Journal of Combinatorial Theory, Series A 213 (2025) 106024

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

journal homepage: www.elsevier.com/locate/jcta

Structure of Terwilliger algebras of quasi-thin association schemes

Zhenxian Chen [a], Changchang Xi [a,b,*]

[a] *School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*
[b] *School of Mathematics and Statistics, Shaanxi Normal University, Xi'an 710062, China*

ARTICLE INFO

Article history:
Received 16 September 2024
Received in revised form 13 January 2025
Accepted 19 January 2025
Available online xxxx

Keywords:
Association scheme
Cellular algebra
Dual extension
Quasi-hereditary algebra
Quasi-thin association scheme
Terwilliger algebra

ABSTRACT

We show that the Terwilliger algebra of a quasi-thin association scheme over a field is always a quasi-hereditary cellular algebra in the sense of Cline-Parshall-Scott and of Graham-Lehrer, respectively, and that the basic algebra of the Terwilliger algebra is the dual extension of a star with all arrows pointing to its center if the field has characteristic 2. Thus many homological and representation-theoretic properties of these Terwilliger algebras can be determined completely. For example, the Nakayama conjecture holds true for Terwilliger algebras of quasi-thin association schemes.

1. Introduction

Association schemes have played an important role in the theory of algebraic combinatorics and designs (see [4,32]). To understand them algebraically, Terwilliger associated each association scheme with an algebra over a field or more generally, over a commuta-

* Corresponding author at: School of Mathematical Sciences, Capital Normal University, Beijing 100048, China.

E-mail addresses: czx18366459216@163.com (Z.X. Chen), xicc@cnu.edu.cn (C.C. Xi).

https://doi.org/10.1016/j.jcta.2025.106024
0097-3165/

tive ring (see [24–26]). This algebra nowadays is termed as the Terwilliger algebra (for example, see [32]). There is a large variety of literature on the study of Terwilliger algebras, especially, for special schemes or ground fields (for example, see [15,19–21] and the references therein). Recently, Hanaki studies the modular case of Terwilliger algebras [10], and Jiang investigates the Jacobson radicals of the Terwilliger algebras of quasi-thin association schemes [16]. The tools used in their study are mainly combination of combinatorics with ring theory. Note that quasi-thin association schemes have been considered in many papers from the view point of combinatorics (see [11,13,14,22]).

The purpose of this note is to understand the Terwilliger algebras of quasi-thin association schemes from the view point of representation theory of algebras, namely we show the following structural and homological result.

Theorem 1.1. *Let R be a field of arbitrary characteristic and S a quasi-thin association scheme on a finite set. Then the Terwilliger R-algebra of S is quasi-hereditary in the sense of Cline-Parshall-Scott, and cellular in the sense of Graham-Lehrer. Moreover, if the field R has characteristic 2, the basic algebra of the Terwilliger R-algebra of S is the dual extension of a star, and has global dimension at most 2.*

For a finite-dimensional algebra A over a field, let $P_1, \cdots, P_n$ be a complete set of non-isomorphic, indecomposable projective A-modules. Then the basic algebra Λ of A is defined to be the endomorphism algebra of the A-module $\bigoplus_{i=1}^{n} P_i$. It is known that A and Λ are Morita equivalent, that is, they have the equivalent module categories.

Observe that the notion of cellular algebras in the sense of Graham-Lehrer in [9] is completely different from the one in [27, Section D, p.23].

The proof of Theorem 1.1 is given in Section 3 after we introduce scheme theoretic terminology needed and provide ring theoretic preliminaries in Section 2.

2. Preliminaries

In this section we first recall basics on Terwilliger algebras of schemes and then provide preliminaries on quasi-hereditary algebras and cellular algebras.

2.1. Terwilliger algebras of association schemes

Throughout this note, R is a field. For $n \in \mathbb{N}$, the symbol $[n]$ denotes the set $\{0, 1, 2, \cdots, n\} \subseteq \mathbb{N}$. The cardinality of a set X is denoted by $|X|$.

Definition 2.1. An *association scheme* or simply a *scheme* of size d on a nonempty finite set X is a partition $S = \{R_0, R_1, \ldots, R_d\}$ of the Cartesian product $X \times X$ with all parts R_i nonempty, satisfying the conditions

$(S1)$ $R_0 = \{(x, x) \mid x \in X\}$.

$(S2)$ For $i \in [d]$, there exists $i' \in [d]$ such that $R_{i'} = \{(x, y) \mid (y, x) \in R_i\}$, and

$(S3)$ For $i, j, \ell \in [d]$ and $(x, y), (u, v) \in R_\ell$, the following holds:

$$|\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}| = |\{w \in X \mid (u, w) \in R_i, (w, v) \in R_j\}|.$$

Now, let $S = \{R_0, R_1, \ldots, R_d\}$ be a scheme of size d. An element $x \in X$ is called a *vertex* of S, and the part R_i is called a *relation* of S. For $i, j, \ell \in [d]$ and $(x, y) \in R_\ell$, we define

$$p_{ij}^{\ell} := |\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}| \in \mathbb{N}.$$

It is known from $(S3)$ that p_{ij}^{ℓ} is independent on the choice of (x, y) in R_ℓ. The number p_{ij}^{ℓ} is called the *intersection number* of S with respect to the triple (R_i, R_j, R_ℓ). By Definition 2.1(S2), the i' is uniquely determined by i. So, the *valency* k_i of R_i is defined by $k_i := p_{ii'}^{0}$. Note that k_i is just the cardinality of the set $xR_i := \{y \in X \mid (x, y) \in R_i\}$ for any $x \in X$. Thus $k_i > 0$ for $i \in [d]$ and $k_0 = 1$. For $j \geq 1$, let $\mathcal{A}_j := \{i \in [d] \mid k_i = j\}$. Then $\mathcal{A}_1 \neq \emptyset$.

Definition 2.2. A scheme S of size d is called a *thin scheme* if $k_i = 1$ for all $i \in [d]$; and a *quasi-thin* scheme if $k_i \leq 2$ for all $i \in [d]$.

Quasi-thin schemes were introduced in [15], but the first result on quasi-thin schemes goes back to [27], where it was proved that any primitive quasi-thin scheme is Schurian.

By definition, if S is thin, then $\mathcal{A}_j = \emptyset$ for $2 \leq j \in \mathbb{N}$. The following properties of schemes are well known.

Lemma 2.3. [12] *Let S be a scheme of size d. If $i, j, \ell \in [d]$, then*

(1) $p_{ji}^{\ell} = p_{i'j'}^{\ell'}$ *and* $k_i = k_{i'}$.
(2) $\sum_{\ell=0}^{d} p_{i\ell}^{j} = k_i$.
(3) $k_i k_j = \sum_{\ell=0}^{d} p_{ij}^{\ell} k_\ell$.
(4) $k_\ell p_{ij}^{\ell} = k_i p_{\ell j'}^{i} = k_j p_{i'\ell}^{j}$.

For any nonempty subsets U, V of S, the multiplication of U and V is defined by

$$UV := \left\{R_\ell \in S \mid \exists\, R_i \in U, \exists\, R_j \in V, \text{ such that } p_{ij}^{\ell} > 0\right\}.$$

For $i, j \in [d]$, $|R_i R_j| \leq \gcd(k_i, k_j)$ for $i, j \in [d]$ by [32, Lemma 1.5.2], where $\gcd(m, n)$ means the greatest common divisor of m and n.

Let $M_X(R)$ be the $X \times X$ matrix algebra over R. We denote by I the identity matrix in $M_X(R)$, E_{xy} the matrix units for $x, y \in X$, and J the matrix with all entries equal to 1.

For a part R_i of a scheme S, there is associated an *adjacency matrix* $A_i := (a_{xy}) \in M_X(R)$ defined by $a_{xy} = 1$ if $(x, y) \in R_i$, and 0 otherwise. Thus $A_0 = I, A_i^t = A_{i'}$ and $\sum_{i=0}^{d} A_i = J$, where A^t is the transpose matrix of A. Moreover, for $i, j \in [d]$,

$$A_iA_j = \sum_{\ell=0}^{d} p_{ij}^{\ell} A_\ell.$$

Fix an $x \in X$ and $i \in [d]$, the *dual idempotent* of R_i with respect to x is defined by $E_i^*(x) := \sum_{y \in xR_i} E_{yy} \in M_X(R)$. Then

$$(\dagger) \quad E_i^*(x)E_j^*(x) = \delta_{ij}E_j^*(x), \ \sum_{i=0}^{d} E_i^*(x) = I, \ \text{ and } JE_i^*(x)J = k_iJ,$$

where δ_{ij} is the Kronecker symbol. Thus $E_0^*, E_1^*, E_2^*, \cdots, E_d^*$ form a complete set of pairwise orthogonal idempotent elements of $M_X(R)$. Moreover, for $M = (m_{xy}) \in M_X(R)$,

$$(\ddagger) \quad E_i^*(x)ME_j^*(x) = \sum_{y \in xR_i} \sum_{z \in xR_j} m_{yz}E_{yz}.$$

This shows $E_i^*(x)A_\ell E_j^*(x)$ is a $(0,1)$-matrix for $i, j, \ell \in [d]$.

Definition 2.4. [16] Let $n \in \mathbb{N}$. If there are numbers $i_b, j_b, \ell_b \in [d]$ for all $b \in [n]$, such that

(1) $k_{i_b} = k_{\ell_b} = 2$ and $p_{i_b j_b}^{\ell_b} = 1$ for all $b \in [n]$, and

(2) $\left|R_{i_0'} R_{\ell_n}\right| = 1$ and $\ell_c = i_{c+1}$ for all $c \in [n-1]$,

then the pair (i_0, ℓ_n) is called a bad pair of S.

Let $\mathcal{S}$ denote the set of all bad pairs of S, $\mathcal{R} := \{(i,j) \in \mathcal{A}_2 \times \mathcal{A}_2 \mid |R_{i'}R_j| = 2\}$, and $\mathcal{U} := \mathcal{R} \cup \mathcal{S}$.

One defines a relation on $\mathcal{A}_2$: For $i, j \in \mathcal{A}_2$, $i \sim j$ if and only if $(i,j) \in \mathcal{U}$. Then $\sim$ is an equivalence relation on $\mathcal{A}_2$ by [16, Lemma 7.2]. The set of equivalence classes of $\sim$ is denoted by $\{\mathcal{C}_1, \ldots, \mathcal{C}_r\}$ for r a natural number. We define $\mathcal{C}_0 := [d]$ and $b_{ij}^0(x) := E_i^*(x)JE_j^*(x)$ for $i, j \in \mathcal{C}_0$ and $x \in X$.

Now, we choose a fixed $x \in X$, a total order $\prec$ for X, and take $i, j \in \mathcal{C}_\ell$ for $1 \le \ell \le r$. If $xR_i = \{y_1, y_2\}$ and $xR_j = \{z_1, z_2\}$ such that $y_1 \prec y_2$ and $z_1 \prec z_2$, then we define $b_{ij}^\ell(x) := E_{y_1z_1} + E_{y_2z_2}$. Clearly, $(b_{ij}^\ell(x))^t = b_{ji}^\ell(x)$ for $\ell \in [r]$, and $b_{jj}^\ell(x) = E_j^*(x)$ for $\ell = 0$ and $j \in \mathcal{A}_1$, or $1 \le \ell \le r$ and $j \in \mathcal{C}_\ell$. Let $\mathcal{B}_\ell(x) := \{b_{ij}^\ell(x) \mid i, j \in \mathcal{C}_\ell\}$ for $0 \le \ell \le r$ and $\mathcal{B}(x) := \bigcup_{\ell=0}^{r} \mathcal{B}_\ell(x)$.

Definition 2.5. [24] Let S be a scheme on a finite set X, and R be a commutative ring R with identity. The *Terwilliger R-algebra* $\mathcal{T}_R(x)$ of a scheme S on X with respect to $x \in X$ is the R-subalgebra of $M_X(R)$ generated by $A_0, A_1, \ldots, A_d$; $E_0^*(x), E_1^*(x), \ldots$, and $E_d^*(x)$.

By a Terwilliger algebra of S, we mean the Terwilliger algebra $\mathcal{T}_R(x)$ with respect to $x \in X$. Terwilliger algebras were first introduced and studied by P. Terwilliger in a series of papers [24–26] for commutative schemes under a different name. These algebras now are called *Terwilliger algebras*. They were studied in [10] over a field of positive characteristic under the name *modular Terwilliger algebras*.

Clearly, the transpose of matrices is an R-linear involution of $\mathcal{T}_R(x)$, and $\mathcal{T}_R(x) = M_X(R)$ if S is thin. In general, for $x, y \in X$ with $x \neq y$, we do not have $\mathcal{T}_R(x) \simeq \mathcal{T}_R(y)$ as algebras [10, Section 5.1]. However, this holds true for quasi-thin scheme, namely if S is a quasi-thin scheme, then $\mathcal{T}_R(x) \simeq \mathcal{T}_R(y)$ as R-algebras for $x, y \in X$ by [16, Theorem D].

From now on we assume that S is a quasi-thin scheme of size d on a finite set X. We fix an $x \in X$ and write $\mathcal{T} := \mathcal{T}_R(x)$ and $E_i^* := E_i^*(x)$ for $i \in [d]$. Similarly, we write b_{ij}^ℓ and $\mathcal{B}_\ell$ for $b_{ij}^\ell(x)$ and $\mathcal{B}_\ell(x)$, respectively.

Proposition 2.6. [16] *Let R be a field of characteristic $p \geq 0$ and S be a quasi-thin scheme. Then*

(1) *$\mathcal{B}$ is an R-basis of $\mathcal{T}$. Thus* $\dim_R(\mathcal{T}) = |\mathcal{R}| + |\mathcal{S}| + (d+1)^2$.

(2) *For $b_{ij}^\ell \in \mathcal{B}_\ell, b_{uv}^w \in \mathcal{B}_w$ with $\ell, w \in [d]$, the following holds*

$$b_{ij}^\ell b_{uv}^w = \begin{cases} \delta_{ju} k_j b_{iv}^0, & \text{if } \ell = w = 0, \\ \delta_{ju}\delta_{\ell w} b_{iv}^\ell, & \text{if } \ell \neq 0 \text{ and } w \neq 0, \\ \delta_{ju} b_{iv}^0, & \text{otherwise.} \end{cases}$$

(3) *$\mathcal{T}$ is semisimple if and only if $p \neq 2$ or $p = 2$ and S is thin.*

(4) *If $p = 2$, the Jacobson radical of $\mathcal{T}$, denoted by* $\operatorname{rad}(\mathcal{T})$, *is spanned R-linearly by* $\{b_{ij}^0 \mid i, j \in [d], \max\{k_i, k_j\} = 2\}$.

2.2. Cellular and quasi-hereditary algebras

Let us recall the definition of cellular algebras in [9].

Definition 2.7. [9] Let R be a commutative noetherian domain. A unitary R-algebra A is called a *cellular algebra* with cell datum (I, M, C, t) if the following are satisfied:

($C1$) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has an R-basis $C_{S,T}^\lambda$ where (S, T) run through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

($C2$) The map t is an R-involution on A, that is, an R-linear anti-automorphism on A of oder 2, such that $C_{S,T}^\lambda$ is sent to $C_{T,S}^\lambda$.

($C3$) For $a \in A$, there holds

$$aC_{S,T}^\lambda = \sum_{U \in M(\lambda)} r_a(U, S) C_{U,T}^\lambda + r'$$

where $r_a(U, S) \in R$ does not depend on T and r' is a linear combination of basis elements $C_{W,V}^\mu$ with $\mu < \lambda$.

A trivial example of cellular algebras is $M_n(R)$ with $I := \{*\}$, $M(*) := [n], C^*_{ij} := E_{ij}$ and t being the matrix transpose. Cellular algebras capture many interesting classes of algebras such as Brauer algebras [9], Hecke algebras of finite type [8], centralizer matrix algebras [31].

The following is an equivalent definition of cellular algebras:

Definition 2.8. [17] Let A be an R-algebra with R a commutative noetherian domain. Assume that there is an involution t on A. A two-sided ideal J in A is called a *cell ideal* if and only if $(J)t = J$ and there exists a left ideal $\Delta \subset J$ of A such that Δ is finitely generated and free over R and that there is an isomorphism of A-bimodules $\alpha : J \simeq \Delta \otimes_R (\Delta)t$ (where $(\Delta)t \subset J$ is the image of Δ under t) making the following diagram commutative:

$$\begin{tikzcd} J \arrow[r, "\alpha"] \arrow[d, "t"'] & \Delta \otimes_R (\Delta)t \arrow[d, "x\otimes y \mapsto (y)t \otimes (x)t"] \\ J \arrow[r, "\alpha"'] & \Delta \otimes_R (\Delta)t \end{tikzcd}$$

The algebra A (with the involution t) is called *cellular* if and only if there is an R-module decomposition $A = J'_1 \oplus J'_2 \oplus \cdots \oplus J'_n$ (for some n) with $(J'_j)t = J'_j$ for each j, such that setting $J_j = \oplus_{l=1}^{j} J'_l$ gives a chain of two-sided ideals of $A : 0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ (each of them fixed by t) and that, for each $1 \leq j \leq n$, the quotient $J'_j = J_j/J_{j-1}$ is a cell ideal (with respect to the involution induced by t on the quotient) of A/J_{j-1}.

Next, we recall the definition of quasi-hereditary algebras introduced in [6].

Definition 2.9. [6] Let A be a finite-dimensional algebra over a field. An ideal J in A is called a *heredity ideal* if J is idempotent, $J(\mathrm{rad}(A))J = 0$ and J is a projective left (or right) A-module. The algebra A is said to be *quasi-hereditary* provided there is a finite chain $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ of ideals in A such that J_j/J_{j-1} is a heredity ideal in A/J_{j-1} for all j. Such a chain is then called a *heredity chain* of the quasi-hereditary algebra A.

To judge whether a cellular algebra is quasi-hereditary, we have the following criteria.

Lemma 2.10. [18, Lemma 2.1] *Let A be a finite-dimensional cellular algebra over a field, with a cell chain $0 = J_0 \subset J_1 \subset \cdots \subset J_n = A$. Then the given cell chain is a heredity chain (make A into a quasi-hereditary algebra) if and only if all J_l satisfy $J_l^2 \nsubseteq J_{l-1}$ if and only if n equals the number of isomorphism classes of simple modules.*

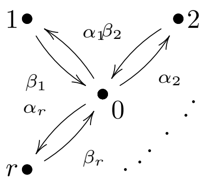
3. Proof of the main result

This section is devoted to the proof of the following main result. Also, representation-theoretic and homological properties of Terwilliger algebras of quasi-thin schemes are deduced.

Theorem 3.1. *Let R be a field of characteristic $p \geq 0$, and assume that S be a quasi-thin scheme on a finite set X. Then the following hold.*

(1) *The Terwilliger R-algebra $\mathcal{T}$ of S is a quasi-hereditary, cellular algebra with respect to the matrix transpose.*

(2) *If R is of characteristic 2 and $\mathcal{A}_2$ has $r \geq 0$ equivalence classes, then the basic algebra Γ of Terwilliger R-algebra $\mathcal{T}$ of S is isomorphic to an R-algebra Λ given by the following quiver*



with relations $\alpha_i\beta_i = 0$ for $1 \leq i \leq r$.

Proof. (1) We apply Proposition 2.6 to show the cellularity of the Terwilliger R-algebra $\mathcal{T}$ of a quasi-thin scheme S.

If $\mathcal{A}_2 = \emptyset$, that is $r = 0$, then $\mathcal{T} = M_X(R)$ with X a finite set. Thus Theorem 3.1 follows.

Now, assume $\mathcal{A}_2 \neq \emptyset$. We define $I := [r]$ with the partial order $\{0 \prec 1, 0 \prec 2, \cdots, 0 \prec r\}$, $M(\ell) := \mathcal{C}_\ell$ and t being the transpose of matrices, namely $a \mapsto a^t, \forall a \in \mathcal{T}$. We show that (I, M, b, t) is a cell datum for $\mathcal{T}$.

Indeed, by Proposition 2.6, $\{b^\ell_{ij} \mid \ell \in I, i, j \in M(\ell)\}$ is an R-basis of $\mathcal{T}$. Clearly, t is an R-linear anti-automorphism of order 2 on $\mathcal{T}$ and $(b^\ell_{ij})^t = b^\ell_{ji}$ for $\ell \in I$ and $i, j \in M(\ell)$. It remains to consider the multiplication of basis elements of $\mathcal{T}$ and verify Definition 2.7(C3). But this follows immediately from Proposition 2.6.

To see that $\mathcal{T}$ is quasi-hereditary, we define a chain of ideals in $\mathcal{T}$. Let $J_0 = 0$ and J_i be the R-span of the elements in $\bigcup_{\ell=0}^{i-1} \mathcal{B}_\ell$ for $0 \neq i \in [r+1]$. Thus the chain $0 = J_0 \subset J_1 \subset \cdots \subset J_{r+1} = \mathcal{T}$ is a cell chain of ideals in $\mathcal{T}$ (see [17]). To show that this cell chain is a heredity chain, we prove the following:

(a) For $0 \in \mathcal{A}_1$, we get $b^0_{00} \in \mathcal{B}_0 \subset J_1$ and $b^0_{00}b^0_{00} = k_0 b^0_{00} = b^0_{00} \neq 0$.

(b) For $\ell \in \{1, 2, \cdots, r\}$, it follows from $\mathcal{C}_\ell \neq \emptyset$ that there is an element $u_\ell \in \mathcal{C}_\ell$, such that $b^\ell_{u_\ell u_\ell} \in \mathcal{B}_\ell \subset J_{\ell+1}$, and

$$b^\ell_{u_\ell u_\ell} b^\ell_{u_\ell u_\ell} = b^\ell_{u_\ell u_\ell} \notin J_\ell.$$

Thus the chain $0 = J_0 \subset J_1 \subset \cdots \subset J_{r+1} = \mathcal{T}$ is a heredity chain by Lemma 2.10, and therefore $\mathcal{T}$ is a quasi-hereditary algebra.

(2) If $\mathcal{T}$ is semisimple, then $r = 0$ and Theorem 3.1 is clearly true. Now suppose that $\mathcal{T}$ is not semisimple, that is, we are in the case (2) of Theorem 3.1 with $r > 0$.

For $\ell \in [r]$, let $\mathcal{D}_0 := \mathcal{A}_1, \mathcal{D}_\ell := \mathcal{C}_\ell$ for $\ell \neq 0$. We have the following

(i) $E := \{b^\ell_{ii} \mid \ell \in [r], i \in \mathcal{D}_\ell\}$ is a complete set of pairwise orthogonal idempotents of $\mathcal{T}$ by (†).

(ii) For $\ell \in [r], i \in \mathcal{D}_\ell$, let $P^\ell_i := \mathcal{T} b^\ell_{ii}$. Then $P^\ell_i \simeq P^\ell_j$ as $\mathcal{T}$-modules for all $i, j \in \mathcal{D}_\ell$.

In fact, P^ℓ_i is R-linearly spanned by $\bigcup_{w\in\{0,\ell\}, j\in\mathcal{C}_w}\{b^w_{ji}\}$. We define a map

$$f : P^\ell_i \longrightarrow P^\ell_j,\ b^w_{ui} \mapsto b^w_{uj}, \forall\, w \in \{0, \ell\}, u \in \mathcal{C}_w.$$

Then f is an isomorphism of $\mathcal{T}$-modules, and therefore $P^\ell_i \simeq P^\ell_j$ as $\mathcal{T}$-modules.

(iii) For $\ell \in [r]$, we fix an element $\ell_0 \in \mathcal{D}_\ell$ and define $P_\ell := P^\ell_{\ell_0}$. Then P_ℓ is indecomposable, $P_m \not\simeq P_n$ for $0 \leq m \neq n \leq r$, and $\mathrm{Hom}_{\mathcal{T}}(P_m, P_n) \simeq Rb^0_{m_0,n_0} + \delta_{mn} Rb^m_{m_0,m_0}$, where δ_{ij} is the Kronecker symbol.

Indeed, for $0 \leq i, j \leq r$, $\mathrm{Hom}_{\mathcal{T}}(P_i, P_j) \simeq b^i_{i_0 i_0} \mathcal{T} b^j_{j_0,j_0}$, that is, $\mathrm{Hom}_{\mathcal{T}}(P_i, P_j)$ is spanned R-linearly by $\{b^0_{i_0,j_0}, \delta_{ij} b^i_{i_0,i_0}\}$. If $i = j$, then $\mathrm{End}_{\mathcal{T}}(P_i)$ has a unique nonzero idempotent element $b^i_{i_0,i_0}$. This implies that P_i is indecomposable.

Now we show that $P_i \not\simeq P_j$ if $i \neq j$. First, we show that no homomorphism $\varphi \in \mathrm{Hom}_{\mathcal{T}}(P_i, P_j)$ is an isomorphism for $i > j$. Actually, $\mathrm{Hom}_{\mathcal{T}}(P_i, P_j)$ is 1-dimensional and φ is given by the right multiplication by $\lambda b^0_{i_0,j_0}$ with $\lambda \in R$. Since $k_{i_0} = 2 = p$ and P_i is spanned R-linearly by $\bigcup_{w\in\{0,i\}, u\in\mathcal{C}_w}\{b^w_{u,i_0}\}$, we get

$$(b^0_{ui_0})\varphi = b^0_{ui_0}(\lambda b^0_{i_0,j_0}) = \lambda k_{i_0} b^0_{u,j_0} = 2\lambda b^0_{u,j_0} = 0$$

for $u \in \mathcal{C}_0$. Thus φ has a nonzero kernel and is not an isomorphism. This shows $P_i \not\simeq P_j$ as $\mathcal{T}$-modules.

If $f : P_i \to P_j$ is an isomorphism of $\mathcal{T}$-modules for $i < j$, then $f^{-1} : P_j \to P_i$ is an isomorphism of $\mathcal{T}$-modules with $j > i$. This contradicts to the foregoing discussion. Thus $P_i \not\simeq P_j$ for all $i \neq j$.

Now, it follows from (i)-(iii) that $\{P_0, P_1, \cdots, P_r\}$ is a complete set of non-isomorphic indecomposable projective $\mathcal{T}$-modules.

(iv) Let $b_\ell := b^\ell_{\ell_0,\ell_0}$. Then the basic algebra of $\mathcal{T}$ is

$$\begin{aligned}\Gamma &:= \mathrm{End}_{\mathcal{T}}(P_0 \oplus P_1 \oplus \cdots \oplus P_r)\\ &= \mathrm{End}_{\mathcal{T}}(\mathcal{T}b_0 \oplus \mathcal{T}b_1 \oplus \cdots \oplus \mathcal{T}b_r) \simeq \begin{pmatrix} b_0\mathcal{T}b_0 & \cdots & b_0\mathcal{T}b_r \\ \vdots & \ddots & \vdots \\ b_r\mathcal{T}b_0 & \cdots & b_r\mathcal{T}b_r \end{pmatrix}\end{aligned}$$

$$\simeq \begin{pmatrix} Rb^0_{0_0,0_0} & Rb^0_{0_0,1_0} & Rb^0_{0_0,2_0} & \cdots & Rb^0_{0_0,r_0} \\ Rb^0_{1_0,0_0} & Rb^0_{1_0,1_0}+Rb^1_{1_0,1_0} & Rb^0_{1_0,2_0} & \cdots & Rb^0_{1_0,r_0} \\ Rb^0_{2_0,0_0} & Rb^0_{2_0,1_0} & Rb^0_{2_0,2_0}+Rb^2_{2_0,2_0} & \cdots & Rb^0_{2_0,r_0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Rb^0_{r_0,0_0} & Rb^0_{r_0,1_0} & Rb^0_{r_0,2_0} & \cdots & Rb^0_{r_0,r_0}+Rb^r_{r_0,r_0} \end{pmatrix}_{(r+1)\times(r+1)}$$

For $u,v \in [r]$, let $E^1_{uv}(a)$ denote the $(r+1)\times(r+1)$ matrix with $(u+1,v+1)$-entry a and other entries 0. Obverse that $\bigcup_{u,v\in[r]}\{E^1_{uv}(b^0_{u_0,v_0})\} \cup \bigcup_{w\in\{1,2,\cdots,r\}}\{E^1_{ww}(b^w_{w_0,w_0})\}$ is an R-basis of Γ and that $\bigcup_{i\in[r]}\{e_i\} \cup \bigcup_{i\in\{1,2,\cdots,r\}}\{\alpha_i,\beta_i\} \cup \bigcup_{i,j\in\{1,2,\cdots,r\}}\{\beta_i\alpha_i\}$ is an R-basis of Λ.

Let $\psi : \Lambda \to \Gamma$ be the R-linear map given by

$$e_i \mapsto E^1_{ii}(b^i_{i_0,i_0}),\ \alpha_m \mapsto E^1_{0m}(b^0_{0_0,m_0}),\ \beta_n \mapsto E^1_{n0}(b^0_{n_0,0_0}),\ \beta_u\alpha_v \mapsto E^1_{uv}(b^0_{u_0,v_0}),$$

for $i \in [r]$ and $1 \le m,n,u,v \le r$. Since ψ sends the R-basis of Λ bijectively to the one of Γ and preserves multiplication of basis elements, ψ is a homomorphism of algebras. This shows that ψ is an isomorphism of algebras. $\square$

The basic algebra Λ of $\mathcal{T}$ is the dual extension of a star with $r+1$ vertices and r arrows directing to the center of the star. For an algebra A over a field R given by a quiver $Q = (Q_0, Q_1)$ with relations $\{\rho_i \mid i \in I\}$, the *dual extension* of A (see [28]) is given be the quiver $(Q_0, Q_1 \cup Q_1')$ with relations $\{\rho_i \mid i \in I\} \cup \{\rho_i' \mid i \in I\} \cup \{\alpha'\beta \mid \alpha,\beta \in Q_1\}$, where Q_1' is the set of opposite arrows in Q_1, that is $Q_1' := \{\alpha' : j \to i \mid \alpha : i \to j \text{ in } Q_1\}$. It was shown that the global dimension of the dual extension of A is the double of the global dimension of A [29].

Theorem 3.1(2) shows that the global dimension of the Terwilliger algebra of a quasi-thin scheme is at most 2. Thus we re-obtain the quasi-heredity of these algebras by a result of Dlab-Ringel which says that finite-dimensional algebras of global dimension at most 2 are always quasi-hereditary [7, Theorem 2].

The representation dimension of an Artin algebra A was introduced by Auslander [2] and defined as follows.

$$\operatorname{repdim}(A) := \inf\{\operatorname{gldim}(\operatorname{End}_A(A \oplus D(A) \oplus M)) \mid M \in A\text{-mod}\}.$$

For the representation dimension of the Terwilliger algebra $\mathcal{T}$, it follows from [30, Theorem 3.5] that $\operatorname{repdim}(\mathcal{T}) = \operatorname{repdim}(\Lambda) \le 3$. If $\mathcal{T}$ is semisimple, then $\operatorname{repdim}(\mathcal{T}) = 0$. Assume that $\mathcal{T}$ is not semisimple. Then $p = 2$ and $r \ge 1$. If $r = 1$, then $\operatorname{repdim}(\mathcal{T}) = \operatorname{repdim}(\Lambda) = 2$. If $r \ge 2$, then Λ is representation-infinite by [28, Lemma 3.4], and therefore $\operatorname{repdim}(\mathcal{T}) = \operatorname{repdim}(\Lambda) \ge 3$. In this case, $\operatorname{repdim}(\mathcal{T}) = \operatorname{repdim}(\Lambda) = 3$.

The dominant dimension of an Artin algebra A, denoted by $\operatorname{domdim}(A)$, is defined to be the minimal nonnegative integer n in a minimal injective resolution

$$0 \longrightarrow {}_AA \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_n \longrightarrow \cdots$$

of ${}_AA$, such that I_n is not projective. If such an integer n does not exist, we define $\operatorname{domdim}(A) = \infty$. Related to dominant dimensions, there is a long-standing, famous conjecture, namely the Nakayama conjecture which states that any finite-dimensional algebra over a field with infinite dominant dimension is self-injective (see [23] and [3, Conjecture (8), p.410]). This conjecture is still open. However, for the Terwilliger algebras of quasi-thin schemes, the Nakayama conjecture holds true. This can be seen from the following.

Corollary 3.2. *Let R be a field of characteristic $p \geq 0$, and assume that S be a quasi-thin scheme on a finite set X, $\mathcal{A}_2$ has $r \geq 0$ equivalence classes. For the Terwilliger R-algebra $\mathcal{T}$ of S, the following holds.*

$$\operatorname{domdim}(\mathcal{T}) = \begin{cases} 0, & \textit{if } p = 2 \textit{ and } r \geq 2, \\ 2, & \textit{if } p = 2 \textit{ and } r = 1, \\ \infty, & \textit{otherwise.} \end{cases}$$

Proof. If $\mathcal{T}$ is semisimple, then $\operatorname{domdim}(\mathcal{T}) = \infty$. By Proposition 2.6(3), $\mathcal{T}$ is semisimple if and only if $p \neq 2$ or $p = 2$ and S is thin. In the latter, $\mathcal{T} = M_X(R)$. Now let $p = 2$ and assume that the quasi-thin scheme S is not thin. In this case, we have $r \geq 1$. By Theorem 3.1(2), $\operatorname{domdim}(\mathcal{T}) = 2$ if $r = 1$, and 0 if $r \geq 2$ because in the latter case, no injective $\mathcal{T}$-modules are projective. □

Thus we have the following.

Corollary 3.3. *The Nakayama conjecture holds true for the class of Terwilliger algebras of quasi-thin schemes over any field.*

As another consequence, we re-obtain the following result in [16].

Corollary 3.4. *The Jacobson radical of the Terwilliger algebra of a quasi-thin scheme over a field has nilpotent index at most* 3.

Proof. Since the statement is true for semisimple Terwilliger algebras of quasi-thin schemes, we have to consider the case of Theorem 3.1(2). In this case, the basic algebra of the Terwilliger algebra of a quasi-thin scheme is radical-cube-zero by Theorem 3.1(2). It is known that Morita equivalent algebras A and B have the isomorphic lattices of two-sided ideals (respectively, lower nil radicals). In fact, if a bimodule ${}_AM_B$ defines a Morita equivalence between A and B, then the isomorphism from the lattice of ideals of A to the one of B is given by $I \mapsto I'$ for I, I' ideals in A and B, respectively, such that $IM = MI'$ (see [5, Theorem (3.5), Chapter 2, p. 65], or [1, Exercise 8, p.267]). Note that for Artin rings, their lower nil radicals coincide with their Jacobson radicals. Thus $\mathrm{rad}(A)^n M = M\mathrm{rad}(B)^n$ for all $n \geq 0$, where $\mathrm{rad}(A)$ stands for the Jacobson radical of A. Hence the nilpotent indices of the radicals of A and B are equal. It follows that the Jacobson radical of the Terwilliger algebra of a quasi-thin scheme over a field has nilpotent index at most 3. $\square$

Declaration of competing interest

There is no competing interest.

Acknowledgments

The research work was partially supported by the National Natural Science Foundation of China (Grant 12031014). Also, the corresponding author C. C. Xi thanks the Tianyuan project of the NSFC for partial support (Grant 12226314). The authors thank Dr. Yu Jiang from Anhui University for his talk at CNU and discussions on Terwilliger algebras, and are grateful to both the editor of JCTA and the anonymous referee for comments that improved the exposition of the article.

Data availability

No data was used for the research described in the article.

References

[1] F.W. Anderson, K.R. Fuller, Rings and Categories of Modules, second edition, Graduate Texts in Mathematics, vol. 13, Springer-Verlag, New York, 1992.
[2] M. Auslander, Representation Dimension of Artin Algebras, Queen Mary College Mathematics Notes, Queen Mary, College, London, 1971.
[3] M. Auslander, I. Reiten, S.O. Smalø, Representation Theory of Artin Algebras, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1995.
[4] E. Bannai, T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin/Cummings, Menlo Park, CA, 1984.
[5] H. Bass, Algebraic K-Theory, W. A. Benjamin, New York, 1968.
[6] E. Cline, B. Parshall, L. Scott, Finite dimensional algebras and highest weight categories, J. reine angew. Math. 391 (1988) 85–99.
[7] V. Dlab, C.M. Ringel, Quasi-hereditary algebras, Ill. J. Math. 33 (2) (1989) 280–291.
[8] M. Geck, Hecke algebras of finite type are cellular, Invent. Math. 169 (3) (2007) 501–517.
[9] J.J. Graham, G.I. Lehrer, Cellular algebras, Invent. Math. 123 (1) (1996) 1–34.
[10] A. Hanaki, Modular Terwilliger algebras of association schemes, Graphs Comb. 37 (5) (2021) 1521–1529.
[11] A. Herman, B. Rahnamai, The character values of commutative quasi-thin schemes, Linear Algebra Appl. 429 (11–12) (2008) 2663–2669.
[12] D.G. Higman, Coherent configurations, part I. Ordinary representation theory, Geom. Dedic. 4 (1) (1975) 1–32.
[13] M. Hirasaka, On quasi-thin association schemes with odd number of points, J. Algebra 240 (2) (2001) 665–679.
[14] M. Hirasaka, M. Muzychuk, On quasi-thin association schemes, J. Comb. Theory, Ser. A 98 (1) (2002) 17–32.

[15] M. Hirasaka, M. Muzychuk, Association schemes generated by a non-symmetric relation of valency 2, Discrete Math. 244 (1–3) (2002) 109–135.

[16] Y. Jiang, On Terwilliger $\mathbb{F}$-algebras of quasi-thin association schemes, Algebraic Combin. 57 (4) (2023) 1219–1251.

[17] S. König, C.C. Xi, On the structure of cellular algebras, in: I. Reiten, S. Smalo, O. Solberg (Eds.), Algebras and Modules II, in: Canad. Math. Soc. Conf. Proc., vol. 24, 1998, pp. 365–386.

[18] S. König, C.C. Xi, When is a cellular algebra quasi-hereditary?, Math. Ann. 315 (1999) 218–293.

[19] F. Levstein, C. Maldonado, The Terwilliger algebra of the Johnson schemes, Discrete Math. 307 (13) (2007) 1621–1635.

[20] B. Lv, C. Maldonado, K. Wang, More on the Terwilliger algebra of Johnson schemes, Discrete Math. 328 (2014) 54–62.

[21] M. Muzychuk, I. Ponomarenko, On quasi-thin association schemes, J. Algebra 351 (1) (2012) 467–489.

[22] M. Muzychuk, P.-H. Zieschang, On association schemes all elements of which have valency 1 or 2, Discrete Math. 308 (14) (2008) 3097–3103.

[23] T. Nakayama, On algebras with complete homology, Abh. Math. Semin. Univ. Hamb. 22 (1958) 300–307.

[24] P. Terwilliger, The subconstituent algebra of an association scheme I, Algebraic Combin. 1 (1992) 363–388.

[25] P. Terwilliger, The subconstituent algebra of an association scheme II, Algebraic Combin. 2 (1993) 73–103.

[26] P. Terwilliger, The subconstituent algebra of an association scheme III, Algebraic Combin. 2 (1993) 177–210.

[27] B. Weisfeiler, On Construction and Identification of Graphs, Lecture Notes in Math., vol. 558, Springer, Berlin, 1976.

[28] C.C. Xi, Quasi-hereditary algebras with a duality, J. reine angew. Math. 449 (1994) 201–215.

[29] C.C. Xi, Global dimensions of dual extension algebras, Manuscr. Math. 88 (1) (1995) 25–31.

[30] C.C. Xi, Representation dimension and quasi-hereditary algebras, Adv. Math. 168 (2) (2002) 193–212.

[31] C.C. Xi, J.B. Zhang, Structures of centralizer matrix algebras, Linear Algebra Appl. 622 (2021) 215–249.

[32] P.-H. Zieschang, Theory of Association Schemes, Spring Monogr. Math., Springer-Verlag, Berlin, 2005.