Singular equivalences and homological conjectures

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Abstract

The fact that each finite-dimensional algebra over a field is isomorphic to the centralizer of two matrices, has suggested to investigate representation theoretical problems of finite-dimensional algebras through centralizer algebras of matrices. The first natural question is to study the problems for the centralizer algebra of one matrix, called a centralizer matrix algebra. In this paper we give complete descriptions of the singularity categories and singular equivalences of centralizer matrix algebras, and verify the Auslander–Reiten (or Gorenstein projective) and Cartan determinant conjectures for centralizer matrix algebras. Consequently, all historical homological conjectures (the finitistic dimension, Wakamatsu tilting, tilting (projective) complement, strong Nakayama, generalized Nakayama and Nakayama conjectures) are true for centralizer matrix algebras over fields. Moreover, we prove some homological invariants of singular equivalences for centralizer matrix algebras.

Contents

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1 Introduction

The familiar fact that every finite-dimensional algebra over a field is isomorphic to the centralizer of two matrices in a full matrix algebra (see [5]), has suggested to contemplate problems on representations of finite-dimensional algebras through the ones of centralizer algebras of matrices, instead of working with quivers and relations. The first fundamental step toward this direction is to investigate the centralizer of one matrix. Such an algebra will be called a centralizer matrix algebra. In this case, a number of useful partial results have been obtained. The earliest result on this subject could trace back to Frobenius, who established an explicit dimension formula for a centralizer matrix algebra in terms of degrees of invariant factors of the given matrix [15] (see [32, Theorems 1 and 2, p.105-106]). Recently, the structure of centralizer matrix algebras was discussed in [35], it was shown that centralizer matrix algebras of Jordan block matrices are cellular in the sense of Graham and Lehrer [17]. By a quite different approach from [35, 36], complete characterizations of derived and stable equivalences were given in terms of new types of matrix equivalence relations [24, 25]. Moreover, some major conjectures in representation theory, namely the finitistic dimension conjecture, Nakayama conjecture and Alperin–Auslander conjecture, are verified for centralizer matrix algebras [24, 37].

In the present paper we have two aims in mind, the first one is to attack the problem of how to describe equivalences of singularity categories of centralizer matrix algebras over fields. Roughly speaking, the singularity category of an Artin algebra is the quotient of its bounded by its perfect derived category. This algebraical construction goes back to Buchweitz (1987) and, independently, the geometrical formulation to Orlov (2004). The second aim, as a consequence of our discussions, is to show that the Auslander–Reiten and Cartan determinant conjectures hold for centralizer matrix algebras. Thus, together with results in [24, 25, 36], we know that all homological conjectures mentioned in [2, Conjectures, p. 409] hold true for centralizer matrix algebras.

To state our results more precisely, we need a few notations. Let R be a field and $M_n(R)$ the full $n\times n$ matrix algebra over R. For a matrix $c\in M_n(R)$, we denote by $S_n(c,R)$ the *centralizer matrix algebra*

$$S_n(c,R) := \{x\in M_n(R) \mid cx = xc\}.$$

To describe singular equivalences of centralizer matrix algebras, we introduce a new type of equivalence relation on all square matrices over a field. This relation is called an Sg-equivalence, defined in terms of equivalence classes of irreducible factors of maximal elementary divisors of matrices together with combinatorial data of multiplicities of elementary divisors. To solve the isomorphism problem of centralizer matrix algebras, we introduce another equivalence relation on matrices, namely the I-equivalence, which is defined by a bijection between maximal elementary divisors of matrices together with combinatorics of some multisets. Both equivalence relations we introduced are in terms of linear algebra. We refer to Section 3 for details.

For an Artin algebra A, we denote by $\mathscr{D}_{sg}(A)$ the singularity category of A, which is the Verdier quotient of the derived category $\mathscr{D}^b(A)$ of A by its perfect subcategory $\mathscr{K}^b(A\text{-proj})$ (see Subsection 2.2 for definition).

With these conventions established we now state our first main result.

Theorem 1.1. [Theorem 5.2] *Let* $A := S_n(c,R)$ *and* $B := S_m(d,R)$ *for* $c\in M_n(R)$ *and* $d\in M_m(R)$. *Then*

(1) *A and B are isomorphic R-algebras if and only if c and d are I-equivalent matrices.*

(2) *The following statements are equivalent.*

(*i*) *$\mathscr{D}_{sg}(A)$ and $\mathscr{D}_{sg}(B)$ are equivalent as triangulated R-categories.*

(*ii*) *$\mathscr{D}_{sg}(A)$ and $\mathscr{D}_{sg}(B)$ are equivalent as R-categories.*

(*iii*) *c and d are Sg-equivalent as matrices.*

The proof of this result is based on a complete description of the singularity category of $S_n(c,R)$ (see Theorem 4.11). For permutation matrices, any singular equivalence between their centralizer matrix algebras induces a singular equivalence of their singular parts. This is given in Corollary 5.8. Furthermore, we give sufficient conditions in Corollary 5.12, such that Morita, derived, stable and singular equivalences induce each other between centralizer matrix algebras of permutation matrices.

In the representation theory and homological algebra of finite-dimensional algebras, homological conjectures have been a core set of problems. Though lots of efforts have been made in the last decades, these important conjectures are still open to date. Applying the ideas in this article, we will verify these major conjectures for centralizer matrix algebras.

An Artin algebra Λ is said to be *CM-finite* if it has only finitely many non-isomorphic indecomposable Gorenstein projective modules, and *n-minimal Auslander-Gorenstein* [22] if $\mathrm{idim}({}_\Lambda\Lambda) \leq n+1 \leq \mathrm{domdim}(\Lambda)$, where $\mathrm{idim}({}_\Lambda M)$ and $\mathrm{domdim}({}_\Lambda M)$ denote the injective and dominant dimensions of a Λ-module M, respectively. For the precise statements of the homological conjectures in the following result, we refer to Section 6 or [2, Conjectures, p.409].

Theorem 1.2. [Proposition 4.8 and Theorem 6.2]

(1) *Let R be a field, $f(x)$ an irreducible polynomial in $R[x]$, and $A := R[x]/(f(x)^n)$. Then, for any finitely generated A-module M, $\mathrm{End}_A(M)$ is a CM-finite 1-minimal Auslander–Gorenstein algebra. Particularly, centralizer matrix algebras over fields are CM-finite.*

(2) *The Auslander–Reiten and Cartan determinant conjectures hold true for centralizer matrix algebras over fields.*

The contents of this article read as follows. In Section 2 we recall notions and terminologies, and prepare basic facts for proofs. In Section 3 we introduce new types of equivalence relations on matrices over fields. We also compare them with some known equivalence relations. In Section 4 we describe the singularity categories of centralizer matrix algebras. In Section 5 we establish characterizations of singular equivalences of centralizer matrix algebras in term of new equivalence relations of matrices. In Section 6 we recall homological conjectures stated in Theorem 1.2(2), prove the validity of these conjectures, and show some homological invariants of singular equivalences between centralizer matrix algebras.

2 Preliminaries

In this section we recall some basic definitions and terminologies on exact categories, Frobenius categories and triangulated categories. For our later proofs, we prove some properties of modules over quotients of polynomial algebras.

2.1 Exact and Frobenius categories

Let $\mathcal{C}$ be an additive category. A full subcategory $\mathcal{B}$ of $\mathcal{C}$ is always assumed to be closed under isomorphisms, that is, if $X \in \mathcal{B}$ and $Y \in \mathcal{C}$ with $Y \simeq X$, then $Y \in \mathcal{B}$.

Let X be an object in $\mathcal{C}$. We denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{C}$ consisting of all direct summands of the coproducts of finitely many copies of X.

The composition of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$ is written as $fg : X \to Z$. The induced morphisms $\mathrm{Hom}_\mathcal{C}(Z, f) : \mathrm{Hom}_\mathcal{C}(Z,X) \to \mathrm{Hom}_\mathcal{C}(Z,Y)$ and $\mathrm{Hom}_\mathcal{C}(f,Z) : \mathrm{Hom}_\mathcal{C}(Y,Z) \to \mathrm{Hom}_\mathcal{C}(X,Z)$ are denoted by f^* and f_*, respectively, while the composition of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ between categories is denoted by $G \circ F$ which is a functor from $\mathcal{C}$ to $\mathcal{E}$.

Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. A morphism $f : X \to Y$ in $\mathcal{C}$ is called a *right minimal* morphism if $\alpha \in \mathrm{End}_\mathcal{C}(X)$ is an isomorphism whenever $f = \alpha f$; a $\mathcal{D}$-*epic* morphism if the induced map $f^* : \mathrm{Hom}_\mathcal{C}(D,X) \to \mathrm{Hom}_\mathcal{C}(D,Y)$ is surjective for any $D \in \mathcal{D}$; and a *right* $\mathcal{D}$-*approximation* of Y if $X \in \mathcal{D}$ and f is $\mathcal{D}$-epic. If

f is both a right minimal morphism and a right $\mathcal{D}$-approximation of Y, then f is called a right *minimal* $\mathcal{D}$*-approximation* of Y. Dually, one defines the notions of $\mathcal{D}$-monic morphisms and left (minimal) $\mathcal{D}$-approximations in $\mathcal{C}$.

A diagram $X \xrightarrow{\lambda} Y \xrightarrow{\pi} Z$ of morphisms in $\mathcal{C}$ is called a *kernel-cokernel pair* (λ, π) if λ is the kernel of π and π is the cokernel of λ. Given another kernel-cokernel pair (λ', π') in $\mathcal{C}$, if there are isomorphisms $a : X \to X', b : Y \to Y'$ and $c : Z \to Z'$ in $\mathcal{C}$ such that $\lambda b = a\lambda'$ and $\pi c = b\pi'$, we say that the pairs (λ, π) and (λ', π') are *isomorphic*. Let $\mathcal{S}$ be a collection of kernel-cokernel pairs in $\mathcal{C}$ which is closed under isomorphisms. A morphism $\lambda : X \to Y$ in $\mathcal{C}$ is called an *admissible monomorphism of* $\mathcal{S}$ if there is a morphism $\pi : Y \to Z$ in $\mathcal{C}$ such that $(\lambda, \pi) \in \mathcal{S}$. Dually, we define admissible epimorphisms of $\mathcal{S}$. The elements of $\mathcal{S}$ are called *admissible exact sequences* of $\mathcal{S}$. If $\mathcal{C}$ together with $\mathcal{S}$ forms an exact category, we often say that $(\mathcal{C}, \mathcal{S})$ is an exact category. We refer to [30] for the precise formulation of exact categories.

Let $(\mathcal{C}, \mathcal{S})$ and $(\mathcal{C}', \mathcal{S}')$ be exact categories. An additive functor $F : \mathcal{C} \to \mathcal{C}'$ is called an *exact functor* if F sends admissible exact sequences of $\mathcal{S}$ to admissible exact sequences of $\mathcal{S}'$. In this case, we often say that $F : (\mathcal{C}, \mathcal{S}) \to (\mathcal{C}', \mathcal{S}')$ is an exact functor.

Let $\mathcal{D}$ be a full additive subcategory of an exact category $(\mathcal{C}, \mathcal{S})$, and let $\tilde{\mathcal{S}}$ be the subclass of $\mathcal{S}$ consisting precisely of the admissible exact sequences of $\mathcal{S}$ whose terms belong to $\mathcal{D}$. The pair $(\mathcal{D}, \tilde{\mathcal{S}})$ is called a *fully exact subcategory* of $(\mathcal{C}, \mathcal{S})$ if $(\mathcal{D}, \tilde{\mathcal{S}})$ is closed under extensions in $(\mathcal{C}, \mathcal{S})$, that is, for an admissible exact sequence $X \to Y \to Z$ of $\mathcal{S}$ with $X, Z \in \mathcal{D}$, we have $Y \in \mathcal{D}$. In this case, the inclusion $(\mathcal{D}, \tilde{\mathcal{S}}) \subseteq (\mathcal{C}, \mathcal{S})$ is a fully faithful exact functor.

Let $(\mathcal{C}, \mathcal{S})$ be an exact category. An object P in $\mathcal{C}$ is said to be $\mathcal{S}$*-projective* if, for every admissible epimorphism $f : Y \to Z$ of $\mathcal{S}$, the induced morphism $f^* : \mathrm{Hom}_{\mathcal{C}}(P, Y) \to \mathrm{Hom}_{\mathcal{C}}(P, Z)$ is surjective. We say that $(\mathcal{C}, \mathcal{S})$ has *enough* $\mathcal{S}$*-projectives* if, for every object $C \in \mathcal{C}$, there exists an admissible epimorphism $P \to X$ of $\mathcal{S}$ such that P is $\mathcal{S}$-projective. Dually, one defines the notions of $\mathcal{S}$*-injective* objects and exact categories with *enough* $\mathcal{S}$*-injectives*. We denote by $\mathrm{Proj}_{\mathcal{S}}(\mathcal{C})$ (respectively, $\mathrm{Inj}_{\mathcal{S}}(\mathcal{C})$) the full subcategory of $\mathcal{C}$ consisting of all $\mathcal{S}$-projective (respectively, $\mathcal{S}$-injective) objects.

A *Frobenius category* is an exact category $(\mathcal{C}, \mathcal{S})$ which has both enough $\mathcal{S}$-projectives and $\mathcal{S}$-injectives, such that $\mathrm{Proj}_{\mathcal{S}}(\mathcal{C})$ and $\mathrm{Inj}_{\mathcal{S}}(\mathcal{C})$ coincide. In this case, the quotient $\mathcal{C}/\mathrm{Inj}_{\mathcal{S}}(\mathcal{C})$ is a triangulated category [18, Theorem 2.6, p.16]. Frobenius categories $(\mathcal{C}, \mathcal{S})$ and $(\mathcal{C}', \mathcal{S}')$ are *equivalent* if there is an exact functor $F : (\mathcal{C}, \mathcal{S}) \to (\mathcal{C}', \mathcal{S}')$ such that $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence of additive categories and induces an equivalence: $\mathrm{Proj}_{\mathcal{S}}(\mathcal{C}) \to \mathrm{Proj}_{\mathcal{S}'}(\mathcal{C}')$ of additive categories. More generally, if $F : (\mathcal{C}, \mathcal{S}) \to (\mathcal{C}', \mathcal{S}')$ is an exact functor between Frobenius categories $(\mathcal{C}, \mathcal{S})$ and $(\mathcal{C}', \mathcal{S}')$ such that F sends objects in $\mathrm{Inj}_{\mathcal{S}}(\mathcal{C})$ to objects in $\mathrm{Inj}_{\mathcal{S}'}(\mathcal{C}')$, then F induces a triangle functor $\underline{F} : \mathcal{C}/\mathrm{Inj}_{\mathcal{S}}(\mathcal{C}) \to \mathcal{C}'/\mathrm{Inj}_{\mathcal{S}'}(\mathcal{C}')$ (see [23, Example 8.1]). Consequently, we get the following result.

Corollary 2.1. *Let* $(\mathcal{C}, \mathcal{S})$ *be a Frobenius category, and let* $(\mathcal{C}', \mathcal{S}')$ *be a full exact subcategory of* $(\mathcal{C}, \mathcal{S})$*. If* $(\mathcal{C}', \mathcal{S}')$ *is a Frobenius category such that* $\mathrm{Inj}_{\mathcal{S}'}(\mathcal{C}') = \mathcal{C}' \cap \mathrm{Inj}_{\mathcal{S}}(\mathcal{C})$*, then the inclusion functor* $F : (\mathcal{C}', \mathcal{S}') \to (\mathcal{C}, \mathcal{S})$ *induces a fully faithful triangle functor* $\underline{F} : \mathcal{C}'/\mathrm{Inj}_{\mathcal{S}'}(\mathcal{C}') \to \mathcal{C}/\mathrm{Inj}_{\mathcal{S}}(\mathcal{C})$.

To get intrinsic descriptions of products of additive categories, we observe the following simple fact.

Lemma 2.2. *Let* $\mathcal{C}$ *be an additive (triangulated) category, and let* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *be full additive (triangulated) subcategories of* $\mathcal{C}$*. Suppose* $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2) = 0 = \mathrm{Hom}_{\mathcal{C}}(C_2, C_1)$ *for* $C_1 \in \mathcal{C}_1$ *and* $C_2 \in \mathcal{C}_2$*. If, for any* $C \in \mathcal{C}$*, there are* $C_1 \in \mathcal{C}_1$ *and* $C_2 \in \mathcal{C}_2$ *such that* $C \simeq C_1 \oplus C_2$ *in* $\mathcal{C}$*, then* $\mathcal{C} \simeq \mathcal{C}_1 \times \mathcal{C}_2$ *as additive (triangulated) categories, where* $\mathcal{C}_1 \times \mathcal{C}_2$ *is the product of categories.*

2.2 Singularity categories of algebras

Throughout this paper, R denote a field unless stated otherwise. By an algebra we mean a finite-dimensional unital associative algebra over R. All modules are finitely generated left modules.

Let Λ be an Artin $\mathbb{K}$-algebra over a commutative Artin ring $\mathbb{K}$. By $\mathrm{rad}(\Lambda)$ and $LL(\Lambda)$ we denote the Jacobson radical and Loewy length of Λ, respectively. Let Λ^{op} stand for the opposite algebra of Λ. We write Λ-mod for the category of all finitely generated Λ-modules, and Λ-proj (respectively, Λ-inj) for the full subcategory of Λ-mod consisting of projective (respectively, injective) Λ-modules. Let Λ-Gproj be the full subcategory of Λ-mod consisting of Gorenstein projective Λ-modules. Then Λ-Gproj contains Λ-proj and is closed under direct summands, extensions and kernels of epimorphisms. It is a resolving subcategory of Λ-mod. We postpone the definition of Gorenstein projective modules to Section 4.

For a Λ-module M, $\ell_\Lambda(M)$ denotes the composition length of M. The *basic module* of M is denoted by $\mathcal{B}(M)$ which is, by definition, the direct sum of all non-isomorphic indecomposable direct summands of M. Let $\mathrm{copres}(M)$ be the full subcategory of Λ-mod consisting of those modules L such that there is an exact sequence: $0 \to L \to M_0 \to M_1$ in Λ-mod, with $M_0, M_1 \in \mathrm{add}(M)$.

A module $M \in \Lambda$-mod is called a *generator* (or *cogenerator*) if $\mathrm{add}({}_\Lambda\Lambda) \subseteq \mathrm{add}(M)$ (or $\mathrm{add}(D(\Lambda_\Lambda)) \subseteq \mathrm{add}(M)$), where $D : \Lambda\text{-mod} \to \Lambda^{\mathrm{op}}\text{-mod}$ is the usual duality of an Artin algebra, and an *additive generator* if $\Lambda\text{-mod} = \mathrm{add}(M)$. A module ${}_\Lambda X$ is said to be *n-self-orthogonal* if $\mathrm{Ext}^i_\Lambda(X,X) = 0$ for all $1 \leq i \leq n$, and *self-orthogonal* if it is n-orthogonal for all $n \geq 1$.

Let $\Lambda\text{-}\underline{\mathrm{mod}} := \Lambda\text{-mod}/\Lambda\text{-proj}$ be the *stable module category* of Λ. Since Λ-Gproj contains Λ-proj, we have the stable subcategory $\Lambda\text{-}\underline{\mathrm{Gproj}} := \Lambda\text{-Gproj}/\Lambda\text{-proj}$ of $\Lambda\text{-}\underline{\mathrm{mod}}$. It is called the *Gorenstein stable category* of Λ. We define

$$\mathcal{S} := \{X \xrightarrow{\lambda} Y \xrightarrow{\pi} Z \mid (\lambda, \pi) \text{ is a kernel-cokernel pair in } \Lambda\text{-mod and } X, Y, Z \in \Lambda\text{-Gproj}\}.$$

It is well known that $(\Lambda\text{-Gproj}, \mathcal{S})$ is a Frobenius $\mathbb{K}$-category with $\mathrm{Inj}_{\mathcal{S}}(\Lambda\text{-Gproj}) = \Lambda\text{-proj}$. Thus $\Lambda\text{-}\underline{\mathrm{Gproj}}$ is a triangulated $\mathbb{K}$-category.

Analogously, replacing module categories and projective modules by derived categories and projective complexes, respectively, one gets the so-called singularity categories. Let $\mathscr{D}^b(\Lambda)$ stand for the bounded derived category of A-mod, and let $\mathscr{K}^b(\Lambda\text{-proj})$ be the full subcategory of $\mathscr{D}^b(\Lambda)$ consisting of the bounded complexes of finitely generated projective Λ-modules. Then $\mathscr{K}^b(\Lambda\text{-proj})$ is a thick triangulated $\mathbb{K}$-subcategory of the triangulated $\mathbb{K}$-category $\mathscr{D}^b(\Lambda)$. Let $\mathscr{D}_{sg}(\Lambda) := \mathscr{D}^b(\Lambda)/\mathscr{K}^b(\Lambda\text{-proj})$ be the Verdier quotient of $\mathscr{D}^b(\Lambda)$ by $\mathscr{K}^b(\Lambda\text{-proj})$. This triangulated $\mathbb{K}$-category is called the *singularity category* of Λ.

Definition 2.3. *Artin $\mathbb{K}$-algebras Λ and Γ are said to be*

(1) stably equivalent *if their stable module categories $\Lambda\text{-}\underline{\mathrm{mod}}$ and $\Gamma\text{-}\underline{\mathrm{mod}}$ are equivalent as $\mathbb{K}$-categories. An equivalence $F : \Lambda\text{-}\underline{\mathrm{mod}} \to \Gamma\text{-}\underline{\mathrm{mod}}$ of $\mathbb{K}$-categories is called a* stable equivalence *between Λ and Γ.*

(2) Singularly equivalent *if their singularity categories $\mathscr{D}_{sg}(\Lambda)$ and $\mathscr{D}_{sg}(\Gamma)$ are equivalent as triangulated $\mathbb{K}$-categories. An equivalence $F : \mathscr{D}_{sg}(\Lambda) \to \mathscr{D}_{sg}(\Gamma)$ of triangulated $\mathbb{K}$-categories is called a* singular equivalence *between Λ and Γ.*

Singularity categories and singular equivalences were introduced and studied in [6] by an algebraical approach, while they were formulated and investigated independently in [29] by geometrical point of view. Since then singularity categories and singular equivalences have been intensively studied.

If an Artin algebra Λ is self-injective, then $\mathscr{D}_{sg}(\Lambda)$ and $\Lambda\text{-}\underline{\mathrm{mod}}$ are equivalent as triangulated $\mathbb{K}$-categories. More generally, if Λ is a Gorenstein algebra (that is, the injective dimensions of ${}_\Lambda\Lambda$ and Λ_Λ are finite), then $\mathscr{D}_{sg}(\Lambda) \simeq \Lambda\text{-}\underline{\mathrm{Gproj}}$ as triangulated $\mathbb{K}$-categories (see [6] or [19]).

2.3 Basic facts on modules over quotients of polynomial algebras

In this subsection we prove some lemmas of modules over quotients of the polynomial algebra $R[x]$.

Let $\mathbb{Z}_{>0}$ be the set of positive integers. For $n \in \mathbb{Z}_{>0}$, let $[n] := \{1, 2, \cdots, n\}$ and Σ_n be the symmetric group of permutations of $[n]$. We write the image of $i \in [n]$ under $\sigma \in \Sigma_n$ as $(i)\sigma$. The cardinality of a set S is denoted by $|S|$.

In the rest of this subsection, we fix an irreducible polynomial $f(x) \in R[x]$ of degree u and set $A := R[x]/(f(x)^n)$ for $n \geq 1$. Then A is a local, commutative, symmetric, Nakayama algebra with n (non-isomorphic) indecomposable modules $M(i)$ for $i \in [n]$, where $M(i)$ equals $R[x]/(f(x)^i) \in A$-mod for $i \in [n]$. We set $M(0) = 0$. Clearly, $M(i) \simeq A/\mathrm{rad}^i(A)$ as A-modules for $0 \leq i \leq n$.

For $0 \leq i \leq j \leq n$, we denote by

$$f_{i,j} : M(i) \to M(j),\ x^k + (f(x)^i) \mapsto f(x)^{j-i} \cdot x^k + (f(x)^j) \text{ for } 0 \leq k \leq ui-1, \text{ and}$$

$$g_{j,i} : M(j) \to M(i),\ x^k + (f(x)^j) \mapsto x^k + (f(x)^i) \text{ for } 0 \leq k \leq uj-1$$

the canonical injective and surjective homomorphisms, respectively. Clearly, $f_{i,j}g_{j,k} = 0$ if $i + k \leq j$.

Lemma 2.4. *Let $i, j, k \in [n]$ with $j, k \geq i$, and let $f : M(i) \to M(k), g : M(k) \to M(i)$ and $h : M(j-i) \to M(k-i)$ be homomorphisms of A-modules.*

(1) *If $k \geq j$, then f and g factorize through $M(j)$.*

(2) *There exists a homomorphism $\tilde{h} : M(j) \to M(k)$ such that $\tilde{h}g_{k,k-i} = g_{j,j-i}h$.*

Proof. (1) Due to $j \geq i$, we have a canonical surjective homomorphism $g_{j,i} : M(j) \to M(i)$. Due to $k \geq j$, the homomorphisms g and $g_{j,i}$ can be regarded as homomorphisms of $R[x]/(f(x)^k)$-modules. Since $M(k)$ is a projective $R[x]/(f(x)^k)$-module, there exists a homomorphism $g_1 : M(k) \to M(j)$ of $R[x]/(f(x)^k)$-modules such that $g_1 g_{j,i} = g$. This shows that g factorizes through $M(j)$. Similarly, we show that f factorizes through $M(j)$.

(2) Assume $j \geq k$. Then $g_{k,k-i}$, $g_{j,j-i}$ and h can be viewed as homomorphisms of $R[x]/(f(x)^j)$-modules. Since $M(j)$ is a projective $R[x]/(f(x)^j)$-module and $g_{k,k-i}$ is an epimorphism, there exists a homomorphism $\tilde{h} : M(j) \to M(k)$ such that $\tilde{h}g_{k,k-i} = g_{j,j-i}h$.

Now we assume $j < k$. Since the map

$$\varphi : R[x]/(f(x)^{j-i}) \longrightarrow \mathrm{Hom}_A(R[x]/(f(x)^{j-i}), R[x]/(f(x)^{k-i})),$$

$$\alpha(x) + f(x)^{j-i} \mapsto [x^s + (f(x)^{j-i}) \mapsto \alpha(x) \cdot f(x)^{k-j} \cdot x^s + (f(x)^{k-i})]$$

for $\alpha(x) \in R[x]$ and $0 \leq s \leq u(j-i)-1$, is an isomorphism of A-modules, we may assume that there exists a polynomial $\alpha(x) \in R[x]$ such that $h = (\alpha(x) + f(x)^{j-i})\varphi$. The surjective map $g_{j,j-i}$ supplies a polynomial $\tilde{g}(x) \in R[x]$ such that $(\tilde{g}(x) + f(x)^j)g_{j,j-i} = \alpha(x) + f(x)^{j-i}$. Define

$$\tilde{h} : R[x]/(f(x)^j) \longrightarrow R[x]/(f(x)^k),\ x^t + (f(x)^j) \mapsto \tilde{g}(x) \cdot f(x)^{k-j} \cdot x^t + (f(x)^k), \text{ for } 0 \leq t \leq uj-1.$$

Then one can check that $\tilde{h}$ is a homomorphism of A-modules and $\tilde{h}g_{k,k-i} = g_{j,j-i}h$. □

Recall that two multisets $\{\{n_1, \cdots, n_s\}\}$ and $\{\{m_1, \cdots, m_t\}\}$ are equal if and only if $s = t$ and there exists a permutation $\sigma \in \Sigma_s$ such that $(m_1, \cdots, m_s)^\sigma := (m_{(1)\sigma}, \cdots, m_{(s)\sigma}) = (n_1, \cdots, n_s)$ in $\mathbb{N}^s$.

Lemma 2.5. *Let $g(x)$ be an irreducible polynomial in $R[x]$ and $B := R[x]/(g(x)^m)$ for some $m \in \mathbb{N}^+$. Suppose that $I := \{\{n_1, \cdots, n_s\}\}$ and $J := \{\{m_1, \cdots, m_t\}\}$ are nonempty multisets with $n_i \in [n]$ for $i \in [s]$ and $m_j \in [m]$ for $j \in [t]$. Let n' and m' be the maximal elements of I and J, respectively. Let ${}_AM := \bigoplus_{i\in[s]} R[x]/(f(x)^{n_i})$ and ${}_BN := \bigoplus_{j\in[t]} R[x]/(g(x)^{m_j})$. Then the following are equivalent.*

(1) $\mathrm{End}_A(M) \simeq \mathrm{End}_B(N)$ *as R-algebras.*

(2) *$I = J$ and $R[x]/(f(x)^{n'}) \simeq R[x]/(g(x)^{m'})$ as R-algebras.*

Proof. We may assume $n' = n$ and $m' = m$. Otherwise we replace A and B by $A' = R[x]/(f(x)^{n'})$ and $B' := R[x]/(g(x)^{m'})$, respectively, and regard M as an A'-module and N as a B'-module. Further, we have $\mathrm{End}_A(M) = \mathrm{End}_{A'}(M)$ and $\mathrm{End}_B(N) = \mathrm{End}_{B'}(N)$.

The implication (2) $\Rightarrow$ (1) is clear. We prove (1) $\Rightarrow$ (2). Indeed, let $\Lambda := \mathrm{End}_A(M)$ and $\Gamma := \mathrm{End}_B(N)$. As left modules, ${}_\Lambda\Lambda = \bigoplus_{i\in[s]} \mathrm{Hom}_B(M, R[x]/(f(x)^{n_i})$ and ${}_\Gamma\Gamma = \bigoplus_{j\in[t]} \mathrm{Hom}_A(N, R[x]/(g(x)^{m_j})$. The summands ${}_\Lambda\mathrm{Hom}_A(M, R[x]/(f(x)^{n_i})$ and ${}_\Gamma\mathrm{Hom}_B(N, R[x]/(g(x)^{m_j})$ are indecomposable projective modules for $i \in [s]$ and $j \in [t]$. Since Λ and Γ are isomorphic R-algebras, we have $s = t$, and there exists $\sigma \in \Sigma_s$ such that $\mathrm{End}_\Lambda(\mathrm{Hom}_A(M, R[x]/(f(x)^{n_i})) \simeq \mathrm{End}_\Gamma(\mathrm{Hom}_B(N, R[x]/(g(x)^{m_{(i)\sigma}}))$ as R-algebras. This implies that

$$R[x]/(f(x)^{n_i}) \simeq \mathrm{End}_\Lambda(\mathrm{Hom}_A(M, R[x]/(f(x)^{n_i})) \simeq \mathrm{End}_\Gamma(\mathrm{Hom}_B(N, R[x]/(g(x)^{m_{(i)\sigma}})) \simeq R[x]/(g(x)^{m_{(i)\sigma}})$$

as R-algebras. Thus $n_i = LL(R[x]/(f(x)^{n_i})) = LL(R[x]/(g(x)^{m_{(i)\sigma}})) = m_{(i)\sigma}$ for $i \in [s]$. This implies that $I = J$ and $A \simeq B$ as R-algebras. $\square$

Lemma 2.6. *Let Λ be an Artin algebra, $\eta : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ an exact sequence in Λ-mod and $i \in \mathbb{N}$. If g is $\mathrm{add}\big(\Lambda/\mathrm{rad}^i(\Lambda)\big)$-epic, then η induces two exact sequences:*

$$0 \longrightarrow \mathrm{soc}^i(X) \xrightarrow{f'} \mathrm{soc}^i(Y) \xrightarrow{g'} \mathrm{soc}^i(Z) \longrightarrow 0 \ \ and$$

$$0 \longrightarrow X/\mathrm{soc}^i(X) \xrightarrow{f^\wedge} Y/\mathrm{soc}^i(Y) \xrightarrow{g^\wedge} Z/\mathrm{soc}^i(Z) \longrightarrow 0,$$

where f' and g' are the restrictions of f and g, respectively, and where $f^\wedge$ and $g^\wedge$ are the induced homomorphisms of f and g, respectively.

Proof. Given $M \in \Lambda$-mod and $i \in \mathbb{N}$, the map $\rho^i_M : \mathrm{Hom}_\Lambda(\Lambda/\mathrm{rad}^i(\Lambda), M) \simeq \mathrm{soc}^i(M)$, $f \mapsto (1)f$ for $f \in \mathrm{Hom}_\Lambda(\Lambda/\mathrm{rad}^i(\Lambda), M)$, is an isomorphism of Λ-modules, and any homomorphism $h : M \to M'$ of Λ-modules restricts to a homomorphism $h' : \mathrm{soc}^i(M) \to \mathrm{soc}^i(M')$ of Λ-modules. We form the following commutative diagram of 0-sequences in Λ-mod:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_\Lambda(\Lambda/\mathrm{rad}^i(\Lambda), X) & \xrightarrow{f^*} & \mathrm{Hom}_\Lambda(\Lambda/\mathrm{rad}^i(\Lambda), Y) & \xrightarrow{g^*} & \mathrm{Hom}_\Lambda(\Lambda/\mathrm{rad}^i(\Lambda), Z) & \longrightarrow & 0 \\
 & & \rho^i_X \downarrow \simeq & & \rho^i_Y \downarrow \simeq & & \rho^i_Z \downarrow \simeq & & \\
0 & \longrightarrow & \mathrm{soc}^i(X) & \xrightarrow{f'} & \mathrm{soc}^i(Y) & \xrightarrow{g'} & \mathrm{soc}^i(Z) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X/\mathrm{soc}^i(X) & \xrightarrow{f^\wedge} & Y/\mathrm{soc}^i(Y) & \xrightarrow{g^\wedge} & Z/\mathrm{soc}^i(Z) & \longrightarrow & 0.
\end{array}$$

As g is $\mathrm{add}(\Lambda/\mathrm{rad}^i(\Lambda))$-epic, the map g^* is surjective. Thus the top row is exact, and therefore the second row is exact. Since all rows and columns are exact except the last row, it follows from 3×3 Lemma that the last row is exact, too. $\square$

Suppose that $\{S_1, \cdots, S_n\}$ is a complete set of non-isomorphic simple Λ-modules and P_i is the projective cover of S_i for $i \in [n]$. For $i, j \in [n]$, let $[P_i : S_j]$ denote the number of composition factors of P_i that are isomorphic to S_j. Then $[P_i : S_j] = \ell_{\mathrm{End}_\Lambda(P_j)}\big(\mathrm{Hom}_\Lambda(P_j, P_i)\big)$ for $i, j \in [n]$. The matrix $C_\Lambda = ([P_i : S_j])_{i,j\in[n]} \in M_n(\mathbb{N})$ is called the *Cartan matrix of* Λ. The determinant of C_Λ is called the *Cartan determinant* of Λ.

The following lemma is an analogy of [12, Section 8], except we remove the assumption on the ground field. For the convienence of the reader, we include here a proof.

Lemma 2.7. *Let $T = \{t_1, \cdots, t_s\}$ be a subset of $[n]$ with $t_1 > t_2 > \cdots > t_s$, ${}_AM := \bigoplus_{i=1}^s M(t_i)$ and $\Gamma := \mathrm{End}_A(M)$. Then the Cartan matrix C_Γ of Γ is equal to*

$$\begin{pmatrix} t_1 & t_2 & \cdots & t_s \\ t_2 & t_2 & \cdots & t_s \\ \vdots & \vdots & \ddots & \vdots \\ t_s & t_s & \cdots & t_s \end{pmatrix}.$$

In particular, C_Γ *is a positive definite matrix and* $\det(C_\Gamma) = t_s\prod_{i=1}^{s-1}(t_i - t_{i+1})$. *Moreover,* $\det(C_\Gamma) = 1$ *if and only if* M *is an additive generator for* $R[x]/(f(x)^{t_1})$-mod.

Proof. We may assume $t_1 = n$. Otherwise we consider the algebra $A' := R[x]/(f(x)^{t_1})$ instead of A. Note that M can be regarded as an A'-module and that the canonical surjective homomorphism $A \to A'$ implies $\mathrm{End}_{A'}(M) = \Gamma$.

The set $\{P_i := \mathrm{Hom}_A(M, M(t_i)) \mid i \in [s]\}$ is a complete set of non-isomorphic indecomposable projective Γ-modules. Since $\mathrm{Hom}_A(M,-) : \mathrm{add}(M) \to \Gamma\text{-proj}$ is an equivalence of additive categories, we have $\mathrm{Hom}_\Gamma(P_j,P_i) \simeq \mathrm{Hom}_\Lambda(M(t_j),M(t_i)) \simeq R[x]/(f(x)^{m_{ji}})$ as $R[x]/(f(x)^{t_j})$-modules for $i, j \in [s]$ with $m_{ji} := \min\{t_j,t_i\}$. Note that $R[x]/(f(x)^{m_{ji}})$-module structure on $\mathrm{Hom}_\Gamma(P_j,P_i)$ is induced by the R-algebra isomorphism $\mathrm{End}_\Gamma(P_j) \simeq R[x]/(f(x)^{t_j})$. Thus $\ell_{\mathrm{End}_\Gamma(P_j)}(\mathrm{Hom}_\Gamma(P_j,P_i)) = \ell_{R[x]/(f(x)^{t_j})}(R[x]/(f(x)^{m_{ji}})) = m_{ji}$ for $i, j \in [s]$. Consequently, the Cartan matrix C_Γ of Γ is just the matrix in Lemma 2.7. Thus C_Γ is positive definite and $\det(C_\Gamma) = t_s\prod_{i=1}^{s-1}(t_i - t_{i+1})$. Clearly, $\det(C_\Gamma) = 1$ if and only if $t_s = 1$ and $t_i - t_{i+1} = 1$ for $i \in [s-1]$ if and only if M is an additive generator for $R[x]/(f(x)^{t_1})$-mod. □

We quote the following two results from [24, 25].

Lemma 2.8. [24, Lemma 2.11] *Given* $\{a,b,c,d\} \subseteq \{0,1,\cdots,n\}$ *with* $b < a < c$, $b < d < c$ *and* $a+d = b+c$, *and* $X \in A$-mod *which does not have indecomposable direct summands* N *with* $b < \ell_A(N) < c$, *let* ${}_AY := {}_AX \oplus M(b) \oplus M(c)$. *Then there is an exact sequence* $0 \to M(a) \xrightarrow{f} M(b) \oplus M(c) \xrightarrow{g} M(d) \to 0$ *in* A-mod, *where* $f = (-g_{a,b}, f_{a,c})$ *is a left minimal* $\mathrm{add}(Y)$*-approximation of* $M(a)$ *and* $g = \begin{pmatrix} f_{b,d} \\ g_{c,d} \end{pmatrix}$ *is a right minimal* $\mathrm{add}(Y)$*-approximation of* $M(d)$.

Lemma 2.9. [25, Lemma 2.10] *Let* $g(x)$ *be an irreducible polynomial in* $R[x]$ *and* $B := R[x]/(g(x)^m)$ *for* $m \geq 2$. *Then the following are equivalent:*

(1) A-$\underline{\mathrm{mod}} \simeq B$-$\underline{\mathrm{mod}}$ *as triangulated* R*-categories.*
(2) A-$\underline{\mathrm{mod}} \simeq B$-$\underline{\mathrm{mod}}$ *as* R*-categories.*
(3) $n = m$ *and* $A/\mathrm{rad}(A) \simeq B/\mathrm{rad}(B)$ *as* R*-algebras.*
(4) $A \simeq B$ *as* R*-algebras.*

2.4 Centralizer algebras of matrices

Centralizer algebras of matrices can realize any finite-dimensional algebras over fields. So it is worthy to investigate finite-dimensional algebras from this point of view. Let us recall some definitions and basic results on centralizer algebras of matrices.

Given a field R and a natural number $n \in \mathbb{Z}_{>0}$, let $M_n(R)$ be the full $n \times n$ matrix algebra over R with the identity matrix I_n. Let e_{ij}, $1 \leq i, j \leq n$, be the matrix units, and $J_n(\lambda) \in M_n(R)$ be the Jordan block matrix with the eigenvalue $\lambda \in R$.

For a nonempty subset X of $M_n(R)$, the *centralizer algebra* $S_n(X,R)$ of X in $M_n(R)$ is defined by

$$S_n(X,R) := \{a \in M_n(R) \mid ax = xa, \ \forall\, x \in X\}.$$

If X is a finite set, Brenner reduced the study of $S_n(X,R)$ to the case that X consists of only two elements [5, Lemma 1]. Furthermore, Brenner showed in [5, Lemma 2] that every finite-dimensional algebra over a field is isomorphic to the centralizer algebra of **two** matrices. Thus it is natural and fundamental to study first the centralizer algebra of a single matrix. This might be an important step to understanding arbitrary finite-dimensional algebras and related topics. For simplicity, we write $S_n(c,R)$ for $S_n(\{c\},R)$. By a *centralizer matrix algebra* we always mean an algebra of the form $S_n(c,R)$.

We have the following property of centralizer matrix algebras.

Lemma 2.10. [24, Lemma 3.2] *For $c \in M_n(R)$, there are the isomorphisms of R-algebras:*

$$S_n(c,R) \simeq S_n(c^{tr},R) \simeq S_n(c,R)^{\mathrm{op}} \simeq \mathrm{End}_{R[c]}(R^n),$$

where $R[c]$ is the subalgebra of $M_n(R)$ generated by c, and where c^{tr} is the transpose of c.

Next, we point out a connection of centralizer matrix algebras with the algebras Γ in Lemma 2.7.

Given a monic polynomial $g(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in R[x]$. The *companion matrix of* $g(x)$ is defined by

$$C[g(x)] := \begin{pmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & \cdots & 0 & -a_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \in M_n(R).$$

It is well known that $m_{C[g(x)]} = \chi_{C[g(x)]} = g(x)$ (see [21, Corollary, p.230]), where $m_{C(g)}(x)$ and $\chi_{C(g)}(x)$ are the minimal and characteristic polynomials of $C[g(x)]$ over R, respectively.

Lemma 2.11. *Let $T = \{t_1, \cdots, t_s\}$ be a subset of $[n]$ with $t_1 > t_2 > \cdots > t_s$, $m := \sum_{i=1}^s ut_i$, $M := \bigoplus_{i=1}^s M(t_i) \in$ A-mod and $\Gamma := \mathrm{End}_A(M)$. Then there is a matrix $c \in M_m(R)$ such that $\Gamma \simeq S_m(c,R)$ as R-algebras.*

Proof. We write $f(x) = a_ux^u + \cdots + a_1x + a_0 \in R[x]$ and consider $A' := R[x]/((a_u^{-1}f(x))^{t_1})$. Then there is a surjective homomorphism $\pi : A \to A'$ of R-algebras with $\mathrm{Ker}(\pi) = (f(x)^{t_1})/(f(x)^n)$, and ${}_AM$ can be regarded as an A'-module. Thus $\mathrm{End}_{A'}(M) = \mathrm{End}_A(M)$. Hence we may assume $t_1 = n$ and $a_u = 1$.

Now, we consider the diagonal block matrix $c := C[f(x)^{t_1}] \oplus \cdots \oplus C[f(x)^{t_s}] \in M_m(R)$. Then $m_c(x) = f(x)^{t_1}$, $R[c] \simeq R[x]/(m_c(x)) \simeq R[x]/(f(x)^{t_1}) = A$ as R-algebras, and $R^m \simeq \bigoplus_{i=1}^s R[x]/(f(x)^{t_i})$ as $R[c]$-modules (see [10, Chapter 4]). Thus it follows from Lemma 2.10 that $S_m(c,R) \simeq \mathrm{End}_{R[c]}(R^m) \simeq \Gamma$ as R-algebras. $\square$

2.5 Basics from number theory

In this subsection we assume that $p = 0$ or $p > 0$ is a prime number, and recall a few results on number theory for our later proofs.

Lemma 2.12. *Suppose $p > 0$ be a prime number and $i, j, k, \ell \in \mathbb{N}$ with $j > k$ and $\ell > i$. Then*

(1) *$p^j - p^k = p^\ell - p^i$ if and only if $j = \ell$ and $k = i$.*

(2) *$p^i - 1 = p^k$ if and only if $p = 2, i = 1$ and $k = 0$.*

(3) *$p^j - p^k = p^i$ if and only if $p = 2$ and $i = k = j - 1$. In particular, $p^j - p^k = 1$ if and only if $p = 2, j = 1$, and $k = 0$.*

For a prime $p > 0$ and $n \in \mathbb{Z}_{>0}$, we denote by $\nu_p(n)$ the maximal power index m such that $p^m \mid n$. For convenience, we define $\nu_p(n) = 0$ for $p = 0$ and all $n \in \mathbb{Z}_{>0}$.

Lemma 2.13. (1) *For $p \geq 0$ and $T := \{n_1, n_2, \cdots, n_t\} \subset \mathbb{Z}_{>0}$ with $t \geq 2$ and $\nu_p(n_i) = 0$ for all $i \in [t]$. If $n_t \nmid n_j$ for all $j \in [t-1]$, then there is an irreducible factor $f(x)$ of $x^{n_t} - 1$ such that $f(x) \nmid x^{n_j} - 1$ for all $j \in [t-1]$.*

(2) *For $n, m \in \mathbb{Z}_{>0}$, $n \mid m$ in $\mathbb{N}$ if and only if $x^n - 1 \mid x^m - 1$ in $R[x]$.*

For a subset $T := \{m_1, m_2, \cdots, m_s\}$ of $\mathbb{Z}_{>0}$ with $m_1 > m_2 > \cdots > m_s$, we associate a set $\mathfrak{J}_T :=$ $\{m_1, m_1 - m_2, \cdots, m_1 - m_s\}$ and a multiset $\mathcal{H}_T := \{\{m_1 - m_2, \cdots, m_{s-1} - m_s, m_s\}\}$. Let $\mathcal{D}_T$ be the multiset obtained from $\mathcal{H}_T$ by removing all 1's.

Lemma 2.14. *For subsets $T := \{m_1, m_2, \cdots, m_s\}$ and $H := \{n_1, n_2, \cdots, n_t\}$ of $\mathbb{Z}_{>0}$ with $m_1 > m_2 > \cdots >$ m_s and $n_1 > n_2 > \cdots > n_t$, if $H = \mathfrak{J}_T$, then $s = t$, $T = \mathfrak{J}_H$ and $\mathcal{H}_T = \mathcal{H}_H$.*

Let $\#_s(I)$ denote the multiplicity of a natural number s in a multiset I. By Lemma 2.12, we have the following lemma.

Lemma 2.15. *Suppose $p > 0$ and $m, t, t' \in \mathbb{N}$. Let $T := \{p^{m_1}, p^{m_2}, \cdots, p^{m_\ell}\}$ and $T^+ := T \cup \{p^m\}$ with integers $m_1 > m_2 > \cdots > m_\ell \geq 0$.*

(1) *If $m > t$, we have $\#_{p^m - p^t}(\mathcal{D}_{T^+}) \geq \#_{p^m - p^t}(\mathcal{D}_T)$ and $\#_{p^m}(\mathcal{D}_{T^+}) \geq \#_{p^m}(\mathcal{D}_T)$. Moreover, if $m > m_\ell$ and $p^m - p^{m_k} > 1$, then $\#_{p^m - p^{m_k}}(\mathcal{D}_{T^+}) > \#_{p^m - p^{m_k}}(\mathcal{D}_T)$, where $k := \max\{i \in [\ell] \mid m \geq m_i\}$.*

(2) *If $t > m$, then $\#_{p^t}(\mathcal{D}_T) \geq \#_{p^t}(\mathcal{D}_{T^+})$. Moreover, if $m_\ell > m$, then $\#_{p^{m_\ell}}(\mathcal{D}_T) > \#_{p^{m_\ell}}(\mathcal{D}_{T^+})$.*

(3) *If $t > m > t'$, then $\#_{p^t - p^{t'}}(\mathcal{D}_T) \geq \#_{p^t - p^{t'}}(\mathcal{D}_{T^+})$. Moreover, if $\ell \geq 2$ and $m_i > m > m_{i+1}$ for some $i \in [\ell - 1]$, then $\#_{p^{m_i} - p^{m_{i+1}}}(\mathcal{D}_T) > \#_{p^{m_i} - p^{m_{i+1}}}(\mathcal{D}_{T^+})$.*

3 New types of equivalence relations of matrices

In this section, we introduce two new equivalence relations on all square matrices and compare them with other known equivalence relations.

Given a matrix $c \in M_n(R)$, we denote by $\tilde{\mathcal{E}}_c$ the multiset of all elementary divisors of c, and $\mathcal{E}_c$ the set of elementary divisors of c, which is obtained from $\tilde{\mathcal{E}}_c$ by removing duplicate elements. Let

$$\mathcal{M}_c := \{f(x) \in \mathcal{E}_c \mid f(x) \text{ is maximal with respect to polynomial divisibility}\},$$

where $f(x) \leq g(x)$ means $f(x) \mid g(x)$ for polynomials $f(x), g(x) \in R[x]$ of positive degree. If $m_c(x)$ is the minimal polynomial of c over R and if $d_1(x), \cdots, d_r(x)$ are invariant factors of c with $d_i(x) \mid d_{i+1}(x)$, $1 \leq i \leq r-1$, then $\mathcal{M}_c$ is determined completely by $d_r(x) = m_c(x)$.

Let $\mathcal{R}_c := \{f(x) \in \mathcal{M}_c \mid f(x) \text{ is reducible}\}$ be the set of all reducible maximal divisors of c. For $f(x) \in \mathcal{M}_c$, let $\tilde{P}_c(f(x))$ be the multiset of *power indices* of $f(x)$ in $\tilde{\mathcal{E}}_c$, defined by

$$\tilde{P}_c(f(x)) := \{\{i \geq 1 \mid \exists \text{ irreducible polynomial } p(x) \text{ such that } p(x) \text{ divides } f(x), p(x)^i \in \tilde{\mathcal{E}}_c\}\},$$

and let $P_c(f(x))$ be the set of *power indices* of $f(x)$ in $\mathcal{E}_c$, it is obtained from $\tilde{P}_c(f(x))$ by deleting duplicate elements.

Let $\mathcal{I}_c := \big\{f(x) \in R[x] \mid f(x) \text{ is irreducible such that } f(x)^i \in \mathcal{M}_c \text{ for some } i \text{ with } |P_c(f(x)^i)| \neq \max\{j \in P_c(f(x)^i)\}\big\}$. For an irreducible polynomial $g(x) \in R[x]$, $P_c(g(x))$ is defined by $P_c(g(x)) := P_c(g(x)^i)$ if $g(x)^i \in \mathcal{M}_c$ for some $i \in \mathbb{N}$, and $P_c(g(x)) := \emptyset$ otherwise.

Now, we define a relation $\sim$ on $\mathcal{I}_c$: For $f(x), g(x) \in \mathcal{I}_c$, we write $f(x) \sim g(x)$ provided $R[x]/(f(x)) \simeq R[x]/(g(x))$ as R-algebras. This equivalence relation on $\mathcal{I}_c$ gives rise to a partition of $\mathcal{I}_c$ into its equivalence classes $\mathcal{I}_{c,1}, \cdots, \mathcal{I}_{c,r_c}$, where r_c is the number of the equivalence classes. For $i \in [r_c]$, let

$$\mathcal{D}_{c,i} := \bigcup_{f(x) \in \mathcal{I}_{c,i}} \mathcal{D}_{P_c(f(x))} \text{ and } Q_{c,i} := R[x]/(p_{c,i}(x)),$$

where $p_{c,i}(x)$ is a fixed representatives of the equivalence class $\mathcal{I}_{c,i}$. Note that $\mathcal{D}_{c,i}$ is a union of multisets and that $Q_{c,i}$ is independent of the choice of the representatives, up to isomorphism of R-algebras.

We define $\mathcal{U}_c := \bigcup_{g(x) \in \mathcal{M}_c} \mathcal{D}_{P_c(g(x))}$, where the union is taken in the sense of multisets. Clearly, $\mathcal{U}_c = \bigcup_{f(x) \in \mathcal{I}_c} \mathcal{D}_{P_c(f(x))}$.

Now, we introduce two new types of equivalence relations on all square matrices over a field.

Definition 3.1. *Two matrices $c \in M_n(R)$ and $d \in M_m(R)$ are said to be*

(1) I-equivalent *if there exists a bijection $\pi : \mathcal{M}_c \to \mathcal{M}_d$, such that $R[x]/(f(x)) \simeq R[x]/((f(x))\pi)$ as R-algebras and $\tilde{P}_c(f(x)) = \tilde{P}_d((f(x))\pi)$ for all $f(x) \in \mathcal{M}_c$. In this case, we simply write $c \overset{I}{\sim} d$.*

(2) Sg-equivalent *if $r_c = r_d$ and there exists a permutation $\sigma \in \Sigma_{r_c}$ such that $Q_{c,i} \simeq Q_{d,(i)\sigma}$ as R-algebras and $\mathcal{D}_{c,i} = \mathcal{D}_{d,(i)\sigma}$ for all $i \in [r_c]$. In this case, we simply write $c \overset{Sg}{\sim} d$.*

Clearly, $c \stackrel{I}{\sim} d$ and $c \stackrel{Sg}{\sim} d$ are equivalence relations on the set of all square matrices over R. If $c \stackrel{I}{\sim} d$, then $\deg m_c(x) = \deg m_d(x)$ and $n = \deg \chi_c(x) = \deg \chi_d(x) = m$. If $c \stackrel{Sg}{\sim} d$, then $\mathcal{U}_c = \mathcal{U}_d$. Moreover, $\mathcal{I}_c = \emptyset$ if and only if $\mathcal{I}_d = \emptyset$.

The two equivalence relations are different from the equivalence relations introduced in [24, Definition 3.1] and [25, Definition 3.1]. Now let us recall these relations.

Definition 3.2. [24, 25] *Two matrices $c \in M_n(R)$ and $d \in M_m(R)$ are said to be*

(1) *M*-equivalent *if there is a bijection $\pi : \mathcal{M}_c \to \mathcal{M}_d$, such that $R[x]/(f(x)) \simeq R[x]/((f(x))\pi)$ as R-algebras and $P_c(f(x)) = P_d((f(x))\pi)$ for all $f(x) \in \mathcal{M}_c$. In this case, we write $c \stackrel{M}{\sim} d$.*

(2) *D*-equivalent *if there is a bijection $\pi : \mathcal{M}_c \to \mathcal{M}_d$, such that $R[x]/(f(x)) \simeq R[x]/((f(x))\pi)$ as R-algebras and $\mathcal{H}_{P_c(f(x))} = \mathcal{H}_{P_d((f(x))\pi)}$ for all $f(x) \in \mathcal{M}_c$. In this case, we write $c \stackrel{D}{\sim} d$.*

(3) *AD*-equivalent *if there is a bijection $\pi : \mathcal{M}_c \to \mathcal{M}_d$, such that $R[x]/(f(x)) \simeq R[x]/((f(x))\pi)$ as R-algebras and either $P_c(f(x)) = P_d((f(x))\pi)$ or $P_c(f(x)) = \mathfrak{J}_{P_d((f(x))\pi)}$ for all $f(x) \in \mathcal{M}_c$. In this case, we write $c \stackrel{AD}{\sim} d$.*

(4) *S*-equivalent *if there is a bijection $\pi : \mathcal{R}_c \to \mathcal{R}_d$, such that $R[x]/(f(x)) \simeq R[x]/((f(x))\pi)$ as R-algebras and either $P_c(f(x)) = P_d((f(x))\pi)$ or $P_c(f(x)) = \mathfrak{J}_{P_d((f(x))\pi)}$ for all $f(x) \in \mathcal{R}_c$. In this case, we write $c \stackrel{S}{\sim} d$.*

The interrelations among these equivalence relations are indicated in the following. In general, the converse of all these implications are not true.

Remark 3.3. Let $c \in M_n(R)$ and $d \in M_m(R)$. Then the following implications hold.

$$\begin{array}{ccccccccc} & & & & & c \stackrel{D}{\sim} d & & & \\ & & & & \nearrow & & \searrow & & \\ c \stackrel{I}{\sim} d & \Longrightarrow & c \stackrel{M}{\sim} d & \Longrightarrow & c \stackrel{AD}{\sim} d & & & & c \stackrel{Sg}{\sim} d. \\ & & & & \searrow & & \nearrow & & \\ & & & & & c \stackrel{S}{\sim} d & & & \end{array}$$

Proof. Clearly, $c \stackrel{I}{\sim} d \Rightarrow c \stackrel{M}{\sim} d \Rightarrow c \stackrel{AD}{\sim} d \Rightarrow c \stackrel{D}{\sim} d$ (by Lemma 2.14), and $c \stackrel{AD}{\sim} d \Rightarrow c \stackrel{S}{\sim} d$. We have the following facts.

(i) If $f(x)$ and $g(x)$ are irreducible polynomials and $i, j \in \mathbb{Z}_{>0}$ such that $R[x]/(f(x)^i) \simeq R[x]/(g(x)^j)$ as algebras, then $i = j$ and $R[x]/(f(x)) \simeq R[x]/(g(x))$ as algebras.

(ii) If $f(x) \in \mathcal{M}_c$, then $|P_c(f(x))| = \max\{i \in P_c(f(x))\}$ if and only if $\mathcal{H}_{P_c(f(x))} = \{\{1, \cdots, 1\}\}$ contains exactly $\max\{i \in P_c(f(x))\}$ elements if and only if $\mathcal{D}_{P_c(f(x))} = \emptyset$.

(iii) If there is a bijection $\pi : \mathcal{I}_c \to \mathcal{I}_d$ such that $R[x]/(f(x)) \simeq R[x]/((f(x))\pi)$ as algebras and $\mathcal{D}_{P_c(f(x))} = \mathcal{D}_{P_d((f(x))\pi)}$ for all $f(x) \in \mathcal{I}_c$, then $c \stackrel{Sg}{\sim} d$.

Suppose $c \stackrel{D}{\sim} d$. Then, by definition, there is a bijection $\pi : \mathcal{M}_c \to \mathcal{M}_d$ such that $R[x]/(f(x)) \simeq R[x]/((f(x))\pi)$ as algebras and $\mathcal{H}_{P_c(f(x))} = \mathcal{H}_{P_d((f(x))\pi)}$ for all $f(x) \in \mathcal{M}_c$. By (ii), $|P_c(f(x))| = \max\{i \in P_c(f(x))\}$ if and only if $|P_c((f(x))\pi)| = \max\{j \in P_c((f(x))\pi)\}$. Hence π induces a bijection $\pi' : \mathcal{I}_c \to \mathcal{I}_d$ such that $R[x]/(f(x)) \simeq R[x]/((f(x))\pi')$ as algebras and $\mathcal{H}_{P_c(f(x))} = \mathcal{H}_{P_d((f(x))\pi')}$ for all $f(x) \in \mathcal{I}_c$ by (i). Hence $\mathcal{D}_{P_c(f(x))} = \mathcal{D}_{P_d((f(x))\pi)}$ for all $f(x) \in \mathcal{I}_c$. It follows from (iii) that $c \stackrel{Sg}{\sim} d$.

Suppose $c \stackrel{S}{\sim} d$. Then, by definition, there is a bijection $\pi : \mathcal{R}_c \to \mathcal{R}_d$ such that $R[x]/(f(x)) \simeq R[x]/((f(x))\pi)$ as R-algebras and either $P_c(f(x)) = P_d((f(x))\pi)$ or $P_c(f(x)) = \mathfrak{J}_{P_d((f(x))\pi)}$ for all $f(x) \in \mathcal{R}_c$. Hence $\mathcal{H}_{P_c(f(x))} = \mathcal{H}_{P_d((f(x))\pi)}$ for all $f(x) \in \mathcal{R}_c$. Furthermore, for an irreducible polynomial $f(x) \in \mathcal{M}_c$, we have $|P_c((f(x))\pi)| = \max\{j \in P_c((f(x))\pi)\} = 1$. Similarly, we can prove $c \stackrel{Sg}{\sim} d$. □

In general, each inverse implications of the above diagram is not true. Finally, we give examples to illustrate I-equivalences and Sg-equivalences. One of the examples shows in general that Sg-equivalences do not have to imply D-equivalence nor S-equivalences.

Example 3.4. Let R be a field and $J_n(\lambda)$ the $n\times n$ Jordan matrix with the eigenvalue $\lambda\in R$.

(1) Let $c := J_3(0)\oplus J_3(0)\in M_6(R)$ and $d := J_3(1)\oplus J_3(1)\in M_6(R)$, where $\oplus$ stands for forming the diagonal block matrix. Then $\tilde{\mathcal{E}}_c = \{\{x^3,x^3\}\}, \tilde{\mathcal{E}}_d = \{\{(x-1)^3,(x-1)^3\}\}$, $\mathcal{M}_c = \{x^3\}, \mathcal{M}_d = \{(x-1)^3\}, \tilde{P}_c(x^3) = \{\{3,3\}\} = \tilde{P}_d((x-1)^3)$, $m_c(x) = x^3, m_d(x) = (x-1)^3$. If we define $\pi : \mathcal{M}_c \to \mathcal{M}_d, x^3 \mapsto (x-1)^3$, then $c \overset{I}{\sim} d$. Since c and d have different minimal polynomials, they are not similar. Thus the I-equivalence is a proper generalization of the similarity relation of matrices.

(2) Let $c := J_2(0)\oplus J_4(0)\in M_6(R)$ and $d := J_2(0)\oplus J_2(1)\in M_4(R)$. Clearly, $\mathcal{E}_c = \{x^2,x^4\}, \mathcal{E}_d = \{x^2,(x-1)^2\} = \mathcal{M}_d = \mathcal{R}_d, \mathcal{M}_c = \{x^4\} = \mathcal{R}_c, \mathcal{I}_c = \{x\}, \mathcal{I}_d = \{x,x-1\}, r_c = 1 = r_d, Q_{c,1} \simeq R[x]/(x) \simeq Q_{d,1}$ and $\mathcal{D}_{c,1} = \{2,2\} = \mathcal{D}_{d,1}$. Thus c and d are Sg-equivalent. Now, it follows from $|\mathcal{M}_c| = |\mathcal{R}_c| = 1 \neq 2 = |\mathcal{M}_d| = |\mathcal{R}_d|$ that c and d are neither D-equivalent nor S-equivalent.

4 Singularity categories

In this section we describe the singularity categories of centralizer matrix algebras over fields. To this purpose, we first study Gorenstein projective modules over the endomorphism algebras of generators, and then pass to the ones of generators over quotients of polynomial algebras. In this way, we describe the singularity categories of centralizer matrix algebras as products of stable module categories.

4.1 Gorenstein projective modules over endomorphism algebras: General theory

This subsection is devoted to the study of the category of Gorenstein projective modules over the endomorphism algebras of generators.

Let $\mathcal{A}$ be an abelian category, and let $\mathcal{C}$ be a full additive subcategories of $\mathcal{A}$. Suppose that $\mathcal{S}$ is a class of kernel-cokernel pairs in $\mathcal{C}$ such that it is closed under isomorphisms. The category $\mathcal{C}\subseteq\mathcal{A}$ is said to be *closed under admissible push-outs of* $\mathcal{S}$ if, for any admissible monomorphism $f : X\to Y$ of $\mathcal{S}$ and homomorphism $\phi : X\to Z$ in $\mathcal{C}$, the push-out of (f,ϕ) in $\mathcal{A}$ belongs to $\mathcal{C}$. Dually, the category $\mathcal{C}\subseteq\mathcal{A}$ is said to be *closed under admissible pull-backs of* $\mathcal{S}$ if, for any admissible epimorphism $g : Y\to Z$ of $\mathcal{S}$ and homomorphism $\psi : X\to Z$ in $\mathcal{C}$, the pull-back of (g,ψ) in $\mathcal{A}$ belongs to $\mathcal{C}$.

Lemma 4.1. *Let $\mathcal{A}$ be an abelian category with a full additive subcategory $\mathcal{D}$, and let $0\to X\xrightarrow{f} Y\xrightarrow{g} Z\to 0$ be an exact sequence in $\mathcal{A}$ such that f and g are $\mathcal{D}$-monic and $\mathcal{D}$-epic, respectively.*

(1) *For any morphism $\phi : X\to X'$ in $\mathcal{A}$, there is the push-out diagram in $\mathcal{A}$:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi'} & & \| & & \\ 0 & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z & \longrightarrow & 0, \end{array}$$

where f' and g' are $\mathcal{D}$-monic and $\mathcal{D}$-epic, respectively.

(2) *Dually, for any morphism $\psi : Z'\to Z$ in $\mathcal{A}$, there is the pull-back diagram in $\mathcal{A}$:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\psi'} & & \downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0, \end{array}$$

where f' and g' are $\mathcal{D}$-monic and $\mathcal{D}$-epic, respectively.

Proof. Since (2) is a dual of (1), we only prove (1).

Since $g = \phi' g'$ is $\mathcal{D}$-epic, g' is $\mathcal{D}$-epic. It remains to show that f' is $\mathcal{D}$-monic. Let $\delta : X' \to D$ be a morphism in $\mathcal{A}$ with $D \in \mathcal{D}$. Since f is $\mathcal{D}$-monic, there is a morphism $\rho : Y \to D$ such that $f\rho = \phi\delta$. By the universal property of push-outs, there is a morphism $\gamma : Y' \to D$ in $\mathcal{A}$ such that $\delta = f'\gamma$. This means that f' is $\mathcal{D}$-monic. $\square$

The following lemma is straightforward and its proof is omitted.

Lemma 4.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be full additive subcategories of an abelian category $\mathcal{A}$ with $\mathcal{D} \subseteq \mathcal{C}$. Define*

$$\mathcal{S}_{\mathcal{D}} := \left\{ C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3 \;\middle|\; \begin{array}{l} (f_1, f_2) \text{ is a kernel-cokernel pair in } \mathcal{A}, f_1 \text{ and } f_2 \text{ are} \\ \mathcal{D}\text{-monic and } \mathcal{D}\text{-epic in } \mathcal{C}, \text{respectively} \end{array} \right\}.$$

Suppose that $\mathcal{C} \subseteq \mathcal{A}$ is closed under admissible push-outs and pull-backs of $\mathcal{S}_{\mathcal{D}}$. Then

(1) *$(\mathcal{C}, \mathcal{S}_{\mathcal{D}})$ is an excat category.*

(2) *If $\mathcal{D} \subseteq \mathcal{A}$ is closed under direct summands and if, for all $C \in \mathcal{C}$, there exist an admissible monomorphism $f : C \to D$ and an admissible epimorphism $g : D' \to C$ with $D, D' \in \mathcal{D}$, then $(\mathcal{C}, \mathcal{S}_{\mathcal{D}})$ is a Frobenius category with $\mathrm{Inj}_{\mathcal{S}_{\mathcal{D}}}(\mathcal{C}) = \mathcal{D}$. Consequently, $\mathcal{C}/\mathcal{D}$ is a triangulated category.*

We recall the definition of Gorenstein projective objects.

Definition 4.3. *Let $\mathcal{A}$ be an abelian category with enough projectives. An object $X \in \mathcal{A}$ is said to be* Gorenstein projective *if there is an exact sequence $P^\bullet : \cdots \to P^{-1} \to P_0 \xrightarrow{d^0} P^1 \to P^2 \to \cdots$ of projective objects in $\mathcal{A}$ such that $X \simeq \mathrm{Ker}(d^0)$ and the complex $\mathrm{Hom}_{\mathcal{A}}(P^\bullet, P)$ is exact for any projective object $P \in \mathcal{A}$.*

Let $\mathcal{A}$-Gproj denote the full subcategory of all Gorenstein projective objects of $\mathcal{A}$. If $\mathcal{A} = \Lambda$-mod for an Artin algebra Λ, then Gorenstein objects in $\mathcal{A}$ are called *Gorenstein projective Λ-modules*. We write Λ-Gproj for the category of all Gorenstein projective modules in Λ-mod.

Example 4.4. Let Λ be an Artin $\mathbb{K}$-algebra and $\mathcal{D} = \mathrm{add}({}_\Lambda\Lambda)$. Then Λ-Gproj $\subseteq \Lambda$-mod is closed under admissible push-outs and admissible pull-backs of $\mathcal{S}_{\mathrm{add}({}_\Lambda\Lambda)}$, and therefore $(\Lambda\text{-Gproj}, \mathcal{S}_{\mathrm{add}(\Lambda)})$ is a Frobenius $\mathbb{K}$-category with $\mathrm{Inj}_{\mathcal{S}_{\mathrm{add}(\Lambda)}} = \mathrm{add}({}_\Lambda\Lambda)$ by Lemma 4.2. Moreover, the exact structure on $(\Lambda\text{-Gproj}, \mathcal{S}_{\mathrm{add}(\Lambda)})$ coincides with the usual exact structure on Λ-Gproj.

The following facts are standard, we leave their proofs to the reader.

Lemma 4.5. *Let Λ be an Artin algebra, ${}_\Lambda M$ a generator for* Λ-mod *and $\Gamma := \mathrm{End}_A(M)$.*

(1) $\mathrm{Hom}_\Lambda(M, -) : \Lambda\text{-mod} \to \Gamma\text{-mod}$ *is a fully faithful functor and induces an equivalence* $\mathrm{add}(M) \to \Gamma$-proj *with a quasi-inverse $M \otimes_\Gamma -$.*

(2) *M_Γ is a projective Γ^{op}-module.*

(3) *If X lies in* $\mathrm{copres}(\Gamma)$*, then the natural homomorphism $\alpha_X : X \to \mathrm{Hom}_\Lambda(M, M \otimes_\Gamma X)$, $x \mapsto [m \mapsto m \otimes x]$ for $x \in X$ and $m \in M$, is an isomorphism of Γ-modules.*

Let Λ be an Artin algebra and $M \in \Lambda$-mod. An exact complex

$$M^\bullet : \cdots \longrightarrow M^{-2} \longrightarrow M^{-1} \longrightarrow M^0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow \cdots$$

of Λ-modules is called a *complete* $\mathrm{add}(M)$*-resolution* if $M^i \in \mathrm{add}(M)$ for $i \in \mathbb{Z}$ and both complexes $\mathrm{Hom}^\bullet_\Lambda(M, M^\bullet)$ and $\mathrm{Hom}^\bullet_\Lambda(M^\bullet, M)$ are exact. A Λ-module X is said to admit *a complete* $\mathrm{add}(M)$*-resolution* if there exists a complete $\mathrm{add}(M)$-resolution $M^\bullet$ such that $X \simeq \mathrm{Ker}(M^0 \to M^1)$.

Let $\mathscr{G}(M)$ be the full subcategory of Λ-mod consisting of all Λ-modules that admit a complete $\mathrm{add}(M)$-resolution. Clearly, $\mathrm{add}(M) \subseteq \mathscr{G}(M)$ and $\mathscr{G}({}_\Lambda\Lambda) = \Lambda$-Gproj.

The M-*stable category* $\Lambda\text{-mod}/[M]$ of Λ-mod is defined to be the quotient category of Λ-mod modulo $\mathrm{add}(M)$. For $X_1, X_2 \in A$-mod, we say that X_1 and X_2 are M-*stably isomorphic* if they are isomorphic in $\Lambda\text{-mod}/[M]$, equivalently, there exist M_1 and M_2 in $\mathrm{add}(M)$ such that $X_1 \oplus M_1 \simeq X_2 \oplus M_2$ as Λ-modules. For a full additive subcategory $\mathcal{C}$ of Λ-mod, we denote by $\mathcal{C}/[M]$ the full subcategory of $\Lambda\text{-mod}/[M]$ consisting of all objects X which are M-stably isomorphic to objects in $\mathcal{C}$.

The following lemma is not hard to prove, so its proof is omitted.

Lemma 4.6. *Let Λ be an Artin $\mathbb{K}$-algebra and $M \in \Lambda\text{-mod}$, $\Gamma := \mathrm{End}_A(M)$. Then*

(1) $\mathscr{G}(M) \subseteq \Lambda\text{-mod}$ *is closed under direct summands and finite direct products.*

(2) *If M is a generator for* Λ-mod, *then*

(*i*) $(\mathscr{G}(M), \mathcal{S}_{\mathrm{add}(M)})$ *is a Frobenius $\mathbb{K}$-category with* $\mathrm{Inj}_{\mathcal{S}_{\mathrm{add}(M)}} = \mathrm{add}(M)$.

(*ii*) $\mathrm{Hom}_\Lambda(M,-) : \Lambda\text{-mod} \to \Gamma\text{-mod}$ *induces an equivalence* $(\mathscr{G}(M), \mathcal{S}_{\mathrm{add}(M)}) \to (\Gamma\text{-Gproj}, \mathcal{S}_{\mathrm{add}(\Gamma)})$ *of Frobenius $\mathbb{K}$-categories with a quasi-inverse $M \otimes_\Gamma -$. In particular, $\mathscr{G}(M)/[M] \simeq \Gamma\text{-}\underline{\mathrm{Gproj}}$ as triangulated $\mathbb{K}$-categories.*

4.2 Special case: Generators over quotients of polynomial algebras

In this subsection we consider the endomorphism algebras of generators over quotients of polynomial algebras.

For a module M over an Artin algebra Λ, we write M as a direct sum of indecomposable modules, say $M = \bigoplus_{i=1}^m M_i$, where all M_i are indecomposable, and denote m by $\#(M)$. Clearly, $\#(M)$ is uniquely determined by M.

Throughout this section, we fix an irreducible polynomial $f(x) \in R[x]$ of positive degree u, and set $A := R[x]/(f(x)^n)$ for a positive integer n. We keep the notation in Section 2.3.

Lemma 4.7. *Let $a, b \in \{0, 1, \cdots, n\}$ with $a < b$, and let $M := \bigoplus_{i=a}^b M(i)$ and $N := M(a) \oplus M(b)$ be A-modules. Then*

(1) $(\mathrm{add}(M), \mathcal{S}_{\mathrm{add}(N)})$ *is a Frobenius R-category with* $\mathrm{Inj}_{\mathcal{S}_{\mathrm{add}(N)}} = \mathrm{add}(N)$.

(2) $\mathrm{add}(M)/[N] \simeq R[x]/(f(x)^{b-a})\text{-}\underline{\mathrm{mod}}$ *as triangulated R-categories.*

Proof. (1) Let $K := M(b) \oplus \bigoplus_{i=0}^a M(i)$. The proof is given by showing the statements (i)-(iii).

(i) For any $X \in \mathrm{add}(M)$, there exist two exact sequences in A-mod:

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \text{ and } 0 \longrightarrow Z' \xrightarrow{f'} Y' \xrightarrow{g'} X \longrightarrow 0$$

with $Y, Y' \in \mathrm{add}(N)$ and $Z, Z' \in \mathrm{add}(M)$, such that f and f' are left $\mathrm{add}(K)$-approximations of X and Z', respectively, and that g and g' are right $\mathrm{add}(K)$-approximations of Z and Z', respectively.

Indeed, it is enough to assume that X is indecomposable in $\mathrm{add}(M)$, that is, $X = M(i)$ for some $a \le i \le b$. If $i = a$ or b, then the trivial exact sequences $0 \to X \xrightarrow{1_X} X \to 0 \to 0$ and $0 \to 0 \to X \xrightarrow{1_X} X \to 0$ satisfy the required conditions.

Assume that $i \ne a$ and $i \ne b$. Then $a < i < b, a < a+b-i < b$ and $a+b = i + (a+b-i)$. It follows from Lemma 2.8 that there are two exact sequences in A-mod:

$$0 \longrightarrow M(i) \xrightarrow{f} M(a) \oplus M(b) \xrightarrow{g} M(a+b-i) \longrightarrow 0, \text{ and}$$

$$0 \longrightarrow M(a+b-i) \xrightarrow{f'} M(a) \oplus M(b) \xrightarrow{g'} M(i) \longrightarrow 0,$$

such that f and f' are left minimal $\mathrm{add}(K)$-approximations and that g and g' are right minimal $\mathrm{add}(K)$-approximations. This shows (i).

(ii) $\mathrm{add}(M) \subseteq A$-mod is closed under admissible push-outs and admissible pull-backs of $\mathcal{S}_{\mathrm{add}(N)}$. This implies that $(\mathrm{add}(M), \mathcal{S}_{\mathrm{add}(N)})$ is an exact R-category by Lemma 4.2.

Actually, since A is a local Nakayama algebra, the following facts hold for an A-module X:

1) $\#(X) = \#(\mathrm{soc}(X))$.

2) $X \in \mathrm{add}(M)$ if and only if $\ell_A(\mathrm{soc}^a(X)) = a(\#(X))$ and $LL(X) \leq b$.

Suppose that $f : X \to Y$ is an admissible monomorphism of $\mathcal{S}_{\mathrm{add}(N)}$, and that $\phi : X \to X'$ is an arbitrary morphism in $\mathrm{add}(M)$. By Lemma 4.1(1), we have the push-out diagram in A-mod:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \phi\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z & \longrightarrow & 0. \end{array}$$

where f' and g' are $\mathrm{add}(N)$-monic and $\mathrm{add}(N)$-epic, respectively. This diagram provides an exact sequence: $0 \to X \xrightarrow{(f,\phi)} Y \oplus X' \to Y' \to 0$ in A-mod, and shows $LL(Y') \leq LL(Y \oplus X') \leq LL(M) \leq b$.

To prove $Y' \in \mathrm{add}(M)$, we only need to show $\ell_A(\mathrm{soc}^a(Y')) = a(\#(Y'))$ by the fact 2). The case $a \leq 1$ is trivial, so we may assume $a > 1$. We first show that g' is $\mathrm{add}(K)$-epic.

Let $h : K' \to Z$ be a homomorphism in A-mod with $K' \in \mathrm{add}(K)$. Since f is an admissible monomorphism of $\mathcal{S}_{\mathrm{add}(N)}$, we have $Z \in \mathrm{add}(M)$. By (i), there is a right $\mathrm{add}(K)$-approximation $h' : K_1 \to Z$ of Z with $K_1 \in \mathrm{add}(N)$. Since g' is $\mathrm{add}(N)$-epic, there is a homomorphism $\phi' : K_1 \to Y'$ such that $\phi' g' = h'$. Since h' is a right $\mathrm{add}(K)$-approximation and $K' \in \mathrm{add}(K)$, there exists a homomorphism $\psi : K' \to K_1$ such that $\psi h' = h$. Hence $\psi\phi' g' = \psi h' = h$, and therefore g' is $\mathrm{add}(K)$-epic.

Since $M(1)$ and $M(a)$ belong to $\mathrm{add}(K)$, g' is both $\mathrm{add}(M(1))$-epic and $\mathrm{add}(M(a))$-epic. By Lemma 2.6, we have two exact sequences in A-mod:

$$(\diamondsuit) \quad 0 \longrightarrow \mathrm{soc}(X') \longrightarrow \mathrm{soc}(Y') \longrightarrow \mathrm{soc}(Z) \longrightarrow 0 \text{ and}$$

$$(\diamondsuit\diamondsuit) \quad 0 \longrightarrow \mathrm{soc}^a(X') \longrightarrow \mathrm{soc}^a(Y') \longrightarrow \mathrm{soc}^a(Z) \longrightarrow 0.$$

It then follows from 1) and $(\diamondsuit)$ that $\#(Y') = \#(\mathrm{soc}(Y')) = \#(\mathrm{soc}(X')) + \#(\mathrm{soc}(Z)) = \#(X') + \#(Z)$. Since both X' and Z lie in $\mathrm{add}(M)$, we infer from $(\diamondsuit\diamondsuit)$ and 2) that $\ell_A(\mathrm{soc}^a(Y')) = \ell_A(\mathrm{soc}^a(X')) + \ell_A(\mathrm{soc}^a(Z)) = a\big(\#(X') + \#(Z)\big)$, and therefore $\ell_A(\mathrm{soc}^a(Y')) = a(\#(X') + \#(Z)) = a(\#(Y'))$. By the fact 2), $Y' \in \mathrm{add}(M)$. Hence $\mathrm{add}(M) \subseteq A$-mod is closed under admissible pull-outs of $\mathcal{S}_{\mathrm{add}(N)}$. Similarly, we can prove that $\mathrm{add}(M) \subseteq A$-mod is closed under admissible pull-backs of $\mathcal{S}_{\mathrm{add}(N)}$.

(iii) $(\mathrm{add}(M), \mathcal{S}_{\mathrm{add}(N)})$ is a Frobenius R-category with $\mathrm{Inj}_{\mathcal{S}_{\mathrm{add}(N)}} = \mathrm{add}(N)$.

Thanks to $N \in \mathrm{add}(K)$, any left $\mathrm{add}(K)$-approximation of $U \in \mathrm{add}(M)$ is $\mathrm{add}(N)$-monic, and any right $\mathrm{add}(K)$-approximation of $V \in \mathrm{add}(M)$ is $\mathrm{add}(N)$-epic. By (i), for any $X \in \mathrm{add}(M)$, there exist an admissible monomorphism $f : X \to Y$ and an admissible epimorphism $g : Y' \to X$ of $\mathcal{S}_{\mathrm{add}(N)}$ with $Y, Y' \in \mathrm{add}(N)$. Thus, by Lemma 4.2(2), $(\mathrm{add}(M), \mathcal{S}_{\mathrm{add}(N)})$ is a Frobenius R-category with $\mathrm{Inj}_{\mathcal{S}_{\mathrm{add}(N)}} = \mathrm{add}(N)$.

(2) Any homomorphism $f : X \to Y$ in A-mod restricts to a homomorphism $f' : \mathrm{soc}^a(X) \to \mathrm{soc}^a(Y)$. Thus there exists a unique homomorphism $f^\wedge : X/\mathrm{soc}^a(X) \to Y/\mathrm{soc}^a(Y)$ of A-modules, fitting into the exact commutative diagram:

$$(\star) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{soc}^a(X) & \longrightarrow & X & \xrightarrow{\pi_X} & X/\mathrm{soc}^a(X) & \longrightarrow & 0 \\ & & f'\downarrow & & f\downarrow & & f^\wedge\downarrow & & \\ 0 & \longrightarrow & \mathrm{soc}^a(Y) & \longrightarrow & Y & \xrightarrow{\pi_Y} & Y/\mathrm{soc}^a(Y) & \longrightarrow & 0, \end{array}$$

where π_X and π_Y are the canonical surjective homomorphisms. Hence we can define a functor

$$F : \ \mathrm{add}(M) \longrightarrow R[x]/(f(x)^{b-a})\text{-mod},$$

$$X \mapsto X/\mathrm{soc}^a(X) \ \text{ for } \ X \in \mathrm{add}(M), \quad f \mapsto f^{\wedge} \ \text{ for } \ f : X \to Y \ \text{ in } \mathrm{add}(M).$$

It is easy to see that F is a well-defined R-functor and $F(\mathrm{add}(N)) = \mathrm{add}(R[x]/(f(x)^{b-a}))$.

Let $B := R[x]/(f(x)^{b-a})$. Since $\mathrm{add}(N)$-epic morphisms are $\mathrm{add}(M(a))$-epic morphisms, the functor F sends admissible exact sequences in $\mathcal{S}_{\mathrm{add}(N)}$ to exact sequences in B-mod by Lemma 2.6. Moreover, F induces a triangle R-functor $\underline{F} : \mathrm{add}(M)/\mathrm{add}(N) \to B\text{-}\underline{\mathrm{mod}}$. Now, we show that $\underline{F}$ is an equivalence.

Clearly, $\underline{F}$ is dense. For $X,Y \in \mathrm{add}(M) \subseteq A$-mod, we denote by $\underline{\mathrm{Hom}}_N(X,Y)$ the Hom-set of X and Y in $\mathrm{add}(M)/\mathrm{add}(N)$, and will show that $\underline{F} : \underline{\mathrm{Hom}}_N(X,Y) \to \underline{\mathrm{Hom}}_B(X,Y)$ is an isomorphism of R-modules.

We may suppose that X and Y are indecomposable. If $X \in \mathrm{add}(N)$ or $Y \in \mathrm{add}(N)$, then $\underline{F}$ is clearly an isomorphism. Now suppose $X = M(i)$ and $Y = M(j)$ with $a < i, j < b$. Then $F(X) = M(i-a), F(Y) = M(j-a)$ and $g g_{j,j-a} = g_{i,i-a} g^{\wedge} = g_{i,i-a} F(g)$ for $g : X \to Y$ in $\mathrm{add}(M)$ (see the diagram $(\star)$). For $\tilde{f} : F(X) \to F(Y)$ in B-mod, Lemma 2.4(2) guarantees a morphism $f : X \to Y \in \mathrm{add}(M)$ such that $\underline{F}(\underline{f}) = \underline{\tilde{f}}$. This shows that $\underline{F}$ is full.

To see that $\underline{F}$ is faithful, we pick up a homomorphism $g : X \to Y$ in $\mathrm{add}(M)$ such that $\underline{F}(\underline{g}) = 0$ in B-$\underline{\mathrm{mod}}$, that is, $\underline{F(g)} = 0$ in B-$\underline{\mathrm{mod}}$. This means that $F(g)$ factorizes through a projective B-module, and therefore through a projective cover $\beta : B \to F(Y)$ of $F(Y)$. Hence there exists a homomorphism $\alpha : F(X) \to B$ of B-modules such that $F(g) = \alpha\beta$. By Lemma 2.4(2), there exist two homomorphisms $\tilde{\alpha} : X \to M(b)$ and $\tilde{\beta} : M(b) \to Y$ of A-modules such that $\tilde{\alpha} g_{b,b-a} = g_{i,i-a}\alpha$ and $\tilde{\beta} g_{j,j-a} = g_{b,b-a}\beta$. Thus we get the following commutative diagram in A-mod:

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{\alpha}} & M(b) & \xrightarrow{\tilde{\beta}} & Y \\ \downarrow{\scriptstyle g_{i,i-a}} & & \downarrow{\scriptstyle g_{b,b-a}} & & \downarrow{\scriptstyle g_{j,j-a}} \\ F(X) & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & F(Y). \end{array}$$

Here, B-mod is regarded as a full subcategory of A-mod. Due to $g g_{j,j-a} = g_{i,i-a} F(g)$, we get $(g - \tilde{\alpha}\tilde{\beta}) g_{j,j-a} = g g_{j,j-a} - \tilde{\alpha}\tilde{\beta} g_{j,j-a} = g_{i,i-a} F(g) - \tilde{\alpha} g_{b,b-a}\beta = g_{i,i-a} F(g) - g_{i,i-a}\alpha\beta = 0$. This shows that $g - \tilde{\alpha}\tilde{\beta}$ factorizes through $\mathrm{Ker}(g_{j,j-a}) \simeq M(a)$. Consequently, there exist two homomorphisms $h_1 : X \to M(a)$ and $h_2 : M(a) \to Y$ in A-mod such that $g - \tilde{\alpha}\tilde{\beta} = h_1 h_2$. Let $\gamma := (h_1, \tilde{\alpha}) : X \to M(a) \oplus M(b)$ and $\delta := \binom{h_2}{\tilde{\beta}} : M(a) \oplus M(b) \to Y$. Then $g = \gamma\delta$, that is, $\underline{g} = 0$ in $\mathrm{add}(M)/\mathrm{add}(N)$. Hence F is faithful. $\square$

Recall that an Artin algebra Λ is said to be *CM-finite* if it has only finitely many non-isomorphic indecomposable Gorenstein projective modules, and *n-minimal Auslander-Gorenstein* [22] if $\mathrm{idim}({}_\Lambda\Lambda) \leq n+1 \leq \mathrm{domdim}(\Lambda)$, where $\mathrm{idim}({}_\Lambda M)$ and $\mathrm{domdim}({}_\Lambda M)$ denote the injective and dominant dimensions of a Λ-module M, respectively. An algebra Λ is n-minimal Auslander-Gorenstein if and only if so is Λ^{op} (see [22, Proposition 4.1(b)]). Thus n-minimal Auslander-Gorenstein algebras are always Gorenstein.

The next lemma characterizes the singularity categories of the endomorphism algebras of a generators over quotient algebras of polynomial algebras.

Proposition 4.8. *Let $T = \{t_1, \cdots, t_s\}$ be a subset of $[n]$ with $t_1 > t_2 > \cdots > t_s$, and set $M := \bigoplus_{i=1}^{s} M(t_i)$, $\Gamma := \mathrm{End}_A(M)$. Then Γ is a CM-finite 1-minimal Auslander-Gorenstein algebra, and there is a triangle equivalence of the triangulated R-categories:*

$$\mathscr{D}_{sg}(\Gamma) \simeq \prod_{i \in \mathcal{D}_T} R[x]/(f(x)^i)\text{-}\underline{\mathrm{mod}}.$$

Proof. By Lemma 2.11, Γ is a centralizer matrix algebra, and therefore Γ is a 1-minimal Auslander-Gorenstein algebra by [36, Theorem 1.1(2)]. This implies that Γ is also a Gorenstein algebra.

Without loss of generality, we may assume $t_1 = n$. Since Γ is a Gorenstein algebra, it follows from Lemma 4.6 that $\mathscr{D}_{sg}(\Gamma) \simeq \Gamma$-$\underline{\mathrm{Gproj}} \simeq \mathscr{G}(M)/[M]$ as triangulated R-categories, where the triangulated R-category structure of $\mathscr{G}(M)/[M]$ is induced by the Frobenius R-category $(\mathscr{G}(M), \mathcal{S}_{\mathrm{add}(M)})$ with $\mathrm{Inj}_{\mathcal{S}_{\mathrm{add}(M)}} = \mathrm{add}(M)$. Now we divide the rest of the proof into three steps (1)-(3).

(1) We first show $\mathscr{G}(M) = A$-mod. Consequently, it follows from Lemma 4.6 that Γ-Gproj $\simeq \mathscr{G}(M) = A$-mod. Moreover, since A is a representation-finite algebra, Γ is a CM-finite algebra.

Indeed, we take an A-module $M(i)$ with $i \in [n]$. If $i \in T$, then $M(i) \in \mathrm{add}(M) \subseteq \mathscr{G}(M)$. If $i \notin T$, then $i < n = t_1$ and there is $b \in [s]$ such that $t_{b+1} < i < t_b$. Here, for convenience, we understand $t_{s+1} = 0$. Let $a := b+1$. Then $t_a < i < t_b,\ t_a < t_a + t_b - i < t_b$ and $t_a + t_b = i + (t_a + t_b - i)$. By Lemma 2.8, there exist two exact sequences in A-mod:

$$0 \longrightarrow M(i) \xrightarrow{f} M(t_a) \oplus M(t_b) \xrightarrow{f'} M(t_a + t_b - i) \longrightarrow 0, \text{ and}$$

$$0 \longrightarrow M(t_a + t_b - i) \xrightarrow{g'} M(t_a) \oplus M(t_b) \xrightarrow{g} M(i) \longrightarrow 0$$

where f and g' are left minimal $\mathrm{add}(M)$-approximations, and where f' and g are right minimal $\mathrm{add}(M)$-approximations. Hence we have a complete $\mathrm{add}(M)$-resolution of $M(i)$:

$$\cdots \xrightarrow{gf} M(t_a) \oplus M(t_b) \xrightarrow{f'g'} M(t_a) \oplus M(t_b) \xrightarrow{gf} M(t_a) \oplus M(t_b) \xrightarrow{f'g'} M(t_a) \oplus M(t_b) \xrightarrow{gf} \cdots,$$

that is, $M(i) \in \mathscr{G}(M)$. Thus $\mathscr{G}(M) = A$-mod.

(2) For $i \in [s]$, $M_{i+1,i} := \bigoplus_{j=t_{i+1}}^{t_i} M(j)$ and $N_{i+1,i} := M(t_{i+1}) \oplus M(t_i)$, we prove that the pair $(\mathrm{add}(M_{i+1,i}), \mathcal{S}_{\mathrm{add}(N_{i+1,i})})$ is an exact R-subcategory of $(A\text{-mod}, \mathcal{S}_{\mathrm{add}(M)})$.

Actually, by Lemma 4.7, $(\mathrm{add}(M_{i+1,i}), \mathcal{S}_{\mathrm{add}(N_{i+1,i})})$ is a Frobenius R-category with $\mathrm{Inj}_{\mathcal{S}_{\mathrm{add}(N_{i+1,i})}} = \mathrm{add}(N_{i+1,i})$. To show that the embedding functor $\lambda_{i+1,i} : \mathrm{add}(M_{i+1,i}) \to A$-mod sends any admissible exact sequence of $\mathcal{S}_{\mathrm{add}(N_{i+1,i})}$ to an admissible exact sequence of $\mathcal{S}_{\mathrm{add}(M)}$, we first prove that $\mathrm{add}(N_{i+1,i})$-monic (or epic) morphisms in $\mathrm{add}(M_{i+1,i})$ are $\mathrm{add}(M)$-monic (or epic) in A-mod.

Indeed, for an $\mathrm{add}(N_{i+1,i})$-monic morphism $f : X \to Y$ in $\mathrm{add}(M_{i+1,i})$, we have a left $\mathrm{add}(M)$-approximation $f_1 : X \to M_1$ of X with $M_1 \in \mathrm{add}(N_{i+1,i})$ (see the proof of (1)). Thus there exists a homomorphism $f_2 : Y \to M_1$ such that $ff_2 = f_1$. For $g_1 : X \to M_2$ of A-modules with $M_2 \in \mathrm{add}(M)$, there exists a homomorphism $g_2 : M_1 \to M_2$ such that $f_1 g_2 = g_1$. Therefore $g_1 = f_1 g_2 = f f_2 g_2$ and f is $\mathrm{add}(M)$-monic in A-mod. Similarly, we show that $\mathrm{add}(N_{i+1,i})$-epic morphisms in $\mathrm{add}(M_{i+1,i})$ are $\mathrm{add}(M)$-epic in A-mod.

It remains to show that admissible exact sequences of $\mathcal{S}_{\mathrm{add}(M)}$ are closed under extensions. Precisely, given an admissible exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of $\mathcal{S}_{\mathrm{add}(M)}$ with $X, Z \in \mathrm{add}(M_{i+1,i})$, we have to show $Y \in \mathrm{add}(M_{i+1,i})$. In fact, by the proof of (1), we have an admissible exact sequence of $\mathcal{S}_{\mathrm{add}(N_{i+1,i})}$: $0 \to Z' \xrightarrow{h} M' \to Z \to 0$ with $M' \in \mathrm{add}(N_{i+1,i})$ and $Z' \in \mathrm{add}(M_{i+1,i})$. Since g is $\mathrm{add}(N_{i+1,i})$-epic, we can get the exact commutative diagram in A-mod:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z' & \xrightarrow{h} & M' & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\phi} & & \downarrow & & \| & & \\
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0.
\end{array}$$

This diagram is a push-out diagram in A-mod. Since h is an admissible monomorphism of $\mathcal{S}_{\mathrm{add}(N_{i+1,i})}$ and $\phi \in \mathrm{add}(M_{i+1,i})$, we deduce from (1)(ii) in the proof of Lemma 4.7 that $Y \in \mathrm{add}(M_{i+1,i})$. This implies that $(\mathrm{add}(M_{i+1,i}), \mathcal{S}_{\mathrm{add}(N_{i+1,i})})$ is closed under extensions in $(A\text{-mod}, \mathcal{S}_{\mathrm{add}(M)})$.

(3) Finally, we show $A\text{-mod}/[M] \simeq \prod_{i \in \mathcal{D}_T} R[x]/(f(x)^i)\text{-}\underline{\mathrm{mod}}$ as triangulated R-categories. As a consequence, we have the triangle equivalence of triangulated R-categories:

$$\mathscr{D}_{sg}(\Gamma) \simeq \prod_{i \in \mathcal{D}_T} R[x]/(f(x)^i)\text{-}\underline{\mathrm{mod}}.$$

Actually, Corollary 2.1 and Lemma 4.7(1) show that the embedding functor $\lambda_{i+1,i}$ induces a fully faithful triangle R-functor

$$\underline{\lambda_{i+1,i}} : \mathrm{add}(M_{i+1,i})/\mathrm{add}(N_{i+1,i}) \longrightarrow A\text{-mod}/[M].$$

Thus we obtain the full triangulated R-subcategories $\mathrm{Im}(\underline{\lambda_{2,1}}), \mathrm{Im}(\underline{\lambda_{3,2}}), \cdots, \mathrm{Im}(\underline{\lambda_{s+1,s}})$ of A-mod$/[M]$. By Lemma 2.4(1), there are no nonzero morphisms between such two distinct subcategories. Trivially, we have $A\text{-mod} = \mathrm{add}(M_{2,1} \oplus \cdots \oplus M_{s+1,s})$. Now, it follows from Lemma 2.2 that there is a triangle equivalence $A\text{-mod}/[M] \simeq \prod_{i=1}^{s} \ \mathrm{add}(M_{i+1,i})/\mathrm{add}(N_{i+1,i})$ of triangulated R-categories. By Lemma 4.7(2), $\mathrm{add}(M_{i+1,i})/\mathrm{add}(N_{i+1,i}) \simeq R[x]/(f(x)^{t_i - t_{i+1}})\text{-}\underline{\mathrm{mod}}$ and $A\text{-mod}/[M] \simeq \prod_{i=1}^{s} R[x]/(f(x)^{t_i-t_{i+1}})\text{-}\underline{\mathrm{mod}}$ as triangulated R-categories. Due to $R[x]/(f(x))\text{-}\underline{\mathrm{mod}} = 0$, we rewrite the latter equivalence as $A\text{-mod}/[M] \simeq \prod_{i\in \mathcal{D}_T} R[x]/(f(x)^i)\text{-}\underline{\mathrm{mod}}$ as triangulated R-categories. □

Corollary 4.9. *Let $T = \{t_1, \cdots, t_s\}$ be a subset of $[n]$ with $t_1 > t_2 > \cdots > t_s$, $M := \bigoplus_{i=1}^{s}(M(t_i))$ and $\Gamma := \mathrm{End}_A(M)$. Then*

$$\mathrm{gldim}(\Gamma) = \begin{cases} 0, & \text{if } t_1 = 1, \\ 2, & \text{if } t_1 \neq 1 \text{ and } M \text{ is an additive generator for } R[x]/(f(x)^{t_1})\text{-mod}, \\ \infty, & \text{otherwise.} \end{cases}$$

In particular, $\mathrm{gldim}(\Gamma) < \infty$ *if and only if* Γ *is the Auslander algebra of* $R[x]/(f(x)^{t_1})$.

Proof. For an Artin algebra Λ, $\mathrm{gldim}(\Lambda) < \infty$ if and only if $\mathscr{D}_{sg}(\Lambda) = 0$. By Proposition 4.8, we know $\mathscr{D}_{sg}(\Gamma) \simeq \prod_{i\in\mathcal{D}_T} R[x]/(f(x)^i)\text{-}\underline{\mathrm{mod}}$ as triangulated categories. Thus $\mathscr{D}_{sg}(\Gamma) = 0$ if and only if $\mathcal{D}_T = \emptyset$ if and only if $T = [t_1]$. Equivalently, $\mathscr{D}_{sg}(\Gamma) = 0$ if and only if M is an additive generator for $R[x]/(f(x)^{t_1})$-mod. In this case, Γ is a semisimple algebra if and only if $R[x]/(f(x)^{t_1})$ is a semisimple algebra if only if $t_1 = 1$. □

Remark that if c has a Jordan normal form then Corollary 4.9 follows from the cellularity of $S_n(c,R)$ by [35]. For an arbitrary field R, however, there may not exist any Jordan normal forms of matrices.

4.3 Singularity categories of centralizer matrix algebras

For $c \in M_n(R)$, let $m_c(x)$ be the minimal polynomial of c over R and $A_c := R[x]/(m_c(x))$. We write

$$m_c(x) := \prod_{i=1}^{l_c} f_i(x)^{n_i} \text{ for } n_i \geq 1 \ \text{ and } \ U_i := R[x]/(f_i(x)^{n_i}) \text{ for } i \in [l_c]$$

where all $f_i(x)$ are distinct irreducible (monic) polynomials in $R[x]$. It is known that U_i is a local, symmetric Nakayama R-algebra for $i \in [l_c]$, and the isomorphism of R-algebras holds: $A_c \simeq U_1 \times U_2 \times \cdots \times U_{l_c}$.

Since $A_c \simeq R[c]$ and the $R[c]$-module $R^n = \{(a_1, a_2, \cdots, a_n)^{tr} \mid a_i \in R, i \in [n]\}$ can be regarded as an A_c-module, we can decompose the A_c-module R^n, according to the blocks of A_c, in the following way:

$$(\star) \qquad R^n \simeq \bigoplus_{i=1}^{l_c} \bigoplus_{j=1}^{s_i} R[x]/(f_i(x)^{e_{ij}}),$$

where s_i and e_{ij} are positive integers. Note that this is also a decomposition of $R[x]$-modules and $\{\{f_i(x)^{e_{ij}} \mid i \in [l_c], j \in [s_i]\}\}$ is the multiset of all elementary divisors of c (see [10, Chapter 4], where $(\star)$ is stated in terms of invariant subspaces of a linear transformation).

Let $M_i := \bigoplus_{j=1}^{s_i} R[x]/(f_i(x)^{e_{ij}})$ be the sum of indecomposable direct summands of the A_c-module R^n belonging to the block U_i, and $A_i := \mathrm{End}_{U_i}(M_i)$ for $i \in [l_c]$. Then all algebras A_i are indecomposable.

Since R^n is a faithful $M_n(R)$-module, R^n is also a faithful $R[c]$-module, and therefore R^n is a generator for $R[c]$-mod, and M_i is a faithful U_i-module for $i \in [l_c]$.

$$S_n(c,R) \simeq \mathrm{End}_{R[c]}(R^n) \simeq \mathrm{End}_{A_c}(\bigoplus_{i=1}^{l_c} M_i) = \prod_{i=1}^{l_c} \mathrm{End}_{U_i}(M_i) = \prod_{i=1}^{l_c} A_i.$$

This is a decomposition of blocks of $S_n(c,R)$. The following lemma is an immediate consequence of $(\star)$.

Lemma 4.10. [24, Lemma 2.16] *There is a bijection π from $\mathcal{E}_c$ to the set of non-isomorphic, indecomposable direct summands of the A_c-module R^n, sending $h(x) \in \mathcal{E}_c$ to the A_c-module $R[x]/(h(x))$.*

Now, we state the description of the singular categories of centralizer matrix algebras.

Theorem 4.11. *For $c \in M_n(R)$, we have the triangle equivalence of triangulated R-categories:*

$$\mathscr{D}_{sg}(S_n(c,R)) \simeq \prod_{f(x)\in\mathcal{I}_c} \prod_{j\in\mathcal{D}_{P_c(f(x))}} R[x]/(f(x)^j)\text{-}\underline{\text{mod}}.$$

Proof. Note that $S_n(c,R) \simeq \prod_{i=1}^{l_c} \text{End}_{U_i}(M_i)$ as R-algebras, where $U_i := R[x]/(f_i(x)^{n_i})$. Clearly, for $i \in [l_c]$, we have $P_c(f_i(x)) = \{e_{ij} \mid j \in [s_i]\}$ and $\mathcal{B}(M_i) \simeq \bigoplus_{r\in P_c(f_i(x))} R[x]/(f_i(x)^r)$ as U_i-modules. Note that $\text{End}_{U_i}(M_i)$ and $\text{End}_{U_i}(\mathcal{B}(M_i))$ are Morita equivalent. Thus $\mathscr{D}_{sg}(\text{End}_{U_i}(M_i)) \simeq \mathscr{D}_{sg}(\text{End}_{U_i}(\mathcal{B}(M_i)))$ as triangulated R-categories. By Lemma 4.8, we know

$$\mathscr{D}_{sg}(\text{End}_{U_i}(M_i)) \simeq \mathscr{D}_{sg}(\text{End}_{U_i}(\mathcal{B}(M_i))) \simeq \prod_{j\in\mathcal{D}_{P_c(f_i(x))}} R[x]/(f_i(x)^j)\text{-}\underline{\text{mod}}$$

as triangulated R-categories. Thus $\mathscr{D}_{sg}(S_n(c,R)) \simeq \prod_{i=1}^{l_c} \prod_{j\in\mathcal{D}_{P_c(f(x))}} R[x]/(f_i(x)^j)\text{-}\underline{\text{mod}}$ as triangulated R-categories. If $|P_c(f_i(x))| = \max\{j \in P_c(f_i(x))\}$ for $i \in [l_c]$, then $\mathcal{D}_{P_c(f_i(x))} = \emptyset$. So we get the triangle equivalence of triangulated R-categories:

$$\prod_{i=1}^{l_c} \prod_{j\in\mathcal{D}_{P_c(f_i(x))}} R[x]/(f_i(x)^j)\text{-}\underline{\text{mod}} \simeq \prod_{f(x)\in\mathcal{I}_c} \prod_{j\in\mathcal{D}_{P_c(f(x))}} R[x]/(f(x)^j)\text{-}\underline{\text{mod}}. \qquad \square$$

5 Singular equivalences

In this section, we characterize the singular equivalences of centralizer matrix algebras.

We follow the notation in [24, Section 4] and keep the notation in Subsection 4.3. Now, let $c \in M_n(R)$ and $d \in M_m(R)$, we write

$$m_d(x) = \prod_{j=1}^{l_d} g_j(x)^{m_j} \text{ for } m_j \geq 1, \quad V_j := R[x]/(g_j(x)^{m_j}) \text{ for } j \in [l_d],$$

where $g_1(x), \cdots, g_{l_d}(x)$ are pairwise distinct monic irreducible polynomials in $R[x]$. Clearly, $\mathcal{M}_c = \{f_i(x)^{n_i} \mid i \in [l_c]\}$ and $\mathcal{M}_d = \{g_i(x)^{n_i} \mid i \in [l_d]\}$. Moreover, $A_c \simeq U_1 \times U_2 \times \cdots \times U_{l_c}$ and $A_d \simeq V_1 \times V_2 \times \cdots \times V_{l_d}$, where U_i and V_j are local, symmetric Nakayama R-algebras.

Due to $A_c \simeq R[c]$, we can view R^n as an A_c-module. According to block decompositions of A_c and A_d, we decompose the A_c-module R^n and the A_d-module R^m as

$$R^n = \bigoplus_{i=1}^{l_c} M_i \text{ and } R^m = \bigoplus_{j=1}^{l_d} N_j,$$

respectively, where M_i is the direct summands of R^n belonging to the block U_i, and where N_j is the direct summands of R^m belonging to the block V_j. Note that M_i and N_j are generators for U_i-mod and for V_j-mod, respectively. It follows from Subsection 4.3 and Lemma 4.10 that

$$M_i = \bigoplus_{j\in\tilde{P}_c(f_i(x)^{n_i})} R[x]/(f_i(x)^j) \text{ and } N_i = \bigoplus_{j\in\tilde{P}_d(g_i(x)^{m_i})} R[x]/(g_i(x)^j),$$

$$(\dagger)\quad \mathcal{B}(M_i)\simeq \bigoplus_{r\in P_c(f_i(x)^{n_i})} R[x]/(f_i(x)^r) \quad\text{and}\quad \mathcal{B}(N_j)\simeq \bigoplus_{s\in P_d(g_j(x)^{m_j})} R[x]/(g_j(x)^s)$$

as U_i-modules and V_j-modules, respectively. We set $A_i := \mathrm{End}_{U_i}(M_i)$ and $B_j := \mathrm{End}_{V_j}(N_j)$ for $i\in[l_c]$ and $j\in[l_d]$. Then A_i and B_j are indecomposable algebras for $i\in[l_c]$ and $j\in[l_d]$. By Lemma 2.10,

$$S_n(c,R)\simeq\prod_{i=1}^{l_c}\mathrm{End}_{U_i}(M_i)=\prod_{i=1}^{l_c}A_i \quad\text{and}\quad S_m(d,R)\simeq\prod_{j=1}^{l_d}\mathrm{End}_{V_j}(N_j)=\prod_{i=1}^{l_d}B_j.$$

5.1 Main result on singular equivalences

In this subsection we study singular equivalences of centralizer matrix algebras.

Firstly, we recall the Auslander-Reiten quiver of stable module category of an Artin algebra Λ. Let Γ_Λ denote the Auslander-Reiten quiver of Λ, and let Γ_Λ^s be the subquiver of Γ_Λ obtained from Γ_Λ by removing all vertices corresponding to indecomposable projective modules and all arrows starting with or ending at projective vertices. For the local, symmetric Nakayama algebra $A=R[x]/(f(x)^n)$ with $f(x)$ an irreducible polynomial and $n\geq 2$ an integer, the quiver Γ_A^s of A is a connected quiver with $n-1$ vertices.

We note the following immediate consequence of [2, Lemma X.1.2(d), p.336].

Lemma 5.1. *Suppose that G is a stable equivalence between Artin algebras $\prod_{i=1}^s C_i$ and $\prod_{j=1}^t D_j$, where C_i and D_j are indecomposable non-semisimple algebras for $i\in[s]$ and $j\in[t]$. Suppose that the quivers $\Gamma_{C_i}^s$ and $\Gamma_{D_j}^s$ are connected for all $i\in[s]$ and all $j\in[t]$. Then G preserves non-semisimple blocks, and therefore $s=t$.*

Now we state our main result on singular equivalences of centralizer matrix algebras.

Theorem 5.2. *Let R be a field, $c\in M_n(R)$ and $d\in M_m(R)$. Then we have the following.*
(1) *$S_n(c,R)$ and $S_m(d,R)$ are isomorphic if and only if c and d are I-equivalent.*
(2) *The following statements are equivalent.*
(*i*) *$\mathscr{D}_{sg}(S_n(c,R))\simeq\mathscr{D}_{sg}(S_m(d,R))$ as triangulated R-categories.*
(*ii*) *$\mathscr{D}_{sg}(S_n(c,R))\simeq\mathscr{D}_{sg}(S_m(d,R))$ as R-categories.*
(*iii*) *c and d are Sg-equivalent.*

Proof. Recall that we have the following isomorphisms of R-algebras:

$$S_n(c,R)\simeq\prod_{i=1}^{l_c}\mathrm{End}_{U_i}(M_i)=\prod_{i=1}^{l_c}A_i \text{ and } S_m(d,R)\simeq\prod_{j=1}^{l_d}\mathrm{End}_{V_j}(N_j)=\prod_{i=1}^{l_d}B_j.$$

(1) If $S_n(c,R)\simeq S_m(d,R)$ as R-algebras, then they have the same number of blocks, that is, $l_c=l_d$. We may assume that A_i and B_i are isomorphic as R-algebras for $i\in[l_c]$. By Lemma 2.5, $A_i\simeq B_i$ if and only if $U_i\simeq V_i$ and $\tilde{P}_c(f_i(x)^{n_i})=\tilde{P}_d(g_i(x)^{m_i})$. Therefore, $S_n(c,R)$ and $S_m(d,R)$ are isomorphic if and only if the matrices c and d are I-equivalent.

(2) By Theorem 4.11, we have the equivalences of triangulated R-categories:

$$\mathscr{D}_{sg}(S_n(c,R))\simeq\prod_{f(x)\in\mathcal{I}_c}\prod_{i\in\mathcal{D}_{P_c(f(x))}} R[x]/(f(x)^i)\text{-}\underline{\mathrm{mod}} \text{ and } \mathscr{D}_{sg}(S_m(d,R))\simeq\prod_{g(x)\in\mathcal{I}_d}\prod_{j\in\mathcal{D}_{P_d(g(x))}} R[x]/(g(x)^j)\text{-}\underline{\mathrm{mod}}.$$

(i) $\Rightarrow$ (ii) This is clear since triangulated R-categories are R-categories.

(ii) $\Rightarrow$ (iii) Suppose $\mathscr{D}_{sg}(S_n(c,R))\simeq\mathscr{D}_{sg}(S_m(d,R))$ as R-categories. Then the algebras

$$\prod_{f(x)\in\mathcal{I}_c}\prod_{i\in\mathcal{D}_{P_c(f(x))}} R[x]/(f(x)^i) \text{ and } \prod_{g(x)\in\mathcal{I}_d}\prod_{j\in\mathcal{D}_{P_d(g(x))}} R[x]/(g(x)^j)$$

are stably equivalent. Suppose

$$F:\prod_{f(x)\in\mathcal{I}_c}\prod_{i\in\mathcal{D}_{P_c(f(x))}}R[x]/(f(x)^i)\text{-}\underline{\mathrm{mod}}\longrightarrow\prod_{g(x)\in\mathcal{I}_d}\prod_{j\in\mathcal{D}_{P_d(g(x))}}R[x]/(g(x)^j)\text{-}\underline{\mathrm{mod}}$$

is a stable equivalence. Since $R[x]/(h(x)^i)$ is not a semisimple algebra for any irreducible polynomial $h(x)\in R[x]$ and integer $i\geq 2$, it follows from Lemma 5.1 that F preserves non-semisimple blocks, that is, for $f(x)\in\mathcal{I}_c$ and $i\in\mathcal{D}_{P_c(f(x))}$, there exist an unique $g_f(x)\in\mathcal{I}_d$ and $j_i\in\mathcal{D}_{P_d(g_f(x))}$ such that $R[x]/(f(x)^i)$-$\underline{\mathrm{mod}}$ and $R[x]/(g_f(x)^{j_i})$-$\underline{\mathrm{mod}}$ are equivalent as R-categories. This implies that $R[x]/(f(x))\simeq R[x]/(g_f(x))$ as R-algebras and $i=j_i$ by Lemma 2.9. Thus $r_c=r_d$ and there is a permutation $\sigma\in\Sigma_{r_c}$ such that $Q_{c,i}\simeq Q_{d,(i)\sigma}$ as R-algebras and $\mathcal{D}_{c,i}=\mathcal{D}_{d,(i)\sigma}$. By definition, c and d are Sg-equivalent.

(iii) $\Rightarrow$ (i) Suppose that c and d are Sg-equivalent, that is, $r_c=r_d$ and there is a permutation $\sigma\in\Sigma_{r_c}$ such that $Q_{c,i}:=R[x]/(p_{c,i}(x))\simeq Q_{d,(i)\sigma}:=R[x]/(p_{d,(i)\sigma}(x))$ as R-algebras and $\mathcal{D}_{c,i}=\mathcal{D}_{d,(i)\sigma}$ for $i\in[r_c]$ (see Section 3 for notation). For $i\in[r_c]$, we deduce from Lemma 2.9 that the following triangle equivalences of triangulated R-categories hold:

$$R[x]/(f(x)^s)\text{-}\underline{\mathrm{mod}}\simeq R[x]/(p_{c,i}(x)^s)\text{-}\underline{\mathrm{mod}}\simeq R[x]/(p_{d,(i)\sigma}(x)^s)\text{-}\underline{\mathrm{mod}}\simeq R[x]/(g(x)^s)\text{-}\underline{\mathrm{mod}}$$

for $f(x)\in\mathcal{I}_{c,i},g(x)\in\mathcal{I}_{d,(i)\sigma}$ and $s\in\mathbb{N}$. It follows from $\mathcal{D}_{c,i}=\mathcal{D}_{d,(i)\sigma}$ for $i\in[r_c]$ that

$$\begin{aligned}\mathscr{D}_{sg}(S_n(c,R))&\simeq\prod_{f(x)\in\mathcal{I}_c}\prod_{i\in\mathcal{D}_{P_c(f(x))}}R[x]/(f(x)^i)\text{-}\underline{\mathrm{mod}}\simeq\prod_{i\in[r_c]}\prod_{j\in\mathcal{D}_{c,i}}R[x]/(p_{c,i}(x)^j)\text{-}\underline{\mathrm{mod}}\\&\simeq\prod_{i\in[r_c]}\prod_{j\in\mathcal{D}_{d,(i)\sigma}}R[x]/(p_{d,(i)\sigma}(x)^j)\text{-}\underline{\mathrm{mod}},\ \text{and}\end{aligned}$$

$$\mathscr{D}_{sg}(S_m(d,R))\simeq\prod_{g(x)\in\mathcal{I}_d}\prod_{j\in\mathcal{D}_{P_d(g(x))}}R[x]/(g(x)^j)\text{-}\underline{\mathrm{mod}}\simeq\prod_{i\in[r_c]}\prod_{j\in\mathcal{D}_{d,i}}R[x]/(p_{d,i}(x)^j)\text{-}\underline{\mathrm{mod}}$$

as triangulated R-categories. Hence $\mathscr{D}_{sg}(S_n(c,R))\simeq\mathscr{D}_{sg}(S_m(d,R))$ as triangulated R-categories. □

Though Morita, derived and stable equivalences preserve the number of non-semisimple blocks of centralizer matrix algebras, singular equivalences may not, in general, preserve the number of non-semisimple blocks of centralizer matrix algebras.

Example 5.3. Let R be a field and $J_n(\lambda)$ the $n\times n$ Jordan matrix with the eigenvalue $\lambda\in R$. We take $c=J_2(0)\oplus J_4(0)\in M_6(R)$ and $d=J_2(0)\oplus J_2(1)\in M_4(R)$. Then $m_c(x)=x^4,m_d(x)=x^2(x-1)^2$, $\mathcal{E}_c=\{x^2,x^4\},\mathcal{E}_d=\{x^2,(x-1)^2\}=\mathcal{M}_d=\mathcal{R}_d,\mathcal{M}_c=\{x^4\}=\mathcal{R}_c,\mathcal{I}_c=\{x\},\mathcal{I}_d=\{x,x-1\},r_c=1=r_d,Q_{c,1}\simeq R[x]/(x)\simeq Q_{d,1}$ and $\mathcal{D}_{c,1}=\{2,2\}=\mathcal{D}_{d,1}$. Thus c and d are Sg-equivalent, but not D-equivalent nor S-equivalent. By Theorem 5.2, $S_6(c,R)$ and $S_4(d,R)$ are singularly equivalent, but they are neither derived equivalent by [24, Theorem 1.1], nor stably equivalent by [25, Theorem 1.1].

In fact, we have $S_6(c,R)\simeq\mathrm{End}_{R[x]/(x^4)}(R[x]/(x^4)\oplus R[x]/(x^2))$ is indecomposable, while $S_4(d,R)\simeq R[x]/(x^2)\times R[x]/((x-1)^2)$ has two blocks of infinite global dimension. This shows that singularly equivalent centralizer matrix algebras may have different numbers of non-semisimple blocks in general.

5.2 Singular equivalences for permutation matrices

In this subsection, we study singular equivalences between centralizer matrix algebras of permutation matrices. We first prepare several useful lemmas.

Throughout this subsection, R denotes a field of characteristic $p\geq 0$.

Let $\sigma=\sigma_1\cdots\sigma_s\in\Sigma_n$ be a product of disjoint cycles, and let $\lambda=(\lambda_1,\ldots,\lambda_s)$ be its cycle type with $\lambda_i\geq 1$ for all $i\in[s]$.

If $p>0$, then a cycle σ_i is said to be *p-regular* if $p \nmid \lambda_i$, and *p-singular* if $p \mid \lambda_i$. If $p=0$, all cycles are regarded as p-regular cycles. Let $r(\sigma)$ (respectively, $s(\sigma)$) be the product of the p-regular (respectively, p-singular) cycles of σ. We consider both $r(\sigma)$ and $s(\sigma)$ as elements in Σ_n. The permutation σ is said to be *p-regular* (or *p-singular*) if $\sigma = r(\sigma)$ (or $\sigma = s(\sigma)$). For $p=0$, we have $r(\sigma)=\sigma$ and $s(\sigma)=id$, the identity permutation.

Let $c_\sigma := \sum_{i=1}^n e_{i,(i)\sigma} \in M_n(R)$ denote the *permutation matrix* of σ over R, where e_{ij} is the matrix with 1 in (i,j)-entry and 0 in all other entries.

The elementary divisors of permutation matrices can be described as follows.

Lemma 5.4. [24, Lemma 2.17] $\mathcal{E}_{c_\sigma} = \{g(x)^{p^{\nu_p(\lambda_i)}} \mid i \in [s],\ g(x)$ *is an irreducible factor of* $x^{\lambda_i}-1\}$.

By Lemma 5.4, the matrix c_σ always has a unique maximal elementary divisor of the form $(x-1)^{p^i}$ for some $i \in \mathbb{N}$. This elementary divisor is called the *exceptional* elementary divisor of c_σ.

Lemma 5.5. *If $p \neq 2$ or both $p=2$ and $P_{c_\sigma}(f(x)) \neq \{1,2\}$ for all $f(x) \in \mathcal{M}_{c_\sigma}$, then*

(1) *$s(\sigma)=id$ if and only if $r(\sigma)=\sigma$ if and only if $\mathcal{I}_{c_\sigma}=\emptyset$.*

(2) *$\mathcal{I}_{c_{s(\sigma)}} = \mathcal{I}_{c_\sigma}$, $r_{c_{s(\sigma)}} = r_{c_\sigma}$ and, for $f(x) \in \mathcal{I}_{c_\sigma}$,*

$$P_{c_{s(\sigma)}}(f(x)) = \begin{cases} P_{c_\sigma}(f(x)), & \text{if } f(x)=x-1, \\ P_{c_\sigma}(f(x)) \setminus \{1\}, & \text{otherwise}. \end{cases}$$

Proof. Without loss of generality, we may assume $p>0$.

(1) We show the statement: For $f(x) \in \mathcal{M}_{c_\sigma}$, the equality $|P_{c_\sigma}(f(x))| = \max\{i \in P_{c_\sigma}(f(x))\}$ holds if and only if $P_{c_\sigma}(f(x)) = \{1\}$.

Indeed, if $P_{c_\sigma}(f(x))=\{1\}$, then $|P_{c_\sigma}(f(x))| = \max\{j \in P_{c_\sigma}(f(x))\}$. Conversely, suppose $|P_{c_\sigma}(f(x))| = \max\{j \in P_{c_\sigma}(f(x))\}$. Then $P_{c_\sigma}(f(x)) = [\,|P_{c_\sigma}(f(x))|\,]$. We write $f(x)=g(x)^j \in \mathcal{M}_{c_\sigma}$ with $g(x) \in R[x]$ an irreducible polynomial and j the maximal power index of $g(x)$. By Lemma 5.4, we have $P_{c_\sigma}(f(x)) = \{p^{\nu_p(\lambda_i)} \mid i \in [s],\ g(x)$ is a divisor of $x^{\lambda_i}-1\}$. Thus it consists of p-powers. By Lemma 2.12(2)-(3), if $p \neq 2$ and $p>0$, then $P_{c_\sigma}(f(x))=\{1\}$; or if $p=2$, then $P_{c_\sigma}(f(x))=\{1,2\}$. Hence it follows from either $p \neq 2$, or $p=2$ and $P_{c_\sigma}(f(x)) \neq \{1,2\}$ that $P_{c_\sigma}(f(x))=\{1\}$.

Therefore $s(\sigma)=id$ if and only if $r(\sigma)=\sigma$ if and only if $P_{c_\sigma}(f(x))=\{1\}$ for all $f(x) \in \mathcal{M}_{c_\sigma}$ if and only if $|P_{c_\sigma}(f(x))| = \max\{j \in P_{c_\sigma}(f(x))\}$ for all $f(x) \in \mathcal{M}_{c_\sigma}$ if and only if $\mathcal{I}_{c_\sigma}=\emptyset$. This shows (1).

(2) This follows from the above statement and Lemma 5.4. $\square$

Given $T := \{n_1, n_2, \cdots, n_s\} \subset \mathbb{Z}_{>0}$ with $n_1 > n_2 > \cdots > n_s$ and $s \geq 2$, we denote by $L_2(T) := \{n_{s-1}, n_s\}$ the last two elements of T, and by $L_{2nd}(T) := \{n_{s-1}\}$ the second to last element of T. We define a multiset $\mathcal{D}'_T := \mathcal{D}_{T \setminus \{n_s\}}$.

For $c \in M_n(R)$ and $i \in [r_c]$, let $\tilde{\mathcal{I}}_{c,i} := \{f(x) \in \mathcal{I}_{c,i} \mid 1 \in P_c(f(x))\}$ and $\tilde{\mathcal{D}}_{c,i} := \bigcup_{f(x) \in \tilde{\mathcal{I}}_{c,i}} \mathcal{D}_{L_2(P_c(f(x)))}$. For the definitions of $\mathcal{D}_T$ and $\mathcal{I}_{c,i}$, we refer to Subsection 2.5 and Section 3, respectively.

Lemma 5.6. *Suppose $p \neq 2$ or $p=2$ and $P_{c_\sigma}(g(x)) \neq \{1,2\}$ for all $g(x) \in \mathcal{M}_{c_\sigma}$. Let $j \in [r_{c_\sigma}]$.*

(1) *If $f(x) \in \mathcal{I}_{c_\sigma,j}$, then* $\mathcal{D}_{P_{c_{s(\sigma)}}(f(x))} = \begin{cases} \mathcal{D}_{P_{c_{s(\sigma)}}(f(x))}, & \text{if } f(x)=x-1 \text{ or } f(x) \notin \tilde{\mathcal{I}}_{c_\sigma,j}, \\ \mathcal{D}'_{P_{c_{s(\sigma)}}(f(x))}, & \text{if } f(x) \neq x-1 \text{ and } f(x) \in \tilde{\mathcal{I}}_{c_\sigma,j}. \end{cases}$

(2) *Assume $\mathcal{I}_{c_{s(\sigma)},j} = \mathcal{I}_{c_\sigma,j}$. Then*

(*i*) *If $x-1 \in \tilde{\mathcal{I}}_{c_\sigma,j}$, then*

$$\mathcal{D}_{c_{s(\sigma)},j} = \mathcal{D}_{L_2(P_{c_\sigma}(x-1))} \cup \bigcup_{x-1 \neq g(x) \in \tilde{\mathcal{I}}_{c_\sigma,1}} L_{2nd}\big(P_{c_\sigma}(g(x))\big) \cup (\mathcal{D}_{c_\sigma,j} \setminus \tilde{\mathcal{D}}_{c_\sigma,j}),$$

where the unions are taken in the sense of multisets.

(ii) If $x-1 \notin \tilde{\mathcal{I}}_{c_\sigma,j}$, then

$$\mathcal{D}_{c_{s(\sigma)},j} = \bigcup_{g(x)\in\tilde{\mathcal{I}}_{c_\sigma,j}} L_{2nd}\big(P_{c_\sigma}(g(x))\big) \cup (\mathcal{D}_{c_\sigma,j} \setminus \tilde{\mathcal{D}}_{c_\sigma,j}),$$

where the unions are taken in the sense of multisets.

Proof. If $\mathcal{I}_{c_\sigma,j} = \emptyset$, then (1) and (2) are trivial. So we suppose $\mathcal{I}_{c_\sigma,j} \neq \emptyset$. In this case, $p > 0$.

(1) This follows from 5.5(2) and the definition of $\mathcal{D}'_{P_{c_{s(\sigma)}}(f(x))}$.

(2) By the definition of $\mathcal{D}_{c_{s(\sigma)},j}$ and the assumption $\mathcal{I}_{c_{s(\sigma)},j} = \mathcal{I}_{c_\sigma,j}$, we know

$$\mathcal{D}_{c_{s(\sigma)},j} = \bigcup_{g(x)\in\mathcal{I}_{c_{s(\sigma)},j}} \mathcal{D}_{P_{c_{s(\sigma)}}(g(x))} = \bigcup_{g(x)\in\mathcal{I}_{c_\sigma,j}} \mathcal{D}_{P_{c_{s(\sigma)}}(g(x))},$$

where the unions are taken in the sense of multisets. To complete the proof, we consider the two cases.

(i) $x-1 \in \tilde{\mathcal{I}}_{c_\sigma,j}$. According to (1), we divide the polynomials in $\mathcal{I}_{c_\sigma,j}$ into the three parts:

$$\begin{aligned}\bigcup_{g(x)\in\mathcal{I}_{c_\sigma,j}} \mathcal{D}_{P_{c_{s(\sigma)}}(g(x))} &= \mathcal{D}_{P_{c_\sigma}(x-1)} \cup \bigcup_{x-1\neq g(x)\in\tilde{\mathcal{I}}_{c_\sigma,j}} \mathcal{D}'_{P_{c_\sigma}(g(x))} \cup \bigcup_{g(x)\in\mathcal{I}_{c_\sigma,j}\setminus\tilde{\mathcal{I}}_{c_\sigma,j}} \mathcal{D}_{P_{c_\sigma}(g(x))}\\ &= \mathcal{D}_{L_2(P_{c_\sigma}(x-1))} \cup \bigcup_{x-1\neq g(x)\in\tilde{\mathcal{I}}_{c_\sigma,j}} L_{2nd}\big(P_{c_\sigma}(g(x))\big) \cup (\mathcal{D}_{c_\sigma,j} \setminus \tilde{\mathcal{D}}_{c_\sigma,j}),\end{aligned}$$

where the unions are taken in the sense of multisets.

(ii) $x-1 \notin \tilde{\mathcal{I}}_{c_\sigma,j}$. By (1), we partition the polynomials in $\mathcal{I}_{c_\sigma,j}$ into two parts:

$$\begin{aligned}\bigcup_{g(x)\in\mathcal{I}_{c_\sigma,j}} \mathcal{D}_{P_{c_{s(\sigma)}}(g(x))} &= \bigcup_{g(x)\in\tilde{\mathcal{I}}_{c_\sigma,j}} \mathcal{D}'_{P_{c_\sigma}(g(x))} \cup \bigcup_{g(x)\in\mathcal{I}_{c_\sigma,j}\setminus\tilde{\mathcal{I}}_{c_\sigma,j}} \mathcal{D}_{P_{c_\sigma}(g(x))}\\ &= \bigcup_{g(x)\in\tilde{\mathcal{I}}_{c_\sigma,j}} L_{2nd}\big(P_{c_\sigma}(g(x))\big) \cup (\mathcal{D}_{c_\sigma,j} \setminus \tilde{\mathcal{D}}_{c_\sigma,j}),\end{aligned}$$

where the unions are taken in the sense of multisets. □

Now, let $\tau = \tau_1 \cdots \tau_t \in \Sigma_m$ be a product of disjoint cycles, and let $\mu = (\mu_1, \ldots, \mu_t)$ be its cycle type with $\mu_i \geq 1$ for all $i \in [t]$. Assume that c_σ and c_τ are Sg-equivalent. Then $r_{c_\sigma} = r_{c_\tau}$ and, by definition, there is a permutation $\delta \in \Sigma_{r_{c_\sigma}}$ such that $Q_{c_\sigma,i} \simeq Q_{c_\tau,(i)\delta}$ as R-algebras and $\mathcal{D}_{c_\sigma,i} = \mathcal{D}_{c_\tau,(i)\delta}$ for $i \in [r_{c_\sigma}]$. Under this assumption, we have the following lemma.

Lemma 5.7. *Assume that c_σ and c_τ are Sg-equivalent. Suppose $p \neq 2$ or $p = 2$ and $\nu_p(\lambda_i) \neq 1 \neq \nu_p(\mu_j)$ for all $i \in [s]$ and $j \in [t]$. Then the following hold.*

(1) For any $f(x) \in \mathcal{I}_{c_\sigma}$ with $1 \in P_{c_\sigma}(f(x))$, there exists $j \in \mathbb{Z}_{>0}$ such that $p^j > 2$ and $\{1, p^j\} = L_2(P_{c_\sigma}(f(x)))$.

(2) Let $j \in [r_{c_\sigma}]$. Then

(i) $\tilde{\mathcal{I}}_{c_\sigma,j} \neq \emptyset$ if and only if $\tilde{\mathcal{I}}_{c_\tau,(j)\delta} \neq \emptyset$.

(ii) $\mathcal{D}_{c_\sigma,j} \setminus \tilde{\mathcal{D}}_{c_\sigma,j} = \mathcal{D}_{c_\tau,(j)\delta} \setminus \tilde{\mathcal{D}}_{c_\tau,(j)\delta}$ and $\tilde{\mathcal{D}}_{c_\sigma,j} = \tilde{\mathcal{D}}_{c_\tau,(j)\delta}$.

(iii) There is a bijection $\pi_j : \tilde{\mathcal{I}}_{c_\sigma,j} \to \tilde{\mathcal{I}}_{c_\tau,(j)\delta}$ such that $L_2\big(P_{c_\sigma}(f(x))\big) = L_2\big(P_{c_\tau}((f(x))\pi_j)\big)$ for all $f(x) \in \tilde{\mathcal{I}}_{c_\sigma,j}$. Moreover, $x-1 \in \tilde{\mathcal{I}}_{c_\sigma,j}$ if and only if $x-1 \in \tilde{\mathcal{I}}_{c_\tau,(j)\delta}$. If $x-1 \in \tilde{\mathcal{I}}_{c_\sigma,j}$, then we may assume $(x-1)\pi_j = x-1$.

Proof. For $p = 0$, it follows from Lemma 5.4 that all maximal elementary divisors of c_σ are irreducible. This shows $\mathcal{I}_{c_\sigma} = \emptyset$. So all conclusions in Lemma 5.7 are trivially true. So we assume $p > 0$.

Suppose $p=2$. Then $\nu_p(\lambda_i)\neq 1$ for all $i\in[s]$ by assumption. According to Lemma 5.4, we know that $P_{c_\sigma}(f(x))$ consists of p-powers for $f(x)\in\mathcal{M}_{c_\sigma}$. This implies $2\notin P_{c_\sigma}(f(x))$ for all $f(x)\in\mathcal{M}_{c_\sigma}$, and therefore $P_{c_\sigma}(f(x))\neq\{1,2\}$ for all $f(x)\in\mathcal{M}_{c_\sigma}$. Hence, both c_σ and c_τ satisfy the conditions of Lemma 5.6.

(1) Suppose $f(x)\in\mathcal{I}_{c_\sigma}$ with $1\in P_{c_\sigma}(f(x))$. In this case, we assume $i\in P_{c_\sigma}(f(x))$ such that $f(x)^i\in\mathcal{M}_{c_\sigma}$. It follows from $|P_{c_\sigma}(f(x))|\neq i$ that $i\neq 1$ and $|P_{c_\sigma}(f(x))|\geq 2$. Then there exists $j\in[s]$ such that $L_2(P_{c_\sigma}(f(x)))=\{1,p^{\nu_p(\lambda_j)}\}$ with $\nu_p(\lambda_j)\neq 0$. By assumption, $p\neq 2$ or $p=2$ and $\nu_p(\lambda_i)\neq 1$ for all $i\in[s]$. This implies $p^{\nu_p(\lambda_j)}>2$. Thus (1) follows.

(2)(i) Suppose $\tilde{\mathcal{I}}_{c_\sigma,j}\neq\emptyset$. Then there is $f(x)\in\mathcal{I}_{c_\sigma,j}$ such that $1\in P_{c_\sigma}(f(x))$. By (1), there exists $j'\in\mathbb{Z}_{>0}$ such that $p^{j'}>2$ and $\{1,p^{j'}\}=L_2(P_{c_\sigma}(f(x)))$. Hence $p^{j'}-1\in\mathcal{D}_{c_\sigma,j}=\mathcal{D}_{c_\tau,(j)\delta}$ (the equality follows by the Sg-equivalence of c_σ and c_τ), and therefore, there is $g(x)\in\mathcal{I}_{c_\tau,(j)\delta}$ such that $p^{j'}-1=p^k-p^{k'}$ with $p^k,p^{k'}\in P_{c_\tau}(g(x))$ for some $k>k'\in\mathbb{N}$ or $p^{j'}-1=p^i\in P_{c_\tau}(g(x))$ for some $i\in\mathbb{Z}_{>0}$. By Lemma 2.12, we have $j'=k$ and $k'=0$; or $p=2$, $j'=1$ and $i=0$. Thanks to $2^1-1=1\notin\mathcal{D}_{c_\sigma,(j)\delta}$ (see the definition of $\mathcal{D}_T$), the latter cannot occur. Hence $L_2\big(P_{c_\tau}(g(x))\big)=\{1,p^{j'}\}$ that is, $\tilde{\mathcal{I}}_{c_\tau,(j)\delta}\neq\emptyset$. Similarly, we can show $\tilde{\mathcal{I}}_{c_\sigma,j}\neq\emptyset$ if $\tilde{\mathcal{I}}_{c_\tau,(j)\delta}\neq\emptyset$.

(2)(ii) It follows from $\mathcal{D}_{c_\sigma,j}=\mathcal{D}_{c_\tau,(j)\delta}$ and the proof of (2)(i) that $\mathcal{D}_{c_\sigma,j}\backslash\tilde{\mathcal{D}}_{c_\sigma,j}=\mathcal{D}_{c_\tau,(j)\delta}\backslash\tilde{\mathcal{D}}_{c_\tau,(j)\delta}$ and $\tilde{\mathcal{D}}_{c_\sigma,j}=\tilde{\mathcal{D}}_{c_\tau,(j)\delta}$.

(2)(iii) By definition, $1\in P_c(f(x))$ for $f(x)\in\tilde{\mathcal{I}}_{c_\sigma,j}$. Due to $\bigcup_{f(x)\in\tilde{\mathcal{I}}_{c_\sigma,j}}\mathcal{D}_{L_2(P_c(f(x))}=\tilde{\mathcal{D}}_{c_\sigma,j}=\tilde{\mathcal{D}}_{c_\tau,(j)\delta}=\bigcup_{g(x)\in\tilde{\mathcal{I}}_{c_\tau,(j)\delta}}\mathcal{D}_{L_2(P_c(g(x))}$ and (1), there is a bijection $\pi_j:\tilde{\mathcal{I}}_{c_\sigma,j}\to\tilde{\mathcal{I}}_{c_\tau,(j)\delta}$ such that, for $f(x)\in\tilde{\mathcal{I}}_{c_\sigma,j}$,

$$L_2\big(P_{c_\sigma}(f(x))\big)=L_2\big(P_{c_\tau}((f(x))\pi_j)\big).$$

Suppose $x-1\in\tilde{\mathcal{I}}_{c_\sigma,j}$. Then $\tilde{\mathcal{I}}_{c_\tau,(j)\delta}\neq\emptyset$ by (2)(i), and therefore there is $g(x)\in\mathcal{I}_{c_\tau,(j)\delta}$ such that $1\in P_{c_\tau}(g(x))$. It then follows from (1) that there exists $j'\in\mathbb{Z}_{>0}$ such that $p^{j'}>2$ and $\{1,p^{j'}\}=L_2(P_{c_\tau}(g(x)))$. This implies that $\nu_p(\mu_i)=0$ and $\nu_p(\mu_{i'})=j'$ for some $i,i'\in[t]$. By Lemma 5.4, we have $\{1,p^{j'}\}\subseteq P_{c_\tau}(x-1)=\{p^{\nu_p(\mu_i)}\mid i\in[t]\}$. Hence $P_{c_\tau}(x-1)\backslash 1\neq\emptyset$. Thanks to $p\neq 2$ or $p=2$ and $\nu_p(\mu_k)\neq 1$ for all $k\in[t]$, we get $\min\{k\in P_{c_\tau}(x-1)\backslash 1\}>2$. This implies $x-1\in\mathcal{I}_{c_\tau}$. By the definition of equivalence classes of $\mathcal{I}_{c_\tau}$, we have $x-1\in\mathcal{I}_{c_\tau,(j)\delta}$. Since $1\in P_{c_\tau}(x-1)$, we have $x-1\in\tilde{\mathcal{I}}_{c_\tau,(j)\delta}$. Similarly, we can show that $x-1\in\tilde{\mathcal{I}}_{c_\sigma,j}$ if $x-1\in\tilde{\mathcal{I}}_{c_\tau,(j)\delta}$.

Now we assume $x-1\in\tilde{\mathcal{I}}_{c_\sigma,j}$ and $(x-1)\pi_j\neq x-1$. Then there are $f(x)\in\tilde{\mathcal{I}}_{c_\sigma,j}$ and $g(x)\in\tilde{\mathcal{I}}_{c_\tau,(j)\delta}$ such that $(x-1)\pi_j=g(x)$ and $(f(x))\pi_j=x-1$. Clearly, $f(x)\neq x-1\neq g(x)$. By (1), there are $k,k'\in\mathbb{Z}_{>0}$ such that $L_2(P_{c_\sigma}(x-1))=\{1,p^k\}=L_2\big(P_{c_\tau}(g(x))\big)$ and $L_2\big(P_{c_\sigma}(f(x))\big)=\{1,p^{k'}\}=L_2(P_{c_\tau}(x-1))$. This implies that $\nu_p(\lambda_i)=k'$ and $\nu_p(\mu_{i'})=k$ for some $i\in[s]$ and $i'\in[t]$. By Lemma 5.4, we have $P_{c_\sigma}(x-1)=\{p^{\nu_p(\lambda_i)}\mid i\in[s]\}$ and $P_{c_\tau}(x-1)=\{p^{\nu_p(\mu_i)}\mid i\in[t]\}$. Thus $p^{k'}\in P_{c_\sigma}(x-1)$ and $p^k\in P_{c_\tau}(x-1)$, and therefore $k=k'$. Now we define a map:

$$\pi'_j:\tilde{\mathcal{I}}_{c_\sigma,j}\longrightarrow\tilde{\mathcal{I}}_{c_\tau,(j)\delta},\quad q(x)\mapsto\begin{cases}(q(x))\pi_j, & \text{if } q(x)\notin\{x-1,f(x)\},\\ x-1, & \text{if } q(x)=x-1,\\ g(x), & \text{if } q(x)=f(x).\end{cases}$$

Then π'_j is a desired map. $\square$

Corollary 5.8. *Suppose $p\neq 2$ or $p=2$ and $\nu_p(\lambda_i)\neq 1\neq\nu_p(\mu_j)$ for all $i\in[s]$ and $j\in[t]$. If $S_n(c_\sigma,R)$ and $S_m(c_\tau,R)$ are singularly equivalent, then so are $S_n(c_{s(\sigma)},R)$ and $S_m(c_{s(\tau)},R)$.*

Proof. Note that all algebras involved in Corollary 5.8 are semisimple if $p=0$. Thus there is nothing to prove. So we may assume $p>0$. By Theorem 5.2(2), it suffices to show that $c_{s(\sigma)}\overset{Sg}{\sim}c_{s(\tau)}$ if $c_\sigma\overset{Sg}{\sim}c_\tau$.

Now suppose $c_\sigma \overset{Sg}{\sim} c_\tau$. Then $r_{c_\sigma} = r_{c_\tau}$ and there is a permutation $\delta \in \Sigma_{r_{c_\sigma}}$ such that $Q_{c_\sigma,i} \simeq Q_{c_\tau,(i)\delta}$ as R-algebras and $\mathcal{D}_{c_\sigma,i} = \mathcal{D}_{c_\tau,(i)\delta}$ for $i \in [r_{c_\sigma}]$. Recall that r_{c_σ} denotes the number of equivalence classes of $\mathcal{I}_{c_\sigma}$ (see Section 3). If $p=2$, then $\nu_p(\lambda_i) \neq 1$ for all $i \in [s]$. By Lemma 5.4, $P_{c_\sigma}(f(x))$ does not contain 2 for all $f(x) \in \mathcal{M}_{c_\sigma}$, and therefore $P_{c_\sigma}(f(x)) \neq \{1,2\}$ for all $f(x) \in \mathcal{M}_{c_\sigma}$. Thus both c_σ and c_τ satisfy the conditions of Lemmas 5.5-5.7. It then follows from Lemma 5.5(2) that $\mathcal{I}_{c_{s(\sigma)}} = \mathcal{I}_{c_\sigma}$ and $\mathcal{I}_{c_{s(\tau)}} = \mathcal{I}_{c_\tau}$. Now we consider the two cases.

(1) $s(\sigma) = id$. Then $\mathcal{I}_{c_\sigma} = \emptyset$ by Lemma 5.5(1). Hence $r_{c_\sigma} = 0$, and therefore $r_{c_\tau} = r_{c_\sigma} = 0$. Thus $\mathcal{I}_{c_\tau} = \emptyset$ and $\mathcal{I}_{c_{s(\tau)}} = \emptyset$. By Lemma 5.5(1), we get $s(\tau) = id$. Hence $c_{s(\sigma)} \overset{Sg}{\sim} c_{s(\tau)}$.

(2) $s(\sigma) \neq id$. Then $\mathcal{I}_{c_\sigma} \neq \emptyset$ by Lemma 5.5(1). Thus $r_{c_\sigma} \neq 0$ and $r_{c_{s(\tau)}} = r_{c_\tau} = r_{c_\sigma} = r_{c_{s(\sigma)}} \neq 0$. Thus $s(\tau) \neq id$. In this case, let $r := r_{c_{s(\sigma)}}$ and assume that $\mathcal{I}_{c_{s(\sigma)},i} = \mathcal{I}_{c_\sigma,i}$ and $\mathcal{I}_{c_{s(\tau)},i} = \mathcal{I}_{c_\tau,i}$ for all $i \in [r]$. We already know $Q_{c_{s(\sigma)},i} \simeq Q_{c_{s(\tau)},(i)\delta}$ as algebras for $i \in [r]$. It remains to show $\mathcal{D}_{c_{s(\sigma)},i} = \mathcal{D}_{c_{s(\tau)},(i)\delta}$ for $i \in [r]$. For this purpose we consider the following two cases:

(i) $x-1 \in \tilde{\mathcal{I}}_{c_\sigma,i}$. Then it follows from Lemma 5.7(2)(iii) that $x-1 \in \tilde{\mathcal{I}}_{c_\tau,(i)\delta}$ and there is a bijection $\pi_i : \tilde{\mathcal{I}}_{c_\sigma,i} \to \tilde{\mathcal{I}}_{c_\tau,(i)\delta}$ such that $(x-1)\pi_i = x-1$ and $L_2\big(P_{c_\sigma}(f(x))\big) = L_2\big(P_{c_\tau}((f(x))\pi_i)\big)$ for all $f(x) \in \tilde{\mathcal{I}}_{c_\sigma,i}$. According to Lemmas 5.6(2)(i) and 5.7(2)(ii), in order to prove $\mathcal{D}_{c_{s(\sigma)},i} = \mathcal{D}_{c_{s(\tau)},(i)\delta}$, it suffices to show:

$$(\sharp) \quad \mathcal{D}_{L_2\left(P_{c_\sigma}(x-1)\right)} \cup \bigcup_{x-1 \neq h(x) \in \tilde{\mathcal{I}}_{c_\sigma,i}} L_{2nd}\big(P_{c_\sigma}(h(x))\big) = \mathcal{D}_{L_2\left(P_{c_\tau}(x-1)\right)} \cup \bigcup_{x-1 \neq g(x) \in \tilde{\mathcal{I}}_{c_\tau,(i)\delta}} L_{2nd}\big(P_{c_\tau}(g(x))\big).$$

In fact, π_i gives rise to the equality $L_2\big(P_{c_\sigma}(x-1)\big) = L_2\big(P_{c_\tau}(x-1)\big)$. Thus $\mathcal{D}_{L_2\left(P_{c_\sigma}(x-1)\right)} = \mathcal{D}_{L_2\left(P_{c_\tau}(x-1)\right)}$. Moreover, due to $L_2\big(P_{c_\sigma}(h(x))\big) = L_2\big(P_{c_\tau}((h(x))\pi_i)\big)$ for $h(x) \in \tilde{\mathcal{I}}_{c_\sigma,i} \setminus \{x-1\}$, we have

$$\bigcup_{x-1 \neq h(x) \in \tilde{\mathcal{I}}_{c_\sigma,i}} L_{2nd}\big(P_{c_\sigma}(h(x))\big) = \bigcup_{x-1 \neq g(x) \in \tilde{\mathcal{I}}_{c_\tau,(i)\delta}} L_{2nd}\big(P_{c_\tau}(g(x))\big).$$

This shows $(\sharp)$.

(ii) $x-1 \notin \tilde{\mathcal{I}}_{c_\sigma,i}$. In this case, by Lemma 5.7(2)(iii), $x-1 \notin \tilde{\mathcal{I}}_{c_\tau,(i)\delta}$ and there is a bijection $\pi_i : \tilde{\mathcal{I}}_{c_\sigma,i} \to \tilde{\mathcal{I}}_{c_\tau,(i)\delta}$ such that $L_2\big(P_{c_\sigma}(f(x))\big) = L_2\big(P_{c_\tau}((f(x))\pi_i)\big)$ for all $f(x) \in \tilde{\mathcal{I}}_{c_\sigma,i}$. Now, we prove $\mathcal{D}_{c_{s(\sigma)},i} = \mathcal{D}_{c_{s(\tau)},(i)\delta}$. By Lemmas 5.6(2)(ii) and 5.7(2)(ii), it suffices to show

$$\bigcup_{h(x) \in \tilde{\mathcal{I}}_{c_\sigma,i}} L_{2nd}\big(P_{c_\sigma}(h(x))\big) = \bigcup_{h'(x) \in \tilde{\mathcal{I}}_{c_\tau,(i)\delta}} L_{2nd}\big(P_{c_\tau}(h'(x))\big).$$

But this follows immediately from the bijection π_i.

Thus we have shown $\mathcal{D}_{c_{s(\sigma)},i} = \mathcal{D}_{c_{s(\tau)},(i)\delta}$ for all $i \in [r]$, and therefore $c_{s(\sigma)} \overset{Sg}{\sim} c_{s(\tau)}$. $\square$

The example shows that some assumptions in Corollary 5.8 cannot be removed.

Example 5.9. Let $R = \mathbb{F}_2$ be the field of two elements, $\sigma \in \Sigma_9$ of the cycle type $(6,3)$, and $\tau \in \Sigma_3$ of the cycle type (3). Then $\nu_2(6) = 1, \nu_2(3) = 0, \mathcal{E}_{c_\sigma} = \{x-1, (x-1)^2, x^2+x+1, (x^2+x+1)^2\}, P_{c_\sigma}((x-1)^2) = \{1,2\} = P_{c_\sigma}((x^2+x+1)^2)$, $|P_{c_\sigma}((x-1)^2)| = 2 = \max\{i \in P_{c_\sigma}((x-1)^2)\}$, and $|P_{c_\sigma}((x^2+x+1)^2)| = 2 = \max\{i \in P_{c_\sigma}((x^2+x+1)^2)\}$. Thus $\mathcal{I}_{c_\sigma} = \emptyset$. Clearly, $s(\sigma) \in \Sigma_9$ is of the cycle type $(6,1,1,1)$ and $\mathcal{E}_{c_{s(\sigma)}} = \{x-1, (x-1)^2, (x^2+x+1)^2\}$ and $\mathcal{I}_{c_{s(\sigma)}} = \{x^2+x+1\}$.

By calculations, $\mathcal{E}_{c_\tau} = \{x-1, x^2+x+1\}$, $\mathcal{I}_{c_\tau} = \emptyset$, $\mathcal{E}_{c_{s(\tau)}} = \{x-1\}$ and $\mathcal{I}_{c_{s(\tau)}} = \emptyset$. Hence $S_9(c_{s(\sigma)}, R)$ and $S_3(c_{s(\tau)}, R)$ are not singularly equivalent, while $S_9(c_\sigma, R)$ and $S_3(c_\tau, R)$ are singularly equivalent by Theorem 5.2(2). Thus Corollary 5.8 may be false without the requirement on $\nu_p(\lambda_i)$ and $\nu_p(\mu_j)$.

Note that $S_n(c_\sigma, R)$ and $S_m(c_\tau, R)$ are Morita equivalent if and only if they are derived equivalent (see [24, Corollary 1.3]). If σ and τ are p-singular permutations for a prime $p > 0$, then $S_n(c_\sigma, R)$ and

$S_m(c_\tau, R)$ are stably equivalent if and only if they are Morita equivalent (see [25, Corollary 4.20]). The next example demonstrates that singular equivalences are substantially different from Morita, derived and stable equivalences even for the centralizer algebras of permutation matrices.

Example 5.10. Let R be an algebraically closed field of characteristic $p \geq 11$, $n = 7p+15p^2+7p^3, m = 8p+15p^2+7p^3$. We take $\sigma \in \Sigma_n$ of the cycle type $(7p^3, 7p^2, 5p^2, 3p^2, 7p)$, and $\tau \in \Sigma_m$ of the cycle type $(7p^3, 7p^2, 5p^2, 3p^2, 5p, 3p)$. Then both $\sigma = s(\sigma)$ and $\tau = s(\tau)$ are p-singular. We will show that $S_n(c_\sigma, R)$ and $S_m(c_\tau, R)$ are singularly equivalent, but not Morita equivalent.

Let ζ_i be a primitive i-th root of unity over R for $i \in \{3,5,7\}$. By Lemma 5.4, we have $\mathcal{E}_{c_\sigma} = \{(x-1)^p, (x-1)^{p^2}, (x-1)^{p^3}, (x-\zeta_3)^{p^2}, (x-\zeta_3^2)^{p^2}, (x-\zeta_5)^{p^2}, \cdots, (x-\zeta_5^4)^{p^2}, (x-\zeta_7)^p, \cdots, (x-\zeta_7^6)^p, (x-\zeta_7)^{p^2}, \cdots, (x-\zeta_7^6)^{p^2}, (x-\zeta_7)^{p^3}, \cdots, (x-\zeta_7^6)^{p^3}\}$ and $\mathcal{E}_{c_\tau} = \{(x-1)^p, (x-1)^{p^2}, (x-1)^{p^3}, (x-\zeta_3)^p, (x-\zeta_3^2)^p, (x-\zeta_3)^{p^2}, (x-\zeta_3^2)^{p^2}, (x-\zeta_5)^p, \cdots, (x-\zeta_5^4)^p, (x-\zeta_5)^{p^2}, \cdots, (x-\zeta_5^4)^{p^2}, (x-\zeta_7)^{p^2}, \cdots, (x-\zeta_7^6)^{p^2}, (x-\zeta_7)^{p^3}, \cdots, (x-\zeta_7^6)^{p^3}\}$. It follows that

$$\mathcal{M}_{c_\sigma} = \{(x-1)^{p^3}, (x-\zeta_3)^{p^2}, (x-\zeta_3^2)^{p^2}, (x-\zeta_5)^{p^2}, \cdots, (x-\zeta_5^4)^{p^2}, (x-\zeta_7)^{p^3}, \cdots, (x-\zeta_7^6)^{p^3}\} = \mathcal{M}_{c_\tau},$$

$P_{c_\sigma}((x-1)^{p^3}) = \{p, p^2, p^3\} = P_{c_\sigma}((x-\zeta_7^k)^{p^3}), P_{c_\sigma}((x-\zeta_3^i)^{p^2}) = P_{c_\sigma}((x-\zeta_5^j)^{p^2}) = \{p^2\}$ for $i \in [2], j \in [4], k \in [6]$; $P_{c_\tau}((x-1)^{p^3}) = \{p, p^2, p^3\}, P_{c_\tau}((x-\zeta_3^i)^{p^2}) = P_{c_\tau}((x-\zeta_5^j)^{p^2}) = \{p, p^2\}, P_{c_\tau}((x-\zeta_7^k)^{p^3}) = \{p^2, p^3\}$ for $i \in [2], j \in [4], k \in [6]$.

Therefore, $\mathcal{I}_{c_\sigma} = \{x-1, x-\zeta_3, x-\zeta_3^2, x-\zeta_5, \cdots, x-\zeta_5^4, x-\zeta_7, \cdots, x-\zeta_7^6\} = \mathcal{I}_{c_\tau}$. Since R is an algebraically closed field, we know $r_{c_\sigma} = 1 = r_{c_\tau}$and $\mathcal{D}_{c_\sigma,1} = \mathcal{D}_{c_\tau,1}$. Thus c_σ and c_τ are Sg-equivalent by Definition 3.1(2), and therefore $S_n(c_\sigma, R)$ and $S_m(c_\tau, R)$ are singularly equivalent by Theorem 5.2(2). Since $P_{c_\sigma}((x-\zeta_3)^{p^2}) \neq P_{c_\tau}(f(x))$ holds for $f(x) \in \mathcal{M}_{c_\tau}$, c_σ and c_τ are not M-equivalent by Definition 3.2(1), and therefore $S_n(c_\sigma, R)$ and $S_m(c_\tau, R)$ are not Morita equivalent by [24, Theorem 1.1].

Finally, we point out a special case where Morita, derived, stable and singular equivalences are mutually implied by each other. We first prepare a couple of lemmas.

For the rest of this subsection, let $\sigma \in \Sigma_n$ be of the cycle type $(\lambda_1, \cdots, \lambda_s)$ and $\sigma^+ \in \Sigma_{n+\lambda_{s+1}}$ be of the cycle type $\lambda^+ := (\lambda_1, \cdots, \lambda_s, \lambda_{s+1})$. We wirte $\lambda_i = p^{\nu_p(\lambda_i)}\lambda_i'$ with $\nu_p(\lambda_i') = 0$ for all $i \in [s+1]$. Set

$$I := \{j \in [s] \mid \lambda_{s+1}' \text{ divides } \lambda_j'\} \text{ and } J := \{p^{\nu_p(\lambda_i)} \mid i \in I\}.$$

Lemma 5.11. *There exists an irreducible factor $f(x)$ of $x^{\lambda_{s+1}'} - 1$ such that $P_{c_\sigma}(f(x)) = J$ and $P_{c_{\sigma^+}}(f(x)) = J \cup \{p^{\nu_p(\lambda_{s+1})}\}$.*

Proof. If $I = [s]$, then $P_{c_\sigma}(x-1) = J$ and $P_{c_{\sigma^+}}(x-1) = J \cup \{p^{\nu_p(\lambda_{s+1})}\}$, according to Lemma 5.4. If $I \neq [s]$, there exists $i \in [s]$ such that $\lambda_{s+1}' \nmid \lambda_i'$, and therefore $|\Phi| \geq 2$, where we define $\Phi := \{\lambda_j' \mid j \in [s+1] \setminus I\}$. By Lemma 2.13(1), there is an irreducible factor $f(x)$ of $x^{\lambda_{s+1}'} - 1$ such that $f(x) \nmid x^{\lambda_j'} - 1$ for all $j \in [s] \setminus I$. For $j \in I$, we have $\lambda_{s+1}' \mid \lambda_j'$, it follows from Lemma 2.13(2) that $x^{\lambda_{s+1}'} - 1 \mid x^{\lambda_j'} - 1$ and $f(x) \mid x^{\lambda_j'} - 1$. Then we deduce from Lemma 5.4 that $P_{c_\sigma}(f(x)) = J$ and $P_{c_{\sigma^+}}(f(x)) = J \cup \{p^{\nu_p(\lambda_{s+1})}\}$. □

Recall that, for $c \in M_n(R)$, we define $\mathcal{U}_c := \bigcup_{g(x) \in \mathcal{M}_c} \mathcal{D}_{P_c(g(x))}$, where the union is taken in the sense of multisets. Clearly, $\mathcal{U}_c = \bigcup_{f(x) \in \mathcal{I}_c} \mathcal{D}_{P_c(f(x))}$.

Corollary 5.12. *Suppose that R is of characteristic $p \geq 0$, $I \neq \emptyset$ and one of the three conditions holds:*

(i) $p \neq 2$;

(ii) $p = 2$ and $\nu_p(\lambda_{s+1}) \neq 1$;

(iii) $p = 2$ and $\nu_p(\lambda_j) > 0$ for some $j \in I$.

Then the following are equivalent:

(1) $S_n(c_\sigma,R)$ *and* $S_{n+\lambda_{s+1}}(c_{\sigma^+},R)$ *are Morita equivalent.*
(2) $S_n(c_\sigma,R)$ *and* $S_{n+\lambda_{s+1}}(c_{\sigma^+},R)$ *are derived equivalent.*
(3) $S_n(c_\sigma,R)$ *and* $S_{n+\lambda_{s+1}}(c_{\sigma^+},R)$ *are stably equivalent.*
(4) $S_n(c_\sigma,R)$ *and* $S_{n+\lambda_{s+1}}(c_{\sigma^+},R)$ *are singularly equivalent.*
(5) *There exists* $k\in[s]$ *such that* $\nu_p(\lambda_k)=\nu_p(\lambda_{s+1})$ *and* $\lambda'_{s+1}\mid\lambda'_k$.

Proof. Clearly, (1) $\Rightarrow$ (2) $\Rightarrow$ (4) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4) by Theorem 5.2(2) and Remark 3.3.

(5) $\Rightarrow$ (1) Suppose that there exists $j\in[s]$ such that $\nu_p(\lambda_j)=\nu_p(\lambda_{s+1})$ and $\lambda'_{s+1}\mid\lambda'_j$. Then $\lambda_{s+1}\mid\lambda_j$. Thus $x^{\lambda_{s+1}}-1\mid x^{\lambda_j}-1$ by Lemma 2.13(2). It follows from Lemma 5.4 that $\mathcal{E}_{c_{\sigma^+}}=\mathcal{E}_{c_\sigma}$. Hence c_σ and c_{σ^+} are M-equivalent, and therefore $S_n(c_\sigma,R)$ and $S_{n+\lambda_{s+1}}(c_{\sigma^+},R)$ are Morita equivalent [24, Theorem 1.1].

Note that (5) always holds if $p=0$. In fact, in this case, $\nu_p(\lambda_j)=0$ for all $j\in[s+1]$. By assumption, $I\neq\emptyset$, that is, $\lambda'_{s+1}\mid\lambda'_t$ for some $t\in[s]$. Therefore (5) holds.

In the following **we assume** $p>0$ and prove that (4) implies (5).

By Theorem 5.2(2), we may suppose that c_σ and c_{σ^+} are Sg-equivalent. Then $\mathcal{U}_{c_\sigma}=\mathcal{U}_{c_{\sigma^+}}$ and $\#_n(\mathcal{U}_{c_\sigma})=\#_n(\mathcal{U}_{c_{\sigma^+}})$ for all $n\in\mathbb{N}$.

Contrarily, suppose that, for any $k\in[s]$, we have $\nu_p(\lambda_k)\neq\nu_p(\lambda_{s+1})$, or $\lambda'_{s+1}\nmid\lambda'_k$. In particular, for $i\in I\subseteq[s]$, we have $\lambda'_{s+1}\mid\lambda'_i$, and therefore $\nu_p(\lambda_i)\neq\nu_p(\lambda_{s+1})$. We write $J=\{p^{\nu_p(\lambda_i)}\mid i\in I\}=\{p^{m_1},p^{m_2},\cdots,p^{m_\ell}\}$ with $m_1>m_2>\cdots>m_\ell\geq 0$. Then $J\neq\emptyset$ by $I\neq\emptyset$, and $\nu_p(\lambda_{s+1})\neq m_i$ for all $i\in[\ell]$, that is, $p^{\nu_p(\lambda_{s+1})}\notin J$.

Note that $x^{\lambda_i}-1=(x^{\lambda'_i}-1)^{p^{\nu_p(\lambda_i)}}$ for all $i\in[s+1]$. By Lemma 5.4, we have

$$\mathcal{E}_{c_\sigma}=\{h(x)^{p^{\nu_p(\lambda_i)}}\mid i\in[s],\ h(x)\text{ is an irreducible factor of }x^{\lambda'_i}-1\}\ \text{ and}$$
$$\mathcal{E}_{c_{\sigma^+}}=\mathcal{E}_{c_\sigma}\cup\{h(x)^{p^{\nu_p(\lambda_{s+1})}}\mid h(x)\text{ is an irreducible factor of }x^{\lambda'_{s+1}}-1\}.$$

Particularly, both $P_{c_\sigma}(f(x))$ and $P_{c_{\sigma^+}}(g(x))$ consist of p-powers for $f(x)\in\mathcal{M}_{c_\sigma}$ and $g(x)\in\mathcal{M}_{c_{\sigma^+}}$.

For any $i\in[s+1]$ and any irreducible factor $f(x)$ of $x^{\lambda'_i}-1$, we have

$$(\heartsuit)\quad P_{c_{\sigma^+}}(f(x))=\begin{cases}P_{c_\sigma}(f(x)), & \text{if } f(x)\nmid x^{\lambda'_{s+1}}-1,\\ P_{c_\sigma}(f(x))\cup\{p^{\nu_p(\lambda_{s+1})}\}, & \text{if } f(x)\mid x^{\lambda'_{s+1}}-1.\end{cases}$$

By ($\heartsuit$), we know that the difference between $\mathcal{U}_{c_\sigma}$ and $\mathcal{U}_{c_{\sigma^+}}$ is determined solely by the sets of power indices of the irreducible factors of $x^{\lambda'_{s+1}}-1$. Now, we choose an irreducible factor $f(x)$ of $x^{\lambda'_{s+1}}-1$ as in Lemma 5.11, that is, $P_{c_\sigma}(f(x))=J$ and $P_{c_{\sigma^+}}(f(x))=J\cup\{p^{\nu_p(\lambda_{s+1})}\}$. We then compare $p^{\nu_p(\lambda_{s+1})}$ with p^{m_ℓ} defined by J and apply Lemma 2.15 to obtain a contradiction in each of cases (i) – (iii).

Suppose $m_\ell>\nu_p(\lambda_{s+1})$. By Lemma 2.15(2), we have $\#_{p^{m_\ell}}(\mathcal{D}_{P_{c_\sigma}(f(x))})>\#_{p^{m_\ell}}(\mathcal{D}_{P_{c_{\sigma^+}}(f(x))})$. Comparing $\#_{p^{m_\ell}}(\mathcal{U}_{c_\sigma})$ with $\#_{p^{m_\ell}}(\mathcal{U}_{c_{\sigma^+}})$, we know from ($\heartsuit$) and Lemma 2.15(2) that $\#_{p^{m_\ell}}(\mathcal{U}_{c_\sigma})>\#_{p^{m_\ell}}(\mathcal{U}_{c_{\sigma^+}})$. This contradicts $\mathcal{U}_{c_\sigma}=\mathcal{U}_{c_{\sigma^+}}$.

Thus $m_\ell\leq\nu_p(\lambda_{s+1})$. Due to $\nu_p(\lambda_{s+1})\neq m_i$ for $i\in[\ell]$, we have $m_\ell<\nu_p(\lambda_{s+1})$. Then either $m_1<\nu_p(\lambda_{s+1})$, or there is $i\in[\ell]$ with $i\geq 2$, such that $m_i<\nu_p(\lambda_{s+1})<m_{i-1}$. In both cases, we have $p^{\nu_p(\lambda_{s+1})}-p^{m_i}\geq 1$, where $i=\max\{i\in[\ell]\mid\nu_p(\lambda_{s+1})\geq m_i\}$.

Assume $p^{\nu_p(\lambda_{s+1})}-p^{m_i}>1$. Then $\#_{p^{\nu_p(\lambda_{s+1})}-p^{m_i}}(\mathcal{D}_{P_{c_{\sigma^+}}(f(x))})>\#_{p^{\nu_p(\lambda_{s+1})}-p^{m_i}}(\mathcal{D}_{P_{c_\sigma}(f(x))})$ by Lemma 2.15(1). Comparing $\#_{p^{\nu_p(\lambda_{s+1})}-p^{m_i}}(\mathcal{U}_{c_\sigma})$ with $\#_{p^{\nu_p(\lambda_{s+1})}-p^{m_i}}(\mathcal{U}_{c_{\sigma^+}})$, we know from ($\heartsuit$) and Lemma 2.15(1) that $\#_{p^{\nu_p(\lambda_{s+1})}-p^{m_i}}(\mathcal{U}_{c_{\sigma^+}})>\#_{p^{\nu_p(\lambda_{s+1})}-p^{m_i}}(\mathcal{U}_{c_\sigma})$. This again contradicts $\mathcal{U}_{c_\sigma}=\mathcal{U}_{c_{\sigma^+}}$.

Thus $p^{\nu_p(\lambda_{s+1})}-p^{m_i}=1$. By Lemma 2.12(3), we have $p=2,\nu_p(\lambda_{s+1})=1$ and $m_i=0$. This shows trivially that both (i) and (ii) lead to a contradiction. Thus we assume the case (iii), that is, $p=2$ and $\nu_p(\lambda_j)>0$ for some $j\in I$. Then $m_1>m_i=0$ and $i\geq 2$. Thanks to $1=\nu_p(\lambda_{s+1})\neq m_j$ for al-l $j\in[\ell]$ and $m_{i-1}>m_i=0$, we have $m_{i-1}\geq 2>\nu_p(\lambda_{s+1})=1>m_i=0$. By Lemma 2.15(3), we have $\#_{p^{m_{i-1}}-p^{m_i}}(\mathcal{D}_{P_{c_\sigma}(f(x))}) > \#_{p^{m_{i-1}}-p^{m_i}}(\mathcal{D}_{P_{c_{\sigma^+}}(f(x))})$. Comparing $\#_{p^{m_{i-1}}-p^{m_i}}(\mathcal{U}_{c_\sigma})$ with $\#_{p^{m_{i-1}}-p^{m_i}}(\mathcal{U}_{c_{\sigma^+}})$, we know from (♡) and Lemma 2.15(3) that $\#_{p^{m_{i-1}}-p^{m_i}}(\mathcal{U}_{c_\sigma}) > \#_{p^{m_{i-1}}-p^{m_i}}(\mathcal{U}_{c_{\sigma^+}})$. This again contradicts $\mathcal{U}_{c_\sigma} = \mathcal{U}_{c_{\sigma^+}}$, and therefore (5) follows. □

Remark that $I \neq \emptyset$ in Corollary 5.12 is fulfilled if one takes $\lambda_{s+1} = 1$. In the next lemma, we show when $I \neq \emptyset$ holds.

Lemma 5.13. *If $\mathcal{U}_{c_\sigma} = \mathcal{U}_{c_{\sigma^+}}$ and $\nu_p(\lambda_{s+1}) > 0$, then $I \neq \emptyset$.*

Proof. We suppose contrarily that $I = \emptyset$ holds. Then $J = \emptyset$. By Lemma 5.11, there is an irreducible factor $f(x)$ of $x^{\lambda'_{s+1}} - 1$ such that $P_{c_\sigma}(f(x)) = \emptyset$ and $P_{c_{\sigma^+}}(f(x)) = \{p^{\nu_p(\lambda_{s+1})}\}$. As $P_{c_\sigma}(f(x)) = \emptyset$, we have $f(x)^i \notin \mathcal{E}_{c_\sigma}$ for $i \in \mathbb{N}$. It follows from $\nu_p(\lambda_{s+1}) > 0$ that $p > 0$, $p^{\nu_p(\lambda_{s+1})} > 1$ and $p^{\nu_p(\lambda_{s+1})} \in \mathcal{D}_{P_{c_{\sigma^+}}(f(x))}$. Now, comparing $\#_{p^{\nu_p(\lambda_{s+1})}}(\mathcal{U}_{c_\sigma})$ with $\#_{p^{\nu_p(\lambda_{s+1})}}(\mathcal{U}_{c_{\sigma^+}})$, we have $\#_{p^{\nu_p(\lambda_{s+1})}}(\mathcal{U}_{c_{\sigma^+}}) > \#_{p^{\nu_p(\lambda_{s+1})}}(\mathcal{U}_{c_\sigma})$ by Lemma 2.15(1) and (♡). This contradicts the assumption $\mathcal{U}_{c_\sigma} = \mathcal{U}_{c_{\sigma^+}}$, and shows $I \neq \emptyset$. □

6 Homological conjectures and invariants of singular equivalences

In this subsection, we prove that the Cartan determinant and Auslander–Reiten conjectures hold true for centralizer matrix algebras. Our results in this section, together with known results, show that all homological conjectures hold true for centralizer matrix algebras. Also, a couple of homological invariants are given for singular equivalences of centralizer matrix algebras.

Let Λ be an Artin algebra. If $\mathrm{gldim}(\Lambda) < \infty$, then $\det(C_\Lambda) = \pm 1$ (see [13]). Furthermore, a well-known conjecture, called the *Cartan determinant conjecture* (see [38]), excludes the case: $\det(C_\Lambda) = -1$.

(CDC) If $\mathrm{gldim}(\Lambda) < \infty$, then $\det(C_\Lambda) = 1$.

This conjecture is known only in a few cases. For example, if Λ is a graded algebra or quasi-hereditary algebras or algebras of radical-cube-zero, then (CDC) holds (see [33, 7, 38, 16]).

We mention another two not yet solved homological conjectures.

Auslander-Reiten conjecture (ARC): If $M \in \Lambda$-mod is a self-orthogonal generator, then M is projective (see [1]).

Gorenstein projective conjecture (GPC): If $M \in \Lambda$-Gproj is self-orthogonal, then M is projective (see [26]).

In general, all of these conjectures are open up to date. We will, however, verify these conjectures for centralizer matrix algebras.

Lemma 6.1. *Let Λ be a Gorenstein Artin algebra.*
(1) *(ARC) holds for Λ if and only if so does (GPC) for Λ.*
(2) *If Λ is CM-finite, then (ARC) holds for Λ.*

Proof. (1) Clearly, if Λ satisfies (ARC), then Λ satisfies (GPC). Suppose Λ satisfies (GPC). Let M be a generator for Λ-mod with $\mathrm{Ext}^i_\Lambda(M,M) = 0$ for $i \geq 1$. Since Λ is a Gorenstein algebra, Λ-Gproj $= \{X \in \Lambda \mid \mathrm{Ext}^i_\Lambda(X,\Lambda) = 0$ for $i \geq 1\}$ by [14, Corollary 11.5.3, p.279]. Thus $M \in \Lambda$-Gproj. By (GPC) for Λ, we infer that M is projective.

(2) Since Λ is a CM-finite Gorenstein Artin algebra, every self-orthogonal module has finite projective dimension by [27, Corollary 1.3]. Note that Gorenstein projective module of finite projective dimension is projective. Hence (GPC) holds for Λ, and therefore (ARC) holds for Λ by (1). □

Theorem 6.2. *Let R be a field, $c \in M_n(R)$. Then*
(1) *$S_n(c,R)$ is a CM-finite 1-minimal Auslander-Gorenstein algebras.*
(2) *The Auslander-Reiten and Gorenstein projective conjectures hold for $S_n(c,R)$.*
(3) *The Cartan determinant conjecture holds for $S_n(c,R)$.*

Proof. (1) By Proposition 4.8, we see that $S_n(c,R)$ is a CM-finite 1-minimal Auslander-Gorenstein algebra for $c \in M_n(R)$ (see also [36, Theorem 1.1(2)] for $S_n(c,R)$ to be 1-minimal Auslander-Gorenstein).

(2) This follows from (1) and Lemma 6.1.

(3) Since we have the algebra isomorphism $S_n(c,R) \simeq \prod_{i=1}^{l_c} \mathrm{End}_{U_i}(M_i)$, where $U_i := R[x]/(f_i(x)^{n_i})$ and $\mathcal{B}(M_i) \simeq \bigoplus_{r\in P_c(f_i(x))} R[x]/(f_i(x)^r)$ as U_i-modules, it suffices to prove that the Cartan determinant conjecture holds for $\mathrm{End}_{U_i}(M_i)$ for $i \in [l_c]$. By Lemma 2.7 and Corollary 4.9, we know $\mathrm{gldim}(\mathrm{End}_{U_i}(M_i)) < \infty$ if and only if M_i is an additive generator for U_i-mod if and only if the Cartan determinant of $\mathrm{End}_{U_i}(M_i)$ is equal to 1 . Thus the Cartan determinant conjecture holds for $S_n(c,R)$. $\square$

Remark 6.3. It was proved that the finitistic dimension conjecture [3] is true for centralizer matrix algebras over fields (see [24, 36]). This implies that the strong Nakayama [9], generalized Nakayama [1], Nakayama [28], Wakamatsu tilting [4, Section IV.3, p.71] and tilting (or projective) complement conjectures [20] hold true for centralizer matrix algebras over fields. Clearly, Gorenstein symmetry conjecture hods for centralizer matrix algebras. Moreover, Tachikawa's first and second conjectures (TC1) and (TC2) (see [31, p.115-116]) hold true for centralizer matrix algebras. In fact, we consider the generator and cogenerator $M := A \oplus D(A)$ for a *CM*-finite Gorenstein algebra A. Then it follows from the condition of (TC1) and Lemma 6.1(2) that M is projective, and therefore $D(A)$ is projective and A is self-injective. Note that (TC2) is a consequence (ARC). Thus all homological conjectures in [2, Cojcetures, p. 409] hold true for centralizer matrix algebras over fields.

At the end of this subsection, we explain another homological property of centralizer matrix algebras, namely the quasi-heredity property. First, we recall the definition of quasi-hereditary algebras introduced in [8].

Definition 6.4. [8] *Let A be a finite-dimensional algebra over R. An ideal J in A is called a* heredity ideal *if J is idempotent, $J\mathrm{rad}(A)J = 0$ and J is a projective left (or right) A-module. The algebra A is said to be* quasi-hereditary *provided there is a finite chain $0 = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n = A$ of ideals in A such that J_j/J_{j-1} is a heredity ideal in A/J_{j-1} for all j.*

Quasi-hereditary algebras were motivated by describing the highest weight category of semisimple Lie algebras. For more information on quasi-hereditary algebras, we refer to [34] and the references therein. The following lemma not only generalizes a result in [35, Theorem 1.1(2)], but also reprove Theorem 6.2(3).

Lemma 6.5. *For $c \in M_n(R)$, let $S := S_n(c,R)$ and C be the Cartan matrix of S. Then*

(1) $\det(C) = \begin{cases} \prod_{f(x)\in\mathcal{I}_c} \prod_{i\in\mathcal{D}_{P_c(f(x))}} i, & \text{if } \mathcal{I}_c \neq \emptyset, \\ 1, & \text{otherwise.} \end{cases}$

(2) *The following are equivalent:*

(i) *S is quasi-hereditary.*

(ii) $\mathrm{gldim}(S) < \infty$.

(iii) $\mathrm{gldim}(S) \leq 2$.

(iv) $\mathcal{I}_c = \emptyset$.

(v) $|P_c(f(x))| = \max\{j \in P_c(f(x))\}$ *for all* $f(x) \in \mathcal{M}_c$.

(vi) $\det(C) = 1$.

Proof. Following the notation in Subsection 4.3, we have $S \simeq \prod_{i=1}^{l_c} \mathrm{End}_{U_i}(M_i)$ as R-algebras, where U_i denotes the algebra $R[x]/(f_i(x)^{n_i})$, $\mathcal{B}(M_i) \simeq \bigoplus_{r\in P_c(f_i(x))} R[x]/(f_i(x)^r)$ is the basic U_i-module of M_i, and l_c is the number of maximal elementary divisors of c. Thus $\det(C_{\mathrm{End}_{U_i}(M_i)}) = \det(C_{\mathrm{End}_{U_i}(\mathcal{B}(M_i))})$ and $\mathrm{gldim}(\mathrm{End}_{U_i}(M_i)) = \mathrm{gldim}(\mathrm{End}_{U_i}(\mathcal{B}(M_i)))$.

(1) By Lemma 2.7, $\det(C_{\mathrm{End}_{U_i}(\mathcal{B}(M_i))}) = \prod_{j\in\mathcal{H}_{P_c(f_i(x)^{n_i})}} j$ for $i\in[l_c]$. It follows from the definitions of $\mathcal{D}_{P_c(f_i(x)^{n_i})}$ and $\mathcal{I}_c$ that

$$\det(C_{\mathrm{End}_{U_i}(M_i)}) = \det(C_{\mathrm{End}_{U_i}(\mathcal{B}(M_i))}) = \begin{cases}\prod_{i\in\mathcal{D}_{P_c(f(x))}} i, & \text{if } f_i(x)\in\mathcal{I}_c,\\ 1, & \text{otherwise.}\end{cases}$$

Thus (1) follows.

(2) By Corollary 4.9 and the definition of $\mathcal{I}_c$, $\mathrm{gldim}(S)<\infty$ if and only if $\mathrm{gldim}(\mathrm{End}_{U_i}(\mathcal{B}(M_i)))<\infty$ for all $i\in[l_c]$ if and only if $\mathrm{gldim}(\mathrm{End}_{U_i}(\mathcal{B}(M_i)))\leq 2$ for all $i\in[l_c]$ if and only if M_i is an additive generator for U_i-mod for all $i\in[l_c]$ if and only if $|P_c(f(x))| = \max\{j\in P_c(f(x))\}$ for all $f(x)\in\mathcal{M}_c$ if and only if $\mathcal{I}_c=\emptyset$. Thus (ii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (iii). Lemma 2.7 shows that $|P_c(f(x))| = \max\{j\in P_c(f(x))\}$ for all $f(x)\in\mathcal{M}_c$ if and only if M_i is an additive generator for U_i-mod for all $i\in[l_c]$ if and only if $\det(C_{\mathrm{End}_{U_i}(\mathcal{B}(M_i))})=1$ for all $i\in[l_c]$. Thus (v) $\Leftrightarrow$ (vi). It follows from [**?**, Theorem 4.3] that (i) $\Rightarrow$ (ii), and from [11, Theorem 2] that (iii) $\Rightarrow$ (i). $\square$

As a consequence of Lemma 6.5, we have the following invariants of Sg-equivalences.

Corollary 6.6. *Let R be a field. If $c\in M_n(R)$ and $d\in M_m(R)$ are Sg-equivalent, then*

(1) *$S_n(c,R)$ is quasi-hereditary if and only if so is $S_m(d,R)$.*

(2) *The Cartan determinants of $S_n(c,R)$ and $S_m(d,R)$ are equal.*

Proof. Since the Sg-equivalence of c and d implies that $\mathcal{I}_c=\emptyset$ if and only $\mathcal{I}_d=\emptyset$, (1) is clear by Lemma 6.5.

We prove (2). Suppose that c and d are Sg-equivalent. Then $\mathcal{U}_c=\mathcal{U}_d$. Note that $\mathcal{I}_c=\emptyset$ if and only if $r_c=0$ if and only if $r_d=0$ if and only if $\mathcal{I}_d=\emptyset$. Suppose $\mathcal{I}_c=\emptyset$. Then $\mathcal{I}_d=\emptyset$ and it follows from Lemma 6.5(1) that $\det(C_{S_n(c,R)})=1=\det(C_{S_m(d,R)})$.

Suppose $\mathcal{I}_c\neq\emptyset$. Then $\mathcal{I}_d\neq\emptyset$. It follows from Lemma 6.5(1) and $\mathcal{U}_c=\mathcal{U}_d$ that

$$\det(C_{S_n(c,R)}) = \prod_{f(x)\in\mathcal{I}_c}\prod_{i\in\mathcal{D}_{P_c(f(x))}} i = \prod_{i\in\mathcal{U}_c} i = \prod_{i\in\mathcal{U}_d} i = \det(C_{S_m(d,R)}). \quad \square$$

Finally, we propose the following question.

Question 6.7. How to classify all (basic) tilting (respectively, silting) modules over $S_n(c,R)$ for R a field?

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