Homological theory of self-orthogonal modules

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Dedicated to Claus Michael Ringel with admiration and affection
on the occasion of his 80th birthday

Abstract

Tachikawa's second conjecture predicts that a finitely generated, self-orthogonal module over a finite-dimensional self-injective algebra is projective. This conjecture is an important part of the Nakayama conjecture. Our principal motivation of this work is a systematic understanding of finitely generated, self-orthogonal generators over a self-injective Artin algebra from the view point of stable module categories. Consequently, we give equivalent characterizations of Tachikawa's second conjecture in terms of M-Gorenstein categories, and establish a recollement of the M-relative stable categories for a self-orthogonal generator M. Further, we show that the Nakayama conjecture holds true for Gorenstein-Morita algebras.

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1 Introduction

Since about half a century homological conjectures form a core set of problems in representation theory and homological algebra of finite-dimensional algebras. One of the most prominent conjectures in this system of closely related conjectures is the Nakayama conjecture posed by Nakayama in [33].

(NC) If a finite-dimensional algebra over a field has infinite dominant dimension, then it is self-injective.

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This conjecture has been verified for a few classes of algebras, for which the conjecture could be checked more or less directly by clever computations. Despite many efforts, taking various approaches, very little is known about the homological conjectures and in particular about Nakayama's conjecture.

To deal with the conjecture, Tachikawa proposed another two homological conjectures (see [41]), they are now called Tachikawa's first and second conjecture.

(TC1) If a finite-dimensional algebra A over a field k satisfies $\mathrm{Ext}_A^n(D(A),A)=0$ for all $n\geq 1$, then A is self-injective, where $D=\mathrm{Hom}_k(-,k)$ is the usual duality.

(TC2) Let A be a finite-dimensional self-injective algebra and M a finitely generated A-module. If M is self-orthogonal, that is, $\mathrm{Ext}_A^n(M,M)=0$ for all $n\geq 1$, then M is projective.

Concerning (TC1) and (TC2), there are only a few cases verified. For instance, (TC1) holds for algebras of radical cube zero [24], special local algebras and commutative algebras (see [2] and [4, 22], respectively), while (TC2) holds true for self-injective algebras of finite representation type, symmetric algebras with radical cube zero, and local self-injective algebras with radical cube zero (see [41, 23, 40], respectively). For further information on these conjectures, we refer to [45].

The validity of both (TC1) and (TC2) is equivalent to the one of (NC). Moreover, by a result of Mueller [32], a pair (A,M), with A a self-injective algebra and M a finitely generated A-module, satisfies (TC2) if and only if the endomorphism algebra $\mathrm{End}_A(A\oplus M)$ satisfies (NC).

Thus it is of significant interest to understand self-orthogonal modules of the form $A\oplus M$ for a self-injective algebra A and finitely generated A-modules M. Generally, self-orthogonal modules of the form $B\oplus Y$ over an Artin algebra B with Y a finitely generated B-module are termed *self-orthogonal generators*.

In this paper, we investigate self-orthogonal generators over self-injective algebras from the point of view of stable categories. More precisely, we first establish a general theory for arbitrary (not necessarily self-orthogonal) generators by constructing two pairs of triangle endofunctors for stable module categories, and then establish specially a recollement of the relative stable categories for a self-orthogonal generator. Finally, we describe compact objects of the right term of the recollement by the heart of a torsion pair in the stable module category. Based on these investigations, we give equivalent characterizations of (TC2) and show that the Nakayama conjecture holds true for Gorenstein-Morita algebras.

1.1 Equivalent characterizations of Tachikawa's second conjecture

In this section, we present our equivalent characterizations of (TC2) in terms of perpendicular categories or special modules associated with self-orthogonal generators. We then introduce the notion of Gorenstein-Morita algebras and state one of our main results, namely (NC) holds for Gorenstein-Morita algebras.

We begin with recalling a few notation and terminology.

Let A be an Artin algebra. We denote by A-Mod (respectively, A-mod) the category of (respectively, finitely generated) left A-modules, by D the usual duality on A-mod and by ν_A the Nakayama functor ${}_AD(A)\otimes_A -$. For $M\in A$-Mod, let $\mathrm{Add}(M)$ (respectively, $\mathrm{add}(M)$) be the full subcategory of A-Mod consisting of direct summands of (respectively, finite) direct sums of copies of M, and let $M^{\perp 1}$ be the full subcategory of A-Mod consisting of modules X with $\mathrm{Ext}_A^1(M,X)=0$. We say that M is *self-orthogonal* if $\mathrm{Ext}_A^i(M,M)=0$ for all $i\geq 1$; *Nakayama-stable* if $\mathrm{add}(M)=\mathrm{add}(\nu_A(M))$; and a *generator* if $A\in\mathrm{add}(M)$. Clearly, ${}_AA$ is Nakayama-stable if A is a self-injective algebra. Every module over a symmetric algebra A (that is, $A\simeq D(A)$ as A-A-bimodules) is Nakayama-stable. Further, we denote by $\Omega_A^-(M)$ the cokernel of an injective envelope $M\hookrightarrow E$ of M with E an injective A-module.

Let A be a self-injective algebra, $M\in A$-mod a self-orthogonal generator and $\Lambda:=\mathrm{End}_A(M)$. Our strategy is to understand the relation between the category of Gorenstein-projective Λ-modules and the perpendicular category $\mathscr{G}$ of M, where

$$\mathscr{G} := \{X \in A\text{-Mod} \mid \mathrm{Ext}^n_A(M,X) = 0 = \mathrm{Ext}^n_A(X,M) \ \text{ for all } \ n \geq 1\}$$

consists of A-modules that are left and right orthogonal to ${}_AM$. These two categories are all Frobenius and equivalent (see Lemma 3.17). This builds a new bridge between (TC2) and (NC). Following [39], $\mathscr{G}$ is called an *M-Gorenstein subcategory* in A-Mod. Then the quotient category of $\mathscr{G}$ modulo $\mathrm{Add}({}_AM)$, denoted by

$$\mathscr{C} := \mathscr{G}/[M],$$

is a triangulated category and equivalent to the stable category of Gorenstein-projective Λ-modules. The category $\mathscr{C}$ is called an *M-Gorenstein stable category*. In particular, if $M = A$, then $\mathscr{C}$ is the usual *stable module category* of A, denoted by A-$\underline{\mathrm{Mod}}$.

Next, we introduce two classes of A-modules determined by M.

Definition 1.1. *Let X be an A-module.*

(i) *X is M-*compact *if it is a compact object in the category $\mathscr{C}$, that is, $X \in \mathscr{G}$ and the functor* $\mathrm{Hom}_{\mathscr{C}}(X,-) : \mathscr{C} \to \mathbb{Z}\text{-Mod}$ *commutes with coproducts.*

(ii) *X is M-*filtered *if it has a filtration $0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X$ in* A-Mod *such that $X = \bigcup_{n=0}^{\infty} X_n$ and the subquotient X_{n+1}/X_n of X is isomorphic to a finite direct sum of A-modules in the set $\{{}_AA\} \cup \{\Omega_A^{-i}(M) \mid i \in \mathbb{N}\}$ for $n \in \mathbb{N}$. If $X = X_n$ for an integer n in the filtration, then X is said to be* finitely M-filtered*.*

Finitely generated modules in $\mathscr{G}$ are M-compact, and finitely M-filtered A-modules are exactly finitely generated, M-filtered A-modules. Further, M-compact, finitely M-filtered A-modules lie in $\mathrm{add}(M)$ by Lemma 4.16. Clearly, A-compact modules are exactly A-modules that are isomorphic in A-$\underline{\mathrm{Mod}}$ to finitely generated modules.

Now, our characterizations of (TC2) for Nakayama-stable generators read as follows.

Theorem 1.2. *Let A be a self-injective Artin algebra and M a self-orthogonal and Nakayama-stable generator for* A-mod*. The following are equivalent.*

(1) *M is a projective A-module.*

(2) *$\mathscr{G}$ coincides with the full subcategory of* A-Mod *consisting of all filtered colimits of finitely generated modules in $\mathscr{G}$.*

(3) *Any M-compact and M-filtered A-module lies in* $\mathrm{Add}(M)$.

(4) *The minimal left $\mathscr{G}$-approximation W of $\Omega_A^-(M)$ is a filtered colimit of finitely generated modules in $\mathscr{G}$.*

(5) *The minimal left $\mathscr{G}$-approximation W of $\Omega_A^-(M)$ has the property: the category $W^{\perp 1}$ is closed under countable direct sums in* A-Mod *of finitely M-filtered A-modules.*

In Theorem 1.2, (2) is equivalent to saying that the algebra Λ is virtually Gorenstein (see Proposition 3.18) in the sense of Beligiannis; $\mathrm{Add}(M)$ contains finitely generated, M-compact and M-filtered A-modules; (4) and (5) hold true if the module W is the direct sum of finitely generated A-modules. Moreover, Theorem 1.2 implies that (TC2) holds for symmetric algebras of finite representation type because (2)-(5) in Theorem 1.2 are satisfied. This can be seen from a classical result, due to Auslander and Ringel–Tachikawa, that any module over an Artin algebra of finite representation type is a direct sum of finitely generated modules. Thus Theorem 1.2 provides a different approach to studying (TC2).

1.2 Nakayama conjecture for Gorenstein-Morita algebras

As indicated by the relation between (TC2) and (NC), we can apply Theorem 1.2 to discuss (NC) for *strongly Morita* algebras which are, by definition, the endomorphism algebras of Nakayama-stable generators over self-injective algebras. For this purpose, we focus on two classes of modules that are associated with compact objects in some stable categories. This leads to introducing the notions of compactly Gorenstein algebras and Gorenstein-Morita algebras in terms of these modules.

Definition 1.3. *Let B be an Artin algebra and Y a B-module.*

(i) *The B-module Y is* compactly filtered *if it has a filtration* $0 = Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y_n \subseteq \cdots \subseteq Y$ *of B-modules such that $Y = \bigcup_{n=0}^{\infty} Y_n$ and the subquotient Y_{n+1}/Y_n of Y is isomorphic to a finitely generated B-module of finite projective dimension for all $n \in \mathbb{N}$; and* compactly Gorenstein-projective *if it is a compact object in the stable category of Gorenstein-projective B-modules.*

(ii) *The algebra B is* compactly Gorenstein *if any compactly filtered, compactly Gorenstein-projective B-module is projective; and* Gorenstein-Morita *if B is both strongly Morita and compactly Gorenstein.*

Clearly, finitely generated and compactly filtered modules are exactly finitely generated modules of finite projective dimension, while finitely generated Gorenstein-projective modules are compactly Gorenstein-projective. Compactly Gorenstein algebras include virtually Gorenstein algebras (see [11, 12]) and algebras of finite finitistic dimension (see Lemma 5.1). Moreover, compactly Gorenstein algebras over a field are closed under derived equivalences and stable equivalences of adjoint type by Corollary 5.3. Since the Nakayama functor of a symmetric algebra is the identity functor, strongly Morita algebras capture *gendo-symmetric algebras* which are, by definition, the endomorphism algebras of generators over symmetric algebras (see [21, 18]). Examples of gendo-symmetric algebras include Hecke algebras, (quantized) Schur algebras, and blocks of the Bernstein-Gelfand-Gelfand category O of semisimple complex Lie algebras.

As a consequence of Theorem 1.2, we show that (NC) holds for Gorenstein-Morita algebras.

Corollary 1.4. *Let B be a Gorenstein-Morita algebra. If B has infinite dominant dimension, then it is self-injective. In particular, any gendo-symmetric, virtually Gorenstein algebra with infinite dominant dimension is symmetric.*

As is known, *not* all Artin algebras are virtually Gorenstein (see [14]), however, we would like to conjecture that *all Artin algebras are compactly Gorenstein*. If this is true, then (NC) holds for strongly Morita algebras, and in particular for gendo-symmetric algebras.

In [19], we discuss (TC2) for symmetric algebras in terms of recollements of derived module categories and stratifying ideals of algebras. Consequently, it is shown that the validity of (TC2) for symmetric algebras is equivalent to saying that no indecomposable symmetric algebras have stratifying ideals apart from themselves and 0.

1.3 Recollements of relative stable categories from self-orthogonal modules

To prove Theorem 1.2, we first investigate an arbitrary (not necessarily self-orthogonal) generator over a self-injective algebra, and construct two pairs of triangle endofunctors for the stable module category of the self-injective algebra. This enables us to establish a recollement of the relative stable categories for a self-orthogonal generator. By employing the heart of a torsion pair in the stable module category, we then characterize the compact objects of the right term of this recollement.

Let A be a self-injective algebra and M a generator for A-mod. For a full subcategory $\mathscr{X}$ of A-Mod, we denote by $\mathscr{X}/[M]$ the quotient category of $\mathscr{X}$ modulo $\mathrm{Add}(M)$. In particular, A-Mod$/[A]$ is the stable module category A-$\underline{\mathrm{Mod}}$. For simplicity, we denote by $\underline{\mathrm{Hom}}(X,Y)$ the Hom-set in A-$\underline{\mathrm{Mod}}$ for A-modules X and Y, and define $\Gamma := \underline{\mathrm{End}}_A(M)$, called the *stable endomorphism algebra* of ${}_AM$. Let

$$\underline{M}^{\perp} := \{X \in A\text{-}\underline{\mathrm{Mod}} \mid \underline{\mathrm{Hom}}_A(M, X[n]) = 0 \ \text{ for all } \ n \in \mathbb{Z}\}.$$

Given the pair (A,M), we construct explicitly two pairs of triangle endofunctors of A-$\underline{\mathrm{Mod}}$ (see Section 3.2 for details):

$$(\Phi, \Psi) \ \text{ and } \ (\Phi', \Psi') : \ A\text{-}\underline{\mathrm{Mod}} \longrightarrow A\text{-}\underline{\mathrm{Mod}},$$

and define $\mathscr{S} := \{X \in A\text{-}\mathrm{Mod} \mid \Psi(\underline{X}) = 0\}$. If ${}_AM$ is additionally self-orthogonal or Ω-periodic (that is, $\Omega_A^n(X) \simeq X$ in A-$\underline{\mathrm{Mod}}$ for a positive integer n), then $\mathscr{S}$ is the smallest thick subcategory of A-Mod containing M and being closed under direct sums (see Corollary 3.13), and the above endofunctors contribute to building the recollement of A-$\underline{\mathrm{Mod}}$ in Theorem 3.14. Note that $\mathscr{S}$ contains all projective A-modules and the quotient $\underline{\mathscr{S}}$, as a full subcategory of A-$\underline{\mathrm{Mod}}$, is well defined.

Now, suppose that M is a self-orthogonal and Nakayama-stable generator for A-mod. Further, we consider the following two categories associated with M:

$$\mathscr{H} := \{X \in \underline{\mathscr{S}} \mid \underline{\mathrm{Hom}}_A(M, X[n]) = 0 \ \text{ for } \ n \neq 0\} \text{ and}$$

$$\mathscr{E} := \{X \in \mathscr{G} \mid \underline{\mathrm{Hom}}_A(M, X), \, \underline{\mathrm{Hom}}_A(M[1], X) \in \Gamma\text{-mod}\},$$

where $\mathscr{H}$ is the heart of a torsion pair in A-$\underline{\mathrm{Mod}}$ determined by M, and thus an abelian category (see the beginning of Section 4).

The main result on constructing recollements of relative stable categories reads as follows.

Theorem 1.5. *Let A be a self-injective Artin algebra and M a self-orthogonal and Nakayama-stable generator for* A-mod*. Then the following hold.*

(1) *There exists a recollement of triangulated categories:*

$$\begin{tikzcd} \underline{M}^{\perp} \arrow[r] & \mathscr{C} \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & (\mathscr{G} \cap \mathscr{S})/[M] \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

(2) *The recollement in* (1) *restricts to a recollement of triangulated categories:*

$$\begin{tikzcd} \underline{M}^{\perp} \arrow[r] & \mathscr{E}/[M] \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & (\mathscr{E} \cap \mathscr{S})/[M]. \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

(3) $\dim\big((\mathscr{E} \cap \mathscr{S})/[M]\big) \leq min\{2\,LL(\Gamma) - 1, 2\,\mathrm{gl.dim}(\Gamma) + 1\}$*, where $LL(\Gamma)$ and* $\mathrm{gl.dim}(\Gamma)$ *denote the Loewy length and global dimension of the algebra Γ, respectively, and where* $\dim(\mathcal{T})$ *is the dimension of a triangulated category $\mathcal{T}$.*

Of importance are the compact objects for triangulated categories. Since $\mathscr{C}$ is compactly generated, it follows from basic properties of recollements that $\underline{M}^{\perp}$ has compact objects. It seems, however, to be unclear that the category $(\mathscr{G} \cap \mathscr{S})/[M]$ has compact objects. In the following, we gives a complete description of its compact objects (see also Corollary 4.18 for details).

Let $\mathcal{T}$ be a triangulated category and $\mathcal{U}$ a set of objects in $\mathcal{T}$. For integers $i \leq j$ and $n \geq 0$, we denote by $\langle \mathcal{U} \rangle_{n+1}^{[i,j]}$ the full subcategory of $\mathcal{T}$ consisting of all objects obtained by taking $(n+1)$-fold extensions of finite direct sums of objects in the set $\{U[-s] \mid U \in \mathcal{U}, s \in \mathbb{Z}, i \leq s \leq j\}$.

Proposition 1.6. *Let A be a self-injective Artin algebra and M a self-orthogonal and Nakayama-stable generator. Then the following hold.*

(1) *Each object X of $\mathscr{E} \cap \mathscr{S}$ is M-compact and isomorphic in* A-$\underline{\mathrm{Mod}}$ *to an M-filtered module. Moreover, $X \in \mathscr{E} \cap \mathscr{S}$ is finitely generated if and only if $X \in$* $\mathrm{add}({}_AM)$.

(2) *The category* $(\mathscr{G}\cap\mathscr{S})/[M]$ *is a compactly generated triangulated category and has* $(\mathscr{E}\cap\mathscr{S})/[M]$ *as its full subcategory consisting of all compact objects.*

(3) *Let* $\mathcal{S}$ *be the set of isomorphism classes of simple objects of* $\mathscr{H}$*, and let n be the Loewy length of the algebra* Γ*. Then* $\mathcal{S}$ *is a finite set,* $\big(\mathscr{G}\cap\mathscr{S}\big)/[M]=\langle\mathrm{Add}(\mathcal{S})\rangle_{2n}^{[-1,0]}$ *and* $(\mathscr{E}\cap\mathscr{S})/[M]=\langle\mathcal{S}\rangle_{2n}^{[-1,0]}$.

Under the assumption of Theorem 1.5, the module ${}_AM$ is projective if and only if $(\mathscr{G}\cap\mathscr{S})/[M]=0$ if and only if $(\mathscr{E}\cap\mathscr{S})/[M]=0$ (see Corollary 4.9). Thus (TC2) is true for the pair (A,M) exactly when the recollements in Theorem 1.5 are trivial. Hence, to construct a counterexample to (TC2), our results, Theorem 1.5 and Proposition 1.6, provide necessary homological information on self-orthogonal modules.

1.4 Overview of the contexts

The contents of this article are sketched as follows. In Section 2 we briefly recall definitions of quotient categories, recollements and Gorenstein-projective modules over algebras. In Section 3 we construct two pairs of triangle endofunctors of the stable module category A-$\underline{\mathrm{Mod}}$ for a self-injective algebra A with a generator ${}_AM$. With these endofunctors, we establish the recollement in Theorem 3.14 of A-$\underline{\mathrm{Mod}}$ determined by M. Moreover, we show that the subcategory $\mathscr{G}$ of A-Mod relative to M is equivalent to the category of Gorenstein-projective modules over the endomorphism algebra of M (see Lemma 3.17). In Section 4 we prove Theorem 1.5 and establish a representability theorem for a series of homological functors (see Theorem 4.14). In Section 5 we show Theorem 1.2 and Corollary 1.4.

2 Preliminaries

In this section we briefly recall definitions, basic facts and notation used in this paper.

2.1 Quotient categories and recollements

Let $\mathcal{C}$ be an additive category.

A full subcategory $\mathcal{B}$ of $\mathcal{C}$ is always assumed to be closed under isomorphisms, that is, if $X\in\mathcal{B}$ and $Y\in\mathcal{C}$ with $Y\simeq X$, then $Y\in\mathcal{B}$.

Let X be an object in $\mathcal{C}$. The full subcategory of $\mathcal{C}$ consisting of all direct summands of finite coproducts of copies of X is denoted by $\mathrm{add}(X)$. If $\mathcal{C}$ admits coproducts (that is, coproducts indexed over sets exist in $\mathcal{C}$), then $\mathrm{Add}(X)$ denotes the full subcategory of $\mathcal{C}$ consisting of all direct summands of coproducts of copies of X. Dually, if $\mathcal{C}$ admits products, then $\mathrm{Prod}(X)$ denotes the full subcategory of $\mathcal{C}$ consisting of all direct summands of products of copies of X.

For morphisms $f:X\to Y$ and $g:Y\to Z$ in $\mathcal{C}$, the composition of f and g is written by fg, a morphism from X to Z. The induced morphisms $\mathrm{Hom}_{\mathcal{C}}(Z,f):\mathrm{Hom}_{\mathcal{C}}(Z,X)\to\mathrm{Hom}_{\mathcal{C}}(Z,Y)$ and $\mathrm{Hom}_{\mathcal{C}}(f,Z):\mathrm{Hom}_{\mathcal{C}}(Y,Z)\to\mathrm{Hom}_{\mathcal{C}}(X,Z)$ are denoted by f^* and f_*, respectively.

For functors $F:\mathcal{C}\to\mathcal{D}$ and $G:\mathcal{D}\to\mathcal{E}$, the composition of F and G is denoted by $G\circ F$ which is a functor from $\mathcal{C}$ to $\mathcal{E}$. Let $\mathrm{Ker}(F)$ and $\mathrm{Im}(F)$ be the kernel and image of the functor F, respectively. In particular, $\mathrm{Ker}(F)$ is closed under isomorphisms in $\mathcal{C}$. In this paper, we require that $\mathrm{Im}(F)$ is closed under isomorphisms in $\mathcal{D}$.

Suppose that $\mathcal{B}$ is a full subcategory of $\mathcal{C}$. A morphism $f:X\to Y$ in $\mathcal{C}$ is called a *right* $\mathcal{B}$*-approximation* of Y if $X\in\mathcal{B}$ and $\mathrm{Hom}_{\mathcal{C}}(B,f):\mathrm{Hom}_{\mathcal{C}}(B,X)\to\mathrm{Hom}_{\mathcal{C}}(B,Y)$ is surjective for any $B\in\mathcal{B}$; and *right minimal* if $\alpha\in\mathrm{End}_{\mathcal{C}}(X)$ is an isomorphism whenever $f=\alpha f$. If f is both a right minimal morphism and a right $\mathcal{B}$-approximation of Y, then f is called a *minimal right* $\mathcal{B}$*-approximation* of Y. In this case, the object X is unique up to isomorphism and is called the minimal right $\mathcal{B}$-approximation of Y (without mentioning f). If each object of $\mathcal{C}$ admits a right $\mathcal{B}$-approximation, then $\mathcal{B}$ is said to be *contravariantly finite* in $\mathcal{C}$.

Dually, there are the notions of (minimal) left approximations and *covariantly finite* subcategories in $\mathcal{C}$. If $\mathcal{B}$ is both contravariantly and covariantly finite in $\mathcal{C}$, then it is said to be *functorially finite* in $\mathcal{C}$.

We recall Wakamatsu's Lemma (see [7, Proposition 1.3]): Let $\mathcal{S}$ be a class of R-modules over a ring R closed under extensions. If $f: C \to X$ is a minimal right $\mathcal{S}$-approximation of an R-module X, then $\mathrm{Ext}^1_R(L, \mathrm{Ker}(f)) = 0$ for $L \in \mathcal{S}$. Dually, if $g: X \to C'$ is a minimal left $\mathcal{S}$-approximation of an R-module X, then $\mathrm{Ext}^1_R(\mathrm{Coker}(g), M) = 0$ for $M \in \mathcal{S}$.

Next, we recall the definition of quotient categories of additive categories.

Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. Denote by $\mathcal{C}/\mathcal{D}$ the *quotient category* of $\mathcal{C}$ modulo $\mathcal{D}$. It has the same objects as $\mathcal{C}$, but its morphism set for any two objects X and Y is given by $\mathrm{Hom}_{\mathcal{C}/\mathcal{D}}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y)/\mathcal{D}(X,Y)$ where $\mathcal{D}(X,Y)$ is the subgroup of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ consisting of all morphisms factorizing through objects in $\mathcal{D}$. The canonical quotient functor $q: \mathcal{C} \to \mathcal{C}/\mathcal{D}$ sends a morphism $f: X \to Y$ in $\mathcal{C}$ to $f + \mathcal{D}(X,Y)$ in $\mathcal{C}/\mathcal{D}$. Clearly, if $\mathcal{C}$ is idempotent complete, then $\mathrm{Ker}(q)$ consists of all direct summands (in $\mathcal{C}$) of objects of $\mathcal{D}$.

Suppose that $\mathcal{C}$ admits coproducts. An object X is said to be *compact* in $\mathcal{C}$ if the functor $\mathrm{Hom}_{\mathcal{C}}(X,-)$ from $\mathcal{C}$ to the category of abelian groups commutes with coproducts. The full subcategory of $\mathcal{C}$ consisting of compact objects is denoted by $\mathcal{C}^{\mathrm{c}}$. A set $\mathcal{U}$ of objects of $\mathcal{C}$ is called a *compact generating set* of $\mathcal{C}$ if each object of $\mathcal{U}$ is compact in $\mathcal{C}$ and an object $X \in \mathcal{C}$ is zero whenever $\mathrm{Hom}_{\mathcal{C}}(U,X) = 0$ for all $U \in \mathcal{U}$. When $\mathcal{C}$ is a triangulated category, it is said to be *compactly generated* if it has a compact generating set. If $\mathcal{U}$ is a set of compact objects of $\mathcal{C}$ closed under shifts, then $\mathcal{U}$ is a compact generating set of $\mathcal{C}$ if and only if $\mathcal{C}$ itself is the smallest full triangulated subcategory of $\mathcal{C}$ containing $\mathcal{U}$ and being closed under coproducts.

The following result is elementary.

Lemma 2.1. (1) *Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories, and let $\mathcal{X} \subseteq \mathcal{C}$ and $\mathcal{Y} \subseteq \mathcal{D}$ be full subcategories. Suppose that $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ form an adjoint pair (F,G) of additive functors. If $F(\mathcal{X}) \subseteq \mathcal{Y}$ and $G(\mathcal{Y}) \subseteq \mathcal{X}$, then the adjoint pair (F,G) induces an adjoint pair (F_0, G_0) of additive functors $F_0: \mathcal{C}/\mathcal{X} \to \mathcal{D}/\mathcal{Y}$ and $G_0: \mathcal{D}/\mathcal{Y} \to \mathcal{C}/\mathcal{X}$.*

(2) *Suppose that an additive category $\mathcal{C}$ admits coproducts and $\mathcal{X}$ is a full subcategory of $\mathcal{C}$ closed under coproducts. Then $\mathcal{C}/\mathcal{X}$ admits coproducts, and the quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{X}$ preserves coproducts and compact objects.*

We denote by $\mathscr{C}(\mathcal{C})$ the category of all complexes over $\mathcal{C}$ with chain maps, and $\mathscr{K}(\mathcal{C})$ the homotopy category of $\mathscr{C}(\mathcal{C})$. For a chain map $f^\bullet: X^\bullet \to Y^\bullet$ in $\mathscr{C}(\mathcal{C})$, we denote by $\mathrm{Con}(f^\bullet)$ the mapping cone of $f^\bullet$. There is a distinguished triangle $X^\bullet \to Y^\bullet \to \mathrm{Con}(f^\bullet) \to X^\bullet[1]$ in $\mathscr{K}(\mathcal{C})$. When $\mathcal{C}$ is a full subcategory of an abelian category $\mathcal{A}$, we denote by $\mathscr{K}_{\mathrm{ac}}(\mathcal{C})$ the full subcategory of $\mathscr{K}(\mathcal{A})$ consisting of acyclic complexes of $\mathcal{C}$. Clearly, if $f^\bullet$ is a quasi-isomorphism, then $\mathrm{Con}(f^\bullet)$ is acyclic. If $\mathcal{C}$ is an abelian category, we denote by $\mathscr{D}(\mathcal{C})$ the *unbounded derived category* of $\mathcal{C}$, which is the localization of $\mathscr{K}(\mathcal{C})$ by inverting all quasi-isomorphisms. Clearly, $\mathscr{K}(\mathcal{A})$ and $\mathscr{D}(\mathcal{C})$ are triangulated categories.

Next, we recall the notion of recollements of triangulated categories, introduced in [10] for studying derived categories of perverse sheaves over singular spaces.

Definition 2.2. Let $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$ be triangulated categories. $\mathcal{T}$ is called a *recollement* of $\mathcal{T}'$ and $\mathcal{T}''$ (or there is a recollement among $\mathcal{T}', \mathcal{T}$ and $\mathcal{T}''$) if there are six triangle functors

$$\begin{tikzcd}
\mathcal{T}'' \arrow[r, "i_*=i_!"] & \mathcal{T} \arrow[l, bend right=50, "i^*"'] \arrow[l, bend left=50, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{T}' \arrow[l, bend right=50, "j_!"'] \arrow[l, bend left=50, "j_*"]
\end{tikzcd}$$

among the three categories such that

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs,

(2) i_*, j_* and $j_!$ are fully faithful functors,

(3) $j^! i_! = 0$ (and thus also $i^! j_* = 0$ and $i^* j_! = 0$), and

(4) for an object $X \in \mathcal{T}$, there are two triangles $i_! i^!(X) \to X \to j_* j^*(X) \to i_! i^!(X)[1]$ and $j_! j^!(X) \to X \to i_* i^*(X) \to j_! j^!(X)[1]$ in $\mathcal{T}$ induced by the counits and units of the adjunctions, where $[1]$ denotes the shift functor of $\mathcal{T}$.

By a *half recollement* among $\mathcal{T}'$, $\mathcal{T}$ and $\mathcal{T}''$, we mean that $i^*, i_*, j_!$ and $j^!$ satisfy the corresponding properties (1)-(4) involved in Definition 2.2. Note that there is a one-to-one correspondence between equivalence classes of half recollements (respectively, recollements) of triangulated categories and hereditary torsion pairs (respectively, TTF triples) of triangulated categories. Recall that a *torsion pair* in $\mathcal{T}$ is a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{T}$ satisfying the three conditions:

(a) $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ for $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$;

(b) $\mathcal{X}[1] \subseteq \mathcal{X}$ and $\mathcal{Y}[-1] \subseteq \mathcal{Y}$; and

(c) for any $M \in \mathcal{T}$, there is a triangle $X \to M \to Y \to X[1]$ in $\mathcal{T}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

A torsion pair $(\mathcal{X}, \mathcal{Y})$ of $\mathcal{T}$ is said to be *hereditary* if $\mathcal{X}$ (or equivalently, $\mathcal{Y}$) is a full triangulated subcategory of $\mathcal{T}$. In this case, the inclusion $\mathcal{X} \to \mathcal{T}$ has a right adjoint, the inclusion $\mathcal{Y} \to \mathcal{T}$ has a left adjoint, and there is a half recollement among $\mathcal{X}, \mathcal{T}$ and $\mathcal{Y}$. If $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are hereditary torsion pairs in $\mathcal{T}$, then $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is called a TTF *(torsion-torsionfree) triple* in $\mathcal{T}$. In this case, there is a recollement among $\mathcal{X}$, $\mathcal{T}$ and $\mathcal{Y}$. Conversely, the recollement in Definition 2.2 gives a TTF triple $(\mathrm{Im}(j_!), \mathrm{Im}(i_*), \mathrm{Im}(j_*))$ in $\mathcal{T}$. For more details, see [34, Chap. 9], [13, Chap. I. 2] or [17, Section 2.3].

Hereditary torsion pairs can be constructed in compactly generated triangulated categories as follows.

Let $\mathcal{T}$ be a compactly generated triangulated category. Then coproducts and products indexed by sets exist in $\mathcal{T}$. Let $\mathcal{S}$ be a set of objects in $\mathcal{T}$. Denote by $\mathrm{Loc}_{\mathcal{T}}(\mathcal{S})$, $\mathrm{Coloc}_{\mathcal{T}}(\mathcal{S})$ and $\mathrm{thick}_{\mathcal{T}}(\mathcal{S})$ the smallest full triangulated subcategories of $\mathcal{T}$ containing $\mathcal{S}$ and being closed under coproducts, products and direct summands, respectively. If $\mathcal{T}$ is clearly understood in the context, we shall write $\mathrm{Loc}(\mathcal{S})$, $\mathrm{Coloc}(\mathcal{S})$ and $\mathrm{thick}(\mathcal{S})$ for $\mathrm{Loc}_{\mathcal{T}}(\mathcal{S})$, $\mathrm{Coloc}_{\mathcal{T}}(\mathcal{S})$ and $\mathrm{thick}_{\mathcal{T}}(\mathcal{S})$, respectively. By $\mathcal{S}^{\perp}$ we denote the right orthogonal full subcategory of $\mathcal{T}$ with respect to $\mathcal{S}$, that is, $\mathcal{S}^{\perp} = \{C \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(S[n], C) = 0, \forall\, S \in \mathcal{S}, n \in \mathbb{Z}\}$. Then $\mathcal{S}^{\perp}$ is a triangulated subcategory of $\mathcal{T}$ closed under products. Similarly, ${}^{\perp}\mathcal{S}$ stands for the left orthogonal full subcategory of $\mathcal{T}$ with respect to $\mathcal{S}$.

In general, the opposite category of a compactly generated triangulated category is not compactly generated, but is perfectly generated in the sense of Krause (see [30]). It is worth mentioning that perfectly generated triangulated categories not only generalize compactly generated triangulated categories, but also satisfy the Brown representability theorem in [30, Theorem A]. This implies the following result:

If a triangle functor from a perfectly generated triangulated category to another triangulated category preserves coproducts, then it has a right adjoint.

The next result seems to be known. For the convenience of the reader, we include here a proof.

Proposition 2.3. *If $\mathcal{T}$ is a compactly generated triangulated category and $\mathcal{S}$ is a set of objects in $\mathcal{T}$, then $(\mathrm{Loc}(\mathcal{S}), \mathcal{S}^{\perp})$ and $({}^{\perp}\mathcal{S}, \mathrm{Coloc}(\mathcal{S}))$ are hereditary torsion pairs in $\mathcal{T}$.*

Proof. The Verdier localization $Q_{\mathcal{S}} : \mathcal{T} \to \mathcal{T}/\mathrm{Loc}(\mathcal{S})$ preserves coproducts and $\mathrm{Ker}(Q_{\mathcal{S}}) = \mathrm{Loc}(\mathcal{S})$. S-ince $\mathcal{T}$ is compactly generated, it is perfectly generated. Thus $Q_{\mathcal{S}}$ has a right adjoint and $(\mathrm{Loc}(\mathcal{S}), \mathrm{Loc}(\mathcal{S})^{\perp})$ is a torsion pair in $\mathcal{T}$ by [34, Theorem 9.1.13]. Hence $(\mathrm{Loc}(\mathcal{S}), \mathcal{S}^{\perp})$ is a torsion pair in $\mathcal{T}$, due to $\mathrm{Loc}(\mathcal{S})^{\perp} = \mathcal{S}^{\perp}$. Moreover, it is hereditary because $\mathrm{Loc}(\mathcal{S})$ is a full triangulated subcategory of $\mathcal{T}$. This proof also implies that $(\mathrm{Loc}(\mathcal{S}), \mathcal{S}^{\perp})$ is a hereditary torsion pair in $\mathcal{T}$ whenever $\mathcal{T}$ is perfectly generated.

Clearly, $\mathrm{Coloc}(\mathcal{S})$ can be regarded as a localizing subcategory of $\mathcal{T}^{\mathrm{op}}$, that is, $\mathrm{Coloc}_{\mathcal{T}}(\mathcal{S}) = \mathrm{Loc}_{\mathcal{T}^{\mathrm{op}}}(\mathcal{S}^{\mathrm{op}})$, where $\mathcal{S}^{\mathrm{op}} := \{X^{\mathrm{op}} \in \mathcal{T}^{\mathrm{op}} \mid X \in \mathcal{S}\}$. Since $\mathcal{T}^{\mathrm{op}}$ is perfectly generated, $(\mathrm{Loc}(\mathcal{S}^{\mathrm{op}}), (\mathcal{S}^{\mathrm{op}})^{\perp})$ is a hereditary torsion pair in $\mathcal{T}^{\mathrm{op}}$, that is, $({}^{\perp}\mathcal{S}, \mathrm{Coloc}(\mathcal{S}))$ is a hereditary torsion pair in $\mathcal{T}$. □

Let F, G and H be triangle endofunctors of $\mathcal{T}$. We say that a sequence of natural transformations

$$F \xrightarrow{\tau} G \xrightarrow{\eta} H \xrightarrow{\sigma} F[1]: \ \mathcal{T} \longrightarrow \mathcal{T}$$

is *exact* if for each $X \in \mathcal{T}$ the sequence $F(X) \xrightarrow{\tau_X} G(X) \xrightarrow{\eta_X} H(X) \xrightarrow{\sigma_X} F(X)[1]$ is a triangle in $\mathcal{T}$.

2.2 Cotorsion pairs and Gorestein-projective modules

In the subsection we recall the definitions of cotorsion pairs in abelian categories and Gorenstein-projective modules over algebras.

Let $\mathcal{A}$ be an abelian category and $n \geq 1$ a natural number. Given a class $\mathcal{S}$ of objects in $\mathcal{A}$, we define

$$^{\perp n}\mathcal{S} := \{X \in \mathcal{A} \mid \mathrm{Ext}^n_{\mathcal{A}}(X,S) = 0 \ \text{ for } \ S \in \mathcal{S}\}, \ \ ^{\perp >0}\mathcal{S} := \bigcap_{n \geq 1} {}^{\perp n}\mathcal{S},$$

$$\mathcal{S}^{\perp n} := \{X \in \mathcal{A} \mid \mathrm{Ext}^n_{\mathcal{A}}(S,X) = 0 \ \text{ for } \ S \in \mathcal{S}\} \ \text{ and } \ \mathcal{S}^{\perp >0} := \bigcap_{n \geq 1} \mathcal{S}^{\perp n}.$$

Definition 2.4. (1) *A pair* $(\mathcal{U}, \mathcal{V})$ *of full subcategories of* $\mathcal{A}$ *is called a* cotorsion pair *in* $\mathcal{A}$ *if the following hold.*

(i) $\mathcal{U} = {}^{\perp 1}\mathcal{V}$ *and* $\mathcal{V} = \mathcal{U}^{\perp 1}$.

(ii) *For each object* $X \in \mathcal{A}$, *there are exact sequences* $0 \to V_X \to U_X \xrightarrow{\pi_X} X \to 0$ *and* $0 \to X \xrightarrow{\lambda_X} V^X \to U^X \to 0$ *in* $\mathcal{A}$ *such that* $U_X, U^X \in \mathcal{U}$ *and* $V_X, V^X \in \mathcal{V}$.

(2) *A cotorsion pair* $(\mathcal{U}, \mathcal{V})$ *in* $\mathcal{A}$ *is* hereditary *if* $\mathcal{U} = {}^{\perp >0}\mathcal{V}$ *and* $\mathcal{V} = \mathcal{U}^{\perp >0}$.

Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair in $\mathcal{A}$. Then π_X is a right $\mathcal{U}$-approximation of X and λ_X is a left $\mathcal{V}$-approximation of X. Let $\mathcal{E} = \mathcal{U} \cap \mathcal{V}$. Then U_X and V^X are unique up to isomorphism in the quotient category $\mathcal{A}/\mathcal{E}$. Further, the inclusion $\mathcal{U}/\mathcal{E} \to \mathcal{A}/\mathcal{E}$ has a right adjoint sending X to U_X, while the inclusion $\mathcal{V}/\mathcal{E} \to \mathcal{A}/\mathcal{E}$ has a left adjoint sending X to V^X. Moreover, X has a $\mathcal{U}$-resolution and a $\mathcal{V}$-coresolution in the following sense.

Definition 2.5. *Let* $\mathcal{C}$ *be a full subcategory of* $\mathcal{A}$. *A* $\mathcal{C}$-resolution *of an object* $U \in \mathcal{A}$ *is a complex* $X^\bullet := (X^i)_{i \in \mathbb{Z}} \in \mathscr{C}(\mathcal{C})$ *with* $X^i = 0$ *for all* $i \geq 1$, *together with a chain map* $\pi^\bullet : X^\bullet \to U$ *(regarded as a stalk complex), such that* $\mathrm{Hom}^\bullet_{\mathcal{A}}(C, \pi^\bullet) : \mathrm{Hom}^\bullet_{\mathcal{A}}(C, X^\bullet) \to \mathrm{Hom}^\bullet_{\mathcal{A}}(C, U)$ *is a quasi-isomorphism for any* $C \in \mathcal{C}$. *In this case,* $\pi^\bullet$ *is called a* $\mathcal{C}$*-resolution of* U *for simplicity. Dually, a* $\mathcal{C}$-coresolution *of* U *can be defined.*

Let A be an Artin algebra. We denote by A-Proj and A-Inj the full subcategories of A-Mod consisting of projective and injective A-modules, respectively. Moreover, we write $\mathscr{K}(A)$, $\mathscr{K}_{\mathrm{ac}}(A)$ and $\mathscr{D}(A)$ for $\mathscr{K}(A\text{-Mod})$, $\mathscr{K}_{\mathrm{ac}}(A\text{-Mod})$ and $\mathscr{D}(A\text{-Mod})$, respectively.

An A-module X is said to be *Gorenstein-projective* if there is an exact complex $P^\bullet : \cdots \to P^{-2} \to P^{-1} \to P^0 \to P^1 \to P^2 \to \cdots$ of projective A-modules such that $X \simeq \mathrm{Ker}(P^0 \to P^1)$ and the complex $\mathrm{Hom}^\bullet_A(P^\bullet, A)$ is exact. The complex $P^\bullet$ is then called a *complete projective resolution* of X. Dually, one can define Gorenstein-injective modules and complete injective coresolutions. Let A-GProj and A-GInj be the full subcategories of all Gorenstein-projective and Gorenstein-injective A-modules, respectively. By [13, Theorem X. 2.4], (A-GProj,A-GProj$^{\perp >0}$) and ($^{\perp >0}A$-GInj,A-GInj) are hereditary cotorsion pairs in A-Mod. Moreover, each A-module admits a minimal right A-GProj-approximation and also a minimal left A-GInj-approximation by [11, Proposition 3.8(iv) and Corollary 6.8].

An Artin algebra A is said to be *virtually Gorenstein* (see [11, Definition 8.1]) if A-GProj$^{\perp >0}$ = $^{\perp >0}A$-GInj. Virtually Gorenstein algebras include Gorenstein algebras and algebras of finite representation type, and are closed under derived equivalences and stable equivalences of Morita type.

By *filtered colimits* of A-modules we mean colimits of filtered diagrams $I \to A$-Mod with I an essentially small, filtered category.

3 Recollements of stable module categories

In this section, we investigate an arbitrary (not necessarily self-orthogonal) generator over a self-injective algebra, and construct two pairs of triangle endofunctors for the stable module category by means of the endomorphism algebra of the generator. When the generator is additionally self-orthogonal or Ω-periodic, we show that these endofunctors coincidentally appear in a recollement of the stable module category determined by the generator (see Theorem 3.14 and Corollary 3.15). This recollement will be restricted to the one of a relative Gorenstein stable category in the next section. Finally, we describe the category of Gorenstein-projective modules over the endomorphism algebra of a self-orthogonal generator in term of a relative Gorenstein category (see Lemma 3.17), and provide equivalent characterizations for the endomorphism algebra to be virtually Gorenstein (see Proposition 3.18).

Throughout this section, let A denote a **self-injective Artin algebra**. Then A-$\underline{\mathrm{Mod}}$ is a triangulated category with a shift functor $[1]: A\text{-}\underline{\mathrm{Mod}} \to A\text{-}\underline{\mathrm{Mod}}$, given by the cosyzygy functor Ω_A^-. Clearly, $\mathrm{Ext}_A^n(X_1,X_2) \simeq \underline{\mathrm{Hom}}_A(X_1,X_2[n])$ for all $n \geq 1$ and $X_1,X_2 \in A$-Mod. Let $q: A\text{-Mod} \to A\text{-}\underline{\mathrm{Mod}}$ be the canonical functor. By Lemma 2.1(2) and its dual, q preserves direct sums and direct products. To emphasize objects in A-$\underline{\mathrm{Mod}}$, the image of $X \in A$-Mod under q is denoted by $\underline{X}$.

A full subcategory $\mathcal{U}$ of A-Mod is called a *thick subcategory* if it is closed under direct summands in A-Mod and has the *two out of three property*: if two terms of an exact sequence $0 \to X \to Y \to Z \to 0$ in A-Mod belong to $\mathcal{U}$, then so does the third. If $\mathcal{U}$ contains all projective A-modules, then $\mathcal{U}$ is a thick subcategory of A-Mod if and only if $\underline{\mathcal{U}}$ is a full triangulated subcategory of A-$\underline{\mathrm{Mod}}$ closed under direct summands.

From now on, let $M = A \oplus M_0$ be a generator in A-mod such that M_0 contains no nonzero projective direct summands. As M is finitely generated, we know $\mathrm{Add}(M) = \mathrm{Prod}(M)$.

3.1 Endomorphism algebras of generators over self-injective algebras

In this subsection, we establish additive equivalences between the subcategories of the module category of a self-injective algebra and the ones of the module category of the endomorphism algebra of a generator over the self-injective algebra.

Let $\Lambda = \mathrm{End}_A(M)$ and e be the idempotent element of Λ such that $\Lambda e = \mathrm{Hom}_A(M,A)$. Then $A \simeq e\Lambda e$ as algebras. Let $S_e: \Lambda\text{-Mod} \to A\text{-Mod}, Y \mapsto eY$ be the *Schur functor* determined by e. This functor has a fully faithful left and right adjoint functors $F := \Lambda e \otimes_A -$ and $G := \mathrm{Hom}_A(e\Lambda, -)$ from A-Mod to Λ-Mod, respectively. Both F and G commute with direct products and direct sums. Due to $e\Lambda \simeq M$ as A-Λ-bimodules, the functor G can be identified with $\mathrm{Hom}_A(M,-)$. Moreover, there is a canonical natural transformation $\delta: F \to G$ defined by

$$\delta_X:\ F(X) \longrightarrow G(X), \quad ae \otimes x \mapsto [eb \mapsto ebaex]$$

where $X \in A$-Mod, $a,b \in \Lambda$ and $x \in X$. For a full subcategory $\mathcal{X} \subseteq A$-Mod and $\mathcal{Y} \subseteq \Lambda$-Mod, we define

$$\mathscr{K}_{\mathrm{ac}}(\mathcal{X}) := \{X^\bullet \in \mathscr{K}(\mathcal{X}) \mid X^\bullet \in \mathscr{K}_{\mathrm{ac}}(A)\} \quad \text{and} \quad \mathscr{K}_{e\text{-ac}}(\mathcal{Y}) := \{Y^\bullet \in \mathscr{K}(\mathcal{Y}) \mid S_e(Y^\bullet) \in \mathscr{K}_{\mathrm{ac}}(A)\}.$$

Since S_e is exact, $\mathscr{K}_{e\text{-ac}}(\mathcal{Y})$ consists of those $Y^\bullet \in \mathscr{K}(\mathcal{Y})$ with $H^i(Y^\bullet) \in (\Lambda/\Lambda e\Lambda)$-Mod for all $i \in \mathbb{Z}$.

Given an Artin algebra B with the usual duality D, we denote by ν_B and ν_B^- the Nakayama functor $D(B) \otimes_B -: B\text{-Mod} \to B\text{-Mod}$ and its right adjoint functor $\mathrm{Hom}_B(D(B),-)$, respectively. By restriction to B-mod, we have natural isomorphisms $\nu_B \simeq D \circ \mathrm{Hom}_B(-,B)$ and $\nu_B^- \simeq \mathrm{Hom}_{B^{\mathrm{op}}}(-,B) \circ D$. As $D(B)$ is finitely presented, $\mathrm{Hom}_B(D(B),-)$ commutes with filtered colimits, while $D(B) \otimes_B -$ commutes always with filtered colimits. Since a projective B-module is always a direct sum of finitely generated projective B-modules (see [1, Theorem 27.11, p.306], we can show by projective presentations of modules that, for a self-injective Artin algebra B, both ν_B and ν_B^- are exact and auto-equivalent on B-Mod.

Lemma 3.1. (1) *The restriction of* δ *to* A-Proj *is a natural isomorphism, that is,* δ_X *is an isomorphism for any* $X \in A$-Proj.

(2) *There are natural isomorphisms of additive functors:*

$$D \circ F \circ \nu_A \simeq \mathrm{Hom}_A(-, M) : A\text{-Mod} \to \Lambda^{\mathrm{op}}\text{-Mod},$$

$$\nu_\Lambda \circ G \simeq F \circ \nu_A \ \ \textit{and} \ \ \nu_\Lambda^- \circ F \simeq G \circ \nu_A^- : \ A\text{-Mod} \to \Lambda\text{-Mod}.$$

In particular, $F(\nu_A(M)) \simeq D(\Lambda)$ *as* Λ-Λ-*bimodules.*

(3) *The functor* S_e *restricts to equivalences of additive categories:*

$$\mathrm{Add}(\Lambda e) \xrightarrow{\simeq} A\text{-Proj}, \ \ \Lambda\text{-Proj} \xrightarrow{\simeq} \mathrm{Add}({}_AM) \ \ \textit{and} \ \ \Lambda\text{-Inj} \xrightarrow{\simeq} \mathrm{Add}(\nu_A(M)).$$

(4) *Let* $\mathcal{X} \subseteq A$-Mod *and* $\mathcal{Y} \subseteq \Lambda$-Mod *be additive, full subcategories. If the functor* S_e *induces an equivalence* $\mathcal{Y} \xrightarrow{\simeq} \mathcal{X}$ *of additive categories, then there is an equivalence* $\mathscr{K}_{\text{e-ac}}(\mathcal{Y}) \xrightarrow{\simeq} \mathscr{K}_{\mathrm{ac}}(\mathcal{X})$ *of triangulated categories.*

Proof. (1) holds because δ_A is an isomorphism and both F and G commute with direct sums.

(2) We need the following result, its proof is left to the reader:

$(*)$ Let A_1, A_2 and A_3 be Artin algebras and let $F_1, F_2 : A_1\text{-Mod} \to A_2\text{-Mod}$ be (covariant) additive functors. If X is an A_1-A_3-bimodule, then $F_1(X)$ is an A_2-A_3-bimodule, where the right A_3-module structure is given by the composition of associated ring homomorphisms $A_3 \to \mathrm{End}_{A_1}(X)$ and $\mathrm{End}_{A_1}(X) \to \mathrm{End}_{A_2}(F_1(X))$. Additionally, if $\eta : F_1 \to F_2$ is a natural transformation, then $\eta_X : F_1(X) \to F_2(X)$ is a homomorphism of A_2-A_3-bimodules.

Note that $D \circ (F \circ \nu_A) = D\big(\Lambda e \otimes_A \nu_A(-)\big) \simeq \mathrm{Hom}_A(\nu_A(-), D(\Lambda e))$ and $D(\Lambda e) = D\mathrm{Hom}_A(M, A) \simeq D(A) \otimes_A M = \nu_A(M)$ as A-Λ-bimodules by $(*)$. Since $\nu_A : A\text{-Mod} \to A\text{-Mod}$ is an auto-equivalence, there are isomorphisms $D \circ (F \circ \nu_A) \simeq \mathrm{Hom}_A\big(\nu_A(-), \nu_A(M)\big) \simeq \mathrm{Hom}_A(-, M) : A\text{-Mod} \to \Lambda^{\mathrm{op}}\text{-Mod}$. This implies $D \circ (F \circ \nu_A)(M) \simeq \Lambda$ as Λ^{op}-modules. Since M is an A-Λ-bimodule, it follows from $(*)$ that $D \circ (F \circ \nu_A)(M) \simeq \Lambda$ as Λ-Λ-bimodules. This leads to $D(\Lambda) \simeq F(\nu_A(M))$ as Λ-Λ-bimodules. Consequently, there are natural isomorphisms:

$$\nu_\Lambda \circ G = D(\Lambda) \otimes_\Lambda G(-) \simeq \Lambda e \otimes_A D(A) \otimes_A M \otimes_\Lambda \mathrm{Hom}_A(M, -) \simeq \Lambda e \otimes_A D(A) \otimes_A - = F \circ \nu_A,$$

$$\nu_\Lambda^- \circ F \simeq \mathrm{Hom}_\Lambda(\Lambda e \otimes_A \nu_A(M), \Lambda e \otimes_A -) \simeq \mathrm{Hom}_A(\nu_A(M), -) \simeq \mathrm{Hom}_A(M, \nu_A^-(-)) = G \circ \nu_A^-.$$

(3) The restrictions of S_e and G to the corresponding subcategories give the first and second equivalences since S_e and G commutes with direct sums. Now, we claim that (F, S_e) restricts to equivalences between Λ-Inj and $\mathrm{Add}(\nu_A(M))$.

In fact, $S_e(D(\Lambda_\Lambda)) = eD(\Lambda) \simeq D(\Lambda e) = \nu_A(M)$. As S_e commutes with direct sums and $\Lambda\text{-Inj} = \mathrm{Add}(D(\Lambda_\Lambda)) = \mathrm{Prod}(D(\Lambda_\Lambda))$, the Schur functor S_e restricts to a functor from Λ-Inj to $\mathrm{Add}(\nu_A(M))$. Moreover, by (2), we have $F(\nu_A(M)) \simeq D(\Lambda)$ as Λ-modules. As F commutes with direct sums, it also restricts to a functor from $\mathrm{Add}(\nu_A(M))$ onto Λ-Inj. Recall that (F, S_e) is an adjoint pair and F is fully faithful. Thus $F : \mathrm{Add}(\nu_A(M)) \to \Lambda$-Inj is an equivalence with the quasi-inverse S_e.

(4) Since $S_e : \mathcal{Y} \to \mathcal{X}$ is an equivalence, it induces an equivalence $\mathscr{K}(\mathcal{Y}) \simeq \mathscr{K}(\mathcal{X})$. Then (4) follows from the definition of $\mathscr{K}_{\text{e-ac}}(\mathcal{Y})$. □

Remark 3.2. (1) Let $\mathbf{P}_1(\Lambda e)$ (respectively, $\mathbf{I}_1(\Lambda e)$) be the full subcategory of Λ-Mod consisting of all modules Y such that there is an exact sequence $E_1 \to E_0 \to Y \to 0$ (respectively, $0 \to Y \to E_0 \to E_1$) in Λ-Mod with $E_0, E_1 \in \mathrm{Add}(\Lambda e)$. By [3, Lemma 3.1], there are additive equivalences $F : A\text{-Mod} \to \mathbf{P}_1(\Lambda e)$ and $G : A\text{-Mod} \to \mathbf{I}_1(\Lambda e)$. Thus, by Lemma 3.1(2), the functor ν_Λ restricts to an additive equivalence $\mathbf{I}_1(\Lambda e) \to \mathbf{P}_1(\Lambda e)$, with the inverse ν_Λ^-.

(2) Each A-module admits a minimal right and left $\mathrm{Add}(M)$-approximations. This property will be used in Section 3.4.

Indeed, since Λ is an Artin algebra, each Λ-module admits a projective cover. For any A-module X, let $f_X : Q \to G(X)$ be a projective cover of ${}_{\Lambda}G(X)$. Since G is fully faithful and restricts to an equivalence from $\mathrm{Add}(M)$ to Λ-Proj by Lemma 3.1(3), there is a homomorphism $r_X : M_X \to X$ of A-modules such that $M_X \in \mathrm{Add}(M)$ and $G(r_X) = f_X$. Then r_X is a minimal right $\mathrm{Add}(M)$-approximation of X.

Let $\overline{F} = F \circ \nu_A : A\text{-Mod} \to \Lambda\text{-Mod}$ be the composition of ν_A with F. Then $\overline{F}$ is fully faithful. By Lemma 3.1(3), $\overline{F}$ restricts to an equivalence from $\mathrm{Add}(M)$ to Λ-Inj. Thus, if $g_X : \overline{F}(X) \to I$ is an injective envelop of $\overline{F}(X)$, then there is a homomorphism $\ell_X : X \to M^X$ in A-Mod with $M^X \in \mathrm{Add}(M)$ such that $g_X = \overline{F}(\ell_X)$. One can check that ℓ_X is a minimal left $\mathrm{Add}(M)$-approximation of X.

Finally, we recall some notation and facts on homotopy and derived categories of rings.

Let $\mathscr{K}(\Lambda)_P$ (respectively, $\mathscr{K}(\Lambda)_I$) be the smallest full triangulated subcategory of $\mathscr{K}(\Lambda)$ which

(i) contains all bounded above (respectively, bounded below) complexes of projective (respectively, injective) Λ-modules, and

(ii) is closed under arbitrary direct sums (respectively, direct products).

Note that $\mathscr{K}(\Lambda)_P \subseteq \mathscr{K}(\Lambda\text{-Proj})$ and $\mathscr{K}(\Lambda)_I \subseteq \mathscr{K}(\Lambda\text{-Inj})$. Moreover, the compositions $\mathscr{K}(\Lambda)_P \hookrightarrow \mathscr{K}(\Lambda) \to \mathscr{D}(\Lambda)$ and $\mathscr{K}(\Lambda)_I \hookrightarrow \mathscr{K}(\Lambda) \to \mathscr{D}(\Lambda)$ are equivalences. This means that, for any $Y^\bullet \in \mathscr{D}(\Lambda)$, there exists a complex ${}_pY^\bullet \in \mathscr{K}(\Lambda)_P$ together with a quasi-isomorphism ${}_pY^\bullet \to Y^\bullet$. Dually, there is a complex ${}_iY^\bullet \in \mathscr{K}(\Lambda)_I$ together with a quasi-isomorphism $Y^\bullet \to {}_iY^\bullet$. Moreover, ${}_pY^\bullet$ and ${}_iY^\bullet$ are unique up to isomorphism in $\mathscr{K}(\Lambda)$. As usual, ${}_pY^\bullet$ and ${}_iY^\bullet$ are called the *projective resolution* and *injective coresolution* of $Y^\bullet$ in $\mathscr{K}(\Lambda)$, respectively. For example, if Y is a Λ-module, then ${}_pY$ is a deleted projective resolution of Y.

Let $Q : \mathscr{K}(\Lambda) \to \mathscr{D}(\Lambda)$ be the localization functor. Then Q has a left adjoint Q_λ and a right adjoint Q_ρ defined by

$$Q_\lambda = {}_p(-) : \ \mathscr{D}(\Lambda) \to \mathscr{K}(\Lambda) \quad \text{and} \quad Q_\rho = {}_i(-) : \ \mathscr{D}(\Lambda) \to \mathscr{K}(\Lambda).$$

It is known that $\mathrm{Im}(Q_\lambda) = \mathscr{K}(\Lambda)_P$ and $\mathrm{Im}(Q_\rho) = \mathscr{K}(\Lambda)_I$.

Lemma 3.3. [31] *$(\mathscr{K}(\Lambda)_P, \mathscr{K}_{\mathrm{ac}}(\Lambda\text{-Proj}))$ and $(\mathscr{K}_{\mathrm{ac}}(\Lambda\text{-Inj}), \mathscr{K}(\Lambda)_I)$ are hereditary torsion pairs in $\mathscr{K}(\Lambda\text{-Proj})$ and $\mathscr{K}(\Lambda\text{-Inj})$, respectively. In other words, there are half recollements of triangulated categories:*

$$\begin{tikzcd}
\mathscr{K}_{\mathrm{ac}}(\Lambda\text{-Proj}) \arrow[r, "I"'] & \mathscr{K}(\Lambda\text{-Proj}) \arrow[l, bend right, "I_\lambda"'] \arrow[r, "Q"'] & \mathscr{D}(\Lambda) \arrow[l, bend right, "Q_\lambda"'] & \textit{and} & \mathscr{D}(\Lambda) \arrow[r, "Q_\rho"'] & \mathscr{K}(\Lambda\text{-Inj}) \arrow[l, bend right, "Q"'] \arrow[r, "J_\rho"'] & \mathscr{K}_{\mathrm{ac}}(\Lambda\text{-Inj}) \arrow[l, bend right, "J"']
\end{tikzcd}$$

where I and J are the inclusion functors, and Q denotes restrictions of the localization functor.

3.2 Construction of triangle endofunctors of stable module categories

In this subsection we introduce two pairs of triangle endofunctors of stable module categories for self-injective algebras, and then discuss two relevant thick subcategories of module categories.

Let $\varepsilon : Q_\lambda \circ Q \to \mathrm{Id}$ and $\eta : \mathrm{Id} \to I \circ I_\lambda$ be the counit and unit adjunctions with respect to the adjoint pairs (Q_λ, Q) and (I_λ, I) in Lemma 3.3, respectively, where Id denotes the identity functor.

Lemma 3.4. (1) *There is a half recollement of triangulated categories:*

$$\begin{tikzcd}
\mathscr{K}_{\mathrm{ac}}(\Lambda\text{-Proj}) \arrow[r, "I"'] & \mathscr{K}_{\text{e-ac}}(\Lambda\text{-Proj}) \arrow[l, bend right, "I_\lambda"'] \arrow[r, "Q_\lambda \circ Q"'] & \mathscr{K}_{\text{e-ac}}(\Lambda\text{-Proj}) \cap \mathscr{K}(\Lambda)_P \arrow[l, bend right, "\mathrm{inc}"']
\end{tikzcd} ,$$

where inc *is the inclusion functor.*

(2) *There is an exact sequence of triangle endofunctors of* $\mathscr{K}_{\text{e-ac}}(\Lambda\text{-Proj})$*:*

$$Q_\lambda \circ Q \xrightarrow{\varepsilon} \mathrm{Id} \xrightarrow{\eta} I \circ I_\lambda \longrightarrow Q_\lambda \circ Q[1]$$

Proof. Since $\mathscr{K}_{\text{ac}}(\Lambda\text{-Proj}) \subseteq \mathscr{K}_{\text{e-ac}}(\Lambda\text{-Proj})$ and $(\mathscr{K}(\Lambda)_P, \mathscr{K}_{\text{ac}}(\Lambda\text{-Proj}))$ is a hereditary torsion pair in $\mathscr{K}(\Lambda\text{-Proj})$, the pair $(\mathscr{K}_{\text{e-ac}}(\Lambda\text{-Proj}) \cap \mathscr{K}(\Lambda)_P, \mathscr{K}_{\text{ac}}(\Lambda\text{-Proj}))$ is a hereditary torsion pair in $\mathscr{K}_{\text{e-ac}}(\Lambda\text{-Proj})$. By Lemma 3.3, each object $Y^\bullet \in \mathscr{K}(\Lambda\text{-Proj})$ is endowed with a canonical triangle

$$Q_\lambda \circ Q(Y^\bullet) \xrightarrow{\varepsilon_{Y^\bullet}} Y^\bullet \xrightarrow{\eta_{Y^\bullet}} I \circ I_\lambda(Y^\bullet) \longrightarrow Q_\lambda \circ Q(Y^\bullet)[1].$$

If $Y^\bullet \in \mathscr{K}_{\text{e-ac}}(\Lambda\text{-Proj})$, then it follows from $I \circ I_\lambda(Y^\bullet) \in \mathscr{K}_{\text{ac}}(\Lambda\text{-Proj})$ and the exactness of S_e that $Q_\lambda \circ Q(Y^\bullet) \in \mathscr{K}_{\text{e-ac}}(\Lambda\text{-Proj}) \cap \mathscr{K}(\Lambda)_P$. Thus (1) and (2) hold. □

Now, let $\ell : \mathscr{K}(A) \to \mathscr{K}(A\text{-Inj})$ be a *left adjoint* of the inclusion $\mathscr{K}(A\text{-Inj}) \hookrightarrow \mathscr{K}(A)$. By [15, Corollary 1.3], ℓ is given by taking the total complexes of Cartan-Eilenberg injective coresolutions of complexes over A-Mod. Moreover, it has the following property.

Lemma 3.5. *The functor* ℓ *restricts to a triangle functor* $\ell_{ac} : \mathscr{K}_{\text{ac}}(A) \to \mathscr{K}_{\text{ac}}(A\text{-Inj})$ *and the composition*

$$\ell_M : \ \mathscr{K}_{\text{ac}}(\mathrm{Add}(M)) \hookrightarrow \mathscr{K}_{\text{ac}}(A) \xrightarrow{\ell_{ac}} \mathscr{K}_{\text{ac}}(A\text{-Inj}) = \mathscr{K}_{\text{ac}}(A\text{-Proj}).$$

is a left adjoint of the inclusion $\mathscr{K}_{\text{ac}}(A\text{-Proj}) \hookrightarrow \mathscr{K}_{\text{ac}}(\mathrm{Add}(M))$.

Proof. Clearly, $(\mathrm{Ker}(\ell), \mathscr{K}(A\text{-Inj}))$ and $(\mathscr{K}_{\text{ac}}(A), \mathscr{K}(A)_I)$ are hereditary torsion pairs in $\mathscr{K}(A)$. Since $\mathscr{K}(A)_I \subseteq \mathscr{K}(A\text{-Inj})$, we have $\mathrm{Ker}(\ell) \subseteq \mathscr{K}_{\text{ac}}(A)$. This implies that $(\mathrm{Ker}(\ell), \mathscr{K}_{\text{ac}}(A\text{-Inj}))$ is a hereditary torsion pair in $\mathscr{K}_{\text{ac}}(A)$. Consequently, ℓ restricts to a functor $\mathscr{K}_{\text{ac}}(A) \to \mathscr{K}_{\text{ac}}(A\text{-Inj})$. Since ${}_AM$ is a generator, $\mathscr{K}_{\text{ac}}(A\text{-Proj}) \subseteq \mathscr{K}_{\text{ac}}(\mathrm{Add}(M))$. Now the second part of Lemma 3.5 holds because ℓ is a left adjoint of the inclusion $\mathscr{K}(A\text{-Inj}) \to \mathscr{K}(A)$. □

Let X be an A-module. We denote by $\pi_X^\bullet : P_X^\bullet \to X$ and $\lambda_X^\bullet : X \to I_X^\bullet$ a minimal projective resolution and injective coresolution of X, respectively. Then there exists a triangle equivalence

$$S : \ A\text{-}\underline{\mathrm{Mod}} \xrightarrow{\simeq} \mathscr{K}_{\text{ac}}(A\text{-Proj}), \quad X \mapsto S(X) := \mathrm{Con}(\pi_X^\bullet \lambda_X^\bullet).$$

This functor S is called the *stabilization functor* of A (for example, see [31]), while $S(X)$ is a complete projective resolution of X. A quasi-inverse of S is given by taking the 0-th cocycle of complexes:

$$Z^0 : \mathscr{K}_{\text{ac}}(A\text{-Proj}) \longrightarrow A\text{-}\underline{\mathrm{Mod}}, \quad I^\bullet \mapsto Z^0(I^\bullet) := \mathrm{Ker}(I^0 \to I^1).$$

Further, let $\ell_M : \mathscr{K}_{\text{ac}}(\mathrm{Add}(M)) \to \mathscr{K}_{\text{ac}}(A\text{-Proj})$ be the triangle functor defined in Lemma 3.5, and let $\mu : \mathscr{K}_{\text{e-ac}}(\mathrm{Add}(\Lambda e)) \to \mathscr{K}_{\text{e-ac}}(\Lambda\text{-Proj})$ be the inclusion induced from $\mathrm{Add}(\Lambda e) \subseteq \Lambda\text{-Proj}$.

By Lemmas 3.4 and 3.5, we can define a pair of triangle endofunctors of A-$\underline{\mathrm{Mod}}$ by

$$\Phi = Z^0 \circ \ell_M \circ S_e \circ (Q_\lambda \circ Q) \circ \mu \circ G \circ S : \ A\text{-}\underline{\mathrm{Mod}} \longrightarrow A\text{-}\underline{\mathrm{Mod}},$$

$$\Psi = Z^0 \circ \ell_M \circ S_e \circ (I \circ I_\lambda) \circ \mu \circ G \circ S : \ A\text{-}\underline{\mathrm{Mod}} \longrightarrow A\text{-}\underline{\mathrm{Mod}}.$$

They are illustrated by the following diagram

$$\begin{tikzcd}
A\text{-}\underline{\mathrm{Mod}} \arrow[r, "S", "\simeq"'] \arrow[d, bend right=50, "\Phi"'] \arrow[d, "\mathrm{Id}"] \arrow[d, bend left=50, "\Psi"] & \mathscr{K}_{\text{ac}}(A\text{-Proj}) \arrow[r, "G", "\simeq"'] & \mathscr{K}_{\text{e-ac}}(\mathrm{Add}(\Lambda e)) \arrow[r, hook, "\mu"] & \mathscr{K}_{\text{e-ac}}(\Lambda\text{-Proj}) \arrow[d, bend right=50, "Q_\lambda \circ Q"'] \arrow[d, "\mathrm{Id}"] \arrow[d, bend left=50, "I \circ I_\lambda"] \\
A\text{-}\underline{\mathrm{Mod}} & \mathscr{K}_{\text{ac}}(A\text{-Proj}) \arrow[l, "Z^0"', "\simeq"] & \mathscr{K}_{\text{ac}}(\mathrm{Add}(M)) \arrow[l, "\ell_M"'] & \mathscr{K}_{\text{e-ac}}(\Lambda\text{-Proj}) \arrow[l, "S_e"', "\simeq"]
\end{tikzcd}$$

where the equivalences of G and S_e follow from Lemma 3.1(3)-(4). There is the natural isomorphism

$$Z^0 \circ \ell_M \circ S_e \circ \mathrm{Id} \circ \mu \circ G \circ S \simeq \mathrm{Id}: \ A\text{-}\underline{\mathrm{Mod}} \longrightarrow A\text{-}\underline{\mathrm{Mod}}.$$

This follows from the equivalence $S_e: \ \mathscr{K}_{\text{e-ac}}(\mathrm{Add}(\Lambda e)) \xrightarrow{\simeq} \mathscr{K}_{\mathrm{ac}}(A\text{-Proj})$ given by Lemma 3.1(3)-(4), together with the fact that the restriction of ℓ_M to $\mathscr{K}_{\mathrm{ac}}(A\text{-Proj})$ is isomorphic to Id.

Dually, we can construct another pair (Φ', Ψ') of endofunctors of A-$\underline{\mathrm{Mod}}$. Here, we only list some key points of this construction, and omit the details.

By Lemma 3.3, there is a half recollement of triangulated categories:

$$\begin{tikzcd}
\mathscr{K}_{\text{e-ac}}(\Lambda\text{-Inj}) \cap \mathscr{K}(\Lambda)_I \arrow[r, "\mathrm{inc}"] & \mathscr{K}_{\text{e-ac}}(\Lambda\text{-Inj}) \arrow[l, bend right, "Q_\rho \circ Q"'] \arrow[r, "J_\rho"] & \mathscr{K}_{\mathrm{ac}}(\Lambda\text{-Inj}) \arrow[l, bend right, "J"'] .
\end{tikzcd}$$

This implies an exact sequence of endofunctors of $\mathscr{K}_{\text{e-ac}}(\Lambda\text{-Inj})$:

$$J \circ J_\rho \xrightarrow{\varepsilon'} \mathrm{Id} \xrightarrow{\eta'} Q_\rho \circ Q \longrightarrow J \circ J_\rho[1]: \ \mathscr{K}_{\text{e-ac}}(\Lambda\text{-Inj}) \longrightarrow \mathscr{K}_{\text{e-ac}}(\Lambda\text{-Inj})$$

where ε' and η' are counit and unit adjunctions of the adjoint pairs (J, J_ρ) and (Q, Q_ρ), respectively (see Lemma 3.3). Let $r: \mathscr{K}(A) \to \mathscr{K}(A\text{-Proj})$ be a right adjoint of the inclusion $\mathscr{K}(A\text{-Proj}) \to \mathscr{K}(A)$. It is given by taking the total direct product complexes of Cartan-Eilenberg projective resolutions of complexes over A-Mod, due to [15, Corollary 1.3]. Moreover, it restricts to a triangle functor $r_{ac}: \mathscr{K}_{\mathrm{ac}}(A) \to \mathscr{K}_{\mathrm{ac}}(A\text{-Proj})$. Let

$$r_M: \ \mathscr{K}_{\mathrm{ac}}(\mathrm{Add}(\nu_A(M))) \hookrightarrow \mathscr{K}_{\mathrm{ac}}(A) \xrightarrow{r_{ac}} \mathscr{K}_{\mathrm{ac}}(A\text{-Proj})$$

be the composition of the inclusion with r_{ac}, and let $\mu': \mathscr{K}_{\text{e-ac}}(\mathrm{Add}(\Lambda e)) \hookrightarrow \mathscr{K}_{\text{e-ac}}(\Lambda\text{-Inj})$ be the inclusion induced from $\mathrm{Add}(\Lambda e) \subseteq \Lambda$-Inj. Now we define the two endofunctors Φ' and Ψ' of A-$\underline{\mathrm{Mod}}$ by

$$\Phi' = Z^0 \circ r_M \circ S_e \circ (Q_\rho \circ Q) \circ \mu' \circ F \circ S: \ A\text{-}\underline{\mathrm{Mod}} \longrightarrow A\text{-}\underline{\mathrm{Mod}},$$

$$\Psi' = Z^0 \circ r_M \circ S_e \circ (J \circ J_\rho) \circ \mu' \circ F \circ S: \ A\text{-}\underline{\mathrm{Mod}} \longrightarrow A\text{-}\underline{\mathrm{Mod}},$$

which are illustrated by the diagram

$$\begin{tikzcd}
A\text{-}\underline{\mathrm{Mod}} \arrow[r, "S", "\simeq"'] \arrow[d, bend right=50, "\Psi'"'] \arrow[d, "\mathrm{Id}"'] \arrow[d, bend left=50, "\Phi'"] & \mathscr{K}_{\mathrm{ac}}(A\text{-Proj}) \arrow[r, "F", "\simeq"'] & \mathscr{K}_{\text{e-ac}}(\mathrm{Add}(\Lambda e)) \arrow[r, hook, "\mu'"] & \mathscr{K}_{\text{e-ac}}(\Lambda\text{-Inj}) \arrow[d, bend right=50, "J \circ J_\rho"'] \arrow[d, "\mathrm{Id}"'] \arrow[d, bend left=50, "Q_\rho \circ Q"] \\
A\text{-}\underline{\mathrm{Mod}} & \mathscr{K}_{\mathrm{ac}}(A\text{-Proj}) \arrow[l, "Z^0"', "\simeq"] & \mathscr{K}_{\mathrm{ac}}(\mathrm{Add}(\nu_A(M))) \arrow[l, "r_M"'] & \mathscr{K}_{\text{e-ac}}(\Lambda\text{-Inj}). \arrow[l, "S_e"', "\simeq"]
\end{tikzcd}$$

Similarly, there is a natural isomorphism of functors:

$$Z^0 \circ r_M \circ S_e \circ \mathrm{Id} \circ \mu' \circ F \circ S \simeq \mathrm{Id}: \ A\text{-}\underline{\mathrm{Mod}} \longrightarrow A\text{-}\underline{\mathrm{Mod}}.$$

By construction, Φ and Ψ commute with direct sums, while Φ' and Ψ' commute with direct products. Clearly, if $M = A$, then $\Phi = \Phi' = 0$ and $\Psi = \mathrm{Id} = \Psi'$. In general, we have the result.

Proposition 3.6. *There exist exact sequences of triangle endofunctors of A-$\underline{\mathrm{Mod}}$:*

$$\Phi \xrightarrow{\widetilde{\varepsilon}} \mathrm{Id} \xrightarrow{\widetilde{\eta}} \Psi \longrightarrow \Phi[1] \quad \textit{and} \quad \Psi' \xrightarrow{\widetilde{\varepsilon'}} \mathrm{Id} \xrightarrow{\widetilde{\eta'}} \Phi' \longrightarrow \Psi'[1],$$

where $\widetilde{\varepsilon}$, $\widetilde{\eta}$, $\widetilde{\varepsilon'}$ and $\widetilde{\eta'}$ are induced from ε, η, ε' and η', respectively.

Next, we investigate two full subcategories of A-Mod associated with the functors Ψ and Ψ'.

$$\mathscr{S} := \{X \in A\text{-Mod} \mid \Psi(\underline{X}) = 0\} \quad \text{and} \quad \mathscr{T} := \{X \in A\text{-Mod} \mid \Psi'(\underline{X}) = 0\}.$$

Since Ψ is a triangle functor commutating with direct sums, $\mathscr{S}$ is a thick subcategory of A-Mod containing all projective modules and being closed under direct sums. Dually, $\mathscr{T}$ is a thick subcategory of A-Mod containing all projective modules and being closed under direct products. Also, $\underline{\mathscr{S}}$ and $\underline{\mathscr{T}}$are full triangulated subcategories of A-$\underline{\text{Mod}}$.

Recall that an A-module X is said to be Ω-*periodic* if $\underline{\Omega_A^n(X)} \simeq \underline{X}$ in A-$\underline{\text{Mod}}$ for a positive integer n.

Lemma 3.7. (1) *If $X \in \text{Add}({}_AM)$ is Ω-periodic, then $X \in \mathscr{S}$ and $\nu_A(X) \in \mathscr{T}$.*

(2) *If Λ has finite global dimension, then both Φ and Φ' are isomorphic to the identity functor, and therefore $\mathscr{S} = \mathscr{T} = A$-*Mod.

Proof. For an A-module X, we write $G \circ S(X) = Y^\bullet := (Y^n, d^n)_{n\in\mathbb{Z}}$ and $F \circ S(\nu_A(X)) = Z^\bullet := (Z^n, h^n)_{n\in\mathbb{Z}}$. Then $Y^\bullet, Z^\bullet \in \mathscr{K}_{\text{e-ac}}(\text{Add}(\Lambda e))$. Since the Λ-module Λe is projective-injective, both Y^n and Z^n are projective-injective. As G is left exact, $\text{Ker}(d^n) = G(\Omega_A^{-n}(X))$. Dually, $\text{Coker}(h^n) = F \circ \Omega_A^{-(n+2)}(\nu_A(X))$ since F is right exact.

(1) Suppose $X \in \text{Add}(M)$ and $X \simeq \Omega_A^s(X)$ in A-$\underline{\text{Mod}}$ for some $s \geq 1$. Then $X \simeq \Omega_A^{-sm}(X)$ in A-$\underline{\text{Mod}}$ for any $m \in \mathbb{N}$ and $G(X) \in \Lambda$-Proj by Lemma 3.1(3). It follows that $\text{Ker}(d^{sm}) = G(\Omega_A^{-sm}(X)) \simeq G(X) = \text{Ker}(d^0) \in \Lambda$-Proj. Here, the isomorphism is regarded in the stable category. Let $\tau_{\leq sm}(Y^\bullet)$ be the subcomplex of $Y^\bullet$:

$$\cdots \longrightarrow Y^{-1} \longrightarrow \cdots \longrightarrow Y^{sm-2} \longrightarrow Y^{sm-1} \longrightarrow \text{Ker}(d^{sm}) \longrightarrow 0.$$

Then $\tau_{\leq sm}(Y^\bullet) \in \mathscr{K}^-(\Lambda\text{-Proj})$. Since $Y^\bullet$ is isomorphic in $\mathscr{K}(\Lambda\text{-Proj})$ to the homotopy colimit of the sequence of inclusions: $\tau_{\leq 0}(Y^\bullet) \hookrightarrow \tau_{\leq s}(Y^\bullet) \hookrightarrow \tau_{\leq 2s}(Y^\bullet) \hookrightarrow \cdots \hookrightarrow \tau_{\leq sm}(Y^\bullet) \hookrightarrow \cdots$, there is a canonical triangle in $\mathscr{K}(\Lambda\text{-Proj})$:

$$\bigoplus_{m=0}^{\infty} \tau_{\leq sm}(Y^\bullet) \longrightarrow \bigoplus_{m=0}^{\infty} \tau_{\leq sm}(Y^\bullet) \longrightarrow Y^\bullet \longrightarrow \bigoplus_{m=0}^{\infty} \tau_{\leq sm}(Y^\bullet).$$

As $\mathscr{K}^-(\Lambda\text{-Proj}) \subseteq \mathscr{K}(\Lambda)_P$ and $\mathscr{K}(\Lambda)_P$ is closed under direct sums in $\mathscr{K}(\Lambda\text{-Proj})$, we get $Y^\bullet \in \mathscr{K}(\Lambda)_P$. Since $(\mathscr{K}(\Lambda)_P, \mathscr{K}_{\text{ac}}(\Lambda\text{-Proj}))$ is a hereditary torsion pair in $\mathscr{K}(\Lambda\text{-Proj})$, we obtain $Q_\lambda \circ Q(Y^\bullet) \simeq Y^\bullet$ and $I \circ I_\lambda(Y^\bullet) = 0$. Thus $\Psi(\underline{X}) = 0$ and $X \in \mathscr{S}$.

Set $U := \nu_A(X)$. It follows from $X \in \text{Add}(M)$ and Lemma 3.1(3) that $F(U) \in \Lambda$-Inj. Since $X \simeq \Omega_A^{-s}(X)$ for some $s \geq 1$ and ν_A is an auto-equivalence on A-Mod, we have $U \simeq \Omega_A^{sm}(U)$ for all $m \in \mathbb{N}$. Thus $\text{Coker}(h^{-2-sm}) = F \circ \Omega_A^{sm}(U) \simeq F(U) = \text{Coker}(h^{-2})$ and the truncated quotient complex of $Z^\bullet$

$$\tau_{\geq -1-sm}(Z^\bullet): \quad \cdots \longrightarrow 0 \longrightarrow \text{Coker}(h^{-2-sm}) \longrightarrow Z^{-sm} \longrightarrow Z^{-sm+1} \longrightarrow \cdots$$

is in $\mathscr{K}^+(\Lambda\text{-Inj})$. Dually, $Z^\bullet$ is isomorphic in $\mathscr{K}(\Lambda\text{-Inj})$ to a homotopy limit of the canonical surjections:

$$\cdots \twoheadrightarrow \tau_{\geq -1-sm}(Z^\bullet) \twoheadrightarrow \cdots \twoheadrightarrow \tau_{\geq -1-2s}(Z^\bullet) \twoheadrightarrow \tau_{\geq -1-s}(Z^\bullet) \twoheadrightarrow \tau_{\geq -1}(Z^\bullet).$$

We can show $Z^\bullet \in \mathscr{K}(\Lambda)_I$. Hence $Q_\rho \circ Q(Z^\bullet) \simeq Z^\bullet$ and $J \circ J_\rho(Z^\bullet) = 0$. Thus $\Psi'(\underline{U}) = 0$ and $U \in \mathscr{T}$.

(2) Suppose Λ has finite global dimension. Then $\mathscr{K}_{\text{ac}}(\Lambda\text{-Proj}) = 0 = \mathscr{K}_{\text{ac}}(\Lambda\text{-Inj})$, $\mathscr{K}(\Lambda\text{-Proj}) = \mathscr{K}(\Lambda)_P$ and $\mathscr{K}(\Lambda\text{-Inj}) = \mathscr{K}(\Lambda)_I$. Hence $Q_\lambda \circ Q$ and $Q_\rho \circ Q$ are naturally isomorphic to Id. This implies that Φ and Φ' are naturally isomorphic to Id, while Ψ and Ψ' are zero functor. □

Proposition 3.8. *Let X be an A-module. Then the following hold.*

(1) *If $X \in M^{\perp>0}$, then $X \in \mathscr{S}$ if and only if for any A-module Y,* $\text{Hom}_{\mathscr{K}(A)}(\text{Con}(\mu_X^\bullet), \text{Con}(\pi_Y^\bullet)) = 0$, *where $\mu_X^\bullet : M_X^\bullet \to X$ is an* Add(M)*-resolution of X.*

(2) *If $X \in {}^{\perp>0}M$, then $\nu_A(X) \in \mathscr{T}$ if and only if for any A-module Y,* $\text{Hom}_{\mathscr{K}(A)}(\text{Con}(\lambda_Y^\bullet), \text{Con}(\sigma_X^\bullet)) = 0$ *where $\sigma_X^\bullet : X \to M_X^\bullet$ is an* Add(M)*-coresolution of X.*

Proof. (1) Let $\pi^\bullet := \pi^\bullet_X$, $\lambda^\bullet := \lambda^\bullet_X$ and $\mu^\bullet := \mu^\bullet_X$. Then $S(X) = \mathrm{Con}(\pi^\bullet\lambda^\bullet)$. Since $\mu^\bullet$ is an $\mathrm{Add}(M)$-resolution of X (see Definition 2.5), $G(\mu^\bullet) : G(M^\bullet_X) \to G(X)$ is a quasi-isomorphism; equivalently, $\mathrm{Con}(G(\mu^\bullet))$ is exact. Moreover, $\mathrm{Con}(G(\mu^\bullet)) = G(\mathrm{Con}(\mu^\bullet))$. As ${}_AM$ is a generator, $\mathrm{Con}(\mu^\bullet)$ is exact. It follows from $P^\bullet_X \in \mathscr{K}^-(A\text{-Proj})$ that the identity map of X can be lifted to a unique morphism $h^\bullet : P^\bullet_X \to M^\bullet_X$ in $\mathscr{K}(A)$ such that $h^\bullet\mu^\bullet = \pi^\bullet$. This implies $G(h^\bullet)G(\mu^\bullet\lambda^\bullet) = G(\pi^\bullet\lambda^\bullet)$. By the octahedral axiom for triangulated categories, there exists a triangle in $\mathscr{K}_{\text{e-ac}}(\Lambda\text{-Proj})$:

$$(*): \;\; \mathrm{Con}(G(h^\bullet)) \longrightarrow \mu\circ G\circ S(X) \longrightarrow \mathrm{Con}(G(\mu^\bullet\lambda^\bullet)) \longrightarrow \mathrm{Con}(G(h^\bullet))[1]$$

where $\mu\circ G\circ S(X) = \mathrm{Con}(G(\pi^\bullet\lambda^\bullet))$. Since X lies in $M^{\perp>0}$, the morphism $G(\lambda^\bullet) : G(X) \to G(I^\bullet_X)$ is a quasi-isomorphism. Thus $G(\mu^\bullet\lambda^\bullet) = G(\mu^\bullet)G(\lambda^\bullet)$ is a composition of two quasi-isomorphisms. This means $\mathrm{Con}(G(\mu^\bullet\lambda^\bullet)) \in \mathscr{K}_{\mathrm{ac}}(\Lambda\text{-Proj})$. It follows from $\mathrm{Con}(G(h^\bullet)) \in \mathscr{K}^-(\Lambda\text{-Proj}) \subseteq \mathscr{K}(\Lambda)_P$ that

$$Q_\lambda\circ Q(\mu\circ G\circ S(X)) \simeq \mathrm{Con}(G(h^\bullet)) \;\text{ and }\; I\circ I_\lambda(\mu\circ G\circ S(X)) \simeq \mathrm{Con}(G(\mu^\bullet\lambda^\bullet)).$$

Since the composition of G with S_e is isomorphic to the identity functor, we apply S_e to the triangle $(*)$ and get another triangle $\mathrm{Con}(h^\bullet) \to S(X) \to \mathrm{Con}(\mu^\bullet\lambda^\bullet) \to \mathrm{Con}(h^\bullet)[1]$ in $\mathscr{K}_{\mathrm{ac}}(\mathrm{Add}(M))$. Further, by applying the functor ℓ_M (see Lemma 3.5) to this triangle, we are led to a triangle in $\mathscr{K}_{\mathrm{ac}}(A\text{-Proj})$:

$$\ell_M(\mathrm{Con}(h^\bullet)) \longrightarrow \ell_M\circ S(X) \longrightarrow \ell_M(\mathrm{Con}(\mu^\bullet\lambda^\bullet)) \longrightarrow \ell_M(\mathrm{Con}(h^\bullet))[1].$$

Clearly, $\ell_M\circ S(X) = S(X)$ since $S(X) \in \mathscr{K}_{\mathrm{ac}}(A\text{-Proj})$. From $Z^0\circ S(X) \simeq X$, we obtain a triangle in $A\text{-}\underline{\mathrm{Mod}}$:

$$Z^0\circ\ell_M(\mathrm{Con}(h^\bullet)) \longrightarrow X \longrightarrow Z^0\circ\ell_M(\mathrm{Con}(\mu^\bullet\lambda^\bullet)) \longrightarrow Z^0\circ\ell_M(\mathrm{Con}(h^\bullet))[1].$$

Thus $\Phi(X) = Z^0\circ\ell_M(\mathrm{Con}(h^\bullet))$ and $\Psi(X) = Z^0\circ\ell_M(\mathrm{Con}(\mu^\bullet\lambda^\bullet))$. Note that $Z^0 : \mathscr{K}_{\mathrm{ac}}(A\text{-Proj}) \to A\text{-}\underline{\mathrm{Mod}}$ is an equivalence. Hence $\Psi(X) = 0$ (equivalently, $X \in \mathscr{S}$) if and only if $\ell_M(\mathrm{Con}(\mu^\bullet\lambda^\bullet)) = 0$ if and only if $\mathrm{Hom}_{\mathscr{K}(A)}(\mathrm{Con}(\mu^\bullet\lambda^\bullet), Q^\bullet) = 0$ for any $Q^\bullet \in \mathscr{K}_{\mathrm{ac}}(A\text{-Proj})$.

Since $\mathrm{Con}(\lambda^\bullet) \in \mathscr{K}^+_{\mathrm{ac}}(A)$ and $\mathscr{K}^+_{\mathrm{ac}}(A) \subseteq {}^\perp\mathscr{K}(A\text{-Inj})$, there holds $\mathrm{Hom}_{\mathscr{K}(A)}(\mathrm{Con}(\lambda^\bullet), Q^\bullet[n]) = 0$ for any $n \in \mathbb{Z}$. Applying $\mathrm{Hom}_{\mathscr{K}(A)}(-, Q^\bullet[n])$ to the triangle

$$\mathrm{Con}(\mu^\bullet) \to \mathrm{Con}(\mu^\bullet\lambda^\bullet) \to \mathrm{Con}(\lambda^\bullet) \to \mathrm{Con}(\mu^\bullet)[1]$$

in $\mathscr{K}_{\mathrm{ac}}(A)$, we get $\mathrm{Hom}_{\mathscr{K}(A)}(\mathrm{Con}(\mu^\bullet\lambda^\bullet), Q^\bullet) \simeq \mathrm{Hom}_{\mathscr{K}(A)}(\mathrm{Con}(\mu^\bullet), Q^\bullet)$. Now, let Y be the kernel of the 0-th differential of $Q^\bullet$. Taking the canonical truncation on $Q^\bullet$ at degree 0, we obtain a subcomplex $\tau_{\leq 0}Q^\bullet$ of $Q^\bullet$, which is acyclic and isomorphic to $\mathrm{Con}(\pi^\bullet_Y)$ in $\mathscr{K}(A)$. Since the inclusion $\tau_{\leq 0}Q^\bullet \subseteq Q^\bullet$ induces $\mathrm{Hom}_{\mathscr{K}(A)}(\mathrm{Con}(\mu^\bullet), \tau_{\leq 0}Q^\bullet) \simeq \mathrm{Hom}_{\mathscr{K}(A)}(\mathrm{Con}(\mu^\bullet), Q^\bullet)$, it follows that $\mathrm{Hom}_{\mathscr{K}(A)}(\mathrm{Con}(\mu^\bullet), \mathrm{Con}(\pi^\bullet_Y)) \simeq \mathrm{Hom}_{\mathscr{K}(A)}(\mathrm{Con}(\mu^\bullet\lambda^\bullet), Q^\bullet)$. So $X \in \mathscr{S}$ if and only if $\mathrm{Hom}_{\mathscr{K}(A)}(\mathrm{Con}(\mu^\bullet), \mathrm{Con}(\pi^\bullet_Y)) = 0$. This shows (1).

(2) Let $\nu := \nu_A$, $Z := \nu(X)$ and $\sigma^\bullet := \sigma^\bullet_X$. Then $F\circ S(Z) = F\big(\mathrm{Con}(\nu(\pi^\bullet\lambda^\bullet))\big) = \mathrm{Con}\big(F(\nu\pi^\bullet)F(\nu\lambda^\bullet)\big)$. Since $X \in {}^{\perp>0}M$ and ν is an auto-equivalence of A-Mod, there hold $D\mathrm{Tor}^A_i(\Lambda e, Z) \simeq \mathrm{Ext}^i_A(Z, D(\Lambda e)) \simeq \mathrm{Ext}^i_A(Z, \nu M) \simeq \mathrm{Ext}^i_A(X, M) = 0$ for any $i \geq 1$. Hence $F(\nu\pi^\bullet)$ is a quasi-isomorphism, that is, $\mathrm{Con}(F(\nu\pi^\bullet))$ is exact. Thus $F\circ S(Z) \simeq \mathrm{Con}(F(\nu\lambda^\bullet))$ in $\mathscr{D}(\Lambda)$, where $\nu\lambda^\bullet : Z \to \nu(I^\bullet_X)$ is an injective coresolution of Z. To calculate $Q_\rho\circ Q\circ\mu'(\mathrm{Con}(F(\nu\lambda^\bullet)))$ in $\mathscr{K}_{\text{e-ac}}(\Lambda\text{-Inj})$, we show that $F(\nu\sigma^\bullet) : F(Z) \to F\circ\nu(M^\bullet_X)$ is an injective coresolution of $F(Z)$, and then replace $F(Z)$ by its deleted injective coresolution.

In fact, by the proof of Lemma 3.1(3), the adjoint pair (F, S_e) induces an equivalence $\mathrm{Add}(\nu M) \xrightarrow{\simeq} \Lambda\text{-Inj}$. This implies $F(\nu(M^\bullet_X)) \in \mathscr{K}^+(\Lambda\text{-Inj})$. It remains to show that $F(\nu\sigma^\bullet)$ is a quasi-isomorphism. Since $D : \Lambda\text{-Mod} \to \Lambda^{\mathrm{op}}\text{-Mod}$ is exact and detects zero objects, we only need to show that $DF(\nu\sigma^\bullet)$ is a quasi-isomorphism. However, by Lemma 3.1(2), $DF(\nu\sigma^\bullet) \simeq \mathrm{Hom}_A(\sigma^\bullet, M)$ which is a quasi-isomorphism by the construction of σ. Thus $F(\nu\sigma^\bullet)$ is an injective coresolution of $F(Z)$.

Now, let $f^\bullet : M_X^\bullet \to I_X^\bullet$ be a chain map which lifts the identity map of X. Then there is a canonical triangle

$$\mathrm{Con}(F\circ\nu(\pi^\bullet\sigma^\bullet)) \longrightarrow \mu'\circ F\circ S(Z) \longrightarrow \mathrm{Con}(F\circ\nu(f^\bullet)) \longrightarrow \mathrm{Con}(F\circ\nu(\pi^\bullet\sigma^\bullet))[1]$$

in $\mathscr{K}_{\text{e-ac}}(\Lambda\text{-Inj})$, where the first term lies in $\mathscr{K}_{\mathrm{ac}}(\Lambda\text{-Inj})$ and the third one lies in $\mathscr{K}^+(\Lambda\text{-Inj})$. Thus

$$J\circ J_\rho\circ\mu'\circ F\circ S(Z)\simeq \mathrm{Con}(F\circ\nu(f^\bullet)) \quad\text{and}\quad Q_\rho\circ Q\circ\mu'\circ F\circ S(Z)\simeq \mathrm{Con}(F\circ\nu(\pi^\bullet\sigma^\bullet)).$$

Since the composition of F with S_e is isomorphic to the identity functor, it follows that

$$\Psi'(Z)=Z^0\circ r_M(\mathrm{Con}(\nu(\pi^\bullet\sigma^\bullet))) \quad\text{and}\quad \Phi'(Z)=Z^0\circ r_M(\mathrm{Con}(\nu f^\bullet)).$$

Dually, by the equivalence of Z^0 and the inclusion $\mathscr{K}^+_{\mathrm{ac}}(A)\subseteq\mathscr{K}(A\text{-Proj})^{\perp}$, we can show that $Z\in\mathscr{T}$ if and only if $\mathrm{Hom}_{\mathscr{K}(A)}(\mathrm{Con}(\lambda_U^\bullet),\mathrm{Con}(\nu\sigma^\bullet))=0$ for any A-module U. Since ν is an auto-equivalence of A-Mod, we have $\mathrm{Hom}_{\mathscr{K}(A)}(\mathrm{Con}(\lambda_U^\bullet),\mathrm{Con}(\nu\sigma^\bullet))\simeq\mathrm{Hom}_{\mathscr{K}(A)}(\mathrm{Con}(\lambda_U^\bullet),\nu(\mathrm{Con}(\sigma^\bullet)))\simeq\mathrm{Hom}_{\mathscr{K}(A)}(\mathrm{Con}(\lambda_Y^\bullet),\mathrm{Con}(\sigma^\bullet))$ with $Y:=\nu^-(U)$. Thus (2) holds. □

The following result is a consequence of Lemma 3.7(1) and Proposition 3.8.

Corollary 3.9. *If ${}_AM$ is self-orthogonal or Ω-periodic, then $M\in\mathscr{S}$ and $\nu_A(M)\in\mathscr{T}$.*

3.3 Recollements of stable module categories induced by self-orthogonal modules

In this subsection, we apply the triangle endofunctors in Section 3.2 to construct a recollement of the stable module category of a self-injective algebra from a generator with conditions that are satisfied for a self-orthogonal or an Ω-periodic generator.

As is known, $A\text{-}\underline{\mathrm{Mod}}$ is compactly generated and the inclusion of A-mod $\to$ A-Mod induces a triangle equivalence from $A\text{-}\underline{\mathrm{mod}}$ to the full subcategory of $A\text{-}\underline{\mathrm{Mod}}$ consisting of all compact objects. Since ${}_AM$ is finitely generated, M is compact in $A\text{-}\underline{\mathrm{Mod}}$. For a set Δ of integers, let

$$\underline{M}^{\perp\Delta}:=\{X\in A\text{-}\underline{\mathrm{Mod}}\mid \underline{\mathrm{Hom}}_A(M,X[n])=0\ \text{ for any }\ n\in\Delta\},$$

$${}^{\perp\Delta}\underline{M}:=\{X\in A\text{-}\underline{\mathrm{Mod}}\mid \underline{\mathrm{Hom}}_A(X,M[n])=0\ \text{ for any }\ n\in\Delta\}.$$

For simplicity, we write $\underline{M}^{\perp}$ and ${}^{\perp}\underline{M}$ for $\underline{M}^{\perp\mathbb{Z}}$ and ${}^{\perp\mathbb{Z}}\underline{M}$, respectively. Then $\underline{M}^{\perp}$ is a full triangulated subcategory of $A\text{-}\underline{\mathrm{Mod}}$ closed under direct sums and direct products.

Lemma 3.10. *Let X be an A-module.*

(1) *$X\in\underline{M}^{\perp}$ if and only if $G\circ S(X)$ (equivalently, $F\circ S(X)$) is an exact complex.*

(2) *If $X\in\underline{M}^{\perp}$, then $\Phi(X)=0$ and $\Phi'(X)=0$.*

Proof. By Lemma 3.1(1), F and G are naturally isomorphic on A-Proj. Thus $G\circ S(X)\simeq F\circ S(X)$ as complexes. So we show (1) for $G\circ S(X)$. Let $0\to X_1\to P\to X_0\to 0$ be an exact sequence of A-modules with $P\in A$-Proj. Then $0\to G(X_1)\to G(P)\to G(X_0)\to 0$ is exact if and only if $\underline{\mathrm{Hom}}_A(M,X_0)=0$. This implies (1). Moreover, (2) follows from (1) and the definitions of Φ and Φ'. □

We need the following result which is concluded from the Auslander-Reiten formula (see [6]).

Lemma 3.11. *If $X\in A$-mod, then there is a natural isomorphism*

$$D\underline{\mathrm{Hom}}_A(X,-)\simeq\underline{\mathrm{Hom}}_A(-,\nu_A(X)[-1]):\ A\text{-}\underline{\mathrm{Mod}}\longrightarrow \underline{\mathrm{End}}_A(X)^{\mathrm{op}}\text{-Mod}.$$

Proof. Since X is finitely generated, it follows from [6, Proposition 2.2] that

$$D\underline{\mathrm{Hom}}_A(X,-)\simeq \mathrm{Ext}^1_A(-,D\mathrm{Tr}(X)):\ A\text{-}\underline{\mathrm{Mod}}\longrightarrow \underline{\mathrm{End}}_A(X)^{\mathrm{op}}\text{-Mod}$$

where $D\mathrm{Tr}$ is the Auslander-Reiten translation on A-$\underline{\mathrm{mod}}$. Since A is self-injective, $D\mathrm{Tr}\simeq \Omega^2_A\circ\nu_A$ as functors on A-$\underline{\mathrm{mod}}$. This implies that

$$\begin{aligned}\mathrm{Ext}^1_A(-,D\mathrm{Tr}(X)) &\simeq \mathrm{Ext}^1_A(-,\Omega^2_A\circ\nu_A(X))\simeq \underline{\mathrm{Hom}}_A(\Omega_A(-),\Omega^2_A\circ\nu_A(X))\\ &\simeq \underline{\mathrm{Hom}}_A(-,\Omega_A\circ\nu_A(X))=\underline{\mathrm{Hom}}_A(-,\nu_A(X)[-1])\end{aligned}$$

on A-$\underline{\mathrm{Mod}}$. Thus Lemma 3.11 holds. □

To construct recollements of A-$\underline{\mathrm{Mod}}$ from compact objects, we establish a result on torsion pairs.

Lemma 3.12. (1) $\big(\mathrm{Loc}(\underline{M}),\underline{M}^{\perp}\big)$ *and* $\big(\underline{M}^{\perp},\mathrm{Coloc}(\underline{\nu_A(M)})\big)$ *are hereditary torsion pairs in* A-$\underline{\mathrm{Mod}}$.

(2) *If* ${}_AM\in\mathscr{S}$*, then* $\underline{\mathscr{S}}=\mathrm{Im}(\Phi)=\mathrm{Loc}(\underline{M})$ *and* $\mathrm{Im}(\Psi)=\underline{M}^{\perp}$.

(3) *If* $\nu_A(M)\in\mathscr{T}$*, then* $\underline{\mathscr{T}}=\mathrm{Im}(\Phi')=\mathrm{Coloc}(\underline{\nu_A(M)})$ *and* $\mathrm{Im}(\Psi')=\underline{M}^{\perp}$.

Proof. (1) Since ${}_AM$ is finitely generated, we have $\underline{M}^{\perp}={}^{\perp}\underline{\nu_A(M)}\subseteq A$-$\underline{\mathrm{Mod}}$ by Lemma 3.11. Clearly, A-$\underline{\mathrm{Mod}}$ is compactly generated by simple modules because each A-module has a radical series of length less than or equal the Loewy length of A. Thus (1) holds by Proposition 2.3.

(2) Let $\mathscr{X}:=\mathrm{Loc}(\underline{M})$ and $\mathscr{Y}:=\underline{M}^{\perp}$. Suppose $M\in\mathscr{S}$. Then $\mathscr{X}\subseteq\underline{\mathscr{S}}$ because $\underline{\mathscr{S}}$ is a full triangulated subcategory of A-$\underline{\mathrm{Mod}}$ closed under direct sums. By Lemma 3.10(2), $\mathscr{Y}\subseteq\mathrm{Ker}(\Phi)$.

Let $N\in A$-Mod. By (1), up to isomorphism, there is a unique triangle $X_N\to N\to Y^N\to X_N[1]$ in A-$\underline{\mathrm{Mod}}$ such that $X_N\in\mathscr{X}$ and $Y^N\in\mathscr{Y}$. This yields $\Psi(X_N)=0=\Phi(Y^N)$. Now, we apply Proposition 3.6 to the triangle and obtain the commutative diagram in A-$\underline{\mathrm{Mod}}$:

$$\begin{tikzcd}
\Phi(X_N) \arrow[r, "\simeq"] \arrow[d, "\widetilde{\varepsilon}_{X_N}"', "\simeq"] & \Phi(N) \arrow[r] \arrow[d, "\widetilde{\varepsilon}_N"'] & 0 \arrow[r] \arrow[d] & \Phi(X_N)[1] \arrow[d, "\simeq"] \\
X_N \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, "\widetilde{\eta}_N"'] & Y^N \arrow[r] \arrow[d, "\widetilde{\eta}_{Y^N}"', "\simeq"] & X_N[1] \arrow[d] \\
0 \arrow[r] \arrow[d] & \Psi(N) \arrow[r, "\simeq"] \arrow[d] & \Psi(Y^N) \arrow[r] \arrow[d] & \Psi(X_N)[1] \arrow[d] \\
\Phi(X_N)[1] \arrow[r, "\simeq"] & \Phi(N)[1] \arrow[r] & 0 \arrow[r] & \Phi(X_N)[2]
\end{tikzcd}$$

where all rows and columns are triangles. Thus $\Phi(N)\simeq X_N\in\mathscr{X}$ and $\Psi(N)\simeq Y^N\in\mathscr{Y}$. This implies $\mathrm{Im}(\Phi)\subseteq\mathscr{X}$ and $\mathrm{Im}(\Psi)\subseteq\mathscr{Y}$. Note that if $N\in\mathscr{X}$, then $N\simeq X_N$, and therefore $N\simeq\Phi(N)\in\mathrm{Im}(\Phi)$. Similarly, if $N\in\mathscr{Y}$, then $N\simeq Y^N$, and therefore $N\simeq\Psi(N)\in\mathrm{Im}(\Psi)$. Thus $\mathrm{Im}(\Phi)=\mathscr{X}$ and $\mathrm{Im}(\Psi)=\mathscr{Y}$. Since $\mathscr{X}\subseteq\underline{\mathscr{S}}\subseteq\mathrm{Im}(\Phi)$, we have $\underline{\mathscr{S}}=\mathrm{Im}(\Phi)$.

(3) Similarly, we can show (3) by Lemma 3.10(2) and by the pair $\big(\underline{M}^{\perp},\mathrm{Coloc}(\underline{\nu_A(M)})\big)$ in (1). □

A consequence of Lemma 3.12 and Corollary 3.9 is the following.

Corollary 3.13. *Suppose that* ${}_AM$ *is self-orthogonal or* Ω*-periodic. Then* $\mathscr{S}$ *is the smallest thick subcategory of* A-Mod *containing* M *and being closed under direct sums, while* $\mathscr{T}$ *is the smallest thick subcategory of* A-Mod *containing* $\nu_A(M)$ *and being closed under direct products.*

Now, we are in position to prove the following main result of this section.

Theorem 3.14. *Suppose A is a self-injective Artin algebra and ${}_AM$ is generator in A*-mod. *If ${}_AM \in \mathscr{S}$ and $\nu_A(M) \in \mathscr{T}$, then there exists a recollement of triangulated categories:*

$$\begin{tikzcd}
\underline{M}^{\perp} \arrow[r, "\mathrm{inc}"] & A\text{-}\underline{\mathrm{Mod}} \arrow[l, bend right, "\widetilde{\Psi}"'] \arrow[l, bend left, "\widetilde{\Psi'}"] \arrow[r, "\widetilde{\Phi}"] & \mathrm{Loc}(\underline{M}) \arrow[l, bend right, "\mathrm{inc}"'] \arrow[l, bend left, "\Phi''"]
\end{tikzcd}$$

such that

$$\Phi = \mathrm{inc} \circ \widetilde{\Phi}, \quad \Psi = \mathrm{inc} \circ \widetilde{\Psi}, \quad \Psi' = \mathrm{inc} \circ \widetilde{\Psi'} \quad \text{and} \quad \Phi'' = \Phi' \circ \mathrm{inc}.$$

Moreover, the functor Φ'' restricts to a triangle equivalence $\mathrm{Loc}(\underline{M}) \xrightarrow{\simeq} \mathrm{Coloc}(\underline{\nu_A(M)})$.

Proof. Suppose ${}_AM \in \mathscr{S}$. By Lemma 3.12(2), we have the factorisation of Φ and Ψ:

$$\Phi: \ A\text{-}\underline{\mathrm{Mod}} \xrightarrow{\widetilde{\Phi}} \mathrm{Loc}(\underline{M}) \hookrightarrow A\text{-}\underline{\mathrm{Mod}} \quad \text{and} \quad \Psi: \ A\text{-}\underline{\mathrm{Mod}} \xrightarrow{\widetilde{\Psi}} \underline{M}^{\perp} \hookrightarrow A\text{-}\underline{\mathrm{Mod}}.$$

By Lemma 3.12(1), $\big(\mathrm{Loc}(\underline{M}), \underline{M}^{\perp}\big)$ is a hereditary torsion pair in A-$\underline{\mathrm{Mod}}$. Then the proof of Lemma 3.12(2) together with [13, Chapter I, Prop. 2.3] implies that $\widetilde{\Phi}$ is a right adjoint of the inclusion $\mathrm{Loc}(\underline{M}) \to A$-$\underline{\mathrm{Mod}}$ and that $\widetilde{\Psi}$ is a left adjoint of the inclusion $\underline{M}^{\perp} \to A$-$\underline{\mathrm{Mod}}$.

Suppose $\nu_A(M) \in \mathscr{T}$. Dually, from the torsion pair $\big(\underline{M}^{\perp}, \mathrm{Coloc}(\underline{\nu_A(M)})\big)$ in Lemma 3.12(1) and from Lemma 3.12(3), we obtain the factorisations of Φ' and Ψ':

$$\Phi': \ A\text{-}\underline{\mathrm{Mod}} \xrightarrow{\widetilde{\Phi'}} \mathrm{Coloc}(\underline{\nu_A(M)}) \hookrightarrow A\text{-}\underline{\mathrm{Mod}} \quad \text{and} \quad \Psi': \ A\text{-}\underline{\mathrm{Mod}} \xrightarrow{\widetilde{\Psi'}} \underline{M}^{\perp} \hookrightarrow A\text{-}\underline{\mathrm{Mod}}$$

such that $\widetilde{\Phi'}$ is a left adjoint of the inclusion $\mathrm{Coloc}(\underline{\nu_A(M)}) \to A$-$\underline{\mathrm{Mod}}$ and $\widetilde{\Psi'}$ is a right adjoint of the inclusion $\underline{M}^{\perp} \to A$-$\underline{\mathrm{Mod}}$. Recall that there is a correspondence between TTF (torsion-torsionfree) triples and recollements of triangulated categories (see, for example, [13, Chapter I. 2] or [17, Section 2.3]). Thus Theorem 3.14 follows from Lemma 3.12(1) and [17, Lemma 2.6]. □

Combining Theorem 3.14 with Corollary 3.9, we obtain the corollary.

Corollary 3.15. *Let A be a self-injective algebra. If ${}_AM$ is self-orthogonal or Ω-periodic, then there exists a recollement of triangulated categories:*

$$\begin{tikzcd}
\underline{M}^{\perp} \arrow[r, "\mathrm{inc}"] & A\text{-}\underline{\mathrm{Mod}} \arrow[l, bend right, "\widetilde{\Psi}"'] \arrow[l, bend left, "\widetilde{\Psi'}"] \arrow[r, "\widetilde{\Phi}"] & \mathrm{Loc}(\underline{M}), \arrow[l, bend right, "\mathrm{inc}"'] \arrow[l, bend left, "\Phi''"]
\end{tikzcd}$$

in which the functors are the same as the ones in Theorem 3.14.

Later we will see that the above recollement restricts to the one of relative stable categories.

3.4 Categories of Gorenstein-projective modules

In this subsection we describe the category of Gorenstein-projective modules over the endomorphism algebra of a self-orthogonal generator.

Recall that the *M-stable category* A-Mod$/[M]$ of A-Mod is defined to be the quotient category of A-Mod modulo $\mathrm{Add}(M)$. From now on, we set

$$\mathscr{D} := A\text{-Mod}/[M] \quad \text{and} \quad \underline{\mathrm{Hom}}_M(X_1, X_2) := \mathrm{Hom}_{\mathscr{D}}(X_1, X_2)$$

for $X_1, X_2 \in A$-Mod. We say that X_1 and X_2 are *M-stably isomorphic* if they are isomorphic in $\mathscr{D}$. For a full subcategory $\mathscr{U}$ of A-Mod, we denote by $\mathscr{U}/[M]$ the full subcategory of $\mathscr{D}$ consisting of all objects X which are M-stably isomorphic to objects of $\mathscr{U}$.

As M is finitely generated and $\mathrm{Add}(M) = \mathrm{Prod}(M)$, we know that $\mathrm{Add}(M)$ is a functorially finite subcategory in A-Mod. According to [13, Chap. II. 1], $\mathscr{D}$ is a *pretriangulated category* mainly consisting of the following data:

(1) An adjoint pair (Ω_M^-, Ω_M) of additive endofunctors $\Omega_M^-, \Omega_M : \mathscr{D} \to \mathscr{D}$.

For an A-module X, $\Omega_M^-(X)$ is defined to be the cokernel of a minimal left $\mathrm{Add}(M)$-approximation $\ell_X : X \to M^X$ of X, while $\Omega_M(X)$ is defined to be the kernel of a minimal right $\mathrm{Add}(M)$-approximation $r_X : M_X \to X$ of X. The existence of minimal approximations follows from Remark 3.2. Moreover, $\Omega_M^-(X)$ and $\Omega_M(X)$ are unique up to isomorphism.

(2) A collection of *right triangles* (up to isomorphism) of the form $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Omega_M^-(X)$ arising from an exact commutative diagram in A-Mod:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X \arrow[r, "\ell_X"] & M^X \arrow[r] & \Omega_M^-(X) \arrow[r] & 0
\end{tikzcd}$$

where $\mathrm{Hom}_A(f, M) : \mathrm{Hom}_A(Y, M) \to \mathrm{Hom}_A(X, M)$ is surjective.

(3) A collection of *left triangles* (up to isomorphism) of the form $\Omega_M(Z) \to X \to Y \to Z$ which is defined in a dual way as in (2).

(4) Right triangles (respectively, left triangles) satisfy all the axioms for triangulated categories, except that Ω_M^- (respectively, Ω_M) is not necessarily an equivalence.

In the following, Ω_M is called the *M-syzygy* functor on $\mathscr{D}$. For an A-module X, we put $\Omega_M^0(X) := X$, and $\Omega_M^n(X) := \Omega_M(\Omega_M^{n-1}(X))$ for $n \geq 1$. The functor Ω_M^n is called the *n-th M-syzygy* functor on $\mathscr{D}$. Dually, Ω_M^- is called the *M-cosyzygy* functor and the *n-th M-cosyzygy* functor Ω_M^{-n} is defined dually. If M is a self-orthogonal generator for A-mod, then $(\Omega_M^{-n}, \Omega_M^n) : \mathscr{D} \to \mathscr{D}$ is an adjoint pair for $n \geq 2$, see [16, Lemmas 3.2 and 3.3].

The following simple observation will be used in later discussions.

Lemma 3.16. (1) *Let X and Y be A-modules. Then X and Y are M-stably isomorphic if and only if there are $M_1, M_2 \in \mathrm{Add}(M)$ such that $X \oplus M_1 \simeq Y \oplus M_2$ in A-Mod.*

(2) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ be a triangle in A-$\underline{\mathrm{Mod}}$. If $Z \in {}^{\perp 1}M$, then there is a right triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \to \Omega_M^-(X)$ in $\mathscr{D}$.*

Proof. (1) This can be proved similarly as done in A-$\underline{\mathrm{Mod}}$.

(2) Up to isomorphism of triangles in A-$\underline{\mathrm{Mod}}$, we can assume that the sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is exact. If $Z \in {}^{\perp 1}M$, then $\mathrm{Hom}_A(f, M)$ is surjective. This implies (2). □

By Lemma 3.16, if $\mathscr{U}$ is a full subcategory of A-Mod containing $\mathrm{Add}(M)$ and being closed under both direct summands and finite direct sums, then $\mathscr{U}/[M]$ is closed under direct summands in $\mathscr{D}$.

Let

$$\mathscr{G} := M^{\perp>0} \cap {}^{\perp>0}M, \quad \mathscr{C} := \mathscr{G}/[M], \quad \mathscr{G}_0 := \mathscr{G} \cap A\text{-mod}.$$

Then $\mathscr{G}$ is always closed under filtered colimits in A-Mod, that is, colimits of filtered diagrams $I \to A$-Mod with I an essentially small, filtered category.

In fact, since M is finitely generated, the functor $\mathrm{Ext}_A^n(M, -)$ commutes with filtered colimits for each $n \geq 0$. Then $M^{\perp>0} \subseteq A$-Mod is closed under filtered colimits. Thanks to $M \simeq D(DM)$ as A-modules,

M is pure-injective. It follows that $\mathrm{Ext}^n_A(-,M)$ sends filtered colimits to filtered limits, and therefore ${}^{\perp>0}M \subseteq A$-Mod is also closed under filtered colimits. Thus $\mathscr{G} \subseteq A$-Mod is closed under filtered colimits.

Let $\varinjlim \mathscr{G}_0$ denote the full subcategory of A-Mod consisting of all filtered colimits of modules in $\mathscr{G}_0$. Then $\varinjlim \mathscr{G}_0 \subseteq \mathscr{G}$.

In the rest of this subsection, we assume that $_AM$ is *self-orthogonal*. The following result is a unbounded version of [16, Lemma 3.5].

Lemma 3.17. (1) *The category $\mathscr{G}$ (respectively, $\mathscr{G}_0$) is a Frobenius category with the shift functor given by Ω_M^-. The full subcategory of projective-injective objects of $\mathscr{G}$ (respectively, $\mathscr{G}_0$) equals* $\mathrm{Add}(M)$ *(respectively,* $\mathrm{add}(M)$*). In particular, $\mathscr{G}/[M]$ is a triangulated category.*

(2) *The functor $G: A$-Mod $\to \Lambda$-Mod restricts to equivalences of Frobenius categories:*

$$\mathscr{G} \xrightarrow{\simeq} \Lambda\text{-GProj} \quad \text{and} \quad \mathscr{G}_0 \xrightarrow{\simeq} \Lambda\text{-Gproj}.$$

In particular, there are equivalences of triangulated categories:

$$\mathscr{C} \xrightarrow{\simeq} \Lambda\text{-}\underline{\text{GProj}} \quad \text{and} \quad \mathscr{G}_0/[M] \xrightarrow{\simeq} \Lambda\text{-}\underline{\text{Gproj}}.$$

Proof. For $n \geq 1$, we define

$$\mathscr{G}_n := \{X \in A\text{-mod} \mid \mathrm{Ext}^i_A(M, \Omega_M^{-j}(X)) = 0 = \mathrm{Ext}^i_A(\Omega_M^j(X), M) \ \text{ for any } \ j \geq 0 \ \text{ and } \ 1 \leq i \leq n\}$$

(see also [16, Definition 3.4]). Then $\mathscr{G}_n$ is a Frobenius category and there is a chain of full subcategories of A-mod: $\mathscr{G}_1 \supseteq \mathscr{G}_2 \supseteq \cdots \supseteq \mathscr{G}_n \supseteq \mathscr{G}_{n+1} \supseteq \cdots$. Moreover, G restricts to an equivalence of Frobenius categories: $\mathscr{G}_n \xrightarrow{\simeq} \Lambda$-Gproj for $n \geq 1$. It follows that the inclusions $\mathscr{G}_{n+1} \to \mathscr{G}_n$ are equivalences of additive categories. Since $\mathscr{G}_n$ are closed under isomorphisms in A-mod, we have $\mathscr{G}_n = \mathscr{G}_{n+1}$ for all $n \geq 1$. As M is self-orthogonal, $\mathscr{G}_0$ is closed under taking Ω_M and Ω_M^- in A-mod. Consequently, $\mathscr{G}_0 \subseteq \bigcap_{n\geq 1}^{\infty} \mathscr{G}_n \subseteq \mathscr{G}_0$. This implies $\mathscr{G}_0 = \mathscr{G}_n$, and thus Lemma 3.17 holds for $\mathscr{G}_0$.

Note that G commutes with direct sums and restricts to an equivalence $\mathrm{Add}(M) \xrightarrow{\simeq} \Lambda$-Proj. As in the proof of [16, Lemma 3.5], we can show that Lemma 3.17 holds first for

$$\mathbf{G}_n := \{X \in A\text{-Mod} \mid \mathrm{Ext}^i_A(M, \Omega_M^{-j}(X)) = 0 = \mathrm{Ext}^i_A(\Omega_M^j(X), M) \ \text{ for } \ j \geq 0 \ \text{ and } \ 1 \leq i \leq n\}$$

and then for $\mathscr{G}$. $\square$

Consequently, we get a characterization of virtually Gorenstein algebras in terms of compact objects.

Proposition 3.18. *Let A be a self-injective algebra and M a self-orthogonal generator for* A-mod. *Then* $\mathrm{End}_A(M)$ *is virtually Gorenstein if and only if $\mathscr{G} = \varinjlim \mathscr{G}_0$ if and only if each compact object of $\mathscr{C}$ is M-stably isomorphic to a finitely generated A-module.*

Proof. By [14, Theorem 5] and [11, Theorem 8.2], $\Lambda := \mathrm{End}_A(M)$ is virtually Gorenstein if and only if Λ-GProj $= \varinjlim(\Lambda$-Gproj$)$ if and only if each compact object of Λ-$\underline{\text{GProj}}$ is isomorphic to an object of Λ-$\underline{\text{Gproj}}$. Note that $G: A$-Mod $\to \Lambda$-Mod is fully faithful and commutes with filtered colimits. Moreover, since $_AM$ is a generator, two A-modules X and Y are isomorphic if and only if $G(X)$ and $G(Y)$ are isomorphic. Now the equivalences in Proposition 3.18 follow from Lemmas 3.17 and 3.16. $\square$

A complex $P^\bullet \in \mathscr{K}(\Lambda$-Proj$)$ is called *totally acyclic* if both $P^\bullet$ and $\mathrm{Hom}^\bullet_\Lambda(P^\bullet, \Lambda)$ are acyclic. Let $\mathscr{K}_{\mathrm{tac}}(\Lambda$-Proj$)$ be the full subcategory of $\mathscr{K}(\Lambda$-Proj$)$ consisting of totally acyclic complexes. It is known that there is a triangle equivalence $\Lambda\text{-}\underline{\mathrm{GProj}} \xrightarrow{\simeq} \mathscr{K}_{\mathrm{tac}}(\Lambda\text{-Proj})$ which sends a Gorenstein-projective Λ-module to its complete projective resolution. Composing this equivalence with the equivalence $\mathscr{C} \xrightarrow{\simeq} \Lambda\text{-}\underline{\mathrm{GProj}}$ in Lemma 3.17(2), we obtain a triangle equivalence

$$\mathrm{Hom}_A(M, M^{\bullet}_{-}) : \ \mathscr{C} \xrightarrow{\simeq} \mathscr{K}_{\mathrm{tac}}(\Lambda\text{-Proj}), \ X \mapsto \mathrm{Hom}_A(M, M^{\bullet}_X)$$

where $M^{\bullet}_X := (M^n_X)_{n\in\mathbb{Z}} \in \mathscr{K}_{\mathrm{ac}}(\mathrm{Add}(M))$ is defined by concatenating an $\mathrm{Add}(M)$-resolution $\cdots \to M^{-2}_X \to M^{-1}_X \to M^0_X \to X \to 0$ of X with an $\mathrm{Add}(M)$-coresolution $0 \to X \to M^1_X \to M^2_X \to M^3_X \to \cdots$ of X at the position X. The complex $M^{\bullet}_X$ is called a *complete* $\mathrm{Add}(M)$*-resolution* of X. For each $n \in \mathbb{Z}$, there is an additive functor

$$\Psi_n := H^n\big(\underline{\mathrm{Hom}}_A(M, M^{\bullet}_{-})\big) : \ \mathscr{C} \longrightarrow \Gamma\text{-Mod}, \ X \mapsto H^n\big(\underline{\mathrm{Hom}}_A(M, M^{\bullet}_X)\big)$$

which is *homological* in the sense that applying Ψ_n to every triangle $X_1 \to X_2 \to X_3 \to \Omega^-_M(X_1)$ in $\mathscr{C}$ yields an exact sequence $\cdots \to \Psi_n(X_1) \to \Psi_n(X_2) \to \Psi_n(X_3) \to \Psi_{n+1}(X_1) \to \cdots$.

Remark 3.19. By Lemma 3.17(1) and Remark 3.2(2), we have the following observation.

For each $X \in \mathscr{G}$, there are isomorphisms $X \simeq \Omega^-_M\Omega_M(X) \oplus M_0 \simeq \Omega_M\Omega^-_M(X) \oplus M_1$ in A-Mod with $M_0, M_1 \in \mathrm{Add}(M)$ such that $\Omega^-_M\Omega_M(X)$ and $\Omega_M\Omega^-_M(X)$ have no nonzero direct summands in $\mathrm{Add}(M)$. Thus, if $X \in \mathscr{G}$ has no nonzero direct summands in $\mathrm{Add}(M)$, then we can choose the n-th differential $d^n_X : M^n_X \to M^{n+1}_X$ such that the induced map $M^n_X \to \mathrm{Im}(d^n_X)$ is a minimal right $\mathrm{Add}(M)$-approximation of $\mathrm{Im}(d^n_X)$.

Lemma 3.20. (1) *The functor* $F \circ \nu_A : A\text{-Mod} \to \Lambda\text{-Mod}$ *restricts to an equivalence:* $\mathscr{G} \xrightarrow{\simeq} \Lambda\text{-GInj}$ *of Frobenius categories.*

(2) $(\mathscr{G}, \mathscr{G}^{\perp 1})$ *and* $({}^{\perp 1}\mathscr{G}, \mathscr{G})$ *are cotorsion pairs in* A-Mod *such that* $\mathscr{G} \cap \mathscr{G}^{\perp 1} = \mathrm{Add}(M) = {}^{\perp 1}\mathscr{G} \cap \mathscr{G}$. *In particular,* $\mathscr{G}$ *is functorially finite in* A-Mod.

(3) *The inclusion* $\mathscr{C} \hookrightarrow \mathscr{D}$ *admits a left adjoint* $T : \mathscr{D} \to \mathscr{C}$ *which is induced from minimal left* $\mathscr{G}$*-approximations of modules. Moreover,* T *preserves compact objects, sends right triangles to triangles and commutes with the functor* Ω^-_M.

(4) *The set* $\{T(X) \mid X \in A\text{-mod}\}$ *is a compact generating set of* $\mathscr{C}$.

Proof. (1) Since the adjoint pair $(\nu_\Lambda, \nu^-_\Lambda)$ induces quasi-inverse equivalences $\Lambda\text{-GProj} \xrightarrow{\simeq} \Lambda\text{-GInj}$, (1) follows from Lemmas 3.17(2) and 3.1(2) together with Remark 3.2(1).

(2) Since $\Omega^-_M(\mathscr{G}) \subseteq \mathscr{G}$ by Lemma 3.17(1), the sequence $0 \to X \xrightarrow{\ell_X} M^X \to \Omega^-_M(X) \to 0$ splits whenever $X \in \mathscr{G} \cap \mathscr{G}^{\perp 1}$. This implies $\mathscr{G} \cap \mathscr{G}^{\perp 1} \subseteq \mathrm{Add}(M)$. Clearly, $\mathrm{Add}(M) \subseteq \mathscr{G} \cap \mathscr{G}^{\perp 1}$. Thus $\mathscr{G} \cap \mathscr{G}^{\perp 1} = \mathrm{Add}(M)$. Dually, ${}^{\perp 1}\mathscr{G} \cap \mathscr{G} = \mathrm{Add}(M)$.

Next, we show that $(\mathscr{G}, \mathscr{G}^{\perp 1})$ is a cotorsion pair in A-Mod. Since A-Mod is an abelian category with enough projectives and injectives, it suffices to show that, for any A-module X, there is an exact sequence $0 \to X_2 \to X_1 \to X \to 0$ in A-Mod such that $X_1 \in \mathscr{G}$ and $X_2 \in \mathscr{G}^{\perp 1}$ (for example, see [13, Lemma V. 3.3]).

Since each Λ-module admits a minimal right Λ-GProj-approximation, we take a minimal right Λ-GProj-approximation of $G(X)$, say $g : Y \to G(X)$. By Lemma 3.17(2), we can assume $Y = G(X_1)$ for some $X_1 \in \mathscr{G}$. As G is fully faithful and M is a generator, there is a surjective map $f : X_1 \to X$ of A-modules such that $g = G(f)$ and f is a minimal right $\mathscr{G}$-approximation of X. Since $\mathscr{G}$ is closed under extensions in A-Mod, it follows from Wakamatsu's Lemma that $\mathrm{Ker}(f) \in \mathscr{G}^{\perp 1}$. Hence the sequence $0 \to \mathrm{Ker}(f) \to X_1 \to X \to 0$ is a desired one.

Similarly, to show that $({}^{\perp 1}\mathscr{G}, \mathscr{G})$ is a cotorsion pair in A-Mod, it is enough to prove that there is an exact sequence $0 \to X \to T^X \to C^X \to 0$ of A-modules such that $T^X \in \mathscr{G}$ and $C^X \in {}^{\perp 1}\mathscr{G}$.

Let $\overline{F} = F \circ \nu_A : A\text{-Mod} \to \Lambda\text{-Mod}$. Then $\overline{F}$ is fully faithful and restricts to an equivalence $\mathscr{G} \xrightarrow{\simeq} \Lambda\text{-GInj}$ by (1). Since each Λ-module admits a minimal left Λ-GInj-approximation, there is a map $h^X : X \to$

T^X in A-Mod with $T^X \in \mathscr{G}$ such that $\overline{F}(h^X)$ is a minimal left Λ-GInj-approximation of $\overline{F}(X)$. This implies that h^X is a minimal left $\mathscr{G}$-approximation of X. Moreover, h^X is injective. This is due to $D(A) \in \mathscr{G}$ and $\mathrm{Coker}(h^X) \in {}^{\perp 1}\mathscr{G}$ by the dual of Wakamatsu's Lemma. Thus the sequence $0 \to X \to T^X \to \mathrm{Coker}(h^X) \to 0$ is the one as desired.

(3) Since $({}^{\perp 1}\mathscr{G}, \mathscr{G})$ is a cotorsion pair with ${}^{\perp 1}\mathscr{G} \cap \mathscr{G} = \mathrm{Add}(M)$ by (2), the functor $T : \mathscr{D} \to \mathscr{C}$ exists (see the comments after Definition 2.4). As the inclusion functor $\mathscr{C} \to \mathscr{D}$ preserves direct sums, we know that T preserves compact objects. It is known that $\mathscr{C}$ is a triangulated category, $\mathscr{D}$ is a pretriangulated category and $\mathscr{C}$ is a pretriangulated subcategory of $\mathscr{D}$. Thus the last assertion follows from the dual version of [13, Proposition II. 2.6].

(4) We show that A-mod$/[M]$ is a compact generating set of $\mathscr{D}$. Clearly, if $X \in A$-mod, then X is compact in A-Mod, and also compact in $\mathscr{D}$ by Lemma 2.1(2). Let $Y \in A$-Mod such that $\underline{\mathrm{Hom}}_M(X,Y) = 0$ for all $X \in A$-mod. Then each map from X to Y factorizes through an object of $\mathrm{Add}(M)$, and particularly, through M_Y via the minimal right $\mathrm{Add}(M)$-approximation $r_Y : M_Y \to Y$ of Y. Recall that, for an Artin algebra B, an exact sequence $0 \to X_1 \to X_2 \xrightarrow{g} X_3 \to 0$ of B-modules is called *pure-exact* if $\mathrm{Hom}_B(Z,g)$ is surjective for any $Z \in B$-mod; equivalently, $0 \to L \otimes_B X_1 \to L \otimes_B X_2 \to L \otimes_B X_3 \to 0$ is exact for any $L \in B^{\mathrm{op}}$-Mod. This implies that $0 \to \Omega_M(Y) \to M_Y \to Y \to 0$ is pure-exact in A-Mod. Note that there is a natural isomorphism $\mathrm{Hom}_\Lambda(U, G(-)) \simeq \mathrm{Hom}_A(M \otimes_\Lambda U, -)$ for any Λ-module U and that ${}_AM \otimes_\Lambda U \in A$-mod if $U \in \Lambda$-mod. Hence $0 \to G(\Omega_M(Y)) \to G(M_Y) \to G(Y) \to 0$ is pure-exact in Λ-Mod. Since $G(M_Y)$ is projective, there holds $\mathrm{Tor}_1^\Lambda(V, G(Y)) = 0$ for all $V \in \Lambda^{\mathrm{op}}$-Mod. Thus $G(Y)$ is flat, and therefore projective since Λ is an Artin algebra. It follows that $G(Y)$ is a direct summand of $G(M_Y)$. Then $Y \in \mathrm{Add}(M)$ due to $A \in \mathrm{add}({}_AM)$. Hence $Y = 0$ in $\mathscr{D}$ and A-mod$/[M]$ is a compact generating set of $\mathscr{D}$.

By (3), if $X \in A$-mod, then $T(X)$ is compact in $\mathscr{C}$. Moreover, since T is a left adjoint of the inclusion $\mathscr{C} \to \mathscr{D}$, one can check that T always preserves generating sets. This shows (4). $\square$

4 Restrictions of recollements to relative stable categories

In this section we prove Theorem 1.5. As a preparation of the proof, we first show that the recollement in Corollary 3.15 restricts to a recollement of $\mathscr{C}$. Throughout this section we set up the following.

Assumption: *Let A be a self-injective algebra and ${}_AM$ a self-orthogonal and Nakayama-stable generator for* A-mod.

We set $\Gamma := \underline{\mathrm{End}}_A(M)$. By Lemma 3.11, there exists a natural isomorphism of additive functors:

$$(\lozenge) \quad D\underline{\mathrm{Hom}}_A(M,-) \simeq \underline{\mathrm{Hom}}_A(-, \nu_A(M)[-1]) : \ A\text{-}\underline{\mathrm{Mod}} \longrightarrow \Gamma^{\mathrm{op}}\text{-Mod}.$$

We define the following categories related to M:

$$\mathscr{Y} := \{Y \in A\text{-}\underline{\mathrm{Mod}} \mid \underline{\mathrm{Hom}}_A(M, Y[n] = 0 \ \text{ for any } \ n \leq 0\},$$

$$\mathscr{X} := \{X \in A\text{-}\underline{\mathrm{Mod}} \mid \underline{\mathrm{Hom}}_A(X,Y) = 0 \ \text{ for any } \ Y \in \mathscr{Y}\},$$

$$\mathscr{H} := \mathscr{X} \cap \mathscr{Y}[1].$$

Then $\mathscr{X}$ is the smallest full subcategory of A-$\underline{\mathrm{Mod}}$ containing M and being closed under $[1]$, extensions and direct sums, $(\mathscr{X}, \mathscr{Y})$ is a torsion pair in A-$\underline{\mathrm{Mod}}$, and $\mathscr{H}$ is an abelian category and called the *heart* of $(\mathscr{X}, \mathscr{Y})$ (see [10]). Clearly, $\underline{M}^\perp \subseteq \mathscr{Y}$ and $\mathscr{X} \subseteq \mathrm{Loc}(\underline{M})$. In general, $\mathscr{X}$ has not to be a triangulated subcategory of A-$\underline{\mathrm{Mod}}$ since it is not necessarily closed under $[-1]$.

Proposition 4.1. *$\mathscr{X} = \underline{M}^{\perp > 0} \cap \underline{\mathscr{S}}$, $\mathscr{Y} = {}^{\perp \geq -1}\underline{M}$ and $\mathscr{H} = \underline{M}^{\perp \neq 0} \cap \underline{\mathscr{S}}$.*

Proof. Since ${}_AM$ is self-orthogonal, it follows from Lemma 3.12(2) that $\underline{\mathscr{S}} = \mathrm{Loc}(\underline{M})$. As $\underline{M}^{\perp>0}$ contains M and is closed under $[1]$, extensions and direct sums in A-$\underline{\mathrm{Mod}}$, we have $\mathscr{X} \subseteq \underline{M}^{\perp>0}$. Thanks to $\mathscr{X} \subseteq \mathrm{Loc}(\underline{M})$, we obtain $\mathscr{X} \subseteq \underline{M}^{\perp>0} \cap \mathrm{Loc}(\underline{M})$. To show the converse inclusion, we pick up $X \in \underline{M}^{\perp>0} \cap \mathrm{Loc}(\underline{M})$ and show $\underline{\mathrm{Hom}}_A(X,Y) = 0$ for any $Y \in \mathscr{Y}$.

Actually, it follows from $\underline{M}^{\perp>0} = \underline{M^{\perp>0}}$ that $X \in M^{\perp>0}$. Note that X lies in $\mathscr{S}$. Let $\mu_X^\bullet : M_X^\bullet \to X$ be an $\mathrm{Add}(M)$-resolution of X. It follows from Proposition 3.8(1) that $\mathrm{Hom}_{\mathscr{K}(A)}(\mathrm{Con}(\mu_X^\bullet), \mathrm{Con}(\pi_Y^\bullet)) = 0$ for any $Y \in \mathscr{Y}$. Since $\underline{\mathrm{Hom}}_A(M, \Omega_A^n(Y)) = \underline{\mathrm{Hom}}_A(M, Y[-n]) = 0$ for any $n \geq 0$, the chain map $\pi_Y^\bullet : P_Y^\bullet \to Y$ is an $\mathrm{Add}(M)$-resolution of Y. Thus each homomorphism $f : X \to Y$ in A-Mod can be lifted to a chain map $f^\bullet : \mathrm{Con}(\mu_X^\bullet) \to \mathrm{Con}(\pi_Y^\bullet)$ in $\mathscr{K}(A)$. From $\mathrm{Hom}_{\mathscr{K}(A)}(\mathrm{Con}(\mu_X^\bullet), \mathrm{Con}(\pi_Y^\bullet)) = 0$, we see that f factorizes through the projective module P_Y^0. Thus $f = 0$ in A-$\underline{\mathrm{Mod}}$ and $\underline{\mathrm{Hom}}_A(X,Y) = 0$.

By definition, $\mathscr{Y} = \underline{M}^{\perp\leq 0}$. It follows from $\mathrm{add}(\nu_A(M)) = \mathrm{add}({}_AM)$ and $(\Diamond)$ that $\mathscr{Y} = {}^{\perp\geq -1}\underline{M}$. Since $\mathscr{H} = \mathscr{X} \cap \mathscr{Y}[1]$ and $\mathscr{Y}[1] = \underline{M}^{\perp<0}$, one gets $\mathscr{H} = \underline{M}^{\perp\neq 0} \cap \mathrm{Loc}(\underline{M}) = \underline{M}^{\perp\neq 0} \cap \underline{\mathscr{S}}$. $\square$

4.1 Restrictions of recollements

In this subsection we consider the restriction of the recollement in Corollary 3.15 to relative stable categories. This leads to a part of the proof of Theorem 1.5.

We begin with the following preparation.

Lemma 4.2. (1) $\underline{M}^{\perp} = {}^{\perp}\underline{M}$, $\quad \underline{\mathscr{G}} = \underline{M}^{\perp\neq 0,-1} = {}^{\perp\neq 0,-1}\underline{M}$ *and* $\quad \mathscr{H} \subseteq \underline{\mathscr{G} \cap \mathscr{S}} \subseteq \mathscr{X}$.

(2) *Let* $\pi : \underline{\mathscr{G}} \to \mathscr{C}$ *be the canonical quotient functor. Then the following are true.*

(a) *The composition* $\underline{M}^{\perp} \hookrightarrow \underline{\mathscr{G}} \xrightarrow{\pi} \mathscr{C}$ *is a fully faithful triangle functor. In particular, the image of this composition is a full triangulated subcategory of* $\mathscr{C}$.

(b) *The composition* $\mathscr{H} \hookrightarrow \underline{\mathscr{G} \cap \mathscr{S}} \xrightarrow{\pi} (\mathscr{G} \cap \mathscr{S})/[M]$ *is a fully faithful additive functor.*

Proof. (1) Since $\mathrm{add}(\nu_A(M)) = \mathrm{add}({}_AM)$, (1) follows from $(\Diamond)$ and Proposition 4.1.

(2) If $X \in \underline{M}^{\perp}$, then $X \in {}^{\perp}\underline{M}$ by (1). In this case, $\underline{\mathrm{Hom}}_A(M,X) = 0 = \underline{\mathrm{Hom}}_A(X,M)$, and therefore $\underline{\mathrm{Hom}}_A(M_0,X) = 0 = \underline{\mathrm{Hom}}_A(X,M_0)$ for $M_0 \in \mathrm{Add}(M)$. This implies that $\Omega_A(X) \simeq \Omega_M(X)$, $\Omega_A^-(X) \simeq \Omega_M^-(X)$ in $\mathscr{D}$ and $\underline{\mathrm{Hom}}_A(X,X') = \underline{\mathrm{Hom}}_M(X,X')$ for any A-module X'. Thus the composition in (a) is fully faithful. It is a triangle functor since $\underline{M}^{\perp}$ is a full triangulated subcategory of A-$\underline{\mathrm{Mod}}$. This shows (a).

Now, let $X \in \mathscr{H}$. Then $\underline{\mathrm{Hom}}_A(M, X[-1]) = 0$. Since $D\underline{\mathrm{Hom}}_A(M, X[-1]) \simeq \underline{\mathrm{Hom}}_A(X, \nu_A(M))$ by $(\Diamond)$ and $\mathrm{add}(\nu_A(M)) = \mathrm{add}(M)$, we have $\underline{\mathrm{Hom}}_A(X,M) = 0$. It then follows from $\mathrm{Add}(\underline{M}) = \mathrm{Prod}(\underline{M})$ that $\underline{\mathrm{Hom}}_A(X,M') = 0$ for $M' \in \mathrm{Add}(M)$. Thus $\underline{\mathrm{Hom}}_A(X,Y) = \underline{\mathrm{Hom}}_M(X,Y)$ for any A-module Y. This implies (b). $\square$

Proposition 4.3. *The recollement in Corollary 3.15 induces a recollement of triangulated categories:*

$$\begin{tikzcd}
\underline{M}^{\perp} \arrow[r, "\pi\circ\mathrm{inc}"] & \mathscr{C} \arrow[l, bend right, "\widetilde{\Psi}"'] \arrow[l, bend left, "\widetilde{\Psi'}"] \arrow[r, "\widetilde{\Phi}"] & (\mathscr{G} \cap \mathscr{S})/[M]. \arrow[l, bend right, "\mathrm{inc}"'] \arrow[l, bend left, "\Phi''"]
\end{tikzcd}$$

Proof. We first show that the recollement in Corollary 3.15 can be restricted to a "recollement" of additive categories with six additive functors:

$$\begin{tikzcd}
(\sharp) \quad \underline{M}^{\perp} \arrow[r, "\mathrm{inc}"] & \underline{\mathscr{G}} \arrow[l, bend right, "\widetilde{\Psi}"'] \arrow[l, bend left, "\widetilde{\Psi'}"] \arrow[r, "\widetilde{\Phi}"] & \underline{\mathscr{G} \cap \mathscr{S}} \arrow[l, bend right, "\mathrm{inc}"'] \arrow[l, bend left, "\Phi''"]
\end{tikzcd}$$

which satisfy the conditions (1)-(3) in Definition 2.2. Obviously, $\mathrm{inc} : \underline{M}^{\perp} \to \underline{\mathscr{G}}$ has left and right adjoints which are the restriction of the functors $\widetilde{\Psi}$ and $\widetilde{\Psi'}$ in Corollary 3.15 to $\underline{\mathscr{G}}$, respectively. Now, we claim that $\widetilde{\Phi}(\underline{\mathscr{G}}) \subseteq \underline{\mathscr{G} \cap \mathscr{S}}$ and $\Phi''(\underline{\mathscr{G} \cap \mathscr{S}}) \subseteq \underline{\mathscr{G}}$. Then $(\mathrm{inc}, \widetilde{\Phi})$ and $(\widetilde{\Phi}, \Phi'')$ in $(\sharp)$ are adjoint pairs.

By Corollary 3.13, $\underline{\mathscr{S}} = \mathrm{Loc}(\underline{M})$. By Theorem 3.14, each A-module X is endowed with a triangle $\Psi(X)[-1] \to \Phi(X) \to X \to \Psi(X)$ in A-$\underline{\mathrm{Mod}}$ such that $\Phi(X) \in \underline{\mathscr{S}}$ and $\Psi(X) \in \underline{M}^{\perp}$. Note that $\underline{\mathscr{G}}$ contains $\underline{M}^{\perp}$ and is closed under extensions of triangles in A-$\underline{\mathrm{Mod}}$, due to Lemma 4.2. Since $\Psi(X)$ and $\Psi(X)[-1]$ lie in $\underline{M}^{\perp}$, we see that $X \in \underline{\mathscr{G}}$ if and only if $\Phi(X) \in \underline{\mathscr{G}}$. Thus $\widetilde{\Phi}(\underline{\mathscr{G}}) = \Phi(\underline{\mathscr{G}}) \subseteq \underline{\mathscr{G} \cap \mathscr{S}}$. Similarly, by the triangle $\Psi'(X) \to X \to \Phi'(X) \to \Psi'(X)[1]$ in Proposition 3.6, $X \in \underline{\mathscr{G}}$ if and only if $\Phi'(X) \in \underline{\mathscr{G}}$. This implies $\Phi''(\underline{\mathscr{G} \cap \mathscr{S}}) = \Phi'(\underline{\mathscr{G} \cap \mathscr{S}}) \subseteq \underline{\mathscr{G}}$.

Next, we show that the functors in ($\sharp$) induce triangle functors among quotient categories.

By Corollary 3.13, $\mathscr{S}$ contains $\mathrm{Add}(M)$ and is closed under taking Ω_M and Ω_M^- in A-Mod. Since $\mathrm{Add}(M) \subseteq \mathscr{G}$ and $\mathscr{C}$ is a triangulated category by Lemma 3.17, $(\mathscr{G} \cap \mathscr{S})/[M]$ is a full triangulated subcategory of $\mathscr{C}$. Note that $\Phi(\mathrm{Add}(\underline{M})) = \mathrm{Add}(\underline{M})$ and $\Psi(\mathrm{Add}(\underline{M})) = 0$ due to $\underline{M} \in \underline{\mathscr{S}}$. By Lemma 2.1(1), the adjoint pairs $(\mathrm{inc}, \widetilde{\Phi})$ and $(\widetilde{\Psi}, \mathrm{inc})$ in ($\sharp$) induce adjoint pairs $(\mathrm{inc}, \widetilde{\Phi}_0)$ and $(\widetilde{\Psi}_0, \pi \circ \mathrm{inc})$ of additive functors among triangulated categories:

$$\begin{tikzcd}
(\natural) \quad \underline{M}^{\perp} \arrow[r, "\pi\circ\mathrm{inc}"] & \mathscr{C} \arrow[l, bend right=50, "\widetilde{\Psi}_0"'] \arrow[r, "\widetilde{\Phi}_0"] & (\mathscr{G} \cap \mathscr{S})/[M] \; . \arrow[l, bend right=50, "\mathrm{inc}"']
\end{tikzcd}$$

In this diagram, both inc and $\pi \circ \mathrm{inc}$ (see Lemma 4.2(2)) are fully faithful triangle functors. It is known that any left or right adjoint of a triangle functor between triangulated categories is again a triangle functor. Thus $\widetilde{\Phi}_0$ and $\widetilde{\Psi}_0$ are triangle functors. In the following, we show that both $\pi \circ \mathrm{inc}$ and $\widetilde{\Phi}_0$ have right adjoints.

According to Lemma 3.12(3), $\underline{\mathscr{T}} = \mathrm{Coloc}(\underline{\nu_A(M)}) = \mathrm{Im}(\Phi')$. By the definition of $\mathscr{T}$, if $X \in \underline{\mathscr{T}}$, then $\Psi'(X) = 0$ and $\Phi'(X) \simeq X$. This means that

$$\Psi'(\mathrm{Prod}(\underline{\nu_A(M)})) = 0 \quad \text{and} \quad \Phi'(\mathrm{Prod}(\underline{\nu_A(M)})) = \mathrm{Prod}(\underline{\nu_A(M)}).$$

Since $\mathrm{add}(\nu_A(M)) = \mathrm{add}(M)$ and ${}_AM \in A$-mod, there holds $\mathrm{Prod}(\underline{\nu_A(M)}) = \mathrm{Prod}(\underline{M}) = \mathrm{Add}(\underline{M})$. It follows that $\Psi'(\mathrm{Add}(\underline{M})) = 0$ and $\Phi'(\mathrm{Add}(\underline{M})) = \mathrm{Add}(\underline{M})$. Since $\overline{\mathrm{Add}(\underline{M})} \subseteq \underline{\mathscr{G} \cap \mathscr{S}}$ and $\Phi'' = \Phi' \circ \mathrm{inc}$, we have $\Phi''(\mathrm{Add}(\underline{M})) = \mathrm{Add}(\underline{M})$. Due to Lemma 2.1(1), $\Phi'' : \underline{\mathscr{G} \cap \mathscr{S}} \to \underline{\mathscr{G}}$ induces a functor $\Phi''_0 : (\mathscr{G} \cap \mathscr{S})/[M] \to \mathscr{C}$ which is a right adjoint of $\widetilde{\Phi}_0$, while $\widetilde{\Psi'} : \underline{\mathscr{G}} \to \underline{M}^{\perp}$ induces a functor $\widetilde{\Psi'}_0 : \mathscr{C} \to \underline{M}^{\perp}$ which is a right adjoint of $\pi \circ \mathrm{inc}$.

When acting on objects, $\widetilde{\Phi}_0$ and $\widetilde{\Phi}$ are the same. So we denote $\widetilde{\Phi}_0$ by $\widetilde{\Phi}$ for simplicity. Similarly, we denote $\widetilde{\Psi}_0$, $\widetilde{\Psi'}_0$ and Φ''_0 by $\widetilde{\Psi}$, $\widetilde{\Psi'}$ and Φ'', respectively.

To show the existence of the recollement of $\mathscr{C}$ in Proposition 4.3, it remains to show the existence of two canonical triangles in Definition 2.2(4). Let $X \in \mathscr{G}$. We have shown that there is a canonical triangle $\mathrm{inc} \circ \widetilde{\Phi}(X) \to X \to \mathrm{inc} \circ \widetilde{\Psi}(X) \to \mathrm{inc} \circ \widetilde{\Phi}(X)[1]$ in A-$\underline{\mathrm{Mod}}$ such that $\widetilde{\Phi}(X) \in \underline{\mathscr{G} \cap \mathscr{S}}$ and $\widetilde{\Psi}(X) \in \underline{M}^{\perp}$. Since $\underline{M}^{\perp} = {}^{\perp}\underline{M} \subseteq {}^{\perp 1}\underline{M}$ by Lemma 4.2(1), it follows from Lemma 3.16(2) that this triangle induces a triangle $\mathrm{inc} \circ \widetilde{\Phi}(X) \to X \to \mathrm{inc} \circ \widetilde{\Psi}(X) \to \Omega_M^-(\mathrm{inc} \circ \widetilde{\Phi}(X))$ in $\mathscr{C}$, which is the required second triangle in Definition 2.2(4). Similarly, by the triangle $\Psi'(X) \to X \to \Phi'(X) \to \Psi'(X)[1]$ with $\Psi'(X) \in \underline{M}^{\perp}$ and $\Phi'(X) \in \underline{\mathscr{G}}$ and by the inclusion $\underline{M}^{\perp} \subseteq \underline{M}^{\perp 1}$, we can obtain the first triangle in Definition 2.2(4). Thus the proof of Proposition 4.3 is completed. $\square$

Now, we define a full subcategory $\mathscr{E}$ of $\mathscr{G}$, which contains both $\mathscr{G}_0$ and A-Proj.

$$\mathscr{E} := \{X \in \mathscr{G} \mid \underline{\mathrm{Hom}}_A(M,X), \underline{\mathrm{Hom}}_A(M[1],X) \in \Gamma\text{-mod}\}.$$

Lemma 4.4. (1) $\mathscr{E} = \{X \in \mathscr{G} \mid \underline{\mathrm{Hom}}_A(M,X) \in \Gamma\text{-mod}, \underline{\mathrm{Hom}}_A(X,M) \in \Gamma^{\mathrm{op}}\text{-mod}\}$.

(2) *$\mathscr{E}$ is a thick subcategory of $\mathscr{G}$ and $\mathscr{E}/[M]$ is a full triangulated subcategory of $\mathscr{C}$.*

Proof. (1) As A is an Artin algebra over a commutative Artin ring R and ${}_AM$ is a finitely generated A-module, Γ is an Artin algebra over R. Moreover, a Γ-module N is finitely generated if and only if ${}_RN$

is finitely generated as an R-module. Thus (1) follows from $D\underline{\mathrm{Hom}}_A(M[1],X) \simeq \underline{\mathrm{Hom}}_A(X,\nu_A(M))$ and $\mathrm{add}(\nu_A(M)) = \mathrm{add}(M)$.

(2) Clearly, $\mathscr{E}$ is closed under direct summands in $\mathscr{G}$. Now, let $0 \to X_1 \to X_2 \to X_3 \to 0$ be an exact sequence in A-Mod with $X_i \in \mathscr{G}$ for $1 \le i \le 3$. Since $\underline{\mathscr{G}} = \underline{M}^{\perp \neq 0,-1}$ by Lemma 4.2, there is a long exact sequence in R-mod:

$$0 \to \underline{\mathrm{Hom}}_A(M[1],X_1) \to \underline{\mathrm{Hom}}_A(M[1],X_2) \to \underline{\mathrm{Hom}}_A(M[1],X_3) \to \underline{\mathrm{Hom}}_A(M,X_1) \to \underline{\mathrm{Hom}}_A(M,X_2) \to \underline{\mathrm{Hom}}_A(M,X_3) \to 0.$$

This implies that $\mathscr{E}$ has the two out of three property in $\mathscr{G}$. Thus $\mathscr{E}$ is a thick subcategory of $\mathscr{G}$.

By definition, $\mathscr{E}/[M]$ consists of all objects $X \in \mathscr{C}$ which is M-stably isomorphic to an object of $\mathscr{E}$. To show that $\mathscr{E}/[M]$ is a full triangulated subcategory of $\mathscr{C}$, it suffices to show that $\mathscr{E}/[M]$ is closed under taking Ω_M and Ω_M^- in $\mathscr{C}$.

Let $X \in \mathscr{E}$. Due to $\underline{\mathrm{Hom}}_A(M,X) \in \Gamma$-mod, we see that X has a right $\mathrm{Add}(M)$-approximation $f : M_0 \oplus P_0 \to X$ such that $M_0 \in \mathrm{add}(M)$ and $P_0 \in A$-Proj. Note that $\mathscr{E}$ contains $\mathrm{add}(M)$ and A-Proj. This forces $M_0 \oplus P_0 \in \mathscr{E}$, and further $\mathrm{Ker}(f) \in \mathscr{E}$. It follows from $\Omega_M(X) \simeq \mathrm{Ker}(f)$ in $\mathscr{C}$ that $\Omega_M(X) \in \mathscr{E}/[M]$. Since $\underline{\mathrm{Hom}}_A(X,M)$ is a Γ^{op}-module, we can show $\Omega_M^-(X) \in \mathscr{E}/[M]$.

The proof of (2) also implies that if $X \in \mathscr{G}$, then $X \in \mathscr{E}$ if and only if it has a complete $\mathrm{Add}(M)$-resolution $M_X^\bullet := (M_X^i)_{i\in\mathbb{Z}}$ satisfying that $M_X^i = N^i \oplus P^i$ with $N^i \in \mathrm{add}(M)$ and $P^i \in A$-Proj. □

Corollary 4.5. *The recollement in Proposition 4.3 can be restricted a recollement of $\mathscr{E}/[M]$:*

$$\begin{tikzcd} \underline{M}^{\perp} \arrow[r, "\pi\circ\mathrm{inc}"] & \mathscr{E}/[M] \arrow[l, bend right, "\widetilde{\Psi}"'] \arrow[l, bend left, "\widetilde{\Psi'}"] \arrow[r, "\widetilde{\Phi}"] & (\mathscr{E}\cap\mathscr{S})/[M]. \arrow[l, bend right, "\mathrm{inc}"'] \arrow[l, bend left, "\Phi''"] \end{tikzcd}$$

Proof. Note that $\big((\mathscr{G}\cap\mathscr{S})/[M]\big) \cap (\mathscr{E}/[M]) = (\mathscr{E}\cap\mathscr{S})/[M]$ and the image of the functor $\pi\circ\mathrm{inc} : \underline{M}^{\perp} \to \mathscr{C}$ is contained in $\mathscr{E}/[M]$. Thus Corollary 4.5 is a consequence of Proposition 4.3. □

4.2 Compacts objects and representability of homological functors

In this subsection, we find out a special compact object in $\mathscr{C}$ (or even in a bigger relative stable category) and establish a series of homological functors from $\mathscr{C}$ to Γ-Mod (see Theorem 4.14).

From now on, let $\mathbf{L} : A\text{-}\underline{\mathrm{Mod}} \to \mathscr{Y}$ be the left adjoint of the inclusion $\mathscr{Y} \subseteq A\text{-}\underline{\mathrm{Mod}}$ and let $\mathbf{R} : A\text{-}\underline{\mathrm{Mod}} \to \mathscr{X}$ be the right adjoint of the inclusion $\mathscr{X} \subseteq A\text{-}\underline{\mathrm{Mod}}$. Define

$$H_*^0 := [1]\circ\mathbf{L}\circ[-1]\circ\mathbf{R} :\ A\text{-}\underline{\mathrm{Mod}} \longrightarrow \mathscr{H},\ \ N := H_*^0(M) \in \mathscr{H}\ \text{ and}$$

$$\mathscr{H}^{\mathrm{fg}} := \{X \in \mathscr{H} \mid \underline{\mathrm{Hom}}_A(M,X) \in \Gamma\text{-mod}\}.$$

Then H_*^0 is a homological functor, that is, for a triangle $X_1 \to X_2 \to X_3 \to X_1[1]$ in A-$\underline{\mathrm{Mod}}$, the sequence $H_*^0(X_1) \to H_*^0(X_2) \to H_*^0(X_3)$ is exact in $\mathscr{H}$. Moreover, $\mathscr{H}^{\mathrm{fg}} = \mathscr{H} \cap \underline{\mathscr{E}}$ by Proposition 4.1 and Lemma 4.2(1). Note that $D\underline{\mathrm{Hom}}_A(M,N[-1]) \simeq \underline{\mathrm{Hom}}_A(N,\nu_A(M))$ by ($\Diamond$). Since $\mathrm{add}(\nu_A(M)) = \mathrm{add}(M)$ and $N \in \mathscr{H}$, we get $\underline{\mathrm{Hom}}_A(N,M) = 0$. This implies $\underline{\mathrm{End}}_A(N) = \mathrm{End}_{\mathscr{C}}(N)$.

Lemma 4.6. *Let $X \in \mathscr{X}$.*

(1) *There is a canonical triangle $[1]\circ\mathbf{R}\circ[-1](X) \to X \xrightarrow{\tau_X} H_*^0(X) \to [2]\circ\mathbf{R}\circ[-1](X)$ in A-$\underline{\mathrm{Mod}}$.*

(2) *If $Y \in \mathscr{Y}[1]$, then $\underline{\mathrm{Hom}}_A(\tau_X,Y) : \underline{\mathrm{Hom}}_A(H_*^0(X),Y) \to \underline{\mathrm{Hom}}_A(X,Y)$ is an isomorphism. In particular, there is a natural isomorphism*

$$\underline{\mathrm{Hom}}_A(\tau_M,-) :\ \underline{\mathrm{Hom}}_A(N,-) \xrightarrow{\simeq} \underline{\mathrm{Hom}}_A(M,-) :\ \mathscr{H} \longrightarrow \Gamma\text{-Mod}.$$

Proof. Since $(\mathscr{X},\mathscr{Y})$ is a torsion pair in A-$\underline{\mathrm{Mod}}$, each A-module W is endowed with a triangle $\mathbf{R}(W)\to W\to \mathbf{L}(W)\to \mathbf{R}(W)[1]$ in A-$\underline{\mathrm{Mod}}$. We take $W=X[-1]$ and get a triangle $\mathbf{R}(X[-1])\to X[-1]\to \mathbf{L}(X[-1])\to \mathbf{R}(X[-1])[1]$ in A-$\underline{\mathrm{Mod}}$. Shifting this triangle by $[1]$ yields another triangle

$$(\dagger)\quad \mathbf{R}(X[-1])[1]\to X\to \mathbf{L}(X[-1])[1]\to \mathbf{R}(X[-1])[2].$$

Due to $X\in\mathscr{X}$, we have $\mathbf{R}(X)\simeq X$. Thus $H^0_*(X)=[1]\circ\mathbf{L}\circ[-1]\circ\mathbf{R}(X)\simeq \mathbf{L}(X[-1])[1]$. Then the triangle $(\dagger)$ can be rewritten as $[1]\circ\mathbf{R}\circ[-1](X)\to X\xrightarrow{\tau_X} H^0_*(X)\to[2]\circ\mathbf{R}\circ[-1](X)$. This shows (1).

Since $\mathscr{X}\subseteq A$-$\underline{\mathrm{Mod}}$ is closed under $[1]$, it follows from $\mathbf{R}(X[-1])\in\mathscr{X}$ that both $[1]\circ\mathbf{R}\circ[-1](X)$ and $[2]\circ\mathbf{R}\circ[-1](X)$ belong to $\mathscr{X}$. As $\underline{\mathrm{Hom}}_A(X,Y)=0$ for $X\in\mathscr{X}$ and $Y\in\mathscr{Y}$, the first part of (2) holds by (1), while the second part of (2) follows from both $M\in\mathscr{X}$ and $\mathscr{H}\subseteq\mathscr{Y}[1]$. □

The next result follows from [13, Chap. III, Lemma 3.3 and Theorem 3.4]; see also [25, Theorem 1.3(3)] for the assertion (3).

Lemma 4.7. (1) *For any $X\in\mathscr{X}$, the morphism τ_X in Lemma 4.6(1) induces an isomorphism of Γ-modules:*

$$\underline{\mathrm{Hom}}_A(M,\tau_X):\underline{\mathrm{Hom}}_A(M,X)\xrightarrow{\simeq}\underline{\mathrm{Hom}}_A(M,H^0_*(X)).$$

(2) *The functor H^0_* induces an isomorphism $\Gamma\xrightarrow{\simeq}\underline{\mathrm{End}}_A(N)$ of algebras such that $\gamma\tau_M=\tau_M H^0_*(\gamma)$ for any $\gamma\in\Gamma$. In this sense, Γ can be identified with $\underline{\mathrm{End}}_A(N)$.*

(3) *The object N is a small projective generator of $\mathscr{H}$ and the functor $\underline{\mathrm{Hom}}_A(N,-):\mathscr{H}\to\Gamma$-Mod is an equivalence of abelian categories.*

An easy observation is the following result, of which (2) conveys that N has only finitely many indecomposable direct summands in $\mathscr{C}$.

Corollary 4.8. (1) *There is a fully faithful functor $\Theta:\Gamma\text{-mod}\to(\mathscr{E}\cap\mathscr{S})/[M]$ which sends Γ to N.*

(2) *Let $M=A\oplus\bigoplus_{i=1}^m M_i$, where $m\in\mathbb{N}$ and M_i are indecomposable and non-projective. Then $N\simeq\bigoplus_{i=1}^m H^0_*(M_i)$ in $\mathscr{C}$ and $H^0_*(M_i)$ are indecomposable in $\mathscr{C}$.*

(3) *Let $X\in\mathscr{G}\cap\mathscr{S}$. Then there is a triangle $X_0\to X\to H^0_*(X)\to\Omega^-_M(X_0)$ in $\mathscr{C}$ such that $X_0\in\mathscr{G}\cap\mathscr{S}$ and $\Omega_M(X_0)\in\mathscr{H}$. Further, if $X\in\mathscr{E}\cap\mathscr{S}$, then $\Omega_M(X_0),H^0_*(X)\in\mathscr{H}^{\mathrm{fg}}$.*

Proof. (1) By Lemma 4.2(2)(b), the composition $U:\mathscr{H}^{\mathrm{fg}}=\mathscr{H}\cap\underline{\mathscr{E}}\hookrightarrow\underline{\mathscr{E}\cap\mathscr{S}}\to(\mathscr{E}\cap\mathscr{S})/[M]$ is fully faithful. Moreover, from Lemmas 4.6(2) and 4.7(3) it follows that $\underline{\mathrm{Hom}}_A(M,-)\simeq\underline{\mathrm{Hom}}_A(N,-):\mathscr{H}^{\mathrm{fg}}\to\Gamma$-mod, which is an equivalence of abelian categories sending N to Γ. In particular, $N\in\mathscr{H}^{\mathrm{fg}}$. Now, let $\Theta:\Gamma\text{-mod}\to(\mathscr{E}\cap\mathscr{S})/[M]$ be the composition of a quasi-inverse of $\underline{\mathrm{Hom}}_A(M,-)$ with U. Then Θ is fully faithful and sends Γ to N.

(2) By Lemmas 4.6(2) and 4.7(1), H^0_* induces an isomorphism $\underline{\mathrm{Hom}}_A(M,M_i)\simeq\underline{\mathrm{Hom}}_A(N,H^0_*(M_i))$. Thus $\underline{\mathrm{End}}_A(M_i)\simeq\underline{\mathrm{End}}_A(H^0_*(M_i))$ as algebras. It then follows from $H^0_*(M_i)\in\mathscr{H}$ that $\underline{\mathrm{End}}_A(H^0_*(M_i))\simeq\mathrm{End}_{\mathscr{C}}(H^0_*(M_i))$ as algebras by Lemma 4.2(2). As M_i is indecomposable and non-projective, it is also indecomposable in A-$\underline{\mathrm{Mod}}$. Thus $H^0_*(M_i)$ is indecomposable in $\mathscr{C}$.

(3) Let $X_0:=\mathbf{R}(X[-1])[1]$. Then $X_0[-1]\in\mathscr{X}$ and there is a triangle $X_0\to X\xrightarrow{\tau_X}H^0_*(X)\to X_0[1]$ in A-$\underline{\mathrm{Mod}}$ by Lemma 4.6(1). Thus $\underline{\mathrm{Hom}}_A(M,X_0[n])=0$ for all $n\geq0$ by Proposition 4.1. In particular, $\mathrm{Ext}^1_A(M,X_0)=0$. Since both X and $H^0_*(X)$ lie in $\mathscr{G}\cap\mathscr{S}$, we see that $X_0\in\mathscr{G}\cap\mathscr{S}$ and the above triangle induces a triangle $X_0\to X\to H^0_*(X)\to\Omega^-_M(X_0)$ in $\mathscr{C}$ by Lemma 3.16(2). Further, $\underline{\mathrm{Hom}}_A(M,X_0[-2])=0$ by Lemma 4.2(1), and $X_0[-1]=\Omega_A(X_0)\simeq\Omega_M(X_0)\in\mathscr{G}$ by $\underline{\mathrm{Hom}}_A(M,X_0)=0$. Thus $X_0[-1]\in\mathscr{H}$ by Proposition 4.1. Suppose that X is in $\mathscr{E}\cap\mathscr{S}$. Then $\underline{\mathrm{Hom}}_A(M[1],X_0)\simeq\underline{\mathrm{Hom}}_A(M,X_0[-1])\simeq\underline{\mathrm{Hom}}_A(M,\mathbf{R}(X[-1]))\simeq\underline{\mathrm{Hom}}_A(M,X[-1])\in\Gamma$-mod, where the last isomorphism follows from the fact that $\mathbf{R}:A$-$\underline{\mathrm{Mod}}\to\mathscr{X}$ is the right adjoint of the inclusion $\mathscr{X}\subseteq A$-$\underline{\mathrm{Mod}}$. Now, it follows from $X_0\in\mathscr{G}$ and $\underline{\mathrm{Hom}}_A(M,X_0)=0$ that $X_0\in\mathscr{E}$. Thus $X_0\in\mathscr{E}\cap\mathscr{S}$. By Lemma 4.4(2), both $\Omega_M(X_0)$ and $H^0_*(X)$ belong to $\mathscr{H}^{\mathrm{fg}}$. □

Corollary 4.9. *${}_AM$ is projective if and only if $(\mathscr{E}\cap\mathscr{S})/[M]=0$ if and only if $(\mathscr{G}\cap\mathscr{S})/[M]=0$.*

Proof. If ${}_AM$ is projective, then $\mathscr{S}$ consists of projective A-modules by Corollary 3.13, which implies $(\mathscr{E}\cap\mathscr{S})/[M]=0=(\mathscr{G}\cap\mathscr{S})/[M]$. If $(\mathscr{E}\cap\mathscr{S})/[M]=0$, then Γ-mod $=0$ by Corollary 4.8(1), that is, $\Gamma=0$, and therefore ${}_AM$ is projective. $\square$

Lemma 4.10. *The object N has a complete* $\mathrm{Add}(M)$*-resolution:*

$$\cdots\longrightarrow M^{-3}\longrightarrow M^{-2}\longrightarrow P^{-1}\xrightarrow{\partial} M\oplus P^{0}\longrightarrow P^{1}\longrightarrow M^{2}\longrightarrow M^{3}\longrightarrow\cdots$$

satisfying

(1) $N=\mathrm{Coker}(\partial)$*,* $M^i\in\mathrm{Add}(M)$ *for all* $|i|\geq 2$ *and* $P^i\in\mathrm{Add}(A)$ *for all* $|i|\leq 1$*; and*

(2) $\mathrm{Ker}(\partial)\in\mathscr{H}$ *and* $\underline{\mathrm{Hom}}_A(M,\mathrm{Ker}(\partial))\simeq D\big(\underline{\mathrm{Hom}}_A(M,\nu_A(M))\big)$ *as* Γ*-modules.*

Proof. (1) Let $\tau_M: M\to N$ be the morphism in A-$\underline{\mathrm{Mod}}$ in Lemma 4.6(1). Up to projective direct summand, we may assume that τ_M is a homomorphism in A-Mod and a preimage of the τ_M in A-$\underline{\mathrm{Mod}}$. Let $f: P^0\to\mathrm{Coker}(\tau_M)$ be a projective cover of $\mathrm{Coker}(\tau_M)$. Then there exists a homomorphism $f_0: P^0\to N$ such that f is the composition of f_0 with the quotient map $N\to\mathrm{Coker}(\tau_M)$. Moreover, the map $\pi_0:=(\tau_M,f_0): M\oplus P^0\to N$ is a right $\mathrm{Add}(M)$-approximation of N. Let $K:=\mathrm{Ker}(\pi_0)$ with a projective cover $\pi_1: P^{-1}\to K$. Then $K\simeq\Omega_M(N)$ in $\mathscr{D}$. By the proof of Corollary 4.8(3), we see that $K\in\mathscr{G}\cap\mathscr{S}$, $\underline{\mathrm{Hom}}_A(M,K)=0$ and $K[-1]=\Omega_A(K)\simeq\Omega_M(K)\in\mathscr{H}$. Consequently, π_1 is a right $\mathrm{Add}(M)$-approximation of K. Now, let $\partial: P^{-1}\to M\oplus P^0$ be the composition of π_1 with the inclusion $\lambda_1: K\to M\oplus P^0$. Then $\mathrm{Coker}(\partial)=N$ and $\mathrm{Ker}(\partial)\simeq K[-1]$ in A-$\underline{\mathrm{Mod}}$. Since $\underline{\mathrm{Hom}}_A(N,M)=0$ and $\mathrm{Add}(M)=\mathrm{Prod}(M)$, a minimal injective envelope $N\to P^1$ is a left $\mathrm{Add}(M)$-approximation of N. Thus (1) holds.

(2) Thanks to $N\in\mathscr{H}$, there holds $\underline{\mathrm{Hom}}_A(M,N[-1])=0=\underline{\mathrm{Hom}}_A(M,N[-2])$ by Proposition 4.1. This implies that $\underline{\mathrm{Hom}}_A(M,\lambda_1[-1]):\underline{\mathrm{Hom}}_A(M,K[-1])\to\underline{\mathrm{Hom}}_A(M,M[-1])$ is an isomorphism. By ($\Diamond$),

$$\underline{\mathrm{Hom}}_A(M,M[-1])\simeq D\big(\underline{\mathrm{Hom}}_A(M[-1],\nu_A(M)[-1])\big)\simeq D\big(\underline{\mathrm{Hom}}_A(M,\nu_A(M))\big)$$

as Γ-modules. Thus $\underline{\mathrm{Hom}}_A(M,\mathrm{Ker}(\partial)\simeq\underline{\mathrm{Hom}}_A(M,K[-1])\simeq D(\underline{\mathrm{Hom}}_A(M,\nu_A(M)))$. $\square$

Corollary 4.11. *If A is a symmetric algebra and Γ is a Frobenius algebra, then $\Omega^2_M(N)\simeq N$ in $\mathscr{C}$.*

Proof. Since A is symmetric, $\nu_A(M)\simeq M$ as A-Λ-bimodules. By Lemma 4.10(2), $\underline{\mathrm{Hom}}_A(M,\mathrm{Ker}(\partial))\simeq D\big(\underline{\mathrm{Hom}}_A(M,M)\big)=D(\Gamma)$ as Γ-modules. Note that $\underline{\mathrm{Hom}}_A(M,N)\simeq\Gamma$ by Lemmas 4.6(2) and 4.7(2). Since Γ is a Frobenius algebra, $\Gamma\simeq D(\Gamma)$ as Γ-modules. Moreover, $\mathrm{Ker}(\partial)\in\mathscr{H}$ and $\mathrm{Ker}(\partial)\simeq\Omega^2_M(N)$ in $\mathscr{C}$ by Lemma 4.10. It follows from Lemmas 4.6(2) and 4.7 that $\mathrm{Ker}(\partial)\simeq N$ in $\mathscr{H}$. Thus $\Omega^2_M(N)\simeq N$ in $\mathscr{C}$. $\square$

Lemma 4.12. *Let $X\in{}^{\perp>0}M$ and $l_X: X\to M^X$ be a left* $\mathrm{Add}(M)$*-approximation of X. Then there is an exact sequence of* Γ*-modules:*

$$0\longrightarrow\underline{\mathrm{Hom}}_A(N,X)\xrightarrow{(\tau_M)_*}\underline{\mathrm{Hom}}_A(M,X)\xrightarrow{(l_X)^*}\underline{\mathrm{Hom}}_A(M,M^X).$$

Proof. Keep all notations introduced in the proof of Lemma 4.10. Recall that $\mathrm{Prod}(M)=\mathrm{Add}(M)$ and $\underline{\mathrm{Hom}}_A(N,M)=0$. If $M_0\in\mathrm{Add}(M)$, then $\underline{\mathrm{Hom}}_A(N,M_0)=0$. It follows from $M^X\in\mathrm{Add}(M)$ that $\underline{\mathrm{Hom}}_A(N,M^X)=0$. This implies that the composition of $(\tau_M)_*$ with $(l_X)^*$ is 0. Thus $\mathrm{Im}((\tau_M)_*)\subseteq\mathrm{Ker}((l_X)^*)$.

Applying $\underline{\mathrm{Hom}}_A(-,X)$ to the triangle $K\to M\xrightarrow{\tau_M}N\to K[1]$ in A-$\underline{\mathrm{Mod}}$ yields an exact sequence $\underline{\mathrm{Hom}}_A(K[1],X)\to\underline{\mathrm{Hom}}_A(N,X)\xrightarrow{(\tau_M)_*}\underline{\mathrm{Hom}}_A(M,X)$. By ($\Diamond$), we have $\underline{{}^{\perp>0}M}=\underline{M}^{\perp\leq -2}=\mathscr{Y}[2]$. Moreover, $\underline{K}[-1]\in\mathscr{H}\subseteq\mathscr{X}$ by the proof of Lemma 4.10. Thus $\underline{K}[1]\in\mathscr{X}[2]$. Since $X\in\underline{{}^{\perp>0}M}$ and $(\mathscr{X},\mathscr{Y})$

is a torsion pair in A-$\underline{\mathrm{Mod}}$, we have $\underline{\mathrm{Hom}}_A(K[1],X)=0$. Thus $(\tau_M)_*$ is injective. It remains to show $\mathrm{Ker}((l_X)^*)\subseteq \mathrm{Im}((\tau_M)_*)$.

Suppose that $g\in \underline{\mathrm{Hom}}_A(M,X)=\underline{\mathrm{Hom}}_A(M\oplus P_0,X)$ with $gl_X=0$. Let $0\to X\xrightarrow{\lambda_0} Q^0\xrightarrow{\mu_0}\Omega_A^-(X)\to 0$ be a short exact sequence of A-modules in which λ_0 is an injective envelope. We claim that the pair $(g\lambda_0,0)$ with $g\lambda_0: M\oplus P^0\to Q^0$ and $0:N\to \Omega_A^-(X)$ can be completed into the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & M^{-3} \arrow[r] \arrow[d, dashed, "f^{-3}"] & M^{-2} \arrow[r] \arrow[d, dashed, "f^{-2}"] & P^{-1} \arrow[r, "\partial"] \arrow[d, dashed, "f^{-1}"] & M\oplus P^0 \arrow[r, "\pi_0"] \arrow[d, "g\lambda_0"] & N \arrow[r] \arrow[d, "0"] & 0 \\
\cdots \arrow[r] & Q^{-3} \arrow[r] & Q^{-2} \arrow[r] & Q^{-1} \arrow[r] & Q^0 \arrow[r, "\mu_0"] & \Omega_A^-(X) \arrow[r] & 0,
\end{tikzcd}$$

in which the first arrow is an $\mathrm{Add}(M)$-resolution of N (see Lemma 4.10) and the second arrow is a minimal projective resolution of $\Omega_A^-(X)$. In other words, there is a chain map $f^\bullet := (\cdots, f^{-3}, f^{-2}, f^{-1}, g\lambda_0, 0)$. To show the existence of the chain map, one splits the long exact sequences into a series of short exact sequences, and then construct relevant homomorphisms together with commutative diagrams between these sequences. We carry out the details as follows.

For $K:=\mathrm{Ker}(\pi_0)$, let $\lambda_1: K\to M\oplus P^0$ be the canonical inclusion, and let $\alpha: K\to X$ be the composition of λ_1 with $g: M\oplus P^0\to X$. Since P^{-1} is projective, there exists a map $f^{-1}: P^{-1}\to Q^{-1}$ making the diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & \Omega_A(K) \arrow[r] \arrow[d, dashed, "\Omega_A(\alpha)"] & P^{-1} \arrow[r] \arrow[d, dashed, "f^{-1}"] & K \arrow[r] \arrow[d, dashed, "\alpha"] & 0 \\
0 \arrow[r] & \Omega_A(X) \arrow[r] & Q^{-1} \arrow[r, "\mu_1"] & X \arrow[r] & 0.
\end{tikzcd}$$

To construct f^{-2}, we will show that $\underline{\mathrm{Hom}}_A(M,\alpha[-1]):\underline{\mathrm{Hom}}_A(M,K[-1])\to \underline{\mathrm{Hom}}_A(M,X[-1])$ is 0, where $[-1]$ denotes Ω_A in A-$\underline{\mathrm{Mod}}$.

By the proof of Lemma 4.10(2), $\underline{\mathrm{Hom}}_A(M,\lambda_1[-1])$ is an isomorphism. It is enough to show that the map $\underline{\mathrm{Hom}}_A(M,g[-1]):\underline{\mathrm{Hom}}_A(M,M[-1])\to \underline{\mathrm{Hom}}_A(M,X[-1])$ is zero. Applying $D:\Gamma\text{-Mod}\to \Gamma^{\mathrm{op}}\text{-Mod}$ to this map, we see that $(\Diamond)$ yields $D\underline{\mathrm{Hom}}_A(M,g[-1])\simeq \underline{\mathrm{Hom}}_A(g,\nu_A(M)):\underline{\mathrm{Hom}}_A(X,\nu_A(M))\to \underline{\mathrm{Hom}}_A(M,\nu_A(M))$. Although D may not be an equivalence in general, it is always exact and reflects zero objects. Thus $\underline{\mathrm{Hom}}_A(M,g[-1])=0$ if and only if $\underline{\mathrm{Hom}}_A(g,\nu_A(M))=0$. Since $\mathrm{add}(\nu_A(M))=\mathrm{add}(M)\subseteq \mathrm{Prod}(M)$, $\underline{\mathrm{Hom}}_A(g,\nu_A(M))=0$ is equivalent to saying that $\underline{\mathrm{Hom}}_A(g,M'):\underline{\mathrm{Hom}}_A(X,M')\to \underline{\mathrm{Hom}}_A(M,M')$ is 0 for any $M'\in \mathrm{Prod}(M)$. By assumption, l_X is a left $\mathrm{Add}(M)$-approximation of X and $gl_X=0$ in A-$\underline{\mathrm{Mod}}$. This leads to $\underline{\mathrm{Hom}}_A(g,M')=0$. Thus $\underline{\mathrm{Hom}}_A(M,\alpha[-1])=0$, and therefore $\underline{\mathrm{Hom}}_A(M^{-2},\alpha[-1])=0$, due to $M^{-2}\in \mathrm{Add}(M)$. Hence there are two homomorphisms f^{-2} and β such that the diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & \Omega_M(\Omega_A(K)) \arrow[r] \arrow[d, dashed, "\beta"] & M^{-2} \arrow[r] \arrow[d, dashed, "f^{-2}"] & \Omega_A(K) \arrow[r] \arrow[d, dashed, "\Omega_A(\alpha)"] & 0 \\
0 \arrow[r] & \Omega_A^2(X) \arrow[r] & Q^{-2} \arrow[r] & \Omega_A(X) \arrow[r] & 0.
\end{tikzcd}$$

Since $X\in {}^{\perp>0}\underline{M}=\underline{M}^{\perp\leq -2}$, we have $\underline{\mathrm{Hom}}_A(M,\Omega_A^n(X))=\underline{\mathrm{Hom}}_A(M,X[-n])=0$ for all $n\geq 2$. Then $\underline{\mathrm{Hom}}_A(M^{-n-1},\Omega_A^n(X))=0$, due to $M^{-n-1}\in \mathrm{Add}(M)$. Consequently, the components $f^{-n-1}: M^{-n-1}\to Q^{-n-1}$ for $n\geq 2$ in $f^\bullet$ can be constructed.

Since $N\in \mathscr{H}\subseteq \underline{M}^{\perp>0}\cap \mathscr{S}$ by Proposition 4.1, it follows from Proposition 3.8(1) that $f^\bullet=0$ in $\mathscr{K}(A)$. Therefore there are homomorphisms $h: N\to Q^0$ and $s^0: M\oplus P^0\to Q^{-1}$ such that $g\lambda_0=s^0\mu_1\lambda_0+\pi_0 h$ and $h\mu_0=0$. Since λ_0 is the kernel of μ_0, there is a unique map $h_0: N\to X$ satisfying $h=h_0\lambda_0$. Since λ_0 is injective, $g=s^0\mu_1+\pi_0 h_0$. This forces $g=\tau_M h_0$ in A-$\underline{\mathrm{Mod}}$ and shows $g\in \mathrm{Im}((\tau_M)_*)$. Thus

$\mathrm{Ker}((l_X)^*) \subseteq \mathrm{Im}((\tau_M)_*)$. Finally, via the isomorphism $\Gamma \simeq \underline{\mathrm{End}}_A(N)$ in Lemma 4.7(2), $\underline{\mathrm{Hom}}_A(N,X)$ becomes a Γ-module and $(\tau_M)_*$ is a homomorphism of Γ-modules. $\square$

Corollary 4.13. *The object N is compact in $\underline{{}^{\perp>0}M}$.*

Proof. It suffices to show that the functor $\underline{\mathrm{Hom}}_A(N,-) : \underline{{}^{\perp>0}M} \to \Gamma\text{-Mod}$ commutes with direct sums. Let $\{X_i\}_{i\in I}$ be a set of A-modules in ${}^{\perp>0}M$ with I an index set. For each i, let $g_i : X_i \to M_i$ be a left $\mathrm{Add}(M)$-approximation of X_i. According to $\mathrm{Add}(M) = \mathrm{Prod}(M)$, the direct sum $g = (g_i)_{i\in I} : \bigoplus_{i\in I} X_i \to \bigoplus_{i\in I} M_i$ of all these g_i is a left $\mathrm{Add}(M)$-approximation of $\bigoplus_{i\in I} X_i$. Note that $\underline{\mathrm{Hom}}_A(M,-) : A\text{-}\underline{\mathrm{Mod}} \to \Gamma\text{-Mod}$ commutes with direct sums since M is compact in $A\text{-}\underline{\mathrm{Mod}}$. By Lemma 4.12, we can construct the following commutative diagram with exact arrows and canonical vertical maps:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i\in I} \underline{\mathrm{Hom}}_A(N,X_i) \arrow[r] \arrow[d] & \bigoplus_{i\in I} \underline{\mathrm{Hom}}_A(M,X_i) \arrow[r] \arrow[d, "\simeq"] & \bigoplus_{i\in I} \underline{\mathrm{Hom}}_A(M,M_i) \arrow[d, "\simeq"] \\
0 \arrow[r] & \underline{\mathrm{Hom}}_A(N,\bigoplus_{i\in I} X_i) \arrow[r] & \underline{\mathrm{Hom}}_A(M,\bigoplus_{i\in I} X_i) \arrow[r] & \underline{\mathrm{Hom}}_A(M,\bigoplus_{i\in I} M_i).
\end{tikzcd}$$

Thus the first vertical map is an isomorphism. $\square$

Theorem 4.14. (1) *The object N belongs to $(\mathscr{E}\cap\mathscr{S})/[M]$ and is compact in $({}^{\perp>0}M)/[M]$. In particular, N is compact in $\mathscr{C}$.*

(2) *For each $n \in \mathbb{Z}$, there exists a natural isomorphism of homological functors:*

$$H^n\big(\underline{\mathrm{Hom}}_A(M,M^\bullet_-)\big) \xrightarrow{\simeq} \underline{\mathrm{Hom}}_M\big(\Omega^n_M(N),-\big) : \ \mathscr{C} \longrightarrow \Gamma\text{-Mod},$$

where $M^\bullet_X$ is a complete $\mathrm{Add}(M)$*-resolution of $X \in \mathscr{C}$.*

Proof (1) By Lemma 4.2, $N \in (\mathscr{G}\cap\mathscr{S})/[M]$. Note that $\underline{\mathrm{Hom}}_A(M[1],N) = 0$ by Proposition 4.1 and $\underline{\mathrm{Hom}}_A(M,N) \simeq \underline{\mathrm{Hom}}_A(M,M)$ by Lemma 4.7(1). This shows $N \in \mathscr{E}$. Since ${}^{\perp>0}M$ contains $\mathrm{Add}(M)$ and is closed under direct sums in A-Mod, we see that ${}^{\perp>0}M$, as an additive category, has coproducts. Moreover, by Lemma 2.1(2), the quotient functor $\underline{{}^{\perp>0}M} \to ({}^{\perp>0}M)/[M]$ preserves coproducts and compact objects. Now (1) follows from Corollary 4.13.

(2) Let $M^\bullet_X := (M^n_X, d^n_X)_{n\in\mathbb{Z}}$ be a complete $\mathrm{Add}(M)$-resolution of X. Then $X \simeq \mathrm{Coker}(d_X^{-1})$ and $\mathrm{Hom}_A(M,M^\bullet_X)$ is acyclic. Clearly, there is a canonical ring homomorphism $\Lambda \twoheadrightarrow \Gamma$ and the functor $\underline{\mathrm{Hom}}_A(M,-) : A\text{-Mod} \to \Gamma\text{-Mod}$ is naturally isomorphic to the composition of G with $\Gamma\otimes_\Lambda -$. Then $\underline{\mathrm{Hom}}_A(M,M^\bullet_X) \simeq \Gamma\otimes_\Lambda \mathrm{Hom}_A(M,M^\bullet_X)$ as complexes. Since $\Gamma\otimes_\Lambda -$ is right exact, the sequence

$$\underline{\mathrm{Hom}}_A(M,M_X^{n-1}) \xrightarrow{(d_X^{n-1})^*} \underline{\mathrm{Hom}}_A(M,M^n_X) \longrightarrow \underline{\mathrm{Hom}}_A(M,\Omega^{-n}_M(X)) \longrightarrow 0$$

is exact for all n. As the inclusion $\lambda_n : \Omega^{-n}_M(X) \to M^{n+1}_X$ is a left $\mathrm{Add}(M)$-approximation of $\Omega^{-n}_M(X)$, it follows from Lemma 4.12 that the sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}_A(N,\Omega^{-n}_M(X)) \xrightarrow{(\tau_M)_*} \underline{\mathrm{Hom}}_A(M,\Omega^{-n}_M(X)) \xrightarrow{(\lambda_n)^*} \underline{\mathrm{Hom}}_A(M,M^{n+1}_X)$$

is exact. Consequently, $H^n(\underline{\mathrm{Hom}}_A(M,M^\bullet_X)) \simeq H^n(\Gamma\otimes_\Lambda \mathrm{Hom}_A(M,M^\bullet_X)) \simeq \underline{\mathrm{Hom}}_A(N,\Omega^{-n}_M(X))$. Since $\underline{\mathrm{Hom}}_A(N,M') = 0$ for any $M' \in \mathrm{Add}(M)$, we have

$$\underline{\mathrm{Hom}}_A(N,\Omega^{-n}_M(X)) = \underline{\mathrm{Hom}}_M(N,\Omega^{-n}_M(X)) \simeq \underline{\mathrm{Hom}}_M(\Omega^n_M(N),X).$$

This shows (2).

4.3 Compact objects from left approximations

In this subsection, we characterize compact objects in $\mathscr{C}$ in terms of M-filtered modules (see Proposition 4.17) and show that all objects of $(\mathscr{E}\cap\mathscr{S})/[M]$ are compact in $\mathscr{C}$. We then establish a connection between $(\mathscr{G}\cap\mathscr{S})/[M]$ and $(\mathscr{E}\cap\mathscr{S})/[M]$ by employing strong generators (see Corollary 4.18).

As a preparation, we recall the constructions of $\mathbf{L}(X)$ and $\mathbf{R}(X)$ from the proof of [13, Theorem III.2.3].

Let $\mathfrak{A} := \{\underline{M}[n] \mid n \geq 0\} \subseteq \mathscr{X}$. Denote by $\mathrm{Add}(\mathfrak{A})$ the full subcategory of A-$\underline{\mathrm{Mod}}$ consisting of direct summands of arbitrary direct sums of objects of $\mathfrak{A}$. For a full subcategory $\mathscr{U}$ of A-$\underline{\mathrm{Mod}}$, we denote by $\mathscr{U}^{\star n} = \mathscr{U} \star \mathscr{U} \star \cdots \star \mathscr{U}$ (n-factors) the category of n-extensions of $\mathscr{U}$ by $\mathscr{U}$ in A-$\underline{\mathrm{Mod}}$.

Let $X \in A$-Mod. We construct a right $\mathrm{Add}(\mathfrak{A})$-approximation $f_1 : Q_1 \to X$ of X as follows: Consider the set I_X of the union of $\underline{\mathrm{Hom}}_A(P,X)$ with P running over $\mathfrak{A}$, define $Q_1 = \bigoplus_{\lambda\in I_X} P_\lambda$ and take f_1 to be the morphism induced by I_X, where λ is a morphism from P_λ to X with $P_\lambda \in \mathfrak{A}$. Then we extend f_1 to a triangle $Q_1 \xrightarrow{f_1} X \xrightarrow{g_1} X_1 \longrightarrow Q_1[1]$ in A-$\underline{\mathrm{Mod}}$. Now, we can repeat this construction by replacing X by $X_1 \in A$-Mod. In general, for each $n \geq 0$, we can inductively construct a triangle

$$Q_{n+1} \xrightarrow{f_{n+1}} X_n \xrightarrow{g_{n+1}} X_{n+1} \longrightarrow Q_{n+1}[1]$$

in A-$\underline{\mathrm{Mod}}$ such that f_{n+1} is a right $\mathrm{Add}(\mathfrak{A})$-approximation of X_n with $X_0 := X$. Setting $T_1 := Q_1$ and $h_1 := f_1$, we then construct inductively a tower of objects $T_1 \xrightarrow{\tau_1} T_2 \xrightarrow{\tau_2} T_3 \longrightarrow \cdots$ which is embedded into the following tower of triangles in A-$\underline{\mathrm{Mod}}$:

$$\begin{tikzcd}
 & T_1 \arrow[r, "h_1"] \arrow[d, "\tau_1"'] & X \arrow[r, "g_1"] \arrow[d, equal] & X_1 \arrow[r] \arrow[d, "g_2"'] & T_1[1] \arrow[d, "\tau_1{[1]}"'] \\
(\star\star) & T_2 \arrow[r, "h_2"] \arrow[d, "\tau_2"'] & X \arrow[r, "g_1g_2"] \arrow[d, equal] & X_2 \arrow[r] \arrow[d, "g_3"'] & T_2[1] \arrow[d, "\tau_2{[1]}"'] \\
 & T_3 \arrow[r, "h_3"] \arrow[d] & X \arrow[r, "g_1g_2g_3"] \arrow[d, equal] & X_3 \arrow[r] \arrow[d] & T_3[1] \arrow[d] \\
 & \vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

Applying the octahedral axiom for triangulated categories for each n yields a series of triangles

$$(\diamond\diamond)\quad T_n \xrightarrow{\tau_n} T_{n+1} \xrightarrow{\sigma_n} Q_{n+1} \longrightarrow T_n[1].$$

This implies $T_n \in \mathrm{Add}(\mathfrak{A})^{\star n} \subseteq \mathscr{X}$ for $n \geq 1$.

Let $\underrightarrow{\underline{\mathrm{Hocolim}}}(T_n)$ be the *homotopy colimit* in A-$\underline{\mathrm{Mod}}$ of the tower of objects

$$T_1 \xrightarrow{\tau_1} T_2 \xrightarrow{\tau_2} T_3 \xrightarrow{\tau_3} \cdots \longrightarrow T_n \xrightarrow{\tau_n} T_{n+1} \longrightarrow \cdots$$

defined by the triangle

$$(\sharp)\qquad \bigoplus_{n\geq 1} T_n \xrightarrow{(1-\tau_*)} \bigoplus_{n\geq 1} T_n \longrightarrow \underrightarrow{\underline{\mathrm{Hocolim}}}(T_n) \longrightarrow \bigoplus_{n\geq 1} T_n[1]$$

where the morphism $(1-\tau_*)$ is induced by $(\mathrm{Id}_{T_n}, -\tau_n) : T_n \to T_n \oplus T_{n+1} \hookrightarrow \bigoplus_{n\geq 1} T_n$. Now, we choose τ_n as a representative in A-Mod and denote by $\varinjlim T_n$ the colimit of the direct system $\{(T_n, \tau_n) \mid n \geq 1\}$ of A-modules. Then there is a short exact sequence

$$0 \to \bigoplus_{n\geq 1} T_n \xrightarrow{(1-\tau_*)} \bigoplus_{n\geq 1} T_n \longrightarrow \varinjlim T_n \longrightarrow 0$$

which induces a canonical triangle in A-$\underline{\mathrm{Mod}}$:

$$\bigoplus_{n\geq 1} T_n \xrightarrow{(1-\tau_*)} \bigoplus_{n\geq 1} T_n \longrightarrow \varinjlim T_n \longrightarrow \bigoplus_{n\geq 1} T_n[1].$$

This implies that $\underrightarrow{\mathrm{Hocolim}}(T_n) \simeq \varinjlim T_n$ in A-$\underline{\mathrm{Mod}}$. Since the homotopy colimit of a tower of triangles in A-$\underline{\mathrm{Mod}}$ is a triangle (for example, see [36, Lemma 4.2]), there is a triangle in A-$\underline{\mathrm{Mod}}$:

$$(\ddagger) \qquad \underrightarrow{\mathrm{Hocolim}}(T_n) \longrightarrow X \longrightarrow \underrightarrow{\mathrm{Hocolim}}(X_n) \longrightarrow \underrightarrow{\mathrm{Hocolim}}(T_n)[1].$$

By the proof of [13, Theorem III.2.3 and Remark III.2.7], we can show $\underrightarrow{\mathrm{Hocolim}}(T_n) \in \mathscr{X}$ and $\underrightarrow{\mathrm{Hocolim}}(X_n) \in \mathscr{Y}$. Since $(\mathscr{X}, \mathscr{Y})$ is a torsion pair in A-$\underline{\mathrm{Mod}}$, there are isomorphisms

$$\mathbf{R}(X) \simeq \underrightarrow{\mathrm{Hocolim}}(T_n) \ \text{ and } \ \mathbf{L}(X) \simeq \underrightarrow{\mathrm{Hocolim}}(X_n).$$

Recall that $T : \mathscr{D} \to \mathscr{C}$ stands for the left adjoint of the inclusion $\mathscr{C} \to \mathscr{D}$ (see Lemma 3.20(3)) and N is defined to be $H^0_*(M)$.

Lemma 4.15. (1) *If* $X \in {}^{\perp>0}M$, *then* $\mathbf{R}(X)$ *lies in* $\underline{\mathscr{G} \cap \mathscr{S}}$ *and is isomorphic in* $\mathscr{C}$ *to* $T(T_3)$.

(2) *If* $X \in \mathscr{G}$, *then* $\mathbf{L}(X) \in \underline{M}^{\perp}$. *If further* $X \in \mathscr{E}$, *then* $T_3 \in A$-$\underline{\mathrm{mod}}$.

(3) $\Omega^-_M(N) \simeq \Omega^-_A(N) \simeq T(\Omega^-_A(M))$ *in* $\mathscr{C}$.

Proof. (1) Recall from Section 4.2 that $\mathscr{X} = \underline{M}^{\perp>0} \cap \underline{\mathscr{S}}$ and $\mathscr{Y} = {}^{\perp\geq -1}\underline{M} \subseteq {}^{\perp>0}\underline{M} = \underline{{}^{\perp>0}M}$. Let $X \in {}^{\perp>0}M$. Since there is a triangle $\mathbf{R}(X) \to X \to \mathbf{L}(X) \to \mathbf{R}(X)[1]$ in A-$\underline{\mathrm{Mod}}$ with $\mathbf{R}(X) \in \mathscr{X}$ and $\mathbf{L}(X) \in \mathscr{Y}$, we see that $\mathbf{R}(X)$ lies in $\underline{{}^{\perp>0}M}$, and therefore in $\underline{\mathscr{G} \cap \mathscr{S}}$.

Set $\Lambda_n := \{i \in \mathbb{N} \mid \underline{\mathrm{Hom}}_A(M[i], X_n) \neq 0\}$ for $n \geq 0$. Since $\mathrm{add}({}_AM) = \mathrm{add}(\nu_A(M))$, it is clear that $\underline{{}^{\perp>0}M} = \underline{M}^{\perp\leq -2}$ by ($\Diamond$). This implies $\Lambda_0 \subseteq \{0,1\}$. So we can choose $Q_1 = M^{(I_{1,0})} \oplus M[1]^{(I_{1,1})}$ in A-$\underline{\mathrm{Mod}}$ for some index sets $I_{1,0}$ and $I_{1,1}$. For a natural number n, we apply $\underline{\mathrm{Hom}}_A(M[i], -)$ for $i \geq 0$ to the triangle

$$Q_{n+1} \xrightarrow{f_{n+1}} X_n \xrightarrow{g_{n+1}} X_{n+1} \longrightarrow Q_{n+1}[1]$$

in A-$\underline{\mathrm{Mod}}$. This yields an exact sequence of abelian groups:

$$\underline{\mathrm{Hom}}_A(M[i], Q_{n+1}) \xrightarrow{(f_{n+1})^*} \underline{\mathrm{Hom}}_A(M[i], X_n) \xrightarrow{(g_{n+1})^*} \underline{\mathrm{Hom}}_A(M[i], X_{n+1}) \longrightarrow \underline{\mathrm{Hom}}_A(M[i], Q_{n+1}[1]).$$

Since f_{n+1} is a right $\mathrm{Add}(\mathfrak{A})$-approximation of X_n, the map $(f_{n+1})^*$ is always surjective, and therefore there is an injection $\underline{\mathrm{Hom}}_A(M[i], X_{n+1}) \hookrightarrow \underline{\mathrm{Hom}}_A(M[i], Q_{n+1}[1])$ for $i \geq 0$. Using the fact $\mathrm{Add}(\underline{M}) \subseteq \underline{\mathscr{G}} = \underline{M}^{\perp\neq 0,-1}$, we then can show $\Lambda_n \subseteq \{i \in \mathbb{N} \mid n \leq i \leq 2n+1\}$ by induction on n and choose $Q_{n+1} = \bigoplus_{j\in\Lambda_n} M[j]^{(I_{n+1,j})}$ for some index sets $I_{n+1,j}$.

Consider $n \geq 3$. Then $\underline{\mathrm{Hom}}_A(Q_{n+1}, M[1]) = 0$. It follows from ($\diamond\diamond$) and Lemma 3.16(2) that there is a triangle $T_n \to T_{n+1} \to Q_{n+1} \to \Omega^-_M(T_n)$ in $\mathscr{D}$. We apply the functor T to this triangle and produce another triangle $T(T_n) \to T(T_{n+1}) \to T(Q_{n+1}) \to \Omega^-_M(T(T_n))$ in $\mathscr{C}$. Since $\underline{\mathrm{Hom}}_A(M[m], \mathscr{G}) = 0$ for any $m \geq 2$, the minimal injective envelope $M[m] \to I^m$ of $M[i]$ is a left $\mathscr{G}$-approximation of $M[m]$. This means $T(M[m]) \simeq I^m = 0$ in $\mathscr{C}$ for any $m \geq 2$. Since T commutes with direct sums, we have $T(Q_{n+1}) \simeq \bigoplus_{j\in\Lambda_n} T(M[j])^{(I_{n+1,j})} = 0$. Thus $T(T_n) \simeq T(T_{n+1})$. Consequently, the homotopy colimit $\underrightarrow{\mathrm{Hocolim}}(T(T_n))$ in $\mathscr{C}$ of $\{(T(T_n), T(\tau_n) \mid n \geq 1\}$ is isomorphic to $T(T_3)$. Moreover, due to $\mathbf{R}(X) \in {}^{\perp 1}M$, we see from Lemma 3.16(2) that $\mathbf{R}(X)$ is also the homotopy colimit in $\mathscr{D}$ of $\{T_n\}$. Since $\mathbf{R}(X) \in \mathscr{G}$ and T commutes with homotopy colimits by Lemma 3.20(3), there are isomorphisms $\mathbf{R}(X) \simeq T(\mathbf{R}(X)) \simeq \underrightarrow{\mathrm{Hocolim}}(T(T_n)) \simeq T(T_3)$ in $\mathscr{C}$.

(2) If $X \in \mathscr{G}$, then $\mathbf{L}(X) \in \underline{M}^{\perp>0}$. This implies $\mathbf{L}(X) \in \underline{M}^{\perp}$ because $\mathbf{L}(X) \in \mathscr{Y} = \underline{M}^{\perp\leq 0}$. Let

$$\mathscr{A} := \{U \in A\text{-}\underline{\mathrm{Mod}} \mid \bigoplus_{n=0}^{\infty} \underline{\mathrm{Hom}}_A(M[n],U) \in \Gamma\text{-mod}\}.$$

Since Γ is an Artin algebra over a commutative Artin ring R, a Γ-module N is finitely generated if and only if N is finitely generated as an R-module. This implies that an A-module U lies in $\mathscr{A}$ if and only if $\bigoplus_{n=0}^{\infty} \underline{\mathrm{Hom}}_A(M[n],U) \in R$-mod. The latter is equivalent to saying that there is a non-negative integer δ_U such that $\underline{\mathrm{Hom}}_A(M[n],U) = 0$ for $n > \delta_U$ and $\underline{\mathrm{Hom}}_A(M[n],U) \in R$-mod for $0 \leq n \leq \delta_U$. Moreover, $\mathscr{A}$ is closed under direct summands, finite direct sums, the shift $[-1]$ and extensions of triangles in A-$\underline{\mathrm{Mod}}$. Now, it follows from $M \in \underline{M}^{\perp\neq 0,-1}$ and $\mathrm{Hom}_A(M[1],M) \simeq D\underline{\mathrm{Hom}}_A(M,\nu_A(M))$ that $M[j] \in \mathscr{A}$ for any $j \in \mathbb{Z}$. Consequently, for $U \in \mathscr{A}$, there is a right $\mathrm{Add}(\mathfrak{A})$-approximation $f_U : Q_U \to U$ of U such that $Q_U \in \mathrm{add}(\bigoplus_{n=0}^{m} M[n])$ for some $m \geq 0$ and the third term C_U of the triangle $Q_U \xrightarrow{f_U} U \to C_U \to Q_U[1]$ in A-$\underline{\mathrm{Mod}}$ still lies in $\mathscr{A}$.

By Lemma 4.2, there holds $\underline{\mathscr{E}} \subseteq \mathscr{A}$. If $X \in \mathscr{E}$, then $I_{n+1,j}$ can be chosen to be finite sets and $Q_{n+1} \in \mathrm{add}(\bigoplus_{j\in\Lambda_n} M[j]) \subseteq A$-$\underline{\mathrm{mod}}$.

(3) In the proof of (1), we take $X = N$. Then $X \in \mathscr{E}$ by Theorem 4.14(1). Recall that $X \in \mathscr{H} \subseteq \underline{M}^{\perp\neq 0}$ by Proposition 4.1 and that $\underline{\mathrm{Hom}}_A(M,M) \simeq \underline{\mathrm{Hom}}_A(M,X)$ by Lemma 4.7(1). Further, we even have

(a) $\Lambda_0 \subseteq \{0\}$, $Q_1 = M$ and $X_1[-2] \in \mathscr{H}$ (see the proof of Lemma 4.10), and

(b) $\Lambda_n \subseteq \{i \in \mathbb{N} \mid n+1 \leq i \leq 2n\}$ and $Q_{n+1} \in \mathrm{add}(\bigoplus_{i=n+1}^{2n} M[i]) \in A$-$\underline{\mathrm{mod}}$ for $n \geq 1$.

Since M lies in $\underline{\mathscr{G}} = \underline{M}^{\perp\neq 0,-1}$, there holds $\underline{\mathrm{Hom}}_A(Q_3, M[1]) = 0$. By Lemma 3.16(2), there exists a right triangle $T_2 \xrightarrow{\tau_2} T_3 \xrightarrow{\sigma_2} Q_3 \to \Omega_M^-(T_2)$ in $\mathscr{D}$. This gives rise to another triangle $T(T_2) \to T(T_3) \to T(Q_3) \to \Omega_M^-(T(T_2))$ in $\mathscr{C}$. Clearly, $T(Q_3) = 0$ by $Q_3 \in \mathrm{add}(M[3] \oplus M[4])$. Thus $T(T_2) \simeq T(T_3)$. It then follows from (1) and $X \in \mathscr{H} \subseteq \mathscr{X}$ that $X \simeq \mathbf{R}(X) \simeq T(T_2)$ in $\mathscr{C}$.

By Lemma 4.10, $X[1] = \Omega_A^-(X) \simeq \Omega_M^-(X)$ in $\mathscr{C}$. Further, we will show $T_2[1] = \Omega_A^-(T_2) \simeq \Omega_M^-(T_2)$ in $\mathscr{C}$. Actually, it suffices to prove that any homomorphism from T_2 to a module in $\mathrm{Add}(M)$ factorizes through the injective envelope of T_2, or equivalently, $\underline{\mathrm{Hom}}_A(T_2, M) = 0$. Thanks to $\mathrm{add}(M) = \mathrm{add}(\nu_A(M))$, we will show $\underline{\mathrm{Hom}}_A(T_2, \nu_A(M)) = 0$.

Indeed, $\underline{\mathrm{Hom}}_A(T_2, \nu_A(M)) \simeq D\underline{\mathrm{Hom}}_A(M[1], T_2)$ by ($\Diamond$). We apply $\underline{\mathrm{Hom}}_A(M[1], -)$ to the triangle $X_2[-1] \to T_2 \to X \to X_2$ (see the diagram ($\star\star$)) and obtain an isomorphism $\underline{\mathrm{Hom}}_A(M[1], X_2[-1]) \simeq \underline{\mathrm{Hom}}_A(M[1], T_2)$. Here, we use the fact $X \in \mathscr{H} \subseteq \underline{M}^{\perp\neq 0}$. So it is enough to show $\underline{\mathrm{Hom}}_A(M[1], X_2[-1]) = 0$. Recall that the triangle $Q_2 \xrightarrow{f_2} X_1 \to X_2 \to Q_2[1]$ has the following properties: $Q_2 \in \mathrm{add}(M[2])$ by (b), $X_1 \in \mathscr{H}[2]$ by (a), and there exists an injection $\underline{\mathrm{Hom}}_A(M[2], X_2) \hookrightarrow \underline{\mathrm{Hom}}_A(M[2], Q_2[1])$ by the proof of (1). It follows from $\underline{\mathrm{Hom}}_A(M, M[1]) = 0$ that $\underline{\mathrm{Hom}}_A(M[1], X_2[-1]) \simeq \underline{\mathrm{Hom}}_A(M[2], X_2) = 0$. Thus $\underline{\mathrm{Hom}}_A(M[1], T_2) = 0$, and therefore $\underline{\mathrm{Hom}}_A(T_2, \nu_A(M)) = 0$.

By the facts that $Q_2 \in \mathrm{add}(M[2])$ and $\underline{\mathrm{Hom}}_A(M[2], M) = 0$, we have

$$\mathrm{Ext}_A^1(Q_2[1], M) \simeq \underline{\mathrm{Hom}}_A(Q_2[1], M[1]) \simeq \underline{\mathrm{Hom}}_A(Q_2, M) = 0.$$

By Lemma 3.16(2), the triangle $M[1] \xrightarrow{\tau_1[1]} T_2[1] \xrightarrow{\sigma_1[1]} Q_2[1] \longrightarrow M[2]$ in A-$\underline{\mathrm{Mod}}$ can be extended to a right triangle $M[1] \xrightarrow{\tau_1[1]} T_2[1] \xrightarrow{\sigma_1[1]} Q_2[1] \longrightarrow \Omega_M^-(M[1])$ in $\mathscr{D}$. This yields the triangle in $\mathscr{C}$

$$T(M[1]) \xrightarrow{T(\tau_1[1])} T(T_2[1]) \xrightarrow{T(\sigma_1[1])} T(Q_2[1]) \longrightarrow \Omega_M^-(T(M[1]))$$

by Lemma 3.20(3). It then follows from $T(M[3]) = 0$ that $T(Q_2[1]) = 0$ and $T(M[1]) \simeq T(T_2[1])$ in $\mathscr{C}$. Moreover, $T(T_2[1]) \simeq T(\Omega_M^-(T_2)) \simeq \Omega_M^-(T(T_2)) \simeq \Omega_M^-(X)$ in $\mathscr{C}$, where the second isomorphism is due to the fact that T commutes with Ω_M^-. Thus $T(\Omega_A^-(M)) = T(M[1]) \simeq \Omega_M^-(X)$ in $\mathscr{C}$. $\square$

Now, we state a property of finitely M-filtered A-modules introduced in Definition 1.1.

Lemma 4.16. *If X is a finitely M-filtered A-module, then $X \in \mathscr{G}^{\perp>0}$ and has an* $\mathrm{add}(M)$*-resolution of finite length. Moreover, if the A-module X lies in $\mathscr{G}$, then $X \in \mathrm{add}(M)$.*

Proof. Since $\mathscr{G}^{\perp>0}$ contains M and is closed under both cosyzygies and extensions in A-Mod, we have $X \in \mathscr{G}^{\perp>0}$. Now, let $\mathscr{M}$ be the full subcategory of A-mod consisting of all those modules having an $\mathrm{add}(M)$-resolution of finite length. Then $Y \in \mathscr{M}$ if and only if $\mathrm{Hom}_A(M,Y) \in \mathscr{P}^{<\infty}(\Lambda)$, the category of all finitely generated Λ-modules of finite projective dimension. Since ${}_AM$ is a self-orthogonal generator, $Y \in \mathscr{M}$ if and only if there is an exact sequence $0 \to M_n \to M_{n-1} \to \cdots \to M_0 \to Y \to 0$ in A-mod for some $n \in \mathbb{N}$ such that $M_j \in \mathrm{add}(M)$ for all $0 \leq j \leq n$. Clearly, $\mathscr{M} \subseteq M^{\perp>0} \subseteq M^{\perp 1}$. As $\mathscr{P}^{<\infty}(\Lambda)$ is always closed under extensions in Λ-mod, $\mathscr{M}$ is closed under extensions in A-mod. So, to show that the finitely M-filtered module X belongs to $\mathscr{M}$, it suffices to show $\Omega_A^{-i}(M) \in \mathscr{M}$ for each $i \geq 0$. However, this follows from the exact sequence $0 \to M \to I^0 \to \cdots \to I^{i-1} \to \Omega_A^{-i}(M) \to 0$, where I^j is injective and therefore in $\mathrm{add}(M)$ for $0 \leq j \leq i-1$. As to the last statement in the lemma, we notice $\mathscr{M} \cap {}^{\perp>0}M = \mathrm{add}(M)$, $\mathscr{G} \subseteq {}^{\perp>0}M$ and $\mathscr{G} \cap \mathscr{M} = \mathrm{add}(M)$. $\square$

In the following, we describe zero objects and compact objects in $\mathscr{C}$. This is related to pure-projective modules. It is known that a module over an Artin algebra is *pure-projective* if and only if it is a direct sum of finitely generated, indecomposable modules.

Proposition 4.17. *The following hold for $X \in \mathscr{E} \cap \mathscr{S}$.*

(1) *X is compact in $\mathscr{C}$ and isomorphic in A-$\underline{\mathrm{Mod}}$ to an M-filtered module.*

(2) *If $X^{\perp 1}$ is closed under direct sums of countably many, finitely M-filtered A-modules in* A-Mod*, then $X \in$* $\mathrm{Add}(M)$*. In particular, if X is pure-projective, then $X \in$* $\mathrm{Add}(M)$.

(3) *If $X \in \mathscr{H} \cap A$-$\underline{\mathrm{mod}}$, then X is projective.*

Proof. Let $X \in \mathscr{E} \cap \mathscr{S}$. We keep all notations in the proof of Lemmas 4.15(1)-(2).

(1) By Lemma 4.2, $X \in \mathscr{X}$ and $X \simeq \mathbf{R}(X)$ in A-$\underline{\mathrm{Mod}}$. Moreover, by Lemmas 4.15(1)-(2), $\mathbf{R}(X) \simeq T(T_3)$ in $\mathscr{C}$ with $T_3 \in A$-$\underline{\mathrm{mod}}$. Each object of A-$\underline{\mathrm{mod}}$ is compact in A-$\underline{\mathrm{Mod}}$ and the quotient functor A-$\underline{\mathrm{Mod}} \to \mathscr{D}$ preserves compact objects by Lemma 2.1(2). Hence T_3 is compact in $\mathscr{D}$. As T preserves compact objects by Lemma 3.20(3), both $T(T_3)$ and X are compact in $\mathscr{C}$.

It follows from $X \in \mathscr{E}$ that the proofs of Lemmas 4.15(1)-(2) yield $Q_{n+1} = \bigoplus_{j\in\Lambda_n} M[j]^{(I_{n+1,j})}$, where $\Lambda_n \subseteq \{i \in \mathbb{N} \mid n \leq i \leq 2n+1\}$, $M[j] = \Omega_A^{-j}(M)$ and $I_{n+1,j}$ are finite sets. This implies $T_n \in A$-$\underline{\mathrm{mod}}$ for $n \geq 1$. Since triangles in A-$\underline{\mathrm{mod}}$ (up to isomorphism) are induced from short exact sequences, we can add finitely generated projective modules to T_n and assume that the associated map $\tau_n : T_n \to T_{n+1}$ is injective and $\mathrm{Coker}(\tau_n)$ is isomorphic to a finite direct sum of modules in $\{{}_AA\} \cup \{\Omega_A^{-i}(M) \mid i \in \mathbb{N}\}$. Now, let X' be the colimit of the direct system $\{(T_n, \tau_n) \mid n \geq 1\}$ in A-Mod. Then the canonical map $T_n \to X'$ is injective. Thus T_n can be regarded as a submodule of X' and $T_n \subseteq T_{n+1}$. Moreover, $X' = \bigcup_{n=0}^{\infty} T_n$ and is M-filtered. Now (1) follows from $X \simeq \mathbf{R}(X) \simeq X'$ in A-$\underline{\mathrm{Mod}}$.

(2) Since all T_n are finitely M-filtered and $X \in \mathscr{G}$, we have $T_n \in X^{\perp 1}$ by Lemma 4.16. Let $Z := \bigoplus_{n\geq 1} T_n$. The assumption of (2) implies $\mathrm{Ext}_A^1(X,Z) = 0$. So the exact sequence $0 \to Z \xrightarrow{(1-\tau_*)} Z \to X' \to 0$ induced by $\{\tau_n \mid n \geq 1\}$ splits, and therefore X' is isomorphic to a direct summand of Z. Since T_n is a finitely generated A-module, it is a direct sum of finitely many indecomposable submodules with the local endomorphism rings. Thus, by [1, Corollary 26.6], we have $X' \simeq \bigoplus_{n\geq 1} T_n'$, where T_n' is a direct summand of T_n. Moreover, by Lemma 4.16, T_n has an $\mathrm{add}(M)$-resolution of finite length, and so does T_n'. It follows from $X \in \mathscr{G}$ and $X \simeq X'$ in A-$\underline{\mathrm{Mod}}$ that $X' \in \mathscr{G}$, and therefore $T_n' \in \mathscr{G}$. Consequently, $T_n' \in \mathrm{add}(M)$ by the proof of Lemma 4.16. Hence $X' \in \mathrm{Add}(M)$ and $X \in \mathrm{Add}(M)$.

For any $V \in A$-mod, V is clearly finitely presented, and therefore $V^{\perp 1}$ is always closed under arbitrary direct sums in A-Mod. This implies that if X is pure-projective, then $X^{\perp 1}$ is closed under arbitrary direct sums in A-Mod, and therefore $X \in \mathrm{Add}(M)$.

(3) By Lemmas 4.2(1) and 4.4(1), $\mathscr{H}\cap A\text{-}\underline{\mathrm{mod}}\subseteq\underline{\mathscr{E}\cap\mathscr{S}}$. It follows from $X\in A\text{-}\underline{\mathrm{mod}}$ that $X^{\perp 1}$ is closed under arbitrary direct sums in A-Mod. Further, $X\in\mathrm{Add}(M)$ by (2), and $\underline{\mathrm{Hom}}_A(N,M)=0$ by Lemma 4.10. Due to $\mathrm{Add}(M)=\mathrm{Prod}(M)$, we have $\underline{\mathrm{Hom}}_A(N,X)=0$. It follows from Lemma 4.7(3) and $X\in\mathscr{H}$ that $X\simeq 0$ in $A\text{-}\underline{\mathrm{Mod}}$. Thus X is projective. $\square$

Now, we describe compact generators of the right-hand side in the recollement of Theorem 1.5.

Let $\mathcal{T}$ be a triangulated category. For a full subcategory $\mathcal{U}$ of $\mathcal{T}$, we denote by $\langle\mathcal{U}\rangle$ the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{U}$ and being closed under finite coproducts, direct summands and shifts. Let $\mathcal{V}$ be another full subcategory of $\mathcal{T}$. Recall that $\mathcal{U}\star\mathcal{V}$ is the full subcategory of $\mathcal{T}$ consisting of objects X such that there is a triangle $U\to X\to V\to U[1]$ with $U\in\mathcal{U}$ and $V\in\mathcal{V}$. Following [38, 3.1], we set $\mathcal{U}\diamond\mathcal{V}:=\langle\mathcal{U}\star\mathcal{V}\rangle$, and then define $\langle\mathcal{U}\rangle_0:=0$ and $\langle\mathcal{U}\rangle_{n+1}:=\langle\mathcal{U}\rangle_n\diamond\langle\mathcal{U}\rangle$ for $n\geq 0$ inductively. Clearly, the objects of $\langle\mathcal{U}\rangle_{n+1}$ are the *direct summands* of the objects obtained by taking $(n+1)$-fold extensions of finite direct sums of shifts of objects of $\mathcal{U}$. Following [38, Definition 3.1], the *dimension* of $\mathcal{T}$, denoted by $\dim(\mathcal{T})$, is defined to be the minimal natural number n such that there is an object $X\in\mathcal{T}$ with $\mathcal{T}=\langle X\rangle_{n+1}$. If no such X exists, one defines $\dim(\mathcal{T})=\infty$. If $\dim(\mathcal{T})$ is finite, then such an object X is called a *strong generator* of $\mathcal{T}$.

For a pair of integers $i\leq j$, we denote by $\langle\mathcal{U}\rangle_{n+1}^{[i,j]}$ the full subcategory of $\langle\mathcal{U}\rangle_{n+1}$ consisting of all objects which are obtained by taking $(n+1)$-fold extensions of finite direct sums of objects in the class $\{U[-s]\mid U\in\mathcal{U}, s\in\mathbb{Z}, i\leq s\leq j\}$. Here, we do not require taking both direct summands and arbitrary shifts in $\langle\mathcal{U}\rangle_{n+1}^{[i,j]}$.

Corollary 4.18. (1) *Let $\mathcal{S}$ be the finite set of isomorphism classes of simple objects of $\mathscr{H}$, and let n be the Loewy length of the algebra Γ. Then $(\mathscr{G}\cap\mathscr{S})/[M]$ is a compactly generated triangulated category:*

$$\big(\mathscr{G}\cap\mathscr{S}\big)/[M]=\langle\mathrm{Add}(\mathcal{S})\rangle_{2n}^{[-1,0]}\ \ \textit{and}\ \ \big((\mathscr{G}\cap\mathscr{S})/[M]\big)^{\mathrm{c}}=(\mathscr{E}\cap\mathscr{S})/[M]=\langle\mathcal{S}\rangle_{2n}^{[-1,0]}.$$

(2) *If A is symmetric and Γ is semisimple, then there are triangle equivalences*

$$(\mathscr{G}\cap\mathscr{S})/[M]=\mathrm{Add}\big(N\oplus\Omega_A^-(N)\big)\xrightarrow{\simeq}(\Gamma\times\Gamma)\text{-Mod},$$

$$(\mathscr{E}\cap\mathscr{S})/[M]=\mathrm{add}\big(N\oplus\Omega_A^-(N)\big)\xrightarrow{\simeq}(\Gamma\times\Gamma)\text{-mod},$$

where $(\Gamma\times\Gamma)$-Mod, as a triangulated category, has the shift functor induced from the automorphism of the algebra $\Gamma\times\Gamma$ by permutating the first and second coordinate.

(3) *If $T(X)$ lies in $\mathscr{E}/[M]$ for all $X\in A$-mod, then the recollement in Proposition 4.3 restricts to a half recollement of triangulated categories:*

$$\begin{tikzcd}(\underline{M}^{\perp})^{\mathrm{c}} \arrow[r, "\pi\circ\mathrm{inc}"] & \mathscr{C}^{\mathrm{c}} \arrow[l, bend right, "\widetilde{\Psi}"'] \arrow[r, "\widetilde{\Phi}"] & (\mathscr{E}\cap\mathscr{S})/[M]. \arrow[l, bend right, "\mathrm{inc}"']\end{tikzcd}$$

Proof. (1) Let $\mathscr{R}:=(\mathscr{G}\cap\mathscr{S})/[M]$. By Lemma 4.2(2)(b), we can identify $\mathscr{H}$ with its (essential) image in $\mathscr{R}$ under the quotient functor $\underline{\mathscr{G}\cap\mathscr{S}}\to\mathscr{R}$. It follows from the first part of Corollary 4.8(3) that $\mathscr{R}=\mathscr{H}[1]\star\mathscr{H}$, where $[1]$ denotes the functor Ω_M^-. To characterize $\mathscr{R}$ in terms of $\mathcal{S}$, we use the functor $\underline{\mathrm{Hom}}_A(N,-):\mathscr{H}\to\Gamma$-Mod which is an equivalence of abelian categories sending N to Γ, due to Lemma 4.7(2)(3). Recall that $\mathscr{H}^{\mathrm{fg}}:=\{X\in\mathscr{H}\mid\underline{\mathrm{Hom}}_A(M,X)\in\Gamma\text{-mod}\}=\mathscr{H}\cap\underline{\mathscr{E}}$. By the proof of Corollary 4.8(1), the functor $\underline{\mathrm{Hom}}_A(N,-)$ restricts to an equivalence: $\mathscr{H}^{\mathrm{fg}}\xrightarrow{\simeq}\Gamma$-mod. This equivalence clearly sends simple objects of $\mathscr{H}$ to simple Γ-modules. Since Γ is an Artin algebra, it has only finitely many isomorphism classes of simple modules and each (respectively, finitely generated) Γ-module is generated by simple modules under arbitrary (respectively, finite) direct sums and taking n-fold extensions. Consequently, $\mathcal{S}$ is a finite set and each object of $\mathscr{H}$ (respectively, $\mathscr{H}^{\mathrm{fg}}$) is generated by $\mathcal{S}$ under arbitrary (respectively, finite) direct sums and taking n-fold extensions. Note that each short exact sequence in $\mathscr{H}$ induces a triangle in $\mathscr{R}$ by Lemma 3.16(2). Now, thanks to $\mathscr{R}=\mathscr{H}[1]\star\mathscr{H}$, the first equality in (1) holds, and therefore $\mathscr{R}=\mathrm{Loc}(\mathcal{S})$. Similarly, from the equality $(\mathscr{E}\cap\mathscr{S})/[M]=\mathscr{H}^{\mathrm{fg}}[1]\star\mathscr{H}^{\mathrm{fg}}$ by the second part of Corollary 4.8(3), we see that $(\mathscr{E}\cap\mathscr{S})/[M]$ is generated by $\mathcal{S}$ under taking the shifts $[i]$ for $i=0,1$ and $(2n)$-fold extensions. This implies the third equality in (1). By Proposition 4.17(1), each object of $\mathscr{H}^{\mathrm{fg}}$ is compact in $\mathscr{C}$ and thus also in $\mathscr{R}$. Hence $\mathscr{R}$ is compactly generated. By [34, Theorem 4.4.9], we have $\mathscr{R}^{\mathrm{c}}=\mathrm{thick}(\mathcal{S})$. Clearly, $\mathrm{thick}(\mathcal{S})\subseteq(\mathscr{E}\cap\mathscr{S})/[M]=\langle\mathcal{S}\rangle_{2n}^{[-1,0]}\subseteq\mathrm{thick}(\mathcal{S})$. Thus $\mathscr{R}^{\mathrm{c}}=\langle\mathcal{S}\rangle_{2n}^{[-1,0]}$.

(2) Suppose that A is symmetric and Γ is semisimple. Then Γ is symmetric. By Corollary 4.11, $\Omega_M^2(N)\simeq N$ in $\mathscr{C}$. Moreover, by Lemma 4.10, $\Omega_M^-(N)\simeq\Omega_A^-(N)$ in $\mathscr{C}$. It follows that $\Omega_M^{2i}(N)\simeq N$ and $\Omega_M^{2i+1}(N)\simeq\Omega_A^-(N)$ for any $i\in\mathbb{Z}$. Let $W:=N\oplus\Omega_A^-(N)$ and $\mathscr{W}:=\mathrm{Add}(W)$. Then $\mathscr{W}=\Omega_M(\mathscr{W})$. Clearly, $\mathscr{W}$ contains N and is closed under direct sums in $\mathscr{C}$. As Γ is semisimple, there holds $\mathrm{add}(N)=\mathrm{add}(\mathcal{S})\subseteq\mathscr{R}$. To show $\mathscr{R}=\mathscr{W}$, it suffices to show that $\mathscr{W}$ is a triangulated subcategory of $\mathscr{C}$. Since $\mathscr{W}$ is closed under Ω_M, we only need to show that, for any morphism $f: X_1\to X_2$ in $\mathscr{W}$ and triangle $X_1\xrightarrow{f}X_2\to X_3\to\Omega_M^-(X_1)$ in $\mathscr{C}$, the term X_3 belongs to $\mathscr{W}$.

Since $\mathscr{H}$ is the heart of the torsion pair $(\mathscr{X},\mathscr{Y})$ in A-$\underline{\mathrm{Mod}}$, $\mathrm{Ext}^j_{\mathscr{H}}(U,V)\simeq\underline{\mathrm{Hom}}_A(U,V[j])$ for any $U,V\in\mathscr{H}$ and $j=0,1$. Since N is a projective object in $\mathscr{H}$ by Lemma 4.7(3), we have $\underline{\mathrm{Hom}}_A(N,N[1])\simeq\mathrm{Ext}^1_{\mathscr{H}}(N,N)=0$. Further, by Lemma 3.17 and [16, Lemma 3.5], if V_1 and V_2 lie in $\mathscr{G}$, then $\underline{\mathrm{Hom}}_A(V_1,V_2[n])\simeq\underline{\mathrm{Hom}}_M(V_1,\Omega_M^{-n}(V_2))$ for all $n\geq 1$. This implies $\underline{\mathrm{Hom}}_A(N,N[1])\simeq\underline{\mathrm{Hom}}_M(N,\Omega_M^-(N))$, and therefore $\underline{\mathrm{Hom}}_M(N,\Omega_M^-(N))=0=\underline{\mathrm{Hom}}_M(\Omega_M^-(N),N)$. Let $B:=\underline{\mathrm{End}}_A(W)$. Then $B\simeq\Gamma\oplus\Gamma$ as algebras. Since B is semisimple and W is compact in $\mathscr{C}$ by Lemma 4.14(1), the functor $\underline{\mathrm{Hom}}_M(W,-):\mathscr{W}\to B$-Mod is an additive equivalence. By this equivalence and the fact that B-Mod is a semisimple abelian category, it can be proved that f as a morphism in $\mathscr{W}$ is isomorphic to a direct sum of the identity map of Z_1 with the zero map $Z_2\to Z_3$, where Z_i lies in $\mathscr{W}$ for $1\leq i\leq 3$. Consequently, $X_3\simeq\Omega_M^-(Z_2)\oplus Z_3$. It then follows from $\mathscr{W}=\Omega_M(\mathscr{W})$ that $X_3\in\mathscr{W}$. Thus $\mathscr{W}$ is a triangulated subcategory of $\mathscr{C}$, and therefore $\mathscr{R}=\mathscr{W}$. By (1), $\mathscr{R}^{\mathrm{c}}=\mathrm{add}(W)$.

(3) By Lemma 3.20(4) and [34, Theorem 4.4.9], the assumption of Corollary 4.18(3) implies $\mathscr{C}^{\mathrm{c}}\subseteq\mathscr{E}/[M]$. Combining Corollary 4.5 with Proposition 4.17(1), we see that $\widetilde{\Phi}$ sends compact objects of $\mathscr{C}$ to objects of $(\mathscr{E}\cap\mathscr{S})/[M]$ which are also compact in $\mathscr{C}$. Note that $\widetilde{\Psi}:\mathscr{C}\to\underline{M}^{\perp}$ always preserves compact generating sets. Thus the first two lines of functors in the recollement of Corollary 4.5 restrict to the half recollement in Corollary 4.18(3). □

Corollary 4.19. (1) $\dim\big((\mathscr{E}\cap\mathscr{S})/[M]\big)\leq 2\,LL(\Gamma)-1$, *where* $LL(\Gamma)$ *is the Loewy length of* Γ.

(2) $\dim\big((\mathscr{E}\cap\mathscr{S})/[M]\big)\leq 2\,\mathrm{gl.dim}(\Gamma)+1$, *where* $\mathrm{gl.dim}(\Gamma)$ *is the global dimension of* Γ.

Proof. (1) follows from Corollary 4.18(1).

(2) If $\mathrm{gl.dim}(\Gamma)$ is infinite, then the inequality in (2) holds trivially. Now, let $m:=\mathrm{gl.dim}(\Gamma)<\infty$. In the following, we follow the notation in the proof of Corollary 4.18(1) and show that $(\mathscr{E}\cap\mathscr{S})/[M]=\langle\mathrm{add}(N)\rangle_{2m+2}^{[-m-1,0]}$. This implies $\dim\big((\mathscr{E}\cap\mathscr{S})/[M]\big)\leq 2m+1$.

By the second part of Corollary 4.8(3), $(\mathscr{E}\cap\mathscr{S})/[M]=\mathscr{H}^{\mathrm{fg}}[1]\star\mathscr{H}^{\mathrm{fg}}$. So, it suffices to control the objects of $\mathscr{H}^{\mathrm{fg}}$ by N. We take an object $X\in\mathscr{H}^{\mathrm{fg}}$. Since the functor $\underline{\mathrm{Hom}}_A(N,-):\mathscr{H}^{\mathrm{fg}}\to\Gamma$-mod is an equivalence of abelian categories sending N to Γ and since each finitely generated Γ-module has a projective resolution of length m by finitely generated projective Γ-modules, there is a long exact sequence $(*):\ 0\to N_m\to\cdots\to N_1\to N_0\to X\to 0$ in $\mathscr{H}^{\mathrm{fg}}$ with $N_i\in\mathrm{add}(N)$ for $0\leq i\leq m$. Note that each short exact sequence $0\to X\to Y\to Z\to 0$ in $\mathscr{H}$ gives rise to a triangle $X\to Y\to Z\to\Omega_A^{-1}(X)$ in A-$\underline{\mathrm{Mod}}$ with the terms $X,Y,Z\in\mathscr{H}$. Since $\mathscr{H}\subseteq\underline{\mathscr{G}\cap\mathscr{S}}\subseteq{}^{\perp_1}\underline{M}$ by Lemma 4.2(1), this triangle induces a triangle $X\to Y\to Z\to X[1]$ in $\mathscr{C}$ by Lemma 3.16(2). So, we divide the sequence $(*)$ into a series of short exact sequences in $\mathscr{H}^{\mathrm{fg}}$ and then obtain $X\in\langle\mathrm{add}(N)\rangle_{m+1}^{[-m,0]}$. Thus $(\mathscr{E}\cap\mathscr{S})/[M]=\langle\mathrm{add}(N)\rangle_{2m+2}^{[-m-1,0]}$. □

Proof of Theorem 1.5. The existence of the recollement in (1) follows from Proposition 4.3, while (2) and (3) are exactly Corollaries 4.5 and 4.19, respectively. $\square$

Proof of Proposition 1.6. (2) and (3) follow from Corollary 4.18(1), while the first part of (1) is Proposition 4.17(1). Let $\mathscr{E}_0 := \mathscr{E} \cap \mathscr{S} \cap A$-mod. Clearly, $\mathrm{add}(M) \subseteq \mathscr{E}_0$. If $X \in \mathscr{E}_0$, then $X \in \mathrm{Add}(M)$ by Proposition 4.17(2) because finitely generated A-modules are pure-projective. As X is finitely generated, it lies in $\mathrm{add}(M)$. Thus $\mathscr{E}_0 = \mathrm{add}(M)$. This implies the second part of (1). $\square$

5 Tachikawa's second conjecture

In this section we prove Theorem 1.2 and Corollary 1.4. As a consequence, we show that the Nakayama conjecture is true for Gorenstein-Morita algebras (see Definition 1.3(ii)). Moreover, we introduce two homological conditions (CI) and (CII)). They are connected with finitistic dimension (see Lemma 5.1). Also, the invariance of (CII) under different types of equivalences between algebras is discussed in Corollary 5.3.

Let B be an Artin algebra. The *dominant dimension* of B, denoted by $\mathrm{dom.dim}(B)$, is by definition the largest natural number n or ∞ such that, in a minimal injective coresolution $0 \to {}_BB \to I^0 \to I^1 \to \cdots \to I^n \to \cdots$, all these module I^i are projective for $0 \leq i < n$. The unsolved Nakayama Conjecture says that an Artin algebra is self-injective whenever its dominant dimension is infinite. Related to this conjecture, Tachikawa proposed two conjectures in [41, p. 115-116].

(TC1): If an Artin algebra B satisfies $\mathrm{Ext}^n_B(D(B), B) = 0$ for all $n \geq 1$, then B is self-injective.

(TC2): Let B be a self-injective algebra and Y a finitely generated B-module. If $\mathrm{Ext}^n_B(Y,Y) = 0$ for all $n \geq 1$, then Y is projective.

As pointed in the introduction, the two conjectures **(TC1)** and **(TC2)** hold true for all algebras if and only if so does the Nakayam conjecture for all algebras. Moreover, it was shown in [32] that, given a pair (B,Y) with B a self-injective algebra and Y a finitely generated, self-orthogonal B-module, the algebra $\mathrm{End}_B(B \oplus Y)$ satisfies the Nakayama conjecture if and only if Y is projective.

By [32], algebras of dominant dimension at least 2 are exactly endomorphism algebras of generator-cogenerators over algebras. Recall that a finitely generated module X over an algebra C is called a *generator-cogenerator* if $C \oplus D(C) \in \mathrm{add}({}_CX)$. If B is the endomorphism algebra of a generator-cogenerator X over an algebra C, then, by Müller's theorem (see [32, Lemma 3]), $\mathrm{dom.dim}(B) = n > 1$ if and only if $\mathrm{Ext}^i_C(X,X) = 0$ for $1 \leq i < n-1$. In particular, $\mathrm{dom.dim}(B) = \infty$ if and only if X is self-orthogonal, that is, $\mathrm{Ext}^i_C(X,X) = 0$ for all $i \geq 1$.

Proof of Theorem 1.2. (1) $\Rightarrow$ (2)-(5). Suppose ${}_AM$ is projective. Then $\mathscr{G} = A$-Mod and $W = \Omega^-_A(M) = 0$. This implies (4) and (5). Since each A-module is always a filtered colimit of finitely generated A-modules, (2) holds. Note that A-filtered modules are projective. Thus (3) holds.

(2) $\Rightarrow$ (4). This is clear since $W \in \mathscr{G}$.

(4) $\Rightarrow$ (5). Let $f : \Omega^-_A(M) \to W$ be the minimal left $\mathscr{G}$-approximation of $\Omega^-_A(M)$ with $W \in \mathscr{G}$. Assume that W is a filtered colimit of $\{W_i \mid i \in I\}$ with I an essentially small, filtered category and with all $W_i \in \mathscr{G}_0$. We show that $W^{\perp 1}$ is closed under countable direct sums in A-Mod. This implies (4).

Let $\lambda_i : W_i \to W$ be the canonical homomorphism of the colimit W. Since $\Omega^-_A(M)$ is finitely generated and even finitely presented, the canonical map $\varinjlim \mathrm{Hom}_A(\Omega^-_A(M), W_i) \to \mathrm{Hom}_A(\Omega^-_A(M), W)$ induced by $\{\mathrm{Hom}_A(\Omega^-_A(M), \lambda_i) \mid i \in I\}$ is an isomorphism. As I is a filtered category, the colimt $\varinjlim \mathrm{Hom}_A(\Omega^-_A(M), W_i)$ of abelian groups is a quotient group of the direct sum $\bigoplus_{i\in I} \mathrm{Hom}_A(\Omega^-_A(M), W_i)$ and each of its elements can be represented by a homomorphism of $\mathrm{Hom}_A(\Omega^-_A(M), W_i)$ for some index $i \in I$. Consequently, there is an index $i \in I$ and a homomorphism $f_i : \Omega^-_A(M) \to W_i$ such that $f = f_i\lambda_i$. Further, due to $W_i \in \mathscr{G}$, the approximation implies that there is a homomorphism $g_i : W \to W_i$ such that $f_i = fg_i$. It follows that

$f = fg_if_i$. Since f is left minimal, g_if_i is an isomorphism. Thus g_i is split-injective. Since W_i is finitely generated, W is also finitely generated (presented). Hence, $W^{\perp 1}$ is closed under arbitrary direct sums in A-Mod.

$(5) \Rightarrow (1)$. By Lemma 4.15(3), $T(\Omega_A^-(M)) \simeq \Omega_M^-(N)$ in $\mathscr{C}$. Thus $W \simeq \Omega_M^-(N)$ in $\mathscr{C}$. By Lemma 3.16(1), there are $M_1, M_2 \in \mathrm{Add}(M)$ such that $W \oplus M_1 \simeq \Omega_M^-(N) \oplus M_2$. Let $\mathscr{M}$ be the full subcategory of A-mod consisting of modules with $\mathrm{add}(M)$-resolutions of finite length. It follows from $M \in \mathscr{G}^{\perp >0}$ and $W \in \mathscr{G}$ that $\mathscr{M} \subseteq \mathscr{G}^{\perp >0} \subseteq W^{\perp 1}$. For $X \in \mathrm{Add}(M)$, since ${}_AM$ is self-orthogonal and finitely presented, $X^{\perp 1}$ contains $\mathscr{M}$ and is closed under arbitrary direct sums in A-Mod. Thus (3) implies that $\Omega_M^-(N)^{\perp 1}$ is closed under countable direct sums in A-Mod of modules in $\mathscr{M}$. By Lemma 4.16, $\Omega_M^-(N)^{\perp 1}$ is closed under countable direct sums in A-Mod of finitely M-filtered A-modules. Further, it follows from $N \in \mathscr{E} \cap \mathscr{S}$, Corollary 3.13 and Lemma 4.4(2) that $\Omega_M^-(N) \in \mathscr{E} \cap \mathscr{S}$. By Proposition 4.17(2), $\Omega_M^-(N) \in \mathrm{Add}(M)$. This shows $\Omega_M^-(N) = 0$ (and thus also $N = 0$) in $\mathscr{C}$. By $\underline{\mathrm{End}}_A(N) = \underline{\mathrm{End}}_M(N)$, we have $N = 0$ in A-$\underline{\mathrm{Mod}}$. Since $\underline{\mathrm{End}}_A(M) \simeq \underline{\mathrm{End}}_A(N)$ by Lemma 4.7(2), $M = 0$ in A-$\underline{\mathrm{Mod}}$. In other words, ${}_AM$ is projective.

$(3) \Rightarrow (1)$. By Proposition 1.6(1), the module N is M-compact (that is, compact in $\mathscr{C}$) and isomorphic in A-$\underline{\mathrm{Mod}}$ to an M-filtered module X. Then there are projective A-modules P and Q with $N \oplus P \simeq X \oplus Q$. Since projective A-modules are zero in $\mathscr{C}$, X is M-compact. By (3), X lies in $\mathrm{Add}(M)$, and therefore $N = 0$ in $\mathscr{C}$. By Lemma 4.7(2), we have $M = 0$ in A-$\underline{\mathrm{Mod}}$. Thus ${}_AM$ is projective. □

Recall that $\mathscr{P}^{<\infty}(B)$ denotes the category of finitely generated B-modules with finite projective dimension. Let B-GProj_ω be the category of *countably generated, compactly Gorenstein-projective* B-modules. Clearly, finitely generated, Gorenstein-projective B-modules are in B-GProj_ω. Note that the category of countably generated B-modules is a Serre subcategory of B-Mod. This is due to the fact: A ring R has the property that each submodule of each countably generated left R-module is countably generated if and only if each left ideal of R is countably generated.

We consider the following two homological conditions:

(CI) The direct sum of countably many B-modules from $\mathscr{P}^{<\infty}(B)$ belongs to B-$\mathrm{GProj}_\omega^{\perp>0}$.

(CII) Any compactly Gorenstein-projective, compactly filtered B-module is projective.

The *finitistic dimension* of an Artin algebra B is the supremum of projective dimensions of all B-modules in $\mathscr{P}^{<\infty}(B)$. The well-known finitistic dimension conjecture says that an Artin algebra B should always have finite finitistic dimension. The validity of this conjecture for B implies the one of the Nakayama conjecture for B. However, the finitistic dimension conjecture is still open.

Lemma 5.1. (1) *If* (CI) *holds, then so does* (CII).

(2) *If an Artin algebra B has finite finitistic dimension or is a virtually Gorenstein algebra, then* (CI) *holds.*

Proof. (1) Let Y be a compactly Gorenstein-projective B-module which is compactly filtered by a sequence $\{Y_i \mid i \in \mathbb{N}\}$ of submodules of Y. Set $X := \bigoplus_{i\in\mathbb{N}} Y_i$. Then there is a canonical exact sequence $0 \to X \to X \to Y \to 0$ in B-Mod. Since Y_i is finitely generated for all i, the module Y is countably generated. This implies $Y \in B\text{-}\mathrm{GProj}_\omega$. Moreover, since $Y_i \in \mathscr{P}^{<\infty}(B)$ for all $i \in \mathbb{N}$, the condition (CI) implies that the sequence splits and Y is isomorphic to a direct summand of X. Note that Y_i is a direct sum of finitely many indecomposable submodules with the local endomorphism rings. By [1, Corollary 26.6], $Y \simeq \bigoplus_{i\in\mathbb{N}} Y_i'$, where Y_i' is a direct summand of Y_i for each i. Since $\mathscr{P}^{<\infty}(B) \cap B\text{-}\mathrm{GProj} = \mathrm{add}(B)$, Y_i' is projective for all i. Thus Y is projective. This shows that (CII) holds.

(2) Clearly, B-$\mathrm{GProj}^{\perp>0}$ contains all B-modules of finite projective dimension. For a virtually Gorenstein algebra B, $B\text{-}\mathrm{GProj}^{\perp>0} = {}^{\perp>0}B\text{-}\mathrm{GInj}$, and therefore B-$\mathrm{GProj}^{\perp>0}$ is closed under arbitrary direct sums in B-Mod. Thus (2) holds. □

Lemma 5.2. *Let B and C be Artin algebras and X a finitely generated C-B-bimodule. Suppose that ${}_CX$ and X_B are projective and the tensor functor $X \otimes_B - : B\text{-Mod} \to C\text{-Mod}$ induces a triangle equivalence $B\text{-}\underline{\mathrm{GProj}} \to C\text{-}\underline{\mathrm{GProj}}$. If C satisfies* (CII)*, then so does B.*

Proof. Let $F := {}_CX \otimes_B -$. Since both ${}_CX$ and X_B are projective, the functor $F : B\text{-Mod} \to C\text{-Mod}$ is exact and preserves projective modules. This yields $F(\mathscr{P}^{<\infty}(B)) \subseteq \mathscr{P}^{<\infty}(C)$. Since F commutes with filtered colimits, it sends compactly filtered B-modules to compactly filtered C-modules. Further, since F induces a triangle equivalence $B\text{-}\underline{\mathrm{GProj}} \to C\text{-}\underline{\mathrm{GProj}}$, it reflects projective modules and sends compactly Gorenstein-projective B-modules to compactly Gorenstein-projective C-modules. This implies Lemma 5.2. □

Next, we point out that the condition (CII) is preserved by several classes of equivalences between algebras. For the unexplained notions below of stable equivalences of adjoint type and singular equivalences of Morita type with level, we refer to [44] and [42], respectively. Given a finitely generated B-module N, we denote by $\mathrm{thick}({}_BN)$ the smallest thick subcategory of B-mod which contains N.

Corollary 5.3. *Let B and C be finite-dimensional algebras over a field. Suppose that*

(a) *B and C are derived equivalent, or*

(b) *B and C are stably equivalent of adjoint type, or*

(c) *B and C are singularly equivalent of Morita type with level defined by a pair of bimodules $({}_CX_B, {}_BY_C)$ such that $\mathrm{Hom}_C(X,C) \in \mathrm{thick}({}_BB \oplus D(B))$ and $\mathrm{Hom}_B(Y,B) \in \mathrm{thick}({}_CC \oplus D(C))$.*

Then B satisfies (CII) *if and only if so does C.*

Proof. (a) follows from Lemma 5.2 and [26, Example 4.7 and Corollary 5.4]. The case (b) is a special case of (c). In the case (c), the modules ${}_CX$ and X_B are finitely generated and projective by the definition of singular equivalences of Morita type with level. By Lemma 5.2, it suffices to show that the functor $F := {}_CX \otimes_B - : B\text{-Mod} \to C\text{-Mod}$ induces a triangle equivalence $B\text{-}\underline{\mathrm{GProj}} \to C\text{-}\underline{\mathrm{GProj}}$. Note that for finitely generated Gorenstein-projective modules, the triangle equivalence $B\text{-}\underline{\mathrm{Gproj}} \to C\text{-}\underline{\mathrm{Gproj}}$ is known in [42, Proposition 4.5] under a stronger assumption that $\mathrm{Hom}_C(X,C) \in \mathscr{P}^{<\infty}(B)$ and $\mathrm{Hom}_B(Y,B) \in \mathscr{P}^{<\infty}(C)$. Since our discussions involve infinitely generated Gorenstin-projective modules, we have to check whether F and G can be regraded as functors between B-GProj and C-GProj.

Let $U \in B\text{-GProj}$ with a complete projective resolution $P^\bullet$. Then $F(P^\bullet) := (F(P^i)_{i\in\mathbb{Z}})$ is an exact complex of projective C-modules. Moreover, $\mathrm{Hom}^\bullet_C(F(P^\bullet),C) \simeq \mathrm{Hom}^\bullet_B(P^\bullet, \mathrm{Hom}_C(X,C))$ as complexes. By [14, Theorem 2], $\mathrm{thick}({}_BB \oplus D(B)) = B\text{-GProj}^{\perp>0} \cap B\text{-mod}$. Since each cocycle of $P^\bullet$ lies in B-GProj and $\mathrm{Hom}_C(X,C) \in \mathrm{thick}({}_BB \oplus D(B))$, the complex $\mathrm{Hom}^\bullet_B(P^\bullet, \mathrm{Hom}_C(X,C))$ is exact. Consequently, $F(P^\bullet)$ is a compete projective resolution of $F(U)$, and therefore $F(U) \in C\text{-GProj}$. In other words, F restricts to a functor $B\text{-GProj} \to C\text{-GProj}$. Similarly, the functor $G := {}_BY \otimes_C - : C\text{-Mod} \to B\text{-Mod}$ also restricts to a functor $C\text{-GProj} \to B\text{-GProj}$.

Recall that $B\text{-}\underline{\mathrm{GProj}}$ is a triangulated category with the shift functor Σ_B which is a quasi-inverse of the syzygy functor Ω_B. This implies $Y \otimes_C X \otimes_B - \simeq \Omega^n_B(-) \simeq \Sigma^{-n}_B : B\text{-}\underline{\mathrm{GProj}} \longrightarrow B\text{-}\underline{\mathrm{GProj}}$, where n is the level of the singular equivalence in (c) and Σ^n_B denotes the n-th shift functor of $B\text{-}\underline{\mathrm{GProj}}$. Similarly, there are equivalences $X \otimes_B Y \otimes_C - \simeq \Omega^n_C(-) \simeq \Sigma^{-n}_C : C\text{-}\underline{\mathrm{GProj}} \to C\text{-}\underline{\mathrm{GProj}}$. Thus F induces a triangle equivalences $B\text{-}\underline{\mathrm{GProj}} \longrightarrow C\text{-}\underline{\mathrm{GProj}}$ with a quasi-inverse $\Sigma^n_B \circ G$: $C\text{-}\underline{\mathrm{GProj}} \to B\text{-}\underline{\mathrm{GProj}}$. □

Recall from Definition 1.3(ii) that an algebra B is said to be *compactly Gorenstein* if (CII) holds; and *Gorenstein-Morita* if B is both strongly Morita and compactly Gorenstein. By Lemma 5.1, a strongly Morita algebra is Gorenstein-Morita if it is virtually Gorenstein or has finite finitistic dimension.

Proof of Corollary 1.4. Suppose that Λ is a Gorenstin-Morita algebra. Then there is a self-injective algebra A and a finitely generated A-module M which is a generator such that $\Lambda = \mathrm{End}_A(M)$ and $\mathrm{add}(M) = \mathrm{add}(\nu_A(M))$. Further, suppose $\mathrm{dom.dim}(\Lambda) = \infty$. Then ${}_AM$ is self-orthogonal by [32, Lemma 3].

In the following, we show that $N := H^0_*(M) = 0$ in $\mathscr{C}$ (see Section 4.2). If it is the case, then $\underline{\mathrm{End}}_A(M) \simeq \underline{\mathrm{End}}_A(N)$ by Lemma 4.7(2), and therefore $M = 0$ in A-$\underline{\mathrm{Mod}}$, that is, ${}_AM$ is projective. Thus Λ is Morita equivalent to A and must be self-injective, and therefore the first part of Corollary 1.4 is proved.

Now, let $M = A \oplus \bigoplus_{i=1}^m M_i$ with $m \in \mathbb{N}$ and M_i indecomposable and non-projective for $1 \le i \le m$. By Lemma 4.8(2), $N \simeq \bigoplus_{i=1}^m H^0_*(M_i)$ in $\mathscr{C}$ and $H^0_*(M_i)$ is indecomposable in $\mathscr{C}$ for $1 \le i \le m$. Clearly, $H^0_*(M_i)$ as an A-module does not contain non-projective direct summands in $\mathrm{Add}(M)$, due to $\underline{\mathrm{Hom}}_A(N,M) = 0$. So we can assume that $H^0_*(M_i) = P_i \oplus N_i$ as A-modules, where P_i is a projective A-module, and N_i is either zero or an indecomposable A-module that does not lie in $\mathrm{Add}(M)$. Now, we turn to showing $N_i = 0$ for all i. This implies that N is projective, and thus $N = 0$ in $\mathscr{C}$.

By Theorem 4.14(1) and Proposition 4.17(1), N_i is isomorphic in A-$\underline{\mathrm{Mod}}$ to an M-filtered A-module X. Now, suppose that X is filtered by a sequence $\{X_n \mid n \in \mathbb{N}\}$ of submodules of X (see Definition 1.1). Then $X = \lim\limits_{n\to\infty} X_n = \bigcup_{n=0}^{\infty} X_n$. By Lemma 4.16, $X_n \in M^{\perp>0}$ and has an $\mathrm{add}(M)$-resolution of finite length. Applying $G := \mathrm{Hom}_A(M,-)$ to the inclusions $X_n \to X_{n+1}$, we obtain $G(X_n) \in \mathscr{P}^{<\infty}(\Lambda)$ and injective homomorphisms $G(X_n) \to G(X_{n+1})$. Since ${}_AM$ is finitely presented, G commutes with filtered colimits. It follows that $G(X) = \lim\limits_{n\to\infty} G(X_n) = \bigcup_{n=0}^{\infty} G(X_n)$. In other words, $G(X)$ is compactly filtered.

Assume $N_i \ne 0$. Then N_i is indecomposable and does not belong to $\mathrm{Add}(M)$. It follows from $X \simeq N_i$ in A-$\underline{\mathrm{Mod}}$ and $N_i \in \mathscr{G}$ that $X \in \mathscr{G}$ and $\Omega_M^-\Omega_M(X) \simeq \Omega_M^-\Omega_M(N_i)$. By Remark 3.19, $\Omega_M^-\Omega_M(N_i) \simeq N_i$, and therefore $X \simeq N_i \oplus M_0$ for some $M_0 \in \mathrm{Add}(M)$. As N is compact in $\mathscr{C}$ by Theorem 4.14(1), X is also compact in $\mathscr{C}$. It follows from Lemma 3.17(2) that $G(X)$ is compactly Gorenstein-projective, but not projective. Since Λ is compactly Gorenstein, $G(X)$ must be projective. This leads to a contradiction. Thus $N_i = 0$.

If Λ is gendo-symmetric, then A can be assumed to be symmetric. Since gendo-symmetric, virtually Gorenstein algebras are Gorenstein-Morita, the second part of Corollary 1.4 follows from the first part of Corollary 1.4. $\square$

To end this section, we mention the following questions related to the results in this paper.

Question 1. Find necessary and sufficient conditions for Artin algebras to be compactly Gorenstein.

Finitely generated Gorenstein-projective modules over an Artin algebra are compactly Gorenstein-projective. The next question is about countably generated Gorenstein-projective modules.

Question 2. Describe the class of countably generated, compactly Gorenstein-projective modules over an Artin algebra.

Question 3. Find more new examples of Gorenstein-Morita algebras.

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