{s \in \Lambda} \sum_{t \in \Gamma} k_s l_t \pi(s, t) \delta_{st}(st) \\ &= \sum_{s \in \Lambda} \sum_{t \in \Gamma} k_s l_t (\delta_{st}(s \bullet t)) = \sum_{s \in \Lambda} \sum_{t \in \Gamma} k_s l_t (\delta_t(t) \bullet \delta_s(s)) \\ &= \left( \sum_{t \in \Gamma} l_t \delta_t(t) \right) \bullet \left( \sum_{s \in \Lambda} k_s \delta_s(s) \right) = \delta(b) \bullet \delta(a). \end{aligned}$$



Observe that  $\text{supp}(\delta_s(s)) \subseteq J_s$ . We have  $\delta_x = \delta_s$  for every  $x \in \text{supp}(\delta_s(s))$  by (1), and hence  $\delta(\delta_s(s)) = \delta_s(\delta_s(s))$ . Thus

$$\begin{aligned}\delta^2(a) &= \delta\left(\sum_{s \in A} k_s(\delta_s(s))\right) = \sum_{s \in A} k_s(\delta(\delta_s(s))) \\ &= \sum_{s \in A} k_s(\delta_s(\delta_s(s))) = \sum_{s \in A} k_s s = a,\end{aligned}$$

that is,  $\delta^2 = \text{id}$ .

Thus  $\delta$  is an involution on  $R^\pi[\mathfrak{S}]$ . The uniqueness of  $\delta$  is clear. By definition,  $\delta$  is of type  $\mathcal{J}$ .

Conversely, assume that  $\delta$  is an involution on  $R^\pi[\mathfrak{S}]$  of type  $\mathcal{J}$ . Take  $a \in \mathfrak{S}$ , we define a map  $\delta_a$  from  $R^\pi[J_a]$  into itself by  $\delta_a(w) = \delta(w)$  for every  $w \in R^\pi[J_a]$ , and prove that  $\delta_a$  is an  $R$ -involution on  $R^\pi[J_a]$ . In fact, if  $x \in J_a$ , then  $\delta(x) \in R[J_a]$  since  $\delta$  is an involution of type  $\mathcal{J}$ . It follows that  $\delta_a$  is well-defined. Moreover, it is easy to see that  $\delta_a$  is  $R$ -linear and that  $\delta_x = \delta_a$  for every  $x \in J_a$  since  $x \in J_a$  implies that  $J_x = J_a$ .

Now, we prove that  $\delta_a(x \odot y) = \delta_a(y) \odot \delta_a(x)$  for all  $x, y \in J_a$ . Suppose  $x, y \in J_a$ ,  $\delta_a(x) = \delta(x) = \sum_{k \in J_a} r_k k$  and  $\delta_a(y) = \delta(y) = \sum_{l \in J_a} r'_l l$  with the coefficients  $r_k, r'_l \in R$ . (Here the summations should be understood as a finite summations). Since  $\delta(x \bullet y) = \delta(y) \bullet \delta(x)$ , we have

$$\begin{aligned}\pi(x, y)\delta(xy) &= \delta(x \bullet y) = \sum_{k, l \in J_a} r_k r'_l l \bullet k \\ &= \sum_{k, l \in J_a; lk \in J_a} r_k r'_l \pi(l, k)lk + \sum_{k, l \in J_a; lk \notin J_a} r_k r'_l \pi(l, k)lk.\end{aligned}$$

Suppose  $xy \in J_a$ . Since  $\delta$  is an involution of type  $\mathcal{J}$ , we have  $\text{supp}(\delta(xy)) \subseteq J_a$ . Thus  $\sum_{k, l \in J_a; lk \notin J_a} r_k r'_l \pi(l, k)lk = 0$ . This means that

$$\delta(x \bullet y) = \pi(x, y)\delta(xy) = \sum_{k, l \in J_a; lk \in J_a} r_k r'_l \pi(l, k)lk = \sum_{k, l \in J_a; kl \in J_a} r_k r'_l l \odot k.$$

By the definition of  $\delta_a$ , we have

$$\begin{aligned}\delta_a(x \odot y) &= \delta_a(\pi_a(x, y)xy) = \pi_a(x, y)\delta_a(xy) = \pi(x, y)\delta(xy) = \delta(x \bullet y) \\ &= \sum_{k, l \in J_a; kl \in J_a} r_k r'_l \pi(l, k)lk = \sum_{k, l \in J_a; kl \in J_a} r_k r'_l l \odot k.\end{aligned}$$

On the other hand, we have  $\sum_{k, l \in J_a; kl \notin J_a} r_k r'_l l \odot k = 0$  in  $(R[J_a], \odot)$ . Therefore

$$\begin{aligned}\delta_a(x \odot y) &= \sum_{k, l \in J_a; kl \in J_a} r_k r'_l \pi(l, k)lk + 0 \\ &= \sum_{k, l \in J_a; kl \in J_a} r_k r'_l l \odot k + \sum_{k, l \in J_a; kl \notin J_a} r_k r'_l l \odot k \\ &= \sum_{k, l \in J_a} r_k r'_l l \odot k = \delta(y) \odot \delta(x) = \delta_a(y) \odot \delta_a(x),\end{aligned}$$

that is,  $\delta_a(x \odot y) = \delta_a(y) \odot \delta_a(x)$ .

Suppose  $xy \notin J_a$ , that is,  $J_{xy} \neq J_a$  and  $J_a \cap J_{xy} = \emptyset$ . In this case,  $x \odot y = 0$  and  $\delta_a(x \odot y) = \delta_a(0) = \delta(0) = 0$ . Since  $\delta$  is an involution of type  $\mathcal{J}$ , we have  $\text{supp}(\delta(xy)) \subseteq J_{xy}$  and  $\text{supp}(\delta(xy)) \cap J_a = \emptyset$ . Thus  $\sum_{k, l \in J_a; lk \in J_a} r_k r'_l \pi(l, k)lk = 0$  in  $R^\pi[\mathfrak{S}]$ , that is,  $\sum_{k, l \in J_a; kl \in J_a} r_k r'_l l \odot k = \sum_{k, l \in J_a; lk \in J_a} r_k r'_l \pi(l, k)lk = 0$  in  $R[J_a]$ . On the other hand, it is clear that  $\sum_{k, l \in J_a; kl \notin J_a} r_k r'_l l \odot k = 0$  in  $R^\pi[J_a]$ . Thus

$$\delta_a(y) \odot \delta_a(x) = \delta(y) \bullet \delta(x) = \sum_{k, l \in J_a} r_k r'_l l \odot k = 0$$

in  $R^\pi[J_a]$ . Consequently,  $\delta_a(x \odot y) = 0 = \delta_a(y) \odot \delta_a(x)$ . Hence we have proved that  $\delta_a$  is an anti-homomorphism from  $R^\pi[J_a]$  into itself. Therefore  $\delta_a$  is an  $R$ -involution on  $R^\pi[J_a]$ .

Finally, we show that the condition (2) is satisfied. Indeed, by the definition of  $\delta_a$ , we have  $\delta_x(x) = \delta(x)$  for every  $x \in \mathfrak{S}$ . Further, since  $\delta$  is an involution on  $R^\pi[\mathfrak{S}]$ , we conclude that  $\delta(x \bullet y) = \delta(y) \bullet \delta(x)$  for all  $x, y \in \mathfrak{S}$ . So

$$\delta_{xy}(x \bullet y) = \delta_{xy}(\pi(x, y)(xy)) = \pi(x, y)\delta_{xy}(xy) = \pi(x, y)\delta(xy) = \delta(x \bullet y) = \delta(y) \bullet \delta(x) = \delta_y(y) \bullet \delta_x(x)$$

for all  $x, y \in \mathfrak{S}$ . Thus the proof is completed.  $\square$

**Proposition 4.4.** Let  $\mathfrak{S} = \mathcal{M}^0(G, I, I; P)$  be a completely 0-simple semigroup, and  $R$  an integral domain. Let  $\pi$  be an  $\mathcal{LR}$ -twisting of  $\mathfrak{S}$  into  $R$  such that  $\pi(e, e) \neq 0$ , and  $*$  an  $R$ -involution on  $R^\pi[G]$  such that  $e^* = e$ , where  $e$  is the identity of  $G$ . Let  $\pi_{i,l}$  stand for  $\pi((e, 0, i), (e, l, 0))$ . Suppose that  $\pi_{i,l}p_{i,l}^* = \pi_{l,i}p_{l,i}$  for all  $i, l \in I$ . Then the map

$$\delta : R_0^\pi[\mathfrak{S}] \rightarrow R_0^\pi[\mathfrak{S}], \quad (a, i, j) \mapsto (a^*, j, i), \quad \text{where } a \in R[G], i, j \in I,$$

is an  $R$ -involution on  $R_0^\pi[\mathfrak{S}]$  of type  $\mathcal{H}$ .

**Proof.** Obviously, the map  $\delta$  can be extended linearly to  $R^\pi[\mathfrak{S}]$  which, as an  $R$ -module, is the same as  $R[\mathfrak{S}]$ . Note that  $*$  can be considered as an  $R$ -linear map on  $R[G]$ . Since  $\pi$  is an  $\mathcal{LR}$ -twisting and since the  $\mathcal{H}$ -class of  $e$  is  $G$ , we have  $\pi(g, h) = \pi(e, e)$  for all  $g, h \in G$  by Definition 2.4(4).

For any  $(g, i, j) \in \mathfrak{S}$ , we have

$$\delta^2(g, i, j) = \delta(g^*, j, i) = ((g^*)^*, i, j) = (g, i, j)$$

and  $\delta^2 = id_{\mathfrak{S}}$ . Let  $(h, k, l) \in \mathfrak{S}$ . Then

$$\begin{aligned} \pi(e, e)^2 \delta((g, i, j) \bullet (h, k, l)) &= \pi(e, e)^2 \pi_{j,k} \delta(gp_{jk}h, i, l) \\ &= \delta(g \bullet p_{jk} \bullet h, i, l) = \pi_{j,k}((g \bullet p_{jk} \bullet h)^*, l, i) \\ &= \pi_{j,k}(h^* \bullet p_{jk}^* \bullet g^*, l, i) \\ &= \pi(e, e)^2 \pi_{k,j}(h^* p_{k,j} g^*, l, i) \quad (\text{since } \pi_{k,j} p_{k,j}^* = \pi_{j,k} p_{j,k}) \\ &= \pi(e, e)^2 \pi_{k,j}(h^*, l, k)(g^*, i, j) \\ &= \pi(e, e)^2 (h^*, l, k) \bullet (g^*, j, i) \\ &= \pi(e, e)^2 \delta(h, k, l) \bullet \delta(g, i, j). \end{aligned}$$

Here we identify  $G$  with the subgroup  $(G, 0, 0)$  of  $\mathfrak{S}$ . Hence  $\delta((g, i, j) \bullet (h, k, l)) = \delta(h, k, l) \bullet \delta(g, i, j)$  since  $R$  is an integral domain. Thus  $\delta$  is an anti-homomorphism. Altogether, we have shown that  $\delta$  is an  $R$ -involution of  $R^\pi[\mathfrak{S}]$ . By the definition of  $\delta$  and Definition 4.1, we see from Lemma 2.1 that  $\delta$  is also of type  $\mathcal{H}$ . The proof is completed.  $\square$

Now, let us consider the converse of Proposition 4.4. For a ring  $T$  with identity, we denote by  $U(T)$  the group of units in  $T$ .

**Proposition 4.5.** Let  $\mathfrak{S} = \mathcal{M}^0(G, I, \Lambda; P)$  be a completely 0-simple semigroup with  $e$  the identity of  $G$ ,  $R$  an integral domain,  $\pi$  an  $\mathcal{LR}$ -twisting of  $\mathfrak{S}$  into  $R$ , and  $\delta$  an  $R$ -involution on  $R_0^\pi[\mathfrak{S}]$  of type  $\mathcal{H}$ . If  $\delta$  fixes the idempotent  $(e, 0, 0)$  and if  $\pi((e, 0, 0), (e, 0, 0)) \neq 0$ , then there exist an involution  $*$  on  $R^\pi[G]$ , a map  $\varphi : I \rightarrow \Lambda$ ,  $i \mapsto \bar{i}$  and a map  $\varepsilon : \Lambda \rightarrow U(R[G])$ ,  $\lambda \mapsto \varepsilon_\lambda$  such that

- (1)  $\varphi$  is bijective and  $\bar{\bar{0}} = 0$ .
- (2)  $\varepsilon_0 = e$ ,  $\varepsilon_\lambda^*$  is invertible in  $R[G]$  for every  $\lambda \in \Lambda$ , and  $\pi_{i,l}p_{i,l}^* = \pi_{l,i}(\varepsilon_i^*)^{-1}p_{l,i}\varepsilon_{\bar{i}}$  for all  $i, l \in I$ , where  $\pi_{i,l} = \pi((e, 0, \bar{i}), (e, l, 0))$ .

Moreover, the  $\delta$  sends  $(g, i, \bar{j})$  to  $(\varepsilon_j g^* (\varepsilon_i^*)^{-1}, j, \bar{i})$ .

**Proof.** Because  $\delta$  is an  $R$ -involution of type  $\mathcal{H}$  on  $R_0^\pi[\mathfrak{S}]$ , there is an  $\mathcal{H}$ -class  $H$  of  $\mathfrak{S}$  such that  $\text{supp}(\delta(g, 0, 0)) \subseteq H$  for every  $g \in G$ . By assumption, we have  $\delta(e, 0, 0) = (e, 0, 0)$ , this shows that  $(e, 0, 0) \in H$  and  $H = (G, 0, 0)$ . Thus  $\delta(g, 0, 0) \in R[G]$  for every  $g \in G$ . It follows that the restriction  $*$  :=  $\delta|_{R[G]}$  is an involution on  $R^\pi[G]$ . Note that  $*$  is also an  $R$ -involution on the group algebra  $R[G]$ . This follows from

$$\begin{aligned} \pi((e, 0, 0), (e, 0, 0))((g, 0, 0)(h, 0, 0))^* &= \delta((g, 0, 0) \bullet (h, 0, 0)) \\ &= \delta(h, 0, 0) \bullet \delta(g, 0, 0) \\ &= \pi((e, 0, 0), (e, 0, 0))(\delta(h, 0, 0) \delta(g, 0, 0)) \\ &= \pi((e, 0, 0), (e, 0, 0))((h, 0, 0)^* (g, 0, 0)^*) \end{aligned}$$

and  $((g, 0, 0)(h, 0, 0))^* = (h, 0, 0)^*(g, 0, 0)^*$  since  $R$  is an integral domain.

Further, we claim that  $\delta(g, 0, 0) = (\delta(g), 0, 0)$  for every  $g \in G$ . To prove this, we pick an element  $g \in G$  and write  $\delta(g) = \sum_{h \in G'} r_h h$  with  $r_h \in R$  and  $G'$  a finite subset of  $G$ . Then

$$\delta(g) = \sum_{h \in G'} r_h (h, 0, 0) = \left( \sum_{h \in G'} r_h h, 0, 0 \right) = (\delta(g), 0, 0).$$

Note that we always identify  $G$  with  $(G, 0, 0)$ . If  $(e, i, 0) \in \mathfrak{S}$ , then we may suppose  $\delta(e, i, 0) = (a, k, \lambda)$  with  $a \in R[G]$  since  $\delta$  is of type  $\mathcal{H}$ . We put  $\pi_{0,0} = \pi((e, 0, 0), (e, 0, 0))$  in  $R$ . Then  $\pi_{0,0} \neq 0$  and, by Proposition 3.3,

$$\begin{aligned} (\pi_{0,0} a, k, \lambda) &= \pi_{0,0} \delta(e, i, 0) = \pi_{0,0} \delta((e, i, 0)(e, 0, 0)) = \delta((e, i, 0) \bullet (e, 0, 0)) \\ &= (e, 0, 0) \bullet (a, k, \lambda) = \pi_{0,k}(p_{0,k} a, 0, \lambda) = (\pi_{0,k} p_{0,k} a, 0, \lambda). \end{aligned}$$



It follows that  $k = 0$ . Thus  $(\pi_{0,0}a, 0, \lambda) = (\pi_{0,0}p_{0,0}a, 0, \lambda)$ . Since  $R$  is an integral domain and  $p_{0,0} = e$ , we have  $\delta(e, i, 0) = (p_{0,0}a, 0, \lambda) = (a, 0, \lambda)$ . This means that  $a$  and  $\lambda$  are dependent only on  $i$ . So we can write  $a = \wp_i$  and  $\lambda = \bar{i}$ . Dually, for  $(e, 0, \lambda) \in \mathfrak{S}$ , there exist  $\varepsilon_\lambda \in R[G]$  and  $\hat{\lambda} \in I$ , which depend only on  $\lambda$ , such that  $\delta(e, 0, \lambda) = (\varepsilon_\lambda, \hat{\lambda}, 0)$ . It follows from

$$\begin{aligned}\pi_{0,0}(e, i, 0) &= \pi_{0,0}\delta^2((e, i, 0)) = \pi_{0,0}\delta((\wp_i, 0, \bar{i})) = \pi_{0,0}\delta((\wp_i, 0, 0)(e, 0, \bar{i})) \\ &= \delta((\wp_i, 0, 0) \bullet (e, 0, \bar{i})) = (\varepsilon_{\bar{i}}, \hat{i}, 0) \bullet (\wp_i^*, 0, 0) = \pi_{0,0}(\varepsilon_{\bar{i}}\wp_i^*, \hat{i}, 0)\end{aligned}$$

that  $(e, i, 0) = (\varepsilon_{\bar{i}}\wp_i^*, \hat{i}, 0)$ ,  $i = \hat{i}$  and  $\varepsilon_{\bar{i}}\wp_i^* = e$ . Similarly, we compute the element  $\pi_{0,0}(e, 0, \lambda)$  for  $\lambda \in \Lambda$  and then get  $\varepsilon_{\bar{i}}^*\wp_i = e$ . Moreover, since  $*$  is an involution on  $R[G]$ , it follows from  $\varepsilon_{\bar{i}}\wp_i^* = e$  that  $\wp_i\varepsilon_{\bar{i}}^* = e$ .

Now we define

$$\varphi : I \longrightarrow \Lambda, \quad i \mapsto \bar{i}$$

and

$$\psi : \Lambda \longrightarrow I, \quad \lambda \mapsto \hat{\lambda}.$$

Thus  $\varphi\psi = id$  and  $\psi\varphi = id$ . It follows that  $\varphi$  is bijective and  $|I| = |\Lambda|$ . It is easy to check that  $\varphi$  satisfies the condition (1).

Next, we define

$$\varepsilon : \Lambda \longrightarrow U(R[G]), \quad \lambda \mapsto \varepsilon_\lambda.$$

Since  $\delta$  fixes  $(e, 0, 0)$ , we have  $0 = \bar{0}$ . From  $(\varepsilon_0, 0, 0) = (e, 0, 0)^* = (e, 0, 0)$  it follows that  $\varepsilon_0 = e$ . On the other hand, we have

$$\begin{aligned}\pi_{i,i}(p_{i,i}^*, 0, 0) &= ((e, 0, \bar{i}) \bullet (e, i, 0))^* = \delta(e, i, 0) \bullet \delta(e, 0, \bar{i}) \\ &= ((\varepsilon_i^*)^{-1}, 0, \bar{i}) \bullet (\varepsilon_{\bar{i}}, i, 0) = \pi_{i,i}((\varepsilon_i^*)^{-1}p_{i,i}\varepsilon_{\bar{i}}, 0, 0).\end{aligned}$$

Hence  $\pi_{i,i}p_{i,i}^* = \pi_{i,i}(\varepsilon_i^*)^{-1}p_{i,i}\varepsilon_{\bar{i}}$ . This proves that  $\varepsilon$  satisfies the condition (2).

Finally, suppose  $(g, i, \bar{j}) \in R_0^\pi[\mathfrak{S}]$  with  $g \in G$ ,  $i, j \in I$ . Then

$$\begin{aligned}\pi_{0,0}^2\delta(g, i, \bar{j}) &= \delta((e, i, 0) \bullet (g, 0, 0) \bullet (e, 0, \bar{j})) = \delta(e, 0, \bar{j}) \bullet (g, 0, 0)^* \bullet \delta(e, i, 0) \\ &= (\varepsilon_{\bar{j}}, j, 0) \bullet (g^*, 0, 0) \bullet ((\varepsilon_i^*)^{-1}, 0, \bar{i}) = \pi_{0,0}^2(\varepsilon_j g^* (\varepsilon_i^*)^{-1}, j, \bar{i})\end{aligned}$$

and  $\delta(g, i, \bar{j}) = (\varepsilon_j g^* (\varepsilon_i^*)^{-1}, j, \bar{i})$ . This finishes the proof.  $\square$

**Remark.** Propositions 4.4 and 4.5 describe the structure of an involution of type  $\mathcal{HL}$  on a twisted contracted completely 0-simple semigroup algebra. By Proposition 4.3, we can establish the structure of an involution of type  $\mathcal{HL}$  and type  $\mathcal{J}$  on a twisted semigroup algebra of a semigroups in which principal factors are completely 0-simple semigroups.

## 5. Cellularity of twisted semigroup algebras

In this section, we consider the cellularity of twisted regular semigroup algebras. One of the differences of our investigation from the one in [12] is that the assumption on the twisting is weaker than the assumption in [12], namely we do not require that a twisting decomposes into a constant part and an invertible part, in order to get cellularity of twisted semigroup algebras (see [12, Assumption 8]). But we strengthen the ground ring to be a domain. This makes sense in the representation theory of orders and Artin algebras, where one often assumes the ground ring to be a discrete valuation ring, or a field. Moreover, since specializations preserve cellularity (see [4, (1.8) Specialization]), one may study the representation theory of cellular twisted semigroup algebras over fields of characteristic zero and characteristic  $p$  in a uniform way by suitable specializations of a cellular twisted semigroup algebra over an integral domain. Another difference is that the involutions in our case are more general than that in [12]. This has been seen in the previous sections.

**Proposition 5.1.** *Let  $R$  be a commutative ring with identity,  $\mathfrak{S}$  a semigroup and  $\delta$  an involution on  $R^\pi[\mathfrak{S}]$ . If  $R^\pi[\mathfrak{S}]$  is a cellular algebra of type  $\mathcal{J}$  with cell datum  $(I, M, C, \delta)$ , then, for each  $a \in \mathfrak{S}$ , the  $R$ -algebra  $R^\pi[J_a]$  is a cellular algebra with cell datum  $(I_a, M_a, C_a, \delta_a)$ , where*

- $I_a = \{i \in I \mid \text{supp}(C_{S,T}^i) \subseteq J_a \text{ for a pair } S, T \in M(i)\},$
- $M_a = \bigcup_{i \in I_a} M(i),$
- $C_a = \bigcup_{i \in I_a} \{C_{S,T}^i \mid S, T \in M(i)\},$
- $\delta_a$  is the restriction of  $\delta$  to  $R^\pi[J_a]$  (see Proposition 4.3).

**Proof.** To show that  $R^\pi[J_a]$  is a cellular algebra with cell datum  $(I_a, M_a, C_a, \delta_a)$ , we verify all conditions in Definition 2.3. By the definition of  $(I_a, M_a, C_a, \delta_a)$ , we have to prove the following two facts:

(1)  $\{C_{S,T}^\alpha \mid \alpha \in I_a, S, T \in M(\alpha)\}$  is an  $R$ -basis of  $R^\pi[J_a]$ .

In fact,  $\bigcup_{\alpha \in I} \{C_{S,T}^\alpha \mid S, T \in M(\alpha)\}$  forms an  $R$ -basis of  $R^\pi[\mathfrak{S}]$ . So, for  $x \in J_a$ , we can write  $x$  as  $\sum_{\lambda \in \Lambda} \sum_{S,T \in N(\lambda)} r^\lambda(S, T) C_{S,T}^\lambda$ , where  $r^\lambda(S, T)$  lies in  $R$ , and where  $\Lambda \subseteq I$  and  $N(\lambda) \subseteq M(\lambda)$ . By assumption,  $R^\pi[\mathfrak{S}]$  is a cellular algebra of type  $\mathcal{J}$ . Thus  $\bigcup_{S,T \in M(\lambda)} \text{supp}(C_{S,T}^\lambda)$  is contained in some  $\mathcal{J}$ -class of  $\mathfrak{S}$ . This means that  $\sum_{\lambda \in \Lambda} \sum_{S,T \in N(\lambda)} r^\lambda(S, T) C_{S,T}^\lambda = 0$ , and therefore  $x = \sum_{\lambda \in \Lambda} \sum_{S,T \in N(\lambda)} r^\lambda(S, T) C_{S,T}^\lambda \in R[J_a]$ . It follows that  $R[J_a]$  can be spanned  $R$ -linearly by elements of  $\{C_{S,T}^\alpha \mid \alpha \in I_a, S, T \in M(\alpha)\}$ , namely we have proved that  $\{C_{S,T}^\alpha \mid \alpha \in I_a, S, T \in M(\alpha)\}$  is an  $R$ -basis of  $R^\pi[J_a]$ .

(2) If  $\alpha \in I_a, S, T \in M(\alpha)$  and  $x \in J_a$ , then

$$x \odot C_{S,T}^\alpha \equiv \sum_{S' \in M(\alpha)} r_x(S', S) C_{S',T}^\alpha \pmod{R[J_a](< \alpha)},$$

where the coefficients  $r_x(S', S) \in R$  do not depend on  $T$ .

In fact, by the definition of a cellular algebra of type  $\mathcal{J}$ , we know that  $x \bullet C_{S,T}^\alpha = \sum_{S' \in M(\alpha)} r_x(S', S) C_{S',T}^\alpha + x'$  for an  $x' \in R^\pi[\mathfrak{S}](< \alpha)$ , where the coefficients  $r_x(S', S) \in R$  do not depend on  $T$ . Now let  $x' = \sum_{\lambda \in \Lambda} \sum_{U,V \in M(\lambda)} r_{x'}(U, V) C_{U,V}^\lambda$  with  $r_{x'}(U, V) \in R$  and  $\Lambda$  a subset of  $\{\lambda \in I \mid \lambda < \alpha\}$ . Since  $R^\pi[\mathfrak{S}]$  is a cellular algebra of type  $\mathcal{J}$ , the element  $x'_1 := \sum_{\lambda \in I_a \cap \Lambda} r_{x'}(U, V) C_{U,V}^\lambda$  belongs to  $R[J_a]$  and the element  $x'_2 := \sum_{\mu \in \Lambda \setminus I_a} r_{x'}(U, V) C_{U,V}^\mu$  with  $\text{supp}(C_{U,V}^\mu) \not\subseteq J_a$  does not belong to  $R[J_a]$ . Note that  $x' = x'_1 + x'_2$ . Thus  $\sum_{S' \in M(\alpha)} r_x(S', S) C_{S',T}^\alpha + \sum_{\lambda \in I_a \cap \Lambda} r_{x'}(U, V) C_{U,V}^\lambda \in R[J_a]$ . Suppose  $C_{S,T}^\alpha = \sum_{t \in J_a} r_t t$ . Since  $xw \in J_a$  or  $xw \in I(a)$  for  $w \in \text{supp}(C_{S,T}^\alpha) \subseteq J_a$ , we find that

$$\begin{aligned} x \bullet C_{S,T}^\alpha &= \sum_{t \in J_a} \pi(x, t) xt = \sum_{t \in J_a, xt \in J_a} r_t x \odot t + \sum_{t \in J_a, xt \notin J_a} r_t \pi(x, t) xt \\ &= x \odot C_{S,T}^\alpha + \sum_{t \in J_a, xt \notin J_a} r_t \pi(x, t) xt. \end{aligned}$$

Thus

$$\left( \sum_{S' \in M(\alpha)} r_x(S', S) C_{S',T}^\alpha + x'_1 \right) + x'_2 = x \bullet C_{S,T}^\alpha = x \odot C_{S,T}^\alpha + \sum_{t \in J_a, xt \notin J_a} r_t \pi(x, t) xt.$$

This means that

$$x \odot C_{S,T}^\alpha = \sum_{S' \in M(\alpha)} r_x(S', S) C_{S',T}^\alpha + x'_1.$$

Note that  $x'_1 \in R^\pi[J_a](< \alpha)$ . This finishes the proof.  $\square$

**Theorem 5.2.** Let  $R$  be an integral domain,  $\mathfrak{S}$  a completely 0-simple semigroup,  $\pi$  an  $\mathcal{LR}$ -twisting of  $\mathfrak{S}$  into  $R$ , and  $\delta$  an  $R$ -involution of type  $\mathcal{H}$  on  $R^\pi[\mathfrak{S}]$ . If  $\delta$  fixes an idempotent  $e$  of  $\mathfrak{S} \setminus \{0\}$  and  $\pi(e, e) \neq 0$ , then  $R_0^\pi[\mathfrak{S}]$  is a cellular algebra of type  $\mathcal{JH}$  for the  $R$ -involution  $\delta$  if and only if the twisted group algebra  $R^\pi[G]$  is a cellular algebra with respect to the restriction of  $\delta$ , where  $G$  is the maximal subgroup of  $\mathfrak{S}$  with the identity  $e$ .

**Proof.** Let  $e$  and  $G$  be the same as in the theorem. We know that the number of  $\mathcal{L}$ -classes in a  $\mathcal{D}$ -class  $\Delta$  of  $\mathfrak{S}$  is the same as the number of  $\mathcal{R}$ -classes in  $\Delta$ . Now, we may assume that  $\mathfrak{S} = \mathcal{M}^0(G, \Lambda, \Lambda; P)$  with  $p_{0,0} = e$ . We denote  $\pi(e, e)$  by  $\pi_{0,0}$ . By the proof of Proposition 4.5, the restriction of  $\delta$  to  $R^\pi[G]$  is an involution of  $R^\pi[G]$ .

Suppose that  $R_0^\pi[\mathfrak{S}]$  is a cellular algebra of type  $\mathcal{JH}$  with cell datum  $(I, M, C, \delta)$ . We define

$K = \{\lambda \in I \mid \text{supp}(C_{S,T}^\lambda) \subseteq G \text{ for a pair } S, T \in M(\lambda)\}$ , with a partial order induced from  $(I, \leq)$ ;

$N(\lambda) = \{S \in M(\lambda) \mid \text{supp}(C_{S,T}^\lambda) \subseteq G \text{ for an element } T \in M(\lambda)\}$  for  $\lambda \in K$ ; and

$D_{S,T}^\lambda = C_{S,T}^\lambda$  for  $\lambda \in K$ , and  $S, T$  in  $N(\lambda)$ .

We claim that  $R^\pi[G]$  is a cellular  $R$ -algebra with cell datum  $(K, N, D, * = \delta|_{R[G]})$ . To prove this, it is sufficient to show the following two statements:

(1)  $R[G]$  can be represented as  $R$ -linear combinations of elements of  $D := \{D_{S,T}^\lambda \mid \lambda \in K, S, T \in N(\lambda)\}$ .

First, we point out that  $D \subseteq R[G]$ . Suppose  $S \in N(\lambda)$ , so  $C_{S,T}^\lambda \in R[G]$  for an element  $T \in M(\lambda)$ . Since  $e$  is a left identity for  $R[G]$ ,

$$e \bullet C_{S,T}^\lambda = \pi(e, e) e C_{S,T}^\lambda = \pi(e, e) C_{S,T}^\lambda.$$

Now for any  $T' \in M(\lambda)$ , Definition 2.3 (C3) implies

$$e \bullet C_{S,T'}^\lambda = \pi(e, e) C_{S,T'}^\lambda + r',$$

where  $r'$  is a linear combination of basis elements  $C_{X,Y}^\mu$  with  $\mu < \lambda$ . The support of the left hand side is contained in  $R_e$ . Since the support of each basis element is contained in an  $\mathcal{H}$ -class, any basis element with non-zero coefficient on the right hand side must be contained in  $R[R_e]$ . In particular,  $C_{S,T'}^\lambda \in R[R_e]$ . Applying  $\delta$ , we have  $C_{S',T}^\lambda \in R[L_e]$  for any  $S' \in M(\lambda)$  and  $T \in N(\lambda)$ . Thus, if  $S, T \in N(\lambda)$ , then

$$C_{S,T}^\lambda \in R[R_e] \cap R[L_e] = R[G].$$

Now we prove (1). If  $a \in R[G]$ , then we write  $a = a_1 + a_2$ , where  $a_1$  is a linear combination of elements in  $D$ , thus in  $R[G]$ , and where  $a_2$  is a linear combination of elements in  $C' := \{C_{S,T}^\lambda \mid \lambda \in I, S, T \in M(\lambda)\} \setminus D$ . Notice that  $\text{supp}(C_{S,T}^\lambda) \subseteq \mathfrak{S} \setminus G$  if  $C_{S,T}^\lambda \in C'$ . Hence  $a_2 \in R[\mathfrak{S} \setminus G]$ . On the other hand,  $a_2 = a - a_1 \in R[G]$ . This implies that  $a_2 = 0$  and that  $a$  can be expressed as an  $R$ -linear combination of elements of  $D$ .

(2) If  $g \in G, \lambda \in K, D_{S,T}^\lambda \in D(\lambda) := \{D_{S,T}^\lambda \mid S, T \in N(\lambda)\}$ , then

$$g \bullet D_{S,T}^\lambda = \sum_{U \in N(\lambda)} r_g(U, S) D_{U,T}^\lambda + a,$$

where  $a \in R^\pi[G]$  is a linear combination of elements of  $D_{U,V}^\mu$  with the upper index  $\mu$  strictly smaller than  $\lambda$ , and where the coefficients  $r_g(U, S) \in R$  are independent of  $T$ .

In fact, since  $R_0^\pi[\mathfrak{S}]$  is a cellular algebra of type  $\mathcal{J}$ , we have

$$g \bullet C_{S,T}^\lambda = \sum_{U \in N(\lambda)} r_g(U, S) C_{U,T}^\lambda + \sum_{U \in M(\lambda) \setminus N(\lambda)} r_g(U, S) C_{U,T}^\lambda + a_0,$$

where  $a_0$  is a linear combination of elements  $C_{U,V}^\mu$  such that the upper indices  $\mu$  belongs to  $N$  and are strictly smaller than  $\lambda$ , and where the coefficients  $r_g(U, S) \in R$  do not depend on  $T$ . Note that both  $\text{supp}(C_{U,T}^\lambda)$  and  $\text{supp}(g \bullet C_{S,T}^\lambda)$  are contained in  $G$ . So, we have  $\text{supp}(\sum_{U \in M(\lambda) \setminus N(\lambda)} r_g(U, S) C_{U,T}^\lambda + a_0) \subseteq G$  and  $\sum_{U \in M(\lambda) \setminus N(\lambda)} r_g(U, S) C_{U,T}^\lambda = 0$  since the upper indices of basis elements in the expression of  $a_0$  is less than  $\lambda$ . It follows from (1) that  $a_0 \in R^\pi[G]$  can be represented as an  $R$ -linear combination of elements of  $D$ . Moreover,  $a_0$  is a linear combination of elements of  $D$  with upper index  $\mu$  strictly smaller than  $\lambda$ . This proves the required claim.

Conversely, suppose that the twisted group algebra  $R^\pi[G]$  of  $G$  is a cellular algebra with cell datum  $(K, M, C, *)$ , where  $*$  is the restriction of  $\delta$ . By assumption, we may suppose that  $\mathfrak{S} = \mathcal{M}^0(G, \Lambda, \Lambda; P)$ . By Proposition 4.5, without loss of generality, we can assume that  $\delta$  sends  $(a, i, j)$  to  $(\varepsilon_j a^* (\varepsilon_i^*)^{-1}, j, i)$ , where  $\varepsilon$  is defined as in Proposition 4.5. For  $\lambda \in K$ , we define  $N(\lambda) = \Lambda \times M(\lambda)$  and  $D_{(x,S),(y,T)}^\lambda = (\varepsilon_x C_{S,T}^\lambda, x, y)$  for  $(x, S), (y, T) \in N(\lambda)$ . We shall prove that  $R_0^\pi[\mathcal{M}^0(G, \Lambda, \Lambda; P)]$  is a cellular algebra with cell datum  $(K, N, D, \delta)$ .

Since  $\varepsilon_x$  is invertible in  $R[G]$  and since  $\{C_{S,T}^\lambda \mid \lambda \in K; S, T \in M(\lambda)\}$  is a basis of  $R[G]$ , we know that  $\{\varepsilon_x C_{S,T}^\lambda \mid \lambda \in K; S, T \in M(\lambda)\}$  is also a basis of  $R[G]$ . Hence each element  $(a, x, y) \in (R[G], x, y)$  can be represented as a linear combination of elements from  $\{D_{(x,S),(y,T)}^\lambda \mid \lambda \in K; (x, S), (y, T) \in N(\lambda)\}$ . Since  $\{\varepsilon_x C_{S,T}^\lambda \mid \lambda \in K; S, T \in M(\lambda)\}$  is a basis for  $R^\pi[G]$ , we have that  $\{D_{(x,S),(y,T)}^\lambda \mid \lambda \in K; (x, S), (y, T) \in N(\lambda)\}$  is  $R$ -linearly independent. Thus  $\{D_{(x,S),(y,T)}^\lambda \mid \lambda \in K, (x, S), (y, T) \in N(\lambda)\}$  is a basis of  $R_0^\pi[\mathfrak{S}]$ .

Next, let us consider the action of the involution  $\delta$  on the basis elements  $D_{(x,S),(y,T)}^\lambda$ . Since

$$\begin{aligned} \pi_{0,0}((\varepsilon_x C_{S,T}^\lambda)^*, 0, 0) &= [\pi_{0,0}(\varepsilon_x, 0, 0)(C_{S,T}^\lambda, 0, 0)]^* = [(\varepsilon_x, 0, 0) \bullet (C_{S,T}^\lambda, 0, 0)]^* \\ &= (C_{S,T}^\lambda, 0, 0)^* \bullet (\varepsilon_x, 0, 0)^* = ((C_{S,T}^\lambda)^*, 0, 0) \bullet (\varepsilon_x^*, 0, 0) = \pi_{0,0}((C_{S,T}^\lambda)^* \varepsilon_x^*, 0, 0), \end{aligned}$$

we have  $(\varepsilon_x C_{S,T}^\lambda)^* = (C_{S,T}^\lambda)^* \varepsilon_x^*$ , where the multiplications in the both sides of the equality are in  $R[G]$ . Furthermore, the following is true:

$$\begin{aligned} \delta(D_{(x,S),(y,T)}^\lambda) &= \delta(\varepsilon_x C_{S,T}^\lambda, x, y) \\ &= (\varepsilon_y (\varepsilon_x C_{S,T}^\lambda)^* (\varepsilon_x^*)^{-1}, y, x) \\ &= (\varepsilon_y (C_{S,T}^\lambda)^* \varepsilon_x^* (\varepsilon_x^*)^{-1}, y, x) \\ &= (\varepsilon_y C_{T,S}^\lambda, y, x) = D_{(y,T),(x,S)}^\lambda. \end{aligned}$$

Now, it remains to show that the condition (C3) in Definition 2.3 holds. In fact, if  $(g, u, v) \in \mathfrak{S}$  with  $g \in G, u, v \in \Lambda$ , and if  $D_{(x,S),(y,T)}^\alpha$  is a basis element, then

$$D_{(x,S),(y,T)}^\alpha \bullet (g, u, v) = (\varepsilon_x C_{S,T}^\alpha, x, y) \bullet (g, u, v) = (\pi_{y,u} \varepsilon_x C_{S,T}^\alpha p_{y,u} g, x, v).$$

On the other hand, since  $(R^\pi[G], \bullet)$  is a cellular algebra, we have

$$C_{S,T}^\alpha \bullet \pi_{y,u} p_{y,u} g = \sum_{T' \in M(\alpha)} r_{\pi_{y,u} p_{y,u} g}(T, T') C_{S,T'}^\alpha + \sum_{\lambda < \alpha; \lambda \in K; U, V \in M(\lambda)} r(\lambda, U, V) C_{U,V}^\lambda,$$

where the coefficients  $r_{\pi_{y,u}p_{y,u}g}(T, T')$  of the first term of the right hand side are in  $R$  and do not depend on  $S$ . By the definition of a twisted product,  $C_{S,T}^\alpha \bullet \pi_{y,u}p_{y,u}g = \pi_{0,0}C_{S,T}^\alpha \pi_{y,u}p_{y,u}g$ , and the element  $C_{S,T}^\alpha \pi_{y,u}p_{y,u}g$  can be expressed as a linear combination of basis elements  $C_{U,V}^\lambda$  of  $R^\pi[G]$ , say  $C_{S,T}^\alpha \pi_{y,u}p_{y,u}g = \sum_{\lambda \in K; U, V \in M(\lambda)} \gamma(\alpha, S, T, y, u, g; \lambda, U, V) C_{U,V}^\lambda$ . Now, by comparing the coefficients of the two expressions of  $C_{S,T}^\alpha \bullet \pi_{y,u}p_{y,u}g$ , we have  $\pi_{0,0}\gamma(\alpha, S, T, y, u, g; \lambda, T, T') = r_{\pi_{y,u}p_{y,u}g}(T, T')$  and  $\pi_{0,0}\gamma(\alpha, S, T, y, u, g; \lambda, U, V) = r(\lambda, U, V)$  for every  $T' \in M(\alpha)$ ,  $\lambda < \alpha$ , and  $U, V \in M(\lambda)$ . This shows that  $\gamma(\alpha, S, T, y, u, g; \lambda, T, T')$  is independent of  $S$  and  $x$  for every  $T' \in M(\alpha)$ ,  $\lambda \leq \alpha$ , and  $U, V \in M(\lambda)$ . Thus

$$\begin{aligned} D_{(x,S),(y,T)}^\alpha \bullet (g, u, v) &\equiv \left( \varepsilon_x \sum_{T' \in M(\alpha)} \gamma(\alpha, S, T, y, u, g; \alpha, T, T') C_{S,T'}^\alpha, x, v \right) \mod R_0^\pi[\mathfrak{S}](< \alpha) \\ &\equiv \sum_{T' \in M(\alpha)} \gamma(\alpha, S, T, y, u, g; \alpha, T, T') (\varepsilon_x C_{S,T'}^\alpha, x, v) \mod R_0^\pi[\mathfrak{S}](< \alpha) \\ &\equiv \sum_{(v,T') \in N(\alpha)} \gamma(\alpha, S, T, y, u, g; \alpha, T, T') D_{(x,S),(v,T')}^\alpha \mod R_0^\pi[\mathfrak{S}](< \alpha). \end{aligned}$$

Note that the coefficients  $\gamma(\alpha, S, T, y, u, g; \alpha, T, T')$  do not depend upon  $(x, S)$ . By applying the involution  $\delta$ , we see that the condition (C3) in Definition 2.3 holds true. Thus the proof is completed.  $\square$

Now we arrive at our main result of the paper.

**Theorem 5.3.** Let  $R$  be an integral domain,  $\mathfrak{S}$  a regular semigroup with principal factors  $\mathcal{M}^0(G_\alpha, I_\alpha, I_\alpha; P_\alpha)$ , where  $\alpha$  runs over  $Y = \mathfrak{S}/\mathcal{J}$ . Suppose that  $\pi$  is an  $\mathcal{LR}$ -twisting of  $\mathfrak{S}$  into  $R$  and that  $\delta$  is an  $R$ -involution of type  $\mathcal{J}$  and type  $\mathcal{H}$  on  $R^\pi[\mathfrak{S}]$ . Let  $E$  be the set  $\{e \in \mathfrak{S} \mid 0 \neq e = e^2, \delta(e) = e, \pi(e, e) \neq 0\}$ . For  $e \in E$ , let  $G_e$  be the maximal subgroup of  $\mathfrak{S}$  with the identity  $e$ . If  $E_\alpha := E \cap E(\mathcal{M}^0(G_\alpha, I_\alpha, I_\alpha; P_\alpha)) \neq \emptyset$  for every  $\alpha \in Y$ , then  $R^\pi[\mathfrak{S}]$  is a cellular algebra of type  $\mathcal{JH}$  with respect to the involution  $\delta$  if and only if for each  $\alpha \in Y$  there is an element  $e \in E_\alpha$  such that  $R^\pi[G_e]$  is a cellular algebra with respect to the  $R$ -involution  $\delta$  restricted.

**Proof.** Suppose  $R^\pi[\mathfrak{S}]$  is a cellular algebra of type  $\mathcal{JH}$  with cell datum  $(I, M, C, \delta)$ . By Proposition 5.1,  $R^\pi[J_a]$  is a cellular algebra with cell datum  $(I_a, M_a, C_a, \delta_a)$  for every  $a \in \mathfrak{S}$ . Pick an  $\alpha \in Y$ . Then there is  $J_\alpha$  such that  $J_\alpha = \mathcal{M}^0(G_\alpha, I_\alpha, I_\alpha; P_\alpha) \setminus \{0\}$ . This implies that  $\mathfrak{S}_\alpha$  is a completely 0-simple semigroup isomorphic to  $\mathcal{M}^0(G_\alpha, I_\alpha, I_\alpha; P_\alpha)$ . From the definition of  $R^\pi[J_a]$  we see that  $R^\pi[J_a]$  is the twisted semigroup algebra of  $\mathfrak{S}_\alpha$  over  $R$  with respect to the twisting  $\pi_\alpha$ . Note that under the condition of type  $\mathcal{HJ}$ , we have  $|I_\alpha| = |\Lambda_\alpha|$  by Proposition 4.5. Since  $E_\alpha \neq \emptyset$ , there is an idempotent  $e \in E_\alpha$ . As pointed out in Section 2,  $\mathfrak{S}_\alpha = \mathcal{M}^0(G_e, I_\alpha, I_\alpha; P_\alpha)$ . It is easy to verify that  $\delta_\alpha$  and  $(e, 0, 0)$  satisfy the conditions in Theorem 5.2 since we identify  $e$  with  $(e, 0, 0)$ . It follows from Theorem 5.2 that  $R^\pi[G_e]$  is a cellular algebra with respect to the  $R$ -involution  $\delta$  restricted.

Conversely, suppose that for each  $\alpha \in Y$ , there is an  $e \in E_\alpha$  such that  $R^\pi[G_e]$  is a cellular algebra with respect to the  $R$ -involution  $\delta$  restricted. Let  $a \in \mathfrak{S}$ . We prove that  $R^\pi[J_a]$  is a cellular algebra of type  $\mathcal{JH}$  for the  $R$ -involution  $\delta_a$ : Since  $\mathfrak{S}$  is a regular semigroup with principal factors  $\mathcal{M}^0(G_\alpha, I_\alpha, I_\alpha; P_\alpha)$ , there is an element  $\alpha \in Y$  such that the principal factor  $(\mathfrak{S}_\alpha := J_\alpha \cup \{0\}, \circ)$  determined by  $a$  is  $\mathcal{M}^0(G_\alpha, I_\alpha, I_\alpha; P_\alpha)$ . Pick an  $e \in E_\alpha$  such that  $R^\pi[G_e]$  is a cellular algebra with the cell datum  $(K, M, C, *)$ , where  $*$  is the restriction of  $\delta$  to  $R^\pi[G_e]$ . By the proof of Theorem 5.2,  $R^\pi[J_a]$  is a cellular algebra of type  $\mathcal{JH}$  with cell datum  $(K_a, M_a, C_a, \delta_a)$ , where  $K_a := K, M_a(\lambda) := \Lambda \times M(\lambda)$ , and  $C_{a;(x,S),(y,T)}^\lambda := (\varepsilon_x C_{S,T}^\lambda, x, y)$  for  $(x, S), (y, T) \in M_a(\lambda)$ , where  $\varepsilon$  is defined as in Proposition 4.5.

Let  $\mathcal{T}$  be a set of representatives of the  $\mathcal{J}$ -classes of  $\mathfrak{S}$ .

In the following, we shall prove  $R^\pi[\mathfrak{S}]$  is a cellular algebra of type  $\mathcal{JH}$  with cell datum  $(\Gamma, N, D, \delta)$ , where we define  $\Gamma = \bigcup_{a \in \mathcal{T}} (a, K_a)$  endowed with a partial order  $\leq$ : For  $a, b \in \mathcal{T}$ ,  $\lambda \in K_a$ ,  $\mu \in K_b$  and  $\lambda \neq \mu$ ,

$$(a, \lambda) < (b, \mu) \Leftrightarrow \text{either } J_a < J_b, \text{ or both } J_a = J_b \text{ and } \lambda < \mu \text{ in } K_a.$$

For  $(a, \lambda) \in (a, K_a)$ , we define  $N(a, \lambda) = M_a(\lambda)$ , and  $D_{U,V}^{(a,\lambda)} = C_{a;U,V}^\lambda$  for  $U, V \in N(a, \lambda)$ .

Because  $R^\pi[J_a]$  is a cellular algebra with cell datum  $(K_a, M_a, C_a, \delta_a)$ , the  $\bigcup_{(a,\lambda) \in (a,K_a)} \{D_{U,V}^{(a,\lambda)} \mid U, V \in N(a, \lambda)\}$  is an  $R$ -basis for  $R^\pi[J_a]$ . Thus  $\bigcup_{a \in \mathcal{T}, (a,\lambda) \in (a,K_a)} \{D_{U,V}^{(a,\lambda)} \mid U, V \in N(a, \lambda)\}$  is an  $R$ -basis for  $R^\pi[\mathfrak{S}]$ . Thus the condition (C1) of Definition 2.3 is satisfied. The condition (C2) of Definition 2.3 is clear from Theorem 5.2. It remains to verify the condition (C3) in Definition 2.3.

We notice the following fact: Suppose  $a, b \in \mathcal{T}$ . If  $b \in I(a)$ ,  $\mu \in K_b$  and  $\alpha \in K_a$ , then  $(b, \mu) < (a, \alpha)$ . Indeed,  $b \in I(a)$  if and only if  $\mathfrak{S}^1 b \mathfrak{S}^1 \subsetneq \mathfrak{S}^1 a \mathfrak{S}^1$  if and only if  $J_b < J_a$ . Thus, if  $b \in I(a)$ , then  $(b, \mu) < (a, \lambda)$  for every  $\mu \in K_b$  and every  $\lambda \in K_a$ .

Now suppose  $\lambda \in K_a$ ,  $(x, S), (y, T) \in M_a(\lambda)$  and  $c \in \mathfrak{S}$ , we calculate  $c \bullet D_{(x,S),(y,T)}^{(a,\lambda)} = c \bullet C_{a;(x,S),(y,T)}^\lambda$ , where  $C_{a;(x,S),(y,T)}^\lambda$  is in  $R^\pi[J_a]$ . It is easy to see that  $\text{supp}(C_{S,T}^\lambda) \subseteq G_e$  and  $\text{supp}(C_{a;(x,S),(y,T)}^\lambda) \subseteq R[(G, x, y)]$ , where  $G_e = (G_e, 0, 0)$ . Since  $\pi$  is an  $\mathcal{LR}$ -twisting of  $\mathfrak{S}$  into  $R$ , we get that  $\pi(c, (g, x, y)) = \pi(c, (e, x, y))$  for every  $g \in G_e$ .

By Lemma 2.1(1), we have either  $c(e, x, 0) \in J_a$  or  $c(e, x, 0) \in I(a)$ . Thus we have to consider the following two cases:

Case 1.  $c(e, x, 0) \in I(a)$ . In this case, by the foregoing fact that  $\pi(c, (g, x, y)) = \pi(c, (e, x, y))$  for every  $g \in G_e$ , we have

$$\begin{aligned} c \bullet D_{(x,S),(y,T)}^{(a,\lambda)} &= c \bullet C_{a;(x,S),(y,T)}^\lambda = \pi(c, (e, x, y)) c(\varepsilon_x C_{S,T}^\lambda, x, y) \\ &= \pi(c, (e, x, y)) c(e, x, 0) (\varepsilon_x C_{S,T}^\lambda, 0, y). \end{aligned}$$

Thus  $c \bullet D_{(x,S),(y,T)}^{(a,\lambda)} \in R[I(a)]$  since  $I(a)$  is an ideal of  $\mathfrak{S}^1 a \mathfrak{S}^1$  and  $\pi(c, (e, x, y))c(e, x, 0)(\varepsilon_x C_{S,T}^\lambda, 0, y) \in R[I(a)]$ . This shows that there exists  $\mathcal{T}_1 \subset \mathcal{T}$  such that  $\mathcal{T}_1 \subseteq I(a)$  and  $c \bullet D_{(x,S),(y,T)}^{(a,\lambda)} \in \bigcup_{b \in \mathcal{T}_1} R[J_b]$ . Note that if  $b \in I(a)$ ,  $\mu \in K_b$  and  $\alpha \in K_a$ , then  $(b, \mu) < (a, \alpha)$ . Hence we have  $\bigcup_{b \in \mathcal{T}_1} R[J_b] \subset R^\pi[\mathfrak{S}](< (a, \lambda))$ , that is,  $c \bullet D_{(x,S),(y,T)}^{(a,\lambda)} \in R^\pi[\mathfrak{S}](< (a, \lambda))$ .

Case 2.  $c(e, x, 0) \in J_a$ . Since  $(e, x, 0)(e, 0, 0) = (e, x, 0)$ , we have  $c(e, x, 0) = [c(e, x, 0)](e, 0, 0) \in (G_e, \Lambda, 0)$ . Note that  $\pi$  is an  $\mathcal{LR}$ -twisting of  $\mathfrak{S}$  into  $R$ . Thus  $\pi((g, x, 0), (h, 0, y)) = \pi((e, 0, 0), (e, 0, 0))$  for all  $g, h \in G_e$ , and  $\pi(c(e, x, y), (g, 0, y)) = \pi(c(e, x, y), (e, 0, 0)) = \pi((e, 0, 0), (e, 0, 0))$ . From  $\pi(e, e) \neq 0$  it follows that  $\pi_{0,0} := \pi((e, 0, 0), (e, 0, 0)) = \pi(e, e) \neq 0$ . Now, we have

$$\begin{aligned} \pi_{0,0} c \bullet C_{a;(x,S),(y,T)}^\lambda &= \pi_{0,0} c \bullet (\varepsilon_x C_{S,T}^\lambda, x, y) = \pi_{0,0} c \bullet [(e, x, 0)(\varepsilon_x C_{S,T}^\lambda, 0, y)] \\ &= c \bullet [\pi_{0,0}(e, x, 0)(\varepsilon_x C_{S,T}^\lambda, 0, y)] = c \bullet [(e, x, 0) \odot (\varepsilon_x C_{S,T}^\lambda, 0, y)] \\ &= \pi(c, (e, x, 0))\pi(c(e, x, 0), (e, 0, 0))c(e, x, 0)(\varepsilon_x C_{S,T}^\lambda, 0, y) \\ &= \pi(c, (e, x, 0))\pi(c(e, x, 0), (e, 0, 0))c(e, x, 0)(\varepsilon_x, 0, 0)(C_{S,T}^\lambda, 0, y) \\ &= \pi(c, (e, x, 0))\pi(c(e, x, 0), (e, 0, 0))c(e, x, 0)(\varepsilon_x, x, 0)D_{(0,S),(y,T)}^{(a,\lambda)} \\ &= \pi(c, (e, x, 0))\pi_{0,0}c(e, x, 0)(\varepsilon_x, x, 0)D_{(0,S),(y,T)}^{(a,\lambda)} \\ &= \pi(c, (e, x, 0))[c(e, x, 0)(\varepsilon_x, x, 0)] \odot D_{(0,S),(y,T)}^{(a,\lambda)}. \end{aligned}$$

That is,

$$(*) \quad \pi_{0,0} c \bullet C_{a;(x,S),(y,T)}^\lambda = \pi(c, (e, x, 0))[c(e, x, 0)(\varepsilon_x, x, 0)] \odot D_{(0,S),(y,T)}^{(a,\lambda)}.$$

Moreover, since  $R^\pi[J_a]$  is a cellular algebra, we get

$$\begin{aligned} (**) \quad \pi(c, (e, x, 0))[c(e, x, 0)(\varepsilon_x, x, 0)] \odot D_{(0,S),(y,T)}^{(a,\lambda)} \\ = \sum_{(s,S') \in N_a(\lambda)} r_{\pi(c, (e, x, 0))[c(e, x, 0)(\varepsilon_x, x, 0)]((s, S'), (0, S))} C_{a;(s,S'),(y,T)}^\lambda + z, \end{aligned}$$

where  $z \in R^\pi[J_a](< \lambda)$ , and where the coefficients  $r_{\pi(c, (e, x, 0))[c(e, x, 0)(\varepsilon_x, x, 0)]((s, S'), (0, S))} \in R$  do not depend on  $(y, T)$ .

Since all  $D_{U,V}^{(a,\lambda)}$  form an  $R$ -basis of  $R^\pi[\mathfrak{S}]$ , we write  $c \bullet C_{a;(x,S),(y,T)}^\lambda$  as an  $R$ -linear combination of elements  $D_{U,V}^{(a,\lambda)}$ , say

$$c \bullet C_{a;(x,S),(y,T)}^\lambda = \sum l(\lambda, s, S', y, T) C_{a;(s,S'),(y,T)}^\lambda + z'$$

with  $z'$  a linear combination of elements  $C_{b;U',V'}^\mu$ , where  $(b, \mu) \neq (a, \lambda)$ . This implies that

$$(***) \quad \pi_{0,0} c \bullet C_{a;(x,S),(y,T)}^\lambda = \sum \pi_{0,0} l(\lambda, s, S', y, T) C_{a;(s,S'),(y,T)}^\lambda + \pi_{0,0} z'.$$

By comparing the coefficients of the right hand sides of (\*\*) and (\*\*\*), we get

$$r_{\pi(c, (e, x, 0))[c(e, x, 0)(\varepsilon_x, 0, 0)]((s, S'), (0, S))} = \pi_{0,0} l(\lambda, s, S', y, T).$$

This means that  $\pi_{0,0}$  divides  $r_{\pi(c, (e, x, 0))[c(e, x, 0)(\varepsilon_x, 0, 0)]((s, S'), (0, S))}$ . Similarly,  $\pi_{0,0}$  divides the coefficients of  $z$ . Hence

$$\begin{aligned} c \bullet C_{a;(x,S),(y,T)}^\lambda &= \sum_{(s,S') \in N_a(\lambda)} (\pi_{0,0})^{-1} r_{\pi(c, (e, x, 0))[c(e, x, 0)(\varepsilon_x, 0, 0)]((s, S'), (0, S))} C_{a;(s,S'),(y,T)}^\lambda + (\pi_{0,0})^{-1} z \\ &= \sum_{(s,S') \in N_a(\lambda)} (\pi_{0,0})^{-1} r_{\pi(c, (e, x, 0))[c(e, x, 0)(\varepsilon_x, 0, 0)]((s, S'), (0, S))} D_{(s,S'),(y,T)}^{(a,\lambda)} + (\pi_{0,0})^{-1} z, \end{aligned}$$

where  $z$  is already in the lower terms. Thus we have proved that the condition (C3) in Definition 2.3 holds. This shows that  $R^\pi[\mathfrak{S}]$  is a cellular algebra with cell datum  $(\Gamma, N, D, \delta)$ . It is not difficult to see that the involution  $\delta$  is of type  $\mathcal{JH}$ . This finishes the proof.  $\square$

**Remarks.** (1) The proof of Theorem 5.3 shows a little bit more: If  $R^\pi[\mathfrak{S}]$  is a cellular algebra of type  $\mathcal{JH}$  with respect to the involution  $\delta$  of both type  $\mathcal{J}$  and type  $\mathcal{H}$  on  $R^\pi[\mathfrak{S}]$ , then  $R^\pi[G_e]$  is a cellular algebra for every idempotent  $e \in E$ .

(2) In the proof of Theorem 5.3 we may replace “ $R$  is a domain” by an arbitrary ring  $R$  and require that  $\pi(e, e) \in R$  is torsion-free in  $R[\mathfrak{S}]$ , that is, if  $\pi(e, e)m = 0$  for some element  $m \in R[\mathfrak{S}]$ , then either  $\pi(e, e) = 0$ , or  $m = 0$ .

As an immediate consequence of Theorem 5.3 and Proposition 4.5, we have the following corollary which generalizes some results in [12,2], respectively.

**Corollary 5.4.** Let  $R$  be an integral domain,  $\mathfrak{S}$  a finite regular semigroup,  $\pi$  an  $\mathcal{LR}$ -twisting of  $\mathfrak{S}$  into  $R$ ,  $\delta$  an  $R$ -involution of both type  $\mathcal{J}$  and type  $\mathcal{H}$  on  $R^\pi[\mathfrak{S}]$  and  $E$  the set of idempotents  $e$  of  $\mathfrak{S}$  such that  $\delta(e) = e$  and  $\pi(e, e) \neq 0$ . For each  $e \in E$  we

denote by  $G_e$  the maximum subgroup of  $\mathfrak{S}$  with the identity  $e$ . If  $E \cap E(\mathcal{M}^0(G_\alpha, I_\alpha, I_\alpha; P_\alpha)) \neq \emptyset$  for all  $\alpha \in Y$ , then  $R^\pi[\mathfrak{S}]$  is a cellular algebra of type  $\mathcal{JH}$  for the involution  $\delta$  if and only if, for every  $e \in E$ , the twisted group algebra  $R^\pi[G_e]$  is a cellular algebra with respect to the restriction of  $\delta$ .

As typical examples of cellular algebras, partition algebras, Brauer algebras, Temperley–Lieb algebras and many other algebras are investigated by many authors. As was pointed out in [12], all these algebras are cellular twisted semigroup algebras with respect to an  $\mathcal{LR}$ -twisting.

For partition algebras and Brauer algebras we can get semigroup structures by just putting the parameter equal to the identity of the ground ring. The corresponding semigroups obtained in this way are called partition semigroups and Brauer semigroups, respectively. Then the usual involution of a Brauer or a partition semigroup fixes all idempotents. Note that each maximal subgroup of a Brauer semigroup or a partition semigroup is isomorphic to a symmetric group. For further information on Brauer algebras and partition algebras we refer to [4,10,11,15].

Thus the following corollary is a direct consequence of Theorem 5.3 since all ingredients for adapting Theorem 5.3 are contained in [12].

**Corollary 5.5.** *All partition algebras, Brauer algebras and Temperley–Lieb algebras over an integral domain are cellular twisted semigroup algebras of partition, Brauer and Temperley–Lieb semigroups, respectively. All of them are of type  $\mathcal{JH}$  with an  $\mathcal{LR}$ -twisting.*

Finally, we remark that Theorem 5.3 enables us to study both the representation theory and homological properties of twisted semigroup algebras by applying the general methods of cellular algebras (see [4,9,16]).

## 6. Semi-simplicity of twisted semigroup algebras

In this section, we shall investigate when a twisted regular semigroup algebra of type  $\mathcal{JH}$  is semisimple. The idea is similar to that in [2,12], namely we use techniques from cellular algebras. First we recall the definition of cell modules for a unitary cellular algebra in [4], and then interpret these cell modules by their matrix representations.

Let  $A$  be a unitary cellular  $R$ -algebra with cell datum  $(\Lambda, M, C, \delta)$ . For each  $\lambda \in \Lambda$ , the cell module  $W(\lambda)$  corresponding to  $\lambda$  is the left  $A$ -module with  $R$ -basis  $\{C_S \mid S \in M(\lambda)\}$  and  $A$ -action

$$aC_S = \sum_{S' \in M(\lambda)} r_a(S', S)C_{S'}$$

for  $a \in A$  and  $S \in M(\lambda)$ , where  $r_a(S', S)$  is defined as in Definition 2.3. Let  $\text{Mat}_{M(\lambda)}(R)$  be the algebra of  $M(\lambda) \times M(\lambda)$  matrices over  $R$ , and

$$\rho^\lambda : A \rightarrow \text{Mat}_{M(\lambda)}(R)$$

the corresponding matrix representation of  $W(\lambda)$  relative to the natural basis, that is,  $\rho^\lambda(a) = (\rho^\lambda(a)_{ST})$  with

$$\rho^\lambda(a)_{ST} = r_a(S, T)$$

for  $a \in A$  and  $S, T \in M(\lambda)$ . For each  $\lambda \in \Lambda$ , there is a bilinear form  $\phi^\lambda$  on  $W(\lambda)$  defined on the basis elements so that  $\phi^\lambda(C_S, C_T)$  is the unique element of  $R$  satisfying

$$C_{S'S}^\lambda C_{TT'}^\lambda \equiv \phi^\lambda(C_S, C_T) C_{S'T'}^\lambda \pmod{A(<\lambda)}$$

for  $S', T' \in M(\lambda)$ . We denote by  $\Phi^\lambda$  the Gram matrix of  $\phi^\lambda$  relative to the natural basis, that is,  $\Phi^\lambda = (\Phi_{ST}^\lambda) \in \text{Mat}_{M(\lambda)}(R)$  with

$$\Phi_{ST}^\lambda = \phi^\lambda(C_S, C_T)$$

for  $S, T \in M(\lambda)$ . The importance of  $\phi^\lambda$  is demonstrated by the following fact.

**Lemma 6.1** ([4]). *Let  $A$  be a finite-dimensional unitary cellular  $R$ -algebra over a field  $R$  with cell datum  $(\Lambda, M, C, \delta)$ . Then the following are equivalent:*

- (1) *The algebra  $A$  is semisimple.*
- (2) *The nonzero cell representations  $W(\lambda)$  are irreducible and pairwise inequivalent.*
- (3) *The form  $\phi^\lambda$  is non-degenerate (that is,  $\det(\Phi^\lambda) \neq 0$ ) for each  $\lambda \in \Lambda$ .*

From now on, we assume the following:

- $R$  is a field with identity 1;  $\mathfrak{S}$  is a monoid with principal factors  $\mathcal{M}^0(G_d, I_d, I_d; P_d = (p_{ij}^d))$ , where  $d$  runs over a set  $\mathcal{T}$  of representatives of the  $\mathcal{J}$ -classes of  $\mathfrak{S}$ ;  $|\mathcal{T}| < \infty$  and  $|I_d| < \infty$ . We denote by  $J_d$  the set  $\mathcal{M}^0(G_d, I_d, I_d; P_d) \setminus \{0\}$ , and  $e_d$  the identity of the group  $G_d$ .
- $\delta$  is an  $R$ -involution on  $R^\pi[\mathfrak{S}]$  of type  $\mathcal{J}$  and type  $\mathcal{H}$  such that  $\delta|_{R^\pi[J_d]} = \delta_d$  as  $R$ -linear maps for every  $d \in \mathcal{T}$ .



- $\pi$  is an  $\mathcal{LR}$ -twisting of  $\mathfrak{S}$  into  $R$ , which satisfies that  $\pi(\iota, x) = 1 = \pi(x, \iota)$  for every idempotent element  $x \in \mathfrak{S}$ , where  $\iota$  is the identity of  $\mathfrak{S}$ , and that  $\pi(e_d, e_d) \neq 0$  for every  $d \in \mathcal{T}$ . We put  $\pi_{i,j}^d = \pi((e_d, 0, i), (e_d, j, 0))$ . In particular,  $\pi_{0,0}^d = \pi((e_d, 0, 0), (e_d, 0, 0)) = \pi(e_d, e_d)$ .
- for each  $d \in \mathcal{T}$ , there is an  $R$ -involution  $\delta_d$  of type  $\mathcal{H}$  on  $(R^\pi[J_d], \odot)$  such that  $\delta_d((e_d, 0, 0)) = (e_d, 0, 0)$  and  $\delta_d((a, i, j)) = (\varepsilon_j a^* (\varepsilon_i^*)^{-1}, j, i)$  for every  $a \in R^\pi[G_d]$  and all  $i, j \in I_d$ , where  $*$  is  $\delta_d|_{R^\pi[G_d]}$  and  $\varepsilon : I_d \rightarrow U(R[G_d])$  is a bijection such that  $\varepsilon_0 = e_d$  and  $\pi_{i,l}^d (p_{i,l}^d)^* = \pi_{l,i}^d (\varepsilon_l^*)^{-1} p_{l,i}^d \varepsilon_i$  for all  $i, l \in I_d$ . (Note that  $U(R[G_d])$  stands for the group of units in the algebra  $R[G_d]$ ).
- $R^\pi[G_d]$  is a cellular algebra with cell datum  $(K_d, M_d, C_d, \delta_d|_{R^\pi[G_d]})$ . Thus the cellular basis of  $R^\pi[G_d]$  is  $\{C_{d;S,T}^\lambda \mid \lambda \in K_d, S, T \in M_d(\lambda)\}$ . The corresponding bilinear form on the cell module  $W_d(\lambda)$  with  $\lambda \in K_d$  is denoted by  $\phi_d^\lambda$ , and its Gram matrix is denoted by  $\Phi_d^\lambda$ .

Since  $\pi$  is an  $\mathcal{LR}$ -twisting of  $\mathfrak{S}$  into  $R$  and since for each  $a \in \mathfrak{S}$  there is an idempotent  $x \in \mathfrak{S}$  with  $a\mathcal{R}x$ , we have that  $\pi(\iota, a) = \pi(\iota, x) = 1$  for every  $a \in \mathfrak{S}$ . Similarly,  $\pi(a, \iota) = 1$  for every  $a \in \mathfrak{S}$ . Now, we can check that  $\iota$  is an identity of  $R^\pi[\mathfrak{S}]$ . On the other hand, since  $\pi$  is an  $\mathcal{LR}$ -twisting of  $\mathfrak{S}$  into  $R$ , the restriction of  $\pi$  to each  $\mathcal{H}$ -class of  $\mathfrak{S}$  is a constant. Thus  $R^\pi[G_d]$  is a unitary algebra with the identity  $\pi(e_d, e_d)^{-1}e_d$  for every  $d \in \mathcal{T}$ . Note that under the above assumptions, we know from Theorem 5.2 that  $(R^\pi[J_d], \odot)$  is a cellular algebra of type  $\mathcal{JH}$  with cell datum  $(X_d, N_d, D_d, \delta_d)$ , where  $X_d = \{d\} \times K_d$ ,  $N_d = I_d \times M_d$  and  $D_{(i,S),(j,T)}^{(d,\lambda)} = (\varepsilon_j C_{d;S,T}^\lambda, i, j)$ . Thus, by Theorem 5.3,  $R^\pi[\mathfrak{S}]$  is a unitary cellular algebra of type  $\mathcal{JH}$  with cell datum  $(I, N, D; \delta)$ , where  $I = \bigcup_{d \in \mathcal{T}} X_d$ ,  $\delta|_{(R^\pi[J_d], \odot)} = \delta_d$ , and where  $N(d, \lambda) = I_d \times M_d(\lambda)$  and  $D_{(i,S),(j,T)}^{(d,\lambda)} = (\varepsilon_j C_{d;S,T}^\lambda, i, j)$  for every  $(d, \lambda) \in X_d$ .

**Lemma 6.2.** Let  $(d, \lambda) \in X_d$  and  $(i, S), (j, T) \in N_d$ . Then the bilinear form of  $\phi^{(d,\lambda)}$  on the cell  $R^\pi[\mathfrak{S}]$ -module corresponding to  $(d, \lambda)$  is

$$\phi^{(d,\lambda)}(D_{(i,S)}^{(d,\lambda)}, D_{(j,T)}^{(d,\lambda)}) = \sum_{U \in M(\lambda)} (\pi_{0,0}^d)^{-2} r_{\pi_{ij}^d p_{ij}^d \varepsilon_j} (U, T) \phi_d^\lambda(S, U).$$

**Proof.** Since  $R^\pi[G_d]$  is a unitary cellular algebra with cell datum  $(K_d, M_d, C_d, *)$ , we have

$$\begin{aligned} \pi_{ij}^d p_{ij}^d \varepsilon_j C_{d;TT'}^\lambda &= (\pi_{0,0}^d)^{-1} (\pi_{ij}^d p_{ij}^d \varepsilon_j) \odot C_{d;TT'}^\lambda \\ &\equiv \sum_{U \in M(\lambda)} (\pi_{0,0}^d)^{-1} r_{\pi_{ij}^d p_{ij}^d \varepsilon_j} (U, T) C_{d;UT'}^\lambda \pmod{R^\pi[G_d](< \lambda)}. \end{aligned}$$

Hence

$$\begin{aligned} D_{(i',S'),(i,S)}^{(d,\lambda)} \bullet D_{(j,T),(j',T')}^{(d,\lambda)} &= (\varepsilon_{i'} C_{S'S}^\lambda, i', i) \bullet (\varepsilon_j C_{TT'}^\lambda, j, j') \\ &= (\varepsilon_{i'} C_{S'S}^\lambda, \pi_{ij}^d p_{ij}^d \varepsilon_j C_{TT'}^\lambda, i', j') \\ &\equiv \sum_{U \in M(\lambda)} (\pi_{0,0}^d)^{-1} r_{\pi_{ij}^d p_{ij}^d \varepsilon_j} (U, T) (\varepsilon_{i'} C_{S'S}^\lambda C_{UT'}^\lambda, i', j') \\ &\equiv \sum_{U \in M(\lambda)} (\pi_{0,0}^d)^{-2} r_{\pi_{ij}^d p_{ij}^d \varepsilon_j} (U, T) (\varepsilon_{i'} C_{S'S}^\lambda \odot C_{UT'}^\lambda, i', j') \\ &\equiv \sum_{U \in M(\lambda)} (\pi_{0,0}^d)^{-2} r_{\pi_{ij}^d p_{ij}^d \varepsilon_j} (U, T) \phi_d^\lambda(S, U) (\varepsilon_{i'} C_{S'T'}^\lambda, i', j') \\ &\equiv \sum_{U \in M(\lambda)} (\pi_{0,0}^d)^{-2} r_{\pi_{ij}^d p_{ij}^d \varepsilon_j} (U, T) \phi_d^\lambda(S, U) D_{(i',S'),(j',T')}^{(d,\lambda)} \pmod{R^\pi[G_d](< \lambda)} \end{aligned}$$

$$\text{and } \phi^{(d,\lambda)}(D_{(i,S)}^{(d,\lambda)}, D_{(j,T)}^{(d,\lambda)}) = \sum_{U \in M(\lambda)} (\pi_{0,0}^d)^{-2} r_{\pi_{ij}^d p_{ij}^d \varepsilon_j} (U, T) \phi_d^\lambda(S, U). \quad \square$$

For  $(d, \lambda) \in N_d$ , we form an  $I_d \times I_d$  block matrix  $\Theta_d = (\theta_{ij})$ , where  $\theta_{ij}$  is an  $M_d(\lambda) \times M_d(\lambda)$  matrix over  $R$  with  $(U, T)$ -entry  $r_{\pi_{ij}^d p_{ij}^d \varepsilon_j} (U, T)$ . Obviously,  $\Theta_d$  is an  $(I_d \times M_d(\lambda)) \times (I_d \times M_d(\lambda))$  matrix over  $R$ . Now, let  $\Gamma^{(d,\lambda)} := \text{diag}(\Phi_d^\lambda, \dots, \Phi_d^\lambda)$ . Then  $\Gamma^{(d,\lambda)}$  is an  $(I_d \times M_d(\lambda)) \times (I_d \times M_d(\lambda))$  matrix over  $R$  and has  $|I_d|$  blocks  $\Phi_d^\lambda$  on the main diagonal.

Thus, by Lemma 6.2, we have the following corollary.

**Corollary 6.3.**  $\Phi^{(d,\lambda)} = (\pi_{0,0}^d)^{-2} \Gamma^{(d,\lambda)} \Theta_d$ , where  $\Phi^{(d,\lambda)}$  is the Gram matrix of  $\phi^{(d,\lambda)}$  with  $(d, \lambda) \in X_d$ .

By the general theory of unitary cellular algebras, we have the following result.

**Theorem 6.4.** With the above assumptions,  $R^\pi[\mathfrak{S}]$  is semi-simple if and only if the following conditions hold:

- (1) For each  $d \in \mathcal{T}$ ,  $\Theta_d$  is invertible.
- (2) For each  $d \in \mathcal{T}$ ,  $R^\pi[G_d]$  is semi-simple.

**Proof.** By Lemma 6.1,  $R^\pi[\mathfrak{S}]$  is semi-simple if and only if  $\det(\Phi^{(d,\lambda)}) \neq 0$  for every  $(d, \lambda)$  if and only if  $\det(\Gamma^{(d,\lambda)}) \neq 0$  and  $\det(\Theta_d) \neq 0$  for every  $(d, \lambda)$  if and only if  $\Theta_d$  is invertible and  $R^\pi[G_d]$  is semi-simple for every  $d \in \mathcal{T}$ . This completes the proof.  $\square$

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## References

- [1] A.H. Clifford, G.B. Preston, The Algebraic Theory of Semigroups, in: Math. Surveys, vol. 7, Amer. Math. Soc., Providenc, RI, 1961.
- [2] J. East, Cellular algebras and inverse semigroups, J. Algebra 296 (2006) 505–519.
- [3] M. Geck, Hecke algebras of finite type are cellular, Invent. Math. 169 (2007) 501–517.
- [4] J.J. Graham, G.I. Lehrer, Cellular algebras, Invent. Math. 123 (1996) 1–34.
- [5] J.J. Graham, G.I. Lehrer, Cellular algebras and diagram algebra in representation theory, Adv. Stud. Pure Math. 40 (2004) 141–173.
- [6] J.A. Green, On the structure of semigroups, Ann. of Math. 54 (2) (1951) 163–172.
- [7] J.M. Howie, An Introduction to Semigroup Theory, Academic Press, London, 1976.
- [8] S. König, C.C. Xi, On the structure of cellular algebras, in: Algebras and Modules II (Geiranger 1996), CMS Conf. Proc. 24 (1998) 365–386.
- [9] S. König, C.C. Xi, When is a cellular algebra quasi-hereditary? Math. Ann. 315 (1999) 281–293.
- [10] S. König, C.C. Xi, A characteristic-free approach to Brauer algebras, Trans. Amer. Math. Soc. 353 (2001) 1489–1505.
- [11] P. Martin, The structure of partition algebras, J. Algebra 183 (1996) 319–358.
- [12] S. Wilcox, Cellularity of diagram algebras and twisted semigroup algebras, J. Algebra 309 (1) (2007) 10–31.
- [13] H. Wenzl, On the structure of Brauer's centralizer algebras, Ann. Math. 128 (1988) 173–193.
- [14] C.C. Xi, On the quasi-hereditary of Birman–Wenzl algebras, Adv. Math. 154 (2000) 280–298.
- [15] C.C. Xi, Partition algebras are cellular, Compositio Math. 119 (1) (1999) 99–109.
- [16] C.C. Xi, Standardly stratified algebras and cellular algebras, Math. Proc. Cambridge Philos. Soc. 133 (1) (2003) 37–53.
- [17] C.C. Xi, D.J. Xiang, Cellular algebras and Cartan matrices, Linear Algebra Appl. 365 (2003) 369–388.