Derived equivalences for mirror-reflective algebras

Hongxing Chen and **Changchang Xi***

Abstract

We show that the construction of mirror-reflective algebras inherits derived equivalences of gendo-symmetric algebras. More precisely, suppose A and B are gendo-symmetric algebras with both Ae and Bf faithful projective-injective left ideals generated by idempotents e in A and f in B, respectively. If A and B are derived equivalent, then the mirror-reflective algebras of (A,e) and (B,f) are derived equivalent.

1 Introduction

Given an (associative) algebra A over a commutative ring k, an idempotent e of A and an element λ in the center of $\Lambda := eAe$, we introduced the mirror-reflective algebra $R(A,e,\lambda)$ of A at level (e,λ) in [4]. Roughly speaking, this algebra has the underlying k-module structure $A \oplus Ae \otimes_\Lambda eA$ such that $Ae \otimes_\Lambda eA$ is an ideal in $R(A,e,\lambda)$. The specialization of $R(A,e,\lambda)$ at $\lambda = e$ is called the mirror-reflective algebra of A at e, denoted by $R(A,e)$. In case that A is a finite-dimensional gendo-symmetric algebra over a field k and e is an idempotent of A such that Ae is a faithful and projective-injective A-module, the algebra $R(A,e)$ is called simply the mirror-reflective algebra of A. Such a construction can be iterated and thus supplies a series of both higher Auslander algebras and recollements of derived module categories. It turns out that a new characterisation of Tachikawa's second conjecture for symmetric algebras can be formulated in terms of stratifying ideals and recollements of derived categories (see [4]).

Our purpose of this note is to show that the construction of mirror-reflective algebras preserves derived equivalences. More precisely, we have the following.

Theorem 1.1. *Suppose that A and B are finite-dimensional gendo-symmetric algebras over a field k and that ${}_AAe$ and ${}_BBf$ are faithful projective-injective modules generated by idempotents $e \in A$ and $f \in B$, respectively. If A and B are derived equivalent, then there is an isomorphism $\sigma : Z(eAe) \to Z(fBf)$ of algebras from the center of eAe to the one of fBf such that, for any $\lambda \in Z(eAe)$, the mirror-reflective algebras $R(A,e,\lambda)$ and $R(B,f,(\lambda)\sigma)$ are derived equivalent.*

During the course of the proof of Theorem 1.1, we will give a general construction of derived equivalences of mirror-reflective algebras of arbitrary algebras at any levels in Theorem 3.1. So Theorem 1.1 is just its consequence.

This note is sketched as follows. In Section 2 we provide preliminaries for the proof of the main result. This includes recalling basic definitions and proving facts on derived equivalences and on mirror-reflective algebras . In Section 3 we prove Theorem 3.1.

Acknowledgements. The research work was supported partially by the National Natural Science Foundation of China (Grants 12031014 and 12122112).

* Corresponding author. Email: xicc@cnu.edu.cn; Fax: 0086 10 68903637.

2010 Mathematics Subject Classification: Primary 16E35, 18E30, 16P10; Secondary 16S10, 16S50

Keywords: Derived equivalence; Gendo-symmetric algebra; Mirror-reflective algebra; Recollement; Symmetric algebra; Tilting complex.

2 Preliminaries

Let k denote a commutative ring with identity. All algebras in the paper are associative k-algebras with identity. For an algebra A, we denote by A-Mod the category of all left A-modules. Let A-mod and A-proj be the full subcategories of A-Mod consisting of finitely generated A-modules and finitely generated projective A-modules, respectively.

Given an additive category $\mathcal{A}$, $\mathscr{C}(\mathcal{A})$ stands for the category of all complexes $X^\bullet = (X^i, d_X^i)$ over $\mathcal{A}$ with cochain maps as morphisms, and $\mathscr{K}(\mathcal{A})$ for the homotopy category of $\mathscr{C}(\mathcal{A})$. We write $\mathscr{C}^b(\mathcal{A})$ and $\mathscr{K}^b(\mathcal{A})$ for the full subcategories of $\mathscr{C}(\mathcal{A})$ and $\mathscr{K}(\mathcal{A})$ consisting of bounded complexes over $\mathcal{A}$, respectively. When $\mathcal{A}$ is abelian, the *(unbounded) derived category* of $\mathcal{A}$ is denoted by $\mathscr{D}(\mathcal{A})$, which is the localization of $\mathscr{K}(\mathcal{A})$ at all quasi-isomorphisms.

For an algebra A, we simply write $\mathscr{K}(A)$ for $\mathscr{K}(A\text{-Mod})$ and $\mathscr{D}(A)$ for $\mathscr{D}(A\text{-Mod})$. Also, A-Mod is often identified with the full subcategory of $\mathscr{D}(A)$ consisting of all stalk complexes concentrated in degree 0. For an idempotent element e in A, the category $\mathscr{K}^b(\mathrm{add}(Ae))$ is identified with its images in $\mathscr{D}(A)$ under the localization functor $\mathscr{K}(A) \to \mathscr{D}(A)$.

The composition of two maps $f: X \to Y$ and $g: Y \to Z$ of sets is written as fg. Thus, for a map $f: X \to Y$, we write $(x)f$ for the image of $x \in X$ under f.

2.1 Derived equivalences of algebras with idempotents

In this subsection, all k-algebras over a commutative ring k are assumed to be projective as k-modules. Let $A^{\mathrm{e}} := A \otimes_k A^{\mathrm{op}}$ be the enveloping algebra of an algebra A, and D be the functor $\mathrm{Hom}_k(-,k)$.

We first recall the definitions of tilting complexes and derived equivalences in [18, 20].

Definition 2.1. *Let A and B be algebras.*

(1) *A complex $P \in \mathscr{K}^b(A\text{-proj})$ is called a* tilting complex *if*

(i) *P is self-orthogonal, that is,* $\mathrm{Hom}_{\mathscr{K}^b(A\text{-proj})}(P, P[n]) = 0$ *for any $n \neq 0$,*

(ii) $\mathrm{add}(P)$ *generates $\mathscr{K}^b(A\text{-proj})$ as a triangulated category, that is, $\mathscr{K}^b(A\text{-proj})$ is the smallest full triangulated subcateory of $\mathscr{K}^b(A\text{-proj})$ containing* $\mathrm{add}(P)$ *and being closed under isomorphisms.*

(2) *A complex $T \in \mathscr{D}(A \otimes_k B^{\mathrm{op}})$ is called a* two-sided tilting complex *if there is a complex $T^\vee \in \mathscr{D}(B \otimes_k A^{\mathrm{op}})$ such that $T \otimes_B^{\mathbb{L}} T^\vee \simeq A$ in $\mathscr{D}(A^{\mathrm{e}})$ and $T^\vee \otimes_A^{\mathbb{L}} T \simeq B$ in $\mathscr{D}(B^{\mathrm{e}})$. The complex $T^\vee$ is called the* inverse *of T.*

(3) *Two algebras A and B are said to be* derived equivalent *if $\mathscr{D}(A)$ and $\mathscr{D}(B)$ are equivalent as triangulated categories, or equivalently, $\mathscr{K}^b(A\text{-proj})$ and $\mathscr{K}^b(B\text{-proj})$ are equivalent as triangulated categories.*

Let T be a two-sided tilting complex in $\mathscr{D}(A \otimes_k B^{\mathrm{op}})$ with the inverse $T^\vee$. By [20, Section 3], we have $T^\vee \simeq \mathbb{R}\mathrm{Hom}_A(T, A) \simeq \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(T, B)$ in $\mathscr{D}(B \otimes_k A^{\mathrm{op}})$. Moreover, the functor $T^\vee \otimes_A^{\mathbb{L}} - : \mathscr{D}(A) \to \mathscr{D}(B)$ is a triangle equivalence with the quasi-inverse $T \otimes_B^{\mathbb{L}} - : \mathscr{D}(B) \to \mathscr{D}(A)$. This implies that ${}_AT$ and T_B are isomorphic to tilting complexes in $\mathscr{D}(A)$ and $\mathscr{D}(B^{\mathrm{op}})$, respectively. By [20, Lemma 4.3], $T^\vee \otimes_k T \in \mathscr{D}(A^{\mathrm{e}} \otimes_k (B^{\mathrm{e}})^{\mathrm{op}})$ is a two-sided tilting complex.

The following theorem is well known (see [8, 13, 18, 20]).

Theorem 2.2. *Let A and B be k-algebras. The following are equivalent.*

(1) *A and B are derived equivalent.*

(2) *There is a tilting complex $P \in \mathscr{K}^b(A\text{-proj})$ such that $B \simeq \mathrm{End}_{\mathscr{D}(A)}(P)$ as algebras.*

(3) *There is a two-sided tilting complex $T \in \mathscr{D}(A \otimes_k B^{\mathrm{op}})$.*

Comparing with recollement-tilting complexes related to idempotents in [15, Definition 3.6], we introduce the definition of derived equivalences of algebras with idempotents.

Definition 2.3. *Let A and B be algebras with idempotent elements $e = e^2 \in A$ and $f = f^2 \in B$. The pairs (A, e) and (B, f) of algebras with idempotents are said to be* derived equivalent *provided that there is a triangle equivalence $\mathscr{D}(A) \to \mathscr{D}(B)$ which restricts to an equivalence $\mathscr{K}^b(\mathrm{add}(Ae)) \to \mathscr{K}^b(\mathrm{add}(Bf))$.*

Clearly, A and B are derived equivalent if and only if so are the pairs $(A, 0)$ and $(B, 0)$ if and only if so are the pairs $(A, 1_A)$ and $(B, 1_B)$. The following result is essentially implied in [15] and provides several equivalent characterizations of derived equivalences of algebras with idempotents. For the convenience of the reader, we provide a proof.

Lemma 2.4. ([15]) *Let A and B be algebras with $e^2 = e \in A$ and $f^2 = f \in B$. The following are equivalent.*

(1) *The pairs (A, e) and (B, f) are derived equivalent.*

(2) *There is a tilting complex $P \in \mathscr{K}^b(A\text{-proj})$ such that $P = P_1 \oplus P_2$ in $\mathscr{K}^b(A\text{-proj})$ satisfying*

(a) *$B \simeq \mathrm{End}_{\mathscr{D}(A)}(P)$ as algebras.*

(b) *P_1 generates $\mathscr{K}^b(\mathrm{add}(Ae))$ as a triangulated category.*

(c) *Under the isomorphism of (a), $f \in B$ corresponds to the composite of the canonical projection $P \to P_1$ with the canonical inclusion $P_1 \to P$.*

(3) *There is a two-sided tilting complex $T \in \mathscr{D}(A \otimes_k B^{\mathrm{op}})$ with the inverse $T^\vee \in \mathscr{D}(B \otimes_k A^{\mathrm{op}})$ such that $eTf \in \mathscr{D}(eAe \otimes_k (fBf)^{\mathrm{op}})$ is a two-sided tilting complex with the inverse $fT^\vee e \in \mathscr{D}(fBf \otimes_k (eAe)^{\mathrm{op}})$ and that all 3 squares in the following diagram are commutative (up to natural isomorphism):*

$$\begin{array}{ccc} \mathscr{D}(A) & \underset{j_{e*}}{\overset{j_{e!}}{\longleftarrow}} \xrightarrow{\;e\cdot\;} & \mathscr{D}(eAe) \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_2} \\ \mathscr{D}(B) & \underset{j_{f*}}{\overset{j_{f!}}{\longleftarrow}} \xrightarrow{\;f\cdot\;} & \mathscr{D}(fBf) \end{array}$$

where $F_1 := T^\vee \otimes^{\mathbb{L}}_A -$, $F_2 := fT^\vee e \otimes^{\mathbb{L}}_{eAe} -$, $j_{e!} := Ae \otimes^{\mathbb{L}}_{eAe} -$, $j_{e*} := \mathbb{R}\mathrm{Hom}_{eAe}(eA, -)$, $j_{f!} := Be \otimes^{\mathbb{L}}_{fBf} -$, $j_{f*} := \mathbb{R}\mathrm{Hom}_{fBf}(fB, -)$, and the functors $e\cdot$ and $f\cdot$ denote the left multiplications by e and f, respectively.

(4) *There is a two-sided tilting complex $T \in \mathscr{D}(A \otimes_k B^{\mathrm{op}})$ with the inverse $T^\vee \in \mathscr{D}(B \otimes_k A^{\mathrm{op}})$ such that*

$$T^\vee \otimes^{\mathbb{L}}_A (Ae \otimes^{\mathbb{L}}_{eAe} eA) \otimes^{\mathbb{L}}_A T \simeq Bf \otimes^{\mathbb{L}}_{fBf} fB \in \mathscr{D}(B^{\mathrm{e}}).$$

Proof. (1) $\Rightarrow$ (2). Assume (1) holds. Then there is a triangle equivalence $F_1 : \mathscr{D}(A) \to \mathscr{D}(B)$ which restricts to an equivalence $\mathscr{K}^b(\mathrm{add}(Ae)) \to \mathscr{K}^b(\mathrm{add}(Bf))$. Let $G_1 : \mathscr{D}(B) \to D(A)$ be the inverse of F_1. Define $P := G_1(B)$, $P_1 := G(Bf)$ and $P_2 := G(B(1-f))$. Then $P = P_1 \oplus P_2$ and $P_1 \in \mathscr{K}^b(\mathrm{add}(Ae))$. Since Bf generates $\mathscr{K}^b(\mathrm{add}(Bf))$ as a triangulated category, all conditions (a), (b) and (c) hold.

(2) $\Rightarrow$ (3). Let $\Lambda := eAe$. Recall that the adjoint pair $(Ae \otimes_\Lambda -, e\cdot)$ between Λ-Mod and A-Mod induces a triangle equivalence $\mathscr{K}^b(\mathrm{add}(Ae)) \xrightarrow{\simeq} \mathscr{K}^b(\Lambda\text{-proj})$. Since P_1 is a direct summand of P and generates $\mathscr{K}^b(\mathrm{add}(Ae))$ as a triangulated category, the complex $eP_1 \in \mathscr{K}^b(\Lambda\text{-proj})$ is a tilting complex. Let T be a two-sided tilting complex in $\mathscr{D}(A \otimes_k B^{\mathrm{op}})$ which is induced by ${}_AP$. Then the argument in the proof of [15, Theorem 3.5] shows that (2) implies (3).

(3) $\Rightarrow$ (1). Let $\Gamma := fBf$. Note that the image of the restriction of $j_{e!}$ to $\mathscr{K}^b(\Lambda\text{-proj})$ coincides with the image of $\mathscr{K}^b(\mathrm{add}(Ae))$ in $\mathscr{D}(A)$. Similarly, the image in $\mathscr{D}(B)$ of the restriction of $j_{f!}$ to $\mathscr{K}^b(\Gamma\text{-proj})$ coincides with the image of $\mathscr{K}^b(\mathrm{add}(Bf))$ in $\mathscr{D}(B)$. Thus the equivalence F_1 in (3) restricts to an equivalence from $\mathscr{K}^b(\mathrm{add}(Ae))$ to $\mathscr{K}^b(\mathrm{add}(Bf))$. Thus (1) holds.

(3) $\Rightarrow$ (4). By [15, Corollaries 3.7 and 3.8], there are isomorphisms in $\mathscr{D}(A \otimes_k B^{\mathrm{op}})$:

$$T \otimes^{\mathbb{L}}_B Bf \otimes^{\mathbb{L}}_\Gamma fB \simeq Tf \otimes^{\mathbb{L}}_\Gamma fB \simeq Ae \otimes^{\mathbb{L}}_\Lambda eT \simeq Ae \otimes^{\mathbb{L}}_\Lambda eA \otimes^{\mathbb{L}}_A T.$$

Applying $T^{\vee}\otimes^{\mathbb{L}}_A - : \mathscr{D}(A\otimes_k B^{\rm op})\to \mathscr{D}(B^{\rm e})$ to these isomorphisms yields

$$Bf\otimes^{\mathbb{L}}_{\Gamma} fB \simeq T^{\vee}\otimes^{\mathbb{L}}_A T\otimes^{\mathbb{L}}_B Bf\otimes^{\mathbb{L}}_{\Gamma} fB\simeq T^{\vee}\otimes^{\mathbb{L}}_A Ae\otimes^{\mathbb{L}}_{\Lambda} eA\otimes^{\mathbb{L}}_A T.$$

(4) $\Rightarrow$ (2). Since ${}_AT_B$ is a two-sided tilting complex, it follows from (4) that there are isomorphisms of complexes

$$\begin{array}{ll}
(i) & Ae\otimes^{\mathbb{L}}_{\Lambda} eA\simeq T\otimes^{\mathbb{L}}_B Bf\otimes^{\mathbb{L}}_{\Gamma} fB\otimes^{\mathbb{L}}_B T^{\vee}\in\mathscr{D}(A^{\rm e}),\\
(ii) & Ae\otimes^{\mathbb{L}}_{\Lambda} eTf\simeq T\otimes^{\mathbb{L}}_B Bf\otimes^{\mathbb{L}}_{\Gamma} fB\otimes^{\mathbb{L}}_B T^{\vee}\otimes^{\mathbb{L}}_A T\otimes^{\mathbb{L}}_B Bf\simeq Tf\in\mathscr{D}(A\otimes\Gamma^{\rm op}),\\
(iii) & fT^{\vee}e\otimes^{\mathbb{L}}_{\Lambda} eTf\simeq fT^{\vee}\otimes^{\mathbb{L}}_A Ae\otimes^{\mathbb{L}}_{\Lambda} eA\otimes^{\mathbb{L}}_A Tf\simeq fBf\otimes^{\mathbb{L}}_{\Gamma} fBf\simeq\Gamma\in\mathscr{D}(\Gamma^{\rm e}),\\
(iv) & eTf\otimes^{\mathbb{L}}_{\Gamma} fT^{\vee}e\simeq eT\otimes^{\mathbb{L}}_B Bf\otimes^{\mathbb{L}}_{\Gamma} fB\otimes^{\mathbb{L}}_B T^{\vee}e\simeq eAe\otimes^{\mathbb{L}}_{\Lambda} eAe\simeq\Lambda\in\mathscr{D}(\Lambda^{\rm e}).
\end{array}$$

Due to (iii) and (iv), ${}_{\Lambda}(eTf)_{\Gamma}$ is a two-sided tilting complex with the inverse $fT^{\vee}e$. In particular, ${}_{\Lambda}eTf$ is isomorphic to a tilting complex. Since $j_{e!}$ induces a triangle equivalence $\mathscr{K}^b(\Lambda\text{-proj})\xrightarrow{\simeq}\mathscr{K}^b(\mathrm{add}(Ae))$, the isomorphisms in (ii) imply that Tf generates $\mathscr{K}^b(\mathrm{add}(Ae))$ as a triangulated category. Clearly, ${}_AT$ is isomorphic to a tilting complex and has a direct summand Tf. Moreover, $\mathrm{End}_{\mathscr{D}(A)}(T)\simeq B$ as algebras and $\mathrm{Hom}_{\mathscr{D}(A)}(T,Tf)\simeq Bf$ as B-modules. Thus (2) holds. $\square$

Corollary 2.5. *Assume that the pairs (A,e) and (B,f) are derived equivalent. Then*

(1) $(A^{\rm op},e^{\rm op})$ *and* $(B^{\rm op},f^{\rm op})$ *are derived equivalent.*

(2) $(A^{\rm e},e\otimes e^{\rm op})$ *and* $(B^{\rm e},f\otimes f^{\rm op})$ *are derived equivalent.*

Proof. Let $(-)^*:=\mathrm{Hom}_A(-,A)$ and P be the tilting complex in Lemma 2.4(2). Then $P^*\in\mathscr{K}^b(A^{\rm op}\text{-proj})$ and $P^*=P_1^*\oplus P_2^*$. By [18, Proposition 9.1], P^* is a tilting complex over $A^{\rm op}$.

(1) Since $(-)^*:\mathscr{K}^b(A\text{-proj})\to\mathscr{K}^b(A^{\rm op}\text{-proj})$ is a triangle equivalence sending Ae to eA, it follows from Lemma 2.4(c) that P_1^* generates $\mathscr{K}^b(\mathrm{add}(eA))$ as a triangulated category. By Lemma 2.4(a) and (c), there is an algebra isomorphism $B^{\rm op}\simeq\mathrm{End}_{\mathscr{D}(A^{\rm op})}(P^*)$ under which $f^{\rm op}$ is the composition of the projection $P^*\to P_1^*$ with the inclusion $P_1^*\to P^*$. Thus $(P^*,e^{\rm op})$ satisfies Lemma 2.4(2). This shows (1).

(2) Let $Q:=P\otimes_k P^*\in\mathscr{K}^b(A^{\rm e}\text{-proj})$. We will show that Q satisfies Lemma 2.4(2) for the pair $(A^{\rm e},e\otimes e^{\rm op})$ and $(B^{\rm e},f\otimes f^{\rm op})$.

In fact, by [20, Theorem 2.1], Q is a tilting complex over $A^{\rm e}$ and $\mathrm{End}_{\mathscr{D}(A^{\rm e})}(Q)\simeq B^{\rm e}$. Clearly, $P_1\otimes_k P_1^*$ is a direct summand of $P\otimes_k P^*$ and there are canonical isomorphisms

$$\mathrm{Hom}_{\mathscr{D}(A^{\rm e})}(Q,P_1\otimes_k P_1^*)\simeq\mathrm{Hom}_{\mathscr{D}(A)}(P,P_1)\otimes_k\mathrm{Hom}_{\mathscr{D}(A^{\rm op})}(P^*,P_1^*)\simeq Bf\otimes_k fB=B^{\rm e}(f\otimes f^{\rm op}).$$

Thus Q satisfies Lemma 2.4(a)-(b). To show Lemma 2.4(c) for Q, we need the following general result:

If $L:\mathcal{C}\to\mathcal{D}$ is a triangle functor between triangulated categories $\mathcal{C}$ and $\mathcal{D}$, then $L(\mathrm{tria}_{\mathcal{C}}(\mathrm{add}(X)))\subseteq\mathrm{tria}_{\mathcal{D}}(\mathrm{add}(L(X)))$ for any $X\in\mathcal{C}$, where $\mathrm{tria}_{\mathcal{C}}(\mathrm{add}(X))$ denotes the smallest full triangulated subcategory of $\mathcal{C}$ containing $\mathrm{add}(X)$.

Since $eA\in\mathscr{K}^b(\mathrm{add}(eA))=\mathrm{tria}_{\mathscr{K}(A^{\rm op})}(\mathrm{add}(P_1^*))$, we apply the functor $Ae\otimes_k-:\mathscr{K}^b(\mathrm{add}(eA))\to\mathscr{K}^b(\mathrm{add}(Ae\otimes_k eA))$ to the k-module eA and obtain $Ae\otimes_k eA\in\mathrm{tria}_{\mathscr{K}(A^{\rm e})}(\mathrm{add}(Ae\otimes_k P_1^*))$. Similarly, we have $Ae\otimes P_1^*\in\mathrm{tria}_{\mathscr{K}(A^{\rm e})}(\mathrm{add}(P_1\otimes_k P_1^*))$ by the functor $-\otimes_k P_1^*:\mathscr{K}^b(\mathrm{add}(Ae))\to\mathscr{K}^b(\mathrm{add}(Ae\otimes_k eA))$. Thus $Ae\otimes_k eA\in\mathrm{tria}(\mathrm{add}(P_1\otimes_k P_1^*))$. By the equivalences of Lemma 2.4(1)-(2), the pairs $(A^{\rm e},e\otimes e^{\rm op})$ and $(B^{\rm e},f\otimes f^{\rm op})$ are derived equivalent. $\square$

A finite-dimensional algebra A over a field k is called a *gendo-symmetric* algebra if $A=\mathrm{End}_{\Lambda}(\Lambda\oplus M)$ with Λ a symmetric algebra and M a finite-dimensional Λ-module. By [5, Theorem 3.2], A is gendo-symmetric if and only if the dominant dimension of A is at least 2 and $D(Ae)\simeq eA$ as eAe-A-bimodules, where $e\in A$ is an idempotent element such that Ae is a faithful projective-injective A-module.

Proposition 2.6. [7, Proposition 3.9] *Suppose that A and B are gendo-symmetric algebras with Ae and Bf faithful projective-injective modules over A and B, respectively. If A and B are derived equivalent, then the pairs (A,e) and (B,f) are derived equivalent of algebras with idempotents.*

2.2 Mirror-reflective algebras

In this section, we recall the construction of mirror-reflective algebras in [4]. Assume that A is a k-algebra over a commutative ring k, $e=e^2\in A$, $\Lambda:=eAe$ and λ lies in the center $Z(\Lambda)$ of Λ. Recall that the mirror-reflective algebra $R(A,e,\lambda)$ of A at level (e,λ), defined in [4], has the underlying k-module $A\oplus Ae\otimes_\Lambda eA$ as its abelian group. Its multiplication $*$ is given explicitly by

$$(a+be\otimes ec)*(a'+b'e\otimes ec'):=aa'+(ab'e\otimes ec'+be\otimes ecd'+becb'e\otimes\lambda ec')$$

for $a,b,c,a',b',c'\in A$. This can be reformulated as follows: Let ω_λ be the composite of the natural maps:

$$(Ae\otimes_\Lambda eA)\otimes_A(Ae\otimes_\Lambda eA)\xrightarrow{\simeq}Ae\otimes_\Lambda(eA\otimes_A Ae)\otimes_\Lambda eA\xrightarrow{\simeq}Ae\otimes_\Lambda\Lambda\otimes_\Lambda eA\overset{\mathrm{Id}\otimes(\cdot\lambda)\otimes\mathrm{Id}}{\longrightarrow}Ae\otimes_\Lambda\Lambda\otimes_\Lambda eA\to Ae\otimes_\Lambda eA,$$

where $(\cdot\lambda):\Lambda\to\Lambda$ is the multiplication map by λ. Then

$$\big((be\otimes ec)\otimes(b'e\otimes ec')\big)\omega_\lambda=(be\otimes ec)*(b'e\otimes ec').$$

Clearly, $R(A,e,0)$ is exactly the trivial extension of A by $Ae\otimes_\Lambda eA$. To understand $R(A,e,\lambda)$, we will employ idealized extensions of algebras.

Definition 2.7. *Let X be an A-A-bimodule. An* idealized extension *of A by X is defined to be an algebra R such that A is a subalgebra (with the same identity) of R, X is an ideal of R, and $R=A\oplus X$ as A-A-bimodules. Two idealized extensions R_1 and R_2 of A by X are said to be* isomorphic *if there exists an algebra isomorphism $\phi:R_1\to R_2$ such that the restriction of ϕ to A is the identity map of A and the one of ϕ to X is an bijection from X to X.*

Clearly, an algebra R is an idealized extension of A by X if and only if R contains A as a subalgebra and there is an algebra homomorphism $\pi:R\to A$ with $X=\mathrm{Ker}(\pi)$ such that the composite of the inclusion $A\to R$ with π is the identity map of A. Hence a mirror-reflective algebra $R(A,e,\lambda)$ is an idealized extension of A by $Ae\otimes_\Lambda eA$.

Let

$$F:=Ae\otimes_\Lambda-\otimes_\Lambda eA:\ \Lambda^{\mathrm{e}}\text{-Mod}\longrightarrow A^{\mathrm{e}}\text{-Mod},\ M\mapsto Ae\otimes_\Lambda M\otimes_\Lambda eA,$$
$$G:=e(-)e:\ A^{\mathrm{e}}\text{-Mod}\longrightarrow\Lambda^{\mathrm{e}}\text{-Mod},\ M\mapsto eMe$$

for $M\in A^{\mathrm{e}}$-Mod. Since $e\otimes e^{\mathrm{op}}$ is an idempotent element of A^{e} and there are natural isomorphisms

$$F\simeq A^{\mathrm{e}}(e\otimes e^{\mathrm{op}})\otimes_{\Lambda^{\mathrm{e}}}-\quad\text{and}\quad G\simeq\mathrm{Hom}_{A^{\mathrm{e}}}(A^{\mathrm{e}}(e\otimes e^{\mathrm{op}}),-),$$

(F,G) is an adjoint pair and F is fully faithful. This implies the following.

Lemma 2.8. *The functor F induces an algebra isomorphism*

$$\rho:\ Z(\Lambda)\longrightarrow\mathrm{End}_{A^{\mathrm{e}}}(Ae\otimes_\Lambda eA),\ \lambda\mapsto\rho_\lambda:=[ae\otimes eb\mapsto ae\lambda\otimes eb]$$

for $\lambda\in Z(\Lambda)$ and $a,b\in A$. Moreover, $\omega_\lambda=\omega_e\rho_\lambda$.

The following result parameterizes the idealized extensions of A by $Ae\otimes_\Lambda eA$.

Proposition 2.9. *Let $Z(\Lambda)^\times$ be the group of units of $Z(\Lambda)$, that is, $Z(\Lambda)^\times$ is the group of all invertible elements in $Z(\Lambda)$. Then there exists a bijection from the quotient of the multiplicative semigroup $Z(\Lambda)$ modulo $Z(\Lambda)^\times$ to the set $\mathscr{S}(A,e)$ of the isomorphism classes of idealized extensions of A by $Ae\otimes_\Lambda eA$:*

$$Z(\Lambda)/Z(\Lambda)^\times\xrightarrow{\simeq}\mathscr{S}(A,e),\quad\lambda Z(\Lambda)^\times\mapsto R(A,e,\lambda)\ \textit{for}\ \lambda\in Z(\Lambda).$$

Proof. Let $Z_0(\Lambda) := Z(\Lambda)/Z(\Lambda)^\times = \{\lambda Z(\Lambda)^\times \mid \lambda \in Z(\Lambda)\}$ and $[\lambda] := \lambda Z(\Lambda)^\times \in Z_0(\Lambda)$ for $\lambda \in Z(\Lambda)$. By [4, Lemma 3.2(2)], if $\mu \in Z(\Lambda)^\times$, then $R(A,e,\lambda) \simeq R(A,e,\lambda\mu)$ as algebras. This means that the map

$$\varphi : Z_0(\Lambda) \longrightarrow \mathscr{S}(A,e), [\lambda] \mapsto R(A,e,\lambda)$$

is well defined. Let R be an idealized extension of A by $X := Ae \otimes_\Lambda eA$. Then the multiplication of R induces a homomorphism $\phi : X \otimes_A X \to X$ of $A^{\rm e}$-modules. Recall that $\omega_e : X \otimes_A X \to X$ is an isomorphism of $A^{\rm e}$-modules. Let $\phi' := \omega_e^{-1}\phi$. Then $\phi' \in \mathrm{End}_{A^{\rm e}}(X)$ and $\phi = \omega_e\phi'$. By Lemma 2.8, $\phi' = \rho_z$ for some $z \in Z(\Lambda)$ and $\phi = \omega_z$. Thus $R = R(A,e,z)$ and φ is surjective.

Now, we show that φ is injective. Suppose $\lambda_i \in Z(\Lambda)$ for $i = 1,2$ and $R(A,e,\lambda_1) \simeq R(A,e,\lambda_2)$ as algebras. Set $R_i := R(A,e,\lambda_i)$. By Definition 2.7, there is an algebra isomorphism $f : R_1 \to R_2$ such that $f|_A = \mathrm{Id}_A$ and $\alpha := f|_X : X \to X$ is an isomorphism of ideals. This implies that α is a homomorphism of $A^{\rm e}$-modules and $(\alpha \otimes_A \alpha)\omega_{\lambda_2} = \omega_{\lambda_1}\alpha : X \otimes_A X \to X$. Since $\omega_{\lambda_i} = \omega_e\rho_{\lambda_i}$ by Lemma 2.8, there holds $(\alpha \otimes_A \alpha)\omega_e\rho_{\lambda_2} = \omega_e\rho_{\lambda_1}\alpha$. Let $\sigma := \omega_e^{-1}(\alpha \otimes_A \alpha)\omega_e \in \mathrm{End}_{A^{\rm e}}(X)$. Then σ is an isomorphism of $A^{\rm e}$-modules and $\rho_{\lambda_1}\alpha = \sigma\rho_{\lambda_2}$. Again by Lemma 2.8, $\alpha = \rho_c$ and $\sigma = \rho_d$ for some $c,d \in Z(\Lambda)^\times$. It follows that $\lambda_1 c = d\lambda_2$, and therefore $[\lambda_1] = [\lambda_2]$. □

3 Derived equivalences of mirror-reflective algebras

In this section, k denotes a commutative ring, all algebras are k-algebras which are projective as k-modules.

Assume that the pairs (A,e) and (B,f) of algebras with idempotents are derived equivalent. By Lemma 2.4, there is a two-sided tilting complex $T \in \mathscr{D}(A \otimes_k B^{\rm op})$ with the quasi-inverse $T^\vee$ such that $T \otimes_k T^\vee \in \mathscr{D}(A^{\rm e} \otimes_k (B^{\rm e})^{\rm op})$ is a two-sided tilting complex with the inverse $T^\vee \otimes_k T$, and there is a derived equivalence:

$$\Phi := T^\vee \otimes_A^{\mathbb{L}} - \otimes_A^{\mathbb{L}} T \simeq (T^\vee \otimes_k T) \otimes_{A^{\rm e}}^{\mathbb{L}} - : \mathscr{D}(A^{\rm e}) \longrightarrow \mathscr{D}(B^{\rm e})$$

which sends A to B up to isomorphism (see [20]). Let $\varepsilon_A : T \otimes_B^{\mathbb{L}} T^\vee \to A$ and $\varepsilon_B : T^\vee \otimes_A^{\mathbb{L}} T \to B$ be the associated isomorphisms in $\mathscr{D}(A^{\rm e})$ and $\mathscr{D}(B^{\rm e})$, respectively. Now, we introduce the notation

$$\Lambda = eAe,\ \Gamma = fBf,\ G_e = e(-)e,\ G_f = f(-)f,$$

$$F_e = Ae \otimes_\Lambda - \otimes_\Lambda eA :\ \Lambda^{\rm e}\text{-Mod} \longrightarrow A^{\rm e}\text{-Mod},\quad F_f = Bf \otimes_\Gamma - \otimes_\Gamma fB :\ \Gamma^{\rm e}\text{-Mod} \longrightarrow B^{\rm e}\text{-Mod},$$

$$\mathbb{L}F_e = Ae \otimes_\Lambda^{\mathbb{L}} - \otimes_\Lambda^{\mathbb{L}} eA :\ \mathscr{D}(\Lambda^{\rm e}) \longrightarrow \mathscr{D}(A^{\rm e}),\quad \mathbb{L}F_f = Bf \otimes_\Gamma^{\mathbb{L}} - \otimes_\Gamma^{\mathbb{L}} fB :\ \mathscr{D}(\Gamma^{\rm e}) \longrightarrow \mathscr{D}(B^{\rm e}),$$

$$\Phi' = fT^\vee e \otimes_\Lambda^{\mathbb{L}} - \otimes_\Lambda^{\mathbb{L}} eTf :\ \mathscr{D}(\Lambda^{\rm e}) \longrightarrow \mathscr{D}(\Gamma^{\rm e}),$$

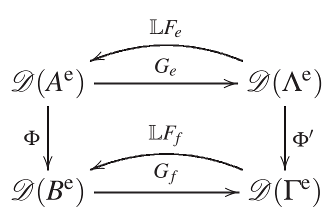
$$\Delta_0 = Ae \otimes_\Lambda eA,\ \Delta = Ae \otimes_\Lambda^{\mathbb{L}} eA,\quad \Theta_0 = Bf \otimes_\Gamma fB,\quad \Theta = Bf \otimes_\Gamma^{\mathbb{L}} fB,$$

together with the identifications (up to isomorphism):

$$\Delta_0 = H^0(\Delta),\ \Theta_0 = H^0(\Theta),\ \Delta = \mathbb{L}F_e(\Lambda),\ \Theta = \mathbb{L}F_f(\Gamma).$$

By Lemma 2.4 and Corollary 2.5, up to natural isomorphism, two squares in the diagram are commutative:

(♯)



where Φ' is the derived equivalence associated with the two-sided tilting complex $eTf \in \mathscr{D}(\Lambda \otimes_k \Gamma)$. Note that $\Phi, \Phi', \mathbb{L}F_e$ and $\mathbb{L}F_f$ commute with derived tensor products. Namely, for $U, V \in \mathscr{D}(A^{\mathrm{e}})$, there are isomorphisms

$$\Phi(U \otimes_A^{\mathbb{L}} V) \simeq T^\vee \otimes_A^{\mathbb{L}} U \otimes_A^{\mathbb{L}} A \otimes_A^{\mathbb{L}} V \otimes_A^{\mathbb{L}} T \simeq T^\vee \otimes_A^{\mathbb{L}} U \otimes_A^{\mathbb{L}} T \otimes_B^{\mathbb{L}} T^\vee \otimes_A^{\mathbb{L}} V \otimes_A^{\mathbb{L}} T = \Phi(U) \otimes_B^{\mathbb{L}} \Phi(V)$$

where the second isomorphism follows from $A \simeq T \otimes_B^{\mathbb{L}} T^\vee$ in $\mathscr{D}(A^{\mathrm{e}})$. This provides a natural isomorphism

$$\phi_{-,-}: \ \Phi(-) \otimes_B^{\mathbb{L}} \Phi(-) \xrightarrow{\simeq} \Phi(- \otimes_A^{\mathbb{L}} -): \quad \mathscr{D}(A^{\mathrm{e}}) \times \mathscr{D}(A^{\mathrm{e}}) \longrightarrow \mathscr{D}(B^{\mathrm{e}}).$$

Since $\Phi'(\Lambda) \simeq \Gamma$, there is an algebra isomorphism

$$\sigma : Z(\Lambda) \longrightarrow Z(\Gamma)$$

defined by the series of isomorphisms $Z(\Lambda) \simeq \mathrm{End}_{\Lambda^{\mathrm{e}}}(\Lambda) \xrightarrow{\simeq} \mathrm{End}_{\Gamma^{\mathrm{e}}}(\Phi'(\Lambda)) \xrightarrow{\simeq} \mathrm{End}_{\Gamma^{\mathrm{e}}}(\Gamma) \simeq Z(\Gamma)$.

Our main result on derived equivalences of mirror-reflective algebras is the following.

Theorem 3.1. *Suppose that there is a derived equivalence between (A, e) and (B, f) of algebras with idempotents, which gives rise to a two-sided tilting complex ${}_AT_B$. If the derived equivalence $\Phi : \mathscr{D}(A^{\mathrm{e}}) \to \mathscr{D}(B^{\mathrm{e}})$ associated with T between the enveloping algebras A^{e} and B^{e} satisfies $\Phi(Ae \otimes_\Lambda eA) \simeq Bf \otimes_\Gamma fB$ in $\mathscr{D}(B^{\mathrm{e}})$, then there is an algebra isomorphism $\sigma : Z(\Lambda) \to Z(\Gamma)$ such that, for each $\lambda \in Z(\Lambda)$, the pairs $\big(R(A, e, \lambda), e\big)$ and $\big(R(B, f, (\lambda)\sigma), f\big)$ of algebras with idempotents are derived equivalent. In particular, $R(A, e)$ and $R(B, f)$ are derived equivalent.*

Before starting with the proof of Theorem 3.1, we first fix notation on derived categories.

Let $\mathcal{A}$ be an abelian category. For each $X := (X^i, d_X^i)_{i\in\mathbb{Z}} \in \mathscr{C}(\mathcal{A})$ and $n \in \mathbb{Z}$, there are two truncated complexes

$$\tau^{\le n}X: \ \cdots \longrightarrow X^{n-3} \xrightarrow{d_X^{n-3}} X^{n-2} \xrightarrow{d_X^{n-2}} X^{n-1} \xrightarrow{d_X^{n-1}} \mathrm{Ker}(d_X^n) \longrightarrow 0,$$

$$\tau^{\ge n}X: \ 0 \longrightarrow \mathrm{Coker}(d_X^{n-1}) \xrightarrow{\overline{d_X^n}} X^{n+1} \xrightarrow{d_X^{n+1}} X^{n+2} \xrightarrow{d_X^{n+2}} X^{n+3} \longrightarrow \cdots,$$

where $\overline{d_X^n}$ is induced from d_X^n. Moreover, there are canonical chain maps in $\mathscr{C}(\mathcal{A})$:

$$\lambda_X^n : \tau^{\le n}X \hookrightarrow X \ \text{ and } \ \pi_X^n : X \twoheadrightarrow \tau^{\ge n}X,$$

and a distinguished triangle in $\mathscr{D}(\mathcal{A})$:

$$\tau^{\le n}X \xrightarrow{\lambda_X^n} X \xrightarrow{\pi_X^{n+1}} \tau^{\ge n+1}X \longrightarrow \tau^{\le n}X[1].$$

Note that $H^n(X) = \tau^{\ge n}\tau^{\le n}X : \mathscr{D}(\mathcal{A}) \to \mathcal{A}$. Let $\mathscr{D}^{\le 0}(\mathcal{A}) := \{X \in \mathscr{D}(\mathcal{A}) \mid H^i(X) = 0, i > 0\}$. For each $X \in \mathscr{D}^{\le 0}(\mathcal{A})$, it is clear that λ_X^0 is an isomorphism in $\mathscr{D}(\mathcal{A})$. In this case, we denote by $\xi_X : X \to H^0(X)$ the composition of the inverse $X \to \tau^{\le 0}X$ of λ_X^0 with $\pi^0_{\tau^{\le 0}X} : \tau^{\le 0}X \to H^0(X)$. Clearly, if $X^i = 0$ for all $i \ge 1$, then $X = \tau^{\le 0}X$ and $\xi_X = \pi_X^0$. Now, there is a natural transformation

$$\xi: \ \mathrm{Id}_{\mathscr{D}^{\le 0}(\mathcal{A})} \longrightarrow H^0: \ \mathscr{D}^{\le 0}(\mathcal{A}) \to \mathscr{D}^{\le 0}(\mathcal{A}).$$

When $\mathcal{A} = A^{\mathrm{e}}$-Mod and $X, Y \in \mathscr{D}^{\le 0}(\mathcal{A})$, we denote the composite of the following morphisms by

$$\theta_{X,Y}: \ X \otimes_A^{\mathbb{L}} Y \xrightarrow{\xi_X \otimes_A^{\mathbb{L}} \xi_Y} H^0(X) \otimes_A^{\mathbb{L}} H^0(Y) \xrightarrow{\xi_{H^0(X) \otimes_A^{\mathbb{L}} H^0(Y)}} H^0(X) \otimes_A H^0(Y).$$

Then $\theta_{X,Y}$ is natural in X and Y. This gives rise to a natural transformation

$$\theta_{-,-}: \ (-) \otimes_A^{\mathbb{L}} (-) \longrightarrow H^0(-) \otimes_A H^0(-): \quad \mathscr{D}^{\le 0}(\mathcal{A}) \times \mathscr{D}^{\le 0}(\mathcal{A}) \longrightarrow A^{\mathrm{e}}\text{-Mod}.$$

We have the following result.

Lemma 3.2. (1) *For $X \in \mathscr{D}^{\leq 0}(\mathcal{A})$, the morphism $H^0(\xi_X)$ is an automorphism of $H^0(X)$.*

(2) *For a morphism $f : X \to Y$ in $\mathscr{D}^{\leq 0}(\mathcal{A})$, there is a unique morphism $f' : H^0(X) \to H^0(Y)$ in $\mathcal{A}$ such that $f\xi_Y = \xi_X f'$. Moreover, $f' = H^0(\xi_X)^{-1}H^0(f)H^0(\xi_Y)$.*

(3) *Let $\mathcal{A} := A^{\mathrm{e}}$-Mod. Then the map $H^0(\theta_{X,Y}) : H^0(X \otimes_A^{\mathbb{L}} Y) \to H^0(X) \otimes_A H^0(Y)$ is an isomorphism and $\theta_{X,Y} = \xi_{X\otimes_A^{\mathbb{L}}Y} H^0(\theta_{X,Y})$. Thus there is a natural isomorphism of functors*

$$H^0(\theta_{-,-}) : \ H^0(- \otimes_A^{\mathbb{L}} -) \xrightarrow{\simeq} H^0(-) \otimes_A H^0(-) : \quad \mathscr{D}^{\leq 0}(\mathcal{A}) \times \mathscr{D}^{\leq 0}(\mathcal{A}) \longrightarrow A^{\mathrm{e}}\text{-Mod}.$$

Proof. (1) and (2) follow from the construction of ξ. Note that $H^0(\xi_X \otimes_A^{\mathbb{L}} \xi_Y)$ and $H^0(\xi_{H^0(X)\otimes_A^{\mathbb{L}}H^0(Y)})$ are isomorphisms. Since $\xi_{H^0(X)\otimes_A H^0(Y)}$ is the identity, (3) follows from (2). $\square$

In the rest of this section, let $\varphi : A \to A'$ be a homomorphism of algebras. Define

$$W := \Phi(A'), \ B' := H^0(W), \ W' := \tau^{\leq 0}W \ \text{ and } \ \varphi' := H^0(\Phi(\varphi)) : B \longrightarrow B'.$$

Lemma 3.3. (1) *$A' \otimes_A^{\mathbb{L}} T$ is isomorphic in $\mathscr{D}(A')$ to a tilting complex if and only if $H^n(W) = 0$ for all $n \neq 0$.*

(2) *B' is an algebra with the multiplication induced from*

$$B' \otimes_B B' \xrightarrow[\simeq]{H^0(\theta_{W',W'})^{-1}} H^0(W' \otimes_B^{\mathbb{L}} W') \xrightarrow{H^0(\lambda_W^0 \otimes_B^{\mathbb{L}} \lambda_W^0)} H^0(W \otimes_B^{\mathbb{L}} W) \xrightarrow[\simeq]{H^0(\phi_{A',A'})} H^0\big(\Phi(A' \otimes_A^{\mathbb{L}} A')\big) \xrightarrow{H^0(\Phi(\pi))} B'$$

where $\pi : A' \otimes_A^{\mathbb{L}} A' \to A'$ is the composite of $\xi_{A'\otimes_A^{\mathbb{L}}A'} : A' \otimes_A^{\mathbb{L}} A' \to A' \otimes_A A'$ with the multiplication map $A' \otimes_A A' \to A'$.

(3) *B' and $\mathrm{End}_{\mathscr{D}(A')}(A' \otimes_A^{\mathbb{L}} T)$ are isomorphic as algebras. Moreover, φ' is a homomorphism of algebras.*

Proof. (1) Since T is isomorphic in $\mathscr{D}(A)$ to a tilting complex P, we have $A' \otimes_A^{\mathbb{L}} T \simeq A' \otimes_A P$ in $\mathscr{D}(A')$ and $A' \otimes_A P \in \mathscr{K}^b(A'\text{-proj})$. As $\mathrm{add}(P)$ generates $\mathscr{K}^b(A\text{-proj})$ as a triangulated category, $\mathrm{add}(A' \otimes_A P)$ generates $\mathscr{K}^b(A'\text{-proj})$ as a triangulated category. This implies that $A' \otimes_A^{\mathbb{L}} T$ is isomorphic in $\mathscr{D}(A')$ to a tilting complex if and only if $A' \otimes_A^{\mathbb{L}} T$ is self-orthogonal in $\mathscr{D}(A')$. Moreover, for $n \in \mathbb{Z}$, it follows from the isomorphism $\varepsilon_B : T^\vee \otimes_A^{\mathbb{L}} T \to B$ in $\mathscr{D}(B^{\mathrm{e}})$ that there is a series of isomorphisms

$$(*) \ \ \mathrm{Hom}_{\mathscr{D}(A')}(A' \otimes_A^{\mathbb{L}} T, A' \otimes_A^{\mathbb{L}} T[n]) \simeq \mathrm{Hom}_{\mathscr{D}(A)}(T, A' \otimes_A^{\mathbb{L}} T[n]) \simeq \mathrm{Hom}_{\mathscr{D}(B)}(T^\vee \otimes_A^{\mathbb{L}} T, T^\vee \otimes_A^{\mathbb{L}} A' \otimes_A^{\mathbb{L}} T[n])$$

$$\simeq \mathrm{Hom}_{\mathscr{D}(B)}(B, T^\vee \otimes_A^{\mathbb{L}} A' \otimes_A^{\mathbb{L}} T[n]) = \mathrm{Hom}_{\mathscr{D}(B)}(B, W[n]) \simeq H^n(W).$$

Thus $A' \otimes_A^{\mathbb{L}} T$ is self-orthogonal if and only if $H^n(W) = 0$ for all $n \neq 0$. This shows (1).

(2) If taking $n = 0$ in $(*)$, we get an isomorphism $\mathrm{End}_{\mathscr{D}(A')}(A' \otimes_A^{\mathbb{L}} T) \simeq B'$ of k-modules. Via the isomorphism, we can transfer the algebra structure of $\mathrm{End}_{\mathscr{D}(A')}(A' \otimes_A^{\mathbb{L}} T)$ to the one of B'.

Let $s_i \in \mathrm{Hom}_{\mathscr{D}(B)}(B, W)$ for $i = 1, 2$. By $(*)$, there are morphisms $t_i : T \to A' \otimes_A^{\mathbb{L}} T$ in $\mathscr{D}(A)$ such that $s_i = \varepsilon_B^{-1}(T^\vee \otimes_A^{\mathbb{L}} t_i)$. By the first isomorphism in $(*)$, we can define a multiplication on the abelian group $\mathrm{Hom}_{\mathscr{D}(A)}(T, A' \otimes_A^{\mathbb{L}} T)$, that is, the multiplication of t_1 with t_2 is given by the composition of the morphisms

$$t_1 \cdot t_2 : \ T \xrightarrow{t_1} A' \otimes_A^{\mathbb{L}} T \xrightarrow{A' \otimes_A^{\mathbb{L}} t_2} A' \otimes_A^{\mathbb{L}} A' \otimes_A^{\mathbb{L}} T \xrightarrow{\pi \otimes_A^{\mathbb{L}} T} A' \otimes_A^{\mathbb{L}} T.$$

This yields the product $s_1 \cdot s_2 \in \mathrm{Hom}_{\mathscr{D}(B)}(B, W)$ of s_1 with s_2, described by the composite of the morphisms

$$s_1 \cdot s_2 : \ B \xrightarrow{\varepsilon_B^{-1}} T^\vee \otimes_A^{\mathbb{L}} T \xrightarrow{T^\vee \otimes_A^{\mathbb{L}} t_1} T^\vee \otimes_A^{\mathbb{L}} A' \otimes_A^{\mathbb{L}} T \xrightarrow{T^\vee \otimes_A^{\mathbb{L}} A' \otimes_A^{\mathbb{L}} t_2} T^\vee \otimes_A^{\mathbb{L}} A' \otimes_A^{\mathbb{L}} A' \otimes_A^{\mathbb{L}} T \xrightarrow{\Phi(\pi)} T^\vee \otimes_A^{\mathbb{L}} A' \otimes_A^{\mathbb{L}} T = W.$$

Since $({}_AT\otimes^{\mathbb{L}}_B -, {}_BT^{\vee}\otimes^{\mathbb{L}}_A -)$ is an adjoint pair of functors between $\mathscr{D}(B)$ and $\mathscr{D}(A)$, the composite of the morphisms

$$T \xrightarrow{\text{can}} T\otimes^{\mathbb{L}}_B B \xrightarrow{T\otimes^{\mathbb{L}}_B\varepsilon_B^{-1}} T\otimes^{\mathbb{L}}_B T^{\vee}\otimes^{\mathbb{L}}_A T \xrightarrow{\varepsilon_A\otimes^{\mathbb{L}}_A T} A\otimes^{\mathbb{L}}_A T \xrightarrow{\text{can}} T$$

is the identity morphism of T, where the first and last morphisms are canonical isomorphisms. It follows that t_2 is the composite of the morphisms

$${}_AT \xrightarrow{\text{can}} T\otimes^{\mathbb{L}}_B B \xrightarrow{T\otimes^{\mathbb{L}}_B\varepsilon_B^{-1}} T\otimes^{\mathbb{L}}_B T^{\vee}\otimes^{\mathbb{L}}_A T \xrightarrow{T\otimes^{\mathbb{L}}_B T^{\vee}\otimes^{\mathbb{L}}_A t_2} T\otimes^{\mathbb{L}}_B T^{\vee}\otimes^{\mathbb{L}}_A A'\otimes^{\mathbb{L}}_A T \xrightarrow{\varepsilon_A\otimes^{\mathbb{L}}_A A'\otimes^{\mathbb{L}}_A T} A\otimes^{\mathbb{L}}_A A'\otimes^{\mathbb{L}}_A T \xrightarrow{\text{can}} {}_AA'\otimes^{\mathbb{L}}_A T,$$

and therefore the multiplication $s_1\cdot s_2$ is the composite of the morphisms

$$B \xrightarrow{s_1} W \underset{\simeq}{\xrightarrow{\text{can}}} W\otimes^{\mathbb{L}}_B B \xrightarrow{W\otimes^{\mathbb{L}}_B s_2} W\otimes^{\mathbb{L}}_B W = \Phi(A')\otimes^{\mathbb{L}}_B \Phi(A') \underset{\simeq}{\xrightarrow{\phi_{A',A'}}} \Phi(A'\otimes^{\mathbb{L}}_A A') \xrightarrow{\Phi(\pi)} W\ .$$

Since the inclusion $\lambda^0_W : W'\to W$ induces an isomorphism $\mathrm{Hom}_{\mathscr{D}(B)}(B,W')\simeq \mathrm{Hom}_{\mathscr{D}(B)}(B,W)$, there are $s'_i\in \mathrm{Hom}_{\mathscr{D}(B)}(B,W')$ for $i=1,2$ such that $s_i = s'_i\lambda^0_W$. Let

$$\widetilde{s_i} := s'_i\pi^0_{W'}\in \mathrm{Hom}_B(B,B').$$

Since H^0 induces an isomorphism $\mathrm{Hom}_{\mathscr{D}(B)}(B,W)\simeq \mathrm{Hom}_B(B,B')\simeq B'$ as k-modules, we have $H^0(s_i)=\widetilde{s_i}$. In the diagram

$$\begin{array}{ccccccccc}
B & \xrightarrow{s_1} & W & \underset{\simeq}{\xrightarrow{\text{can}}} & W\otimes^{\mathbb{L}}_B B & \xrightarrow{W\otimes^{\mathbb{L}}_B s_2} & W\otimes^{\mathbb{L}}_B W & \xrightarrow{\phi_{A',A'}\ \Phi(\pi)} & W \\
\| & & \uparrow{\scriptstyle \lambda^0_W} & & \uparrow{\scriptstyle \lambda^0_W\otimes^{\mathbb{L}}_B B} & & \uparrow{\scriptstyle \lambda^0_W\otimes^{\mathbb{L}}_B\lambda^0_W} & & \uparrow \\
B & \xrightarrow{s'_1} & W' & \underset{\simeq}{\xrightarrow{\text{can}}} & W'\otimes^{\mathbb{L}}_B B & \xrightarrow{W'\otimes^{\mathbb{L}}_B s'_2} & W'\otimes^{\mathbb{L}}_B W' & = & W'\otimes^{\mathbb{L}}_B W' \\
\| & & \downarrow{\scriptstyle \pi^0_{W'}} & & \downarrow{\scriptstyle \pi^0_{W'}\otimes^{\mathbb{L}}_B B} & & \downarrow{\scriptstyle \pi^0_{W'}\otimes^{\mathbb{L}}_B\pi^0_{W'}} & & \downarrow{\scriptstyle \theta_{W',W'}} \\
B & \xrightarrow{\widetilde{s_1}} & B' & \underset{\simeq}{\xrightarrow{\text{can}}} & B'\otimes^{\mathbb{L}}_B B & \xrightarrow{B'\otimes^{\mathbb{L}}_B\widetilde{s_2}} & B'\otimes^{\mathbb{L}}_B B' & & \\
\| & & \| & & \downarrow{\scriptstyle \pi^0_{B'\otimes^{\mathbb{L}}_B B}} & & \downarrow{\scriptstyle \pi^0_{B'\otimes^{\mathbb{L}}_B B'}} & & \\
B & \xrightarrow{\widetilde{s_1}} & B' & \underset{\simeq}{\xrightarrow{\text{can}}} & B'\otimes_B B & \xrightarrow{B'\otimes_B\widetilde{s_2}} & B'\otimes_B B' & = & B'\otimes_B B'
\end{array}$$

of morphisms in $\mathscr{D}(B)$, all the squares are commutative and $H^0(\theta_{W',W'})$ is an isomorphism. Let $\mu : B\to B'\otimes_B B'$ be the composite of the morphisms in the bottom line of the diagram. Then

$$H^0(s_1\cdot s_2) = \mu H^0(\theta_{W',W'})^{-1}H^0(\lambda^0_W\otimes^{\mathbb{L}}_B\lambda^0_W)H^0(\phi_{A',A'})H^0(\Phi(\pi)) :\ B\longrightarrow B'.$$

Note that μ sends the identity 1 of B to $(1)\widetilde{s_1}\otimes(1)\widetilde{s_2}$. Now, by identifying B' with $\mathrm{Hom}_{\mathscr{D}(B)}(B,W)$ and also with $\mathrm{Hom}_B(B,B')$, (2) can be proved.

(3) The isomorphism $\mathrm{End}_{\mathscr{D}(A')}(A'\otimes^{\mathbb{L}}_A T)\simeq B'$ as algebras has been shown in the proof of (2). Now, we denote by $\mu_{B'} : B'\otimes_B B'\to B'$ the composite of the morphisms in (2). Recall that $\Phi(A)\simeq B$ and $H^0(\Phi(A)) = B$. If $A' = A$ and $\varphi = \mathrm{Id}_A$, then $B' = B$ and $\mu_B : B\otimes_B B\to B$ is the canonical isomorphism induced by the multiplication of B. For a general $\varphi : A\to A'$, there is an equality $\mu_B\varphi' = (\varphi'\otimes_B\varphi')\mu_{B'}$ which means that φ' is an algebra homomorphism. Note that if B and B' are identified with $\mathrm{End}_{\mathscr{D}(A)}(T)$ and $\mathrm{End}_{\mathscr{D}(A')}(A'\otimes^{\mathbb{L}}_A T)$, respectively, then $\varphi' : B\to B'$ is exactly the algebra homomorphism induced from $A'\otimes^{\mathbb{L}}_A - : \mathscr{D}(A)\to\mathscr{D}(A')$. $\square$

The following result provides a method for constructing derived equivalences of algebras with idempotents from given ones. It also generalizes derived equivalences of trivial extensions of algebras by bimodules (see [19, Corollary 5.4]).

Proposition 3.4. *Suppose $H^n(W)=0$ for all $n\neq 0$. Then the pairs $(A',(e)\varphi)$ and $(B',(f)\varphi')$ of algebras with idempotents are derived equivalent. In particular, A' and B' are derived equivalent.*

Proof. By Lemma 3.3, $A'\otimes^{\mathbb{L}}_A T$ is isomorphic in $\mathscr{D}(A')$ to a tilting complex and $\mathrm{End}_{\mathscr{D}(A')}(A'\otimes^{\mathbb{L}}_A T)\simeq B'$ as algebras. It follows from Theorem 2.2 that A' and B' are derived equivalent. Clearly, $T\otimes^{\mathbb{L}}_B Bf\simeq Tf\in \mathrm{add}({}_AT)$ and $A'\otimes^{\mathbb{L}}_A Tf\in \mathrm{add}({}_{A'}A'\otimes^{\mathbb{L}}_A T)$. Since (A,e) and (B,f) are derived equivalent, Tf generates $\mathscr{K}^b(\mathrm{add}(Ae))$ in $\mathscr{D}(A)$ by Lemma 2.4(2). Applying the functor $A'\otimes^{\mathbb{L}}_A - : \mathscr{D}(A)\to\mathscr{D}(A')$ to Tf and $\mathscr{K}^b(\mathrm{add}(Ae))$, we see from the general result in the proof of Lemma 2.5(2) that $A'\otimes^{\mathbb{L}}_A Tf$ generates $\mathscr{K}^b(\mathrm{add}(A'(e)\varphi))$ in $\mathscr{D}(A')$. Note that

$$\mathrm{Hom}_{\mathscr{D}(A')}(A'\otimes^{\mathbb{L}}_A T, A'\otimes^{\mathbb{L}}_A Tf)\simeq \mathrm{Hom}_{\mathscr{D}(A)}(T, A'\otimes^{\mathbb{L}}_A Tf)\simeq \mathrm{Hom}_{\mathscr{D}(B)}(T^\vee\otimes^{\mathbb{L}}_A T, T^\vee\otimes^{\mathbb{L}}_A A'\otimes^{\mathbb{L}}_A Tf)$$

$$\simeq \mathrm{Hom}_{\mathscr{D}(B)}(B, W\otimes_B Bf)\simeq H^0(W)\otimes_B Bf\simeq B'(f)\varphi'.$$

By Lemma 2.4(2), the pairs $(A',(e)\varphi)$ and $(B',(f)\varphi')$ are derived equivalent. $\square$

Now, we turn to mirror-reflective algebras at any levels. Recall that, for each $\lambda\in Z(\Lambda)$, the multiplication map $(\cdot\lambda):\Lambda\to\Lambda$ induces a homomorphism $\omega_\lambda:\Delta_0\otimes_A\Delta_0\to\Delta_0$ in A^{e}-Mod, which is the composite of the maps

$$\Delta_0\otimes_A\Delta_0\xrightarrow{\omega_e}\Delta_0\xrightarrow{F_e(\cdot\lambda)}\Delta_0.$$

We define the derived version of ω_λ to be the composite of the maps in $\mathscr{D}(A^{\mathrm{e}})$:

$$\mathbb{L}\omega_\lambda:\ \Delta\otimes^{\mathbb{L}}_A\Delta\xrightarrow{\simeq}\Delta\xrightarrow{\mathbb{L}F_e(\cdot\lambda)}\Delta$$

where the first isomorphism is canonical, due to $eA\otimes^{\mathbb{L}}_A Ae\simeq\Lambda$ in $\mathscr{D}(\Lambda^{\mathrm{e}})$. Note that both ω_e and $\mathbb{L}\omega_e$ are isomorphisms since $(\cdot e)$ is the identity map of Λ, and that F_e and $\mathbb{L}F_e$ are fully faithful functors. Thus we have the following result.

Lemma 3.5. *There are isomorphisms*

$$\omega_{(-)}:\ Z(\Lambda)\xrightarrow{\simeq}\mathrm{Hom}_{A^{\mathrm{e}}}(\Delta_0\otimes_A\Delta_0,\Delta_0),\ \ \lambda\mapsto\omega_\lambda=\omega_eF_e(\cdot\lambda),$$

$$\mathbb{L}\omega_{(-)}:\ Z(\Lambda)\xrightarrow{\simeq}\mathrm{Hom}_{\mathscr{D}(A^{\mathrm{e}})}(\Delta\otimes^{\mathbb{L}}_A\Delta,\Delta),\ \ \lambda\mapsto\mathbb{L}\omega_\lambda=(\mathbb{L}\omega_e)\mathbb{L}F_e(\cdot\lambda).$$

Moreover, ω_λ is an isomorphism if and only if λ is invertible if and only if $\mathbb{L}\omega_\lambda$ is an isomorphism.

Similarly, for $\mu\in Z(\Gamma)$, there is a homomorphism $\omega_\mu:\Theta_0\otimes_A\Theta_0\to\Theta_0$ in B^{e}-Mod with its derived version

$$\mathbb{L}\omega_\mu:\ \Theta\otimes^{\mathbb{L}}_B\Theta\xrightarrow{\simeq}\Theta\xrightarrow{\mathbb{L}F_f(\cdot\mu)}\Theta$$

in $\mathscr{D}(B^{\mathrm{e}})$. Following the diagram $(\sharp)$, let $\eta:\Phi\circ\mathbb{L}F_e\to\mathbb{L}F_f\circ\Phi'$ be a natural isomorphism of functors from $\mathscr{D}(\Lambda^{\mathrm{e}})$ to $\mathscr{D}(B^{\mathrm{e}})$ and let $\tau:\Phi'(\Lambda)\to\Gamma$ be an isomorphism in $\mathscr{D}(\Gamma^{\mathrm{e}})$.

Lemma 3.6. *The following hold for $\lambda\in Z(\Lambda)$ and $\mu:=(\lambda)\sigma\in Z(\Gamma)$.*

(1) *There are commutative diagrams*

$$\begin{array}{ccc}\Delta\otimes^{\mathbb{L}}_A\Delta & \xrightarrow{\mathbb{L}\omega_\lambda} & \Delta\\ \Big\downarrow{\scriptstyle\theta_{\Delta,\Delta}} & & \Big\downarrow{\scriptstyle\xi_\Delta}\\ \Delta_0\otimes_A\Delta_0 & \xrightarrow{\omega_{\lambda\lambda_1}} & \Delta_0,\end{array}\qquad\qquad \begin{array}{ccc}\Phi(\Delta)\otimes^{\mathbb{L}}_B\Phi(\Delta) & \xrightarrow{\phi_{\Delta,\Delta}\Phi(\mathbb{L}\omega_\lambda)} & \Phi(\Delta)\\ \Big\downarrow{\scriptstyle\tau_1\otimes^{\mathbb{L}}_B\tau_1} & & \Big\downarrow{\scriptstyle\tau_1}\\ \Theta\otimes^{\mathbb{L}}_B\Theta & \xrightarrow{\mathbb{L}\omega_{\mu\mu_1}} & \Theta\end{array}$$

where $\lambda_1 \in Z(\Lambda)$ *and* $\mu_1 \in Z(\Gamma)$ *are invertible, and* $\tau_1 := \eta_\Lambda \mathbb{L}F_f(\tau) : \Phi(\Delta) \to \Theta$ *is an isomorphism.*

(2) *Let*

$$W_0 := \Phi(\Delta_0),\ \ \tau_2 := \tau_1^{-1}\Phi(\xi_\Delta) :\ \ \Theta \longrightarrow W_0,\ \ \psi_\lambda := \xi_{\Delta_0 \otimes^{\mathbb{L}}_A \Delta_0}\omega_{\lambda\lambda_1} :\ \ \Delta_0 \otimes^{\mathbb{L}}_A \Delta_0 \longrightarrow \Delta_0.$$

Then there is a commutative diagram in $\mathscr{D}(B^{\mathrm{e}})$:

$$\begin{array}{ccc}
\Theta \otimes^{\mathbb{L}}_B \Theta & \xrightarrow{\ \mathbb{L}\omega_{\mu\mu_1}\ } & \Theta \\
{\scriptstyle \tau_2 \otimes^{\mathbb{L}}_B \tau_2}\downarrow & & \downarrow{\scriptstyle \tau_2} \\
W_0 \otimes^{\mathbb{L}}_B W_0 & \xrightarrow{\ \phi_{\Delta_0,\Delta_0}\Phi(\psi_\lambda)\ } & W_0.
\end{array}$$

(3) *If* W_0 *lies in* $\mathscr{D}^{\leq 0}(B^{\mathrm{e}})$, *then there is a commutative diagram in* B^{e}-Mod :

$$\begin{array}{ccc}
\Theta_0 \otimes_B \Theta_0 & \xrightarrow{\ \omega_{\mu\mu'}\ } & \Theta_0 \\
{\scriptstyle H^0(\tau_2)\otimes_B H^0(\tau_2)}\downarrow & & \downarrow{\scriptstyle H^0(\tau_2)} \\
H^0(W_0) \otimes_B H^0(W_0) & \xrightarrow{\ H^0(\theta_{W_0,W_0})^{-1} H^0(\phi_{\Delta_0,\Delta_0}) H^0(\Phi(\psi_\lambda))\ } & H^0(W_0),
\end{array}$$

where $\mu' \in Z(\Gamma)$ *is invertible. If, in addition,* $H^0(\Phi(\xi_\Delta))$ *is an isomorphism, then there is an algebra isomorphism*

$$H^0\big(\Phi(R(A,e,\lambda))\big) \simeq R(B,f,\mu).$$

(4) *If* $W_0 \simeq \Theta_0$ *in* $\mathscr{D}(B^{\mathrm{e}})$, *then* $H^0(\Phi(\xi_\Delta))$ *is an isomorphism of* B^e*-modules.*

Proof. (1) Note that $\mathbb{L}\omega_e$ is an isomorphism. By Lemma 3.2(2)-(3), $\mathbb{L}F_e(\cdot\lambda)\xi_\Delta = \xi_\Delta F_e(\cdot\lambda)$ and there is a unique isomorphism $\alpha : \Delta_0 \otimes_A \Delta_0 \to \Delta_0$ such that $(\mathbb{L}\omega_e)\xi_\Delta = \theta_{\Delta,\Delta}\alpha$. By the first isomorphism in Lemma 3.5, there is an element $\lambda_1 \in Z(\Lambda)^\times$ such that $\alpha = \omega_{\lambda_1} = \omega_e F_e(\cdot\lambda_1)$. Thus

$$(\mathbb{L}\omega_\lambda)\xi_\Delta = (\mathbb{L}\omega_e)\mathbb{L}F_e(\cdot\lambda)\xi_\Delta = \theta_{\Delta,\Delta}\alpha F_e(\cdot\lambda) = \theta_{\Delta,\Delta}\omega_e F_e(\cdot\lambda_1)F_e(\cdot\lambda) = \theta_{\Delta,\Delta}\omega_e F_e(\cdot(\lambda\lambda_1)) = \theta_{\Delta,\Delta}\omega_{\lambda\lambda_1}.$$

Hence the diagram in the left-hand side of (1) is commutative.

Since $\mathbb{L}\omega_e$ and τ are isomorphisms and η is a natural isomorphism, there is a unique isomorphism $\beta : \Theta \otimes^{\mathbb{L}}_B \Theta \to \Theta$ such that all the squares in the diagram are commutative:

$$\begin{array}{ccccccc}
\Phi(\Delta) \otimes^{\mathbb{L}}_B \Phi(\Delta) & \xrightarrow{\ \phi_{\Delta,\Delta}\ } & \Phi(\Delta \otimes^{\mathbb{L}}_A \Delta) & \xrightarrow{\ \Phi(\mathbb{L}\omega_e)\ } & \Phi(\Delta) & \xrightarrow{\ \Phi\circ\mathbb{L}F_e(\cdot\lambda)\ } & \Phi(\Delta) \\
{\scriptstyle \eta_\Lambda \otimes^{\mathbb{L}}_B \eta_\Lambda}\downarrow & & & & \downarrow{\scriptstyle \eta_\Lambda} & & \downarrow{\scriptstyle \eta_\Lambda} \\
\mathbb{L}F_f\circ\Phi'(\Lambda) \otimes^{\mathbb{L}}_B \mathbb{L}F_f\circ\Phi'(\Lambda) & & & & \mathbb{L}F_f\circ\Phi'(\Lambda) & \xrightarrow{\ \mathbb{L}F_f\circ\Phi'(\cdot\lambda)\ } & \mathbb{L}F_f\circ\Phi'(\Lambda) \\
{\scriptstyle \mathbb{L}F_f(\tau)\otimes^{\mathbb{L}}_B \mathbb{L}F_f(\tau)}\downarrow & & & & \downarrow{\scriptstyle \mathbb{L}F_f(\tau)} & & \downarrow{\scriptstyle \mathbb{L}F_f(\tau)} \\
\Theta \otimes^{\mathbb{L}}_B \Theta & & \xrightarrow{\ \ \ \ \beta\ \ \ \ } & & \Theta & \xrightarrow{\ \mathbb{L}F_f(\cdot\mu)\ } & \Theta
\end{array}$$

Further, by applying a similar isomorphism in Lemma 3.5 to the pair (B,f), we have $\beta = \mathbb{L}\omega_{\mu_1} = (\mathbb{L}\omega_f)\mathbb{L}F_f(\cdot\mu_1)$ for some invertible element $\mu_1 \in Z(\Gamma)$. This implies

$$\beta\mathbb{L}F_f(\cdot\mu) = (\mathbb{L}\omega_f)\mathbb{L}F_f(\cdot\mu_1)\mathbb{L}F_f(\cdot\mu) = (\mathbb{L}\omega_f)\mathbb{L}F_f(\cdot(\mu\mu_1)) = \mathbb{L}\omega_{\mu\mu_1}.$$

Thus the second diagram in (1) is commutative.

(2) It follows from $\theta_{\Delta,\Delta} = (\xi_\Delta \otimes^{\mathbb{L}}_A \xi_\Delta)\xi_{\Delta_0 \otimes^{\mathbb{L}}_A \Delta_0}$ and the first diagram in (1) that there is the commutative diagram:

$$\begin{array}{ccc}
\Delta \otimes^{\mathbb{L}}_A \Delta & \xrightarrow{\mathbb{L}\omega_\lambda} & \Delta \\
\downarrow{\scriptstyle \xi_\Delta \otimes^{\mathbb{L}}_A \xi_\Delta} & & \downarrow{\scriptstyle \xi_\Delta} \\
\Delta_0 \otimes^{\mathbb{L}}_A \Delta_0 & \xrightarrow{\psi_\lambda} & \Delta_0.
\end{array}$$

Applying Φ and the natural isomorphism $\phi_{-,-}: \ \Phi(-) \otimes^{\mathbb{L}}_B \Phi(-) \xrightarrow{\simeq} \Phi(- \otimes^{\mathbb{L}}_A -)$ to this diagram, we get another commutative diagram:

$$\begin{array}{ccc}
\Phi(\Delta) \otimes^{\mathbb{L}}_B \Phi(\Delta) & \xrightarrow{\phi_{\Delta,\Delta}\Phi(\mathbb{L}\omega_\lambda)} & \Phi(\Delta) \\
\downarrow{\scriptstyle \Phi(\xi_\Delta) \otimes^{\mathbb{L}}_B \Phi(\xi_\Delta)} & & \downarrow{\scriptstyle \Phi(\xi_\Delta)} \\
W_0 \otimes^{\mathbb{L}}_A W_0 & \xrightarrow{\phi_{\Delta_0,\Delta_0}\Phi(\psi_\lambda)} & W_0.
\end{array}$$

Now, the commutative diagram in (2) follows from the second commutative diagram in (1).

(3) Note that $H^0 \circ \mathbb{L}F_e = F_e$. Applying H^0 to the diagram in (2), we see from Lemma 3.2(3) that the squares in the diagram

$$(\natural_1) \qquad \begin{array}{ccccc}
\Theta_0 \otimes_B \Theta_0 & \xleftarrow[\simeq]{H^0(\theta_{\Theta,\Theta})} & H^0(\Theta \otimes^{\mathbb{L}}_B \Theta) & \xrightarrow{H^0(\mathbb{L}\omega_{\mu\mu_1})} & H^0(\Theta) \\
\downarrow{\scriptstyle H^0(\tau_2)\otimes_B H^0(\tau_2)} & & \downarrow{\scriptstyle H^0(\tau_2 \otimes^{\mathbb{L}}_B \tau_2)} & & \downarrow{\scriptstyle H^0(\tau_2)} \\
H^0(W_0) \otimes_B H^0(W_0) & \xleftarrow[\simeq]{H^0(\theta_{W_0,W_0})} & H^0(W_0 \otimes^{\mathbb{L}}_B W_0) & \xrightarrow{H^0(\phi_{\Delta_0,\Delta_0})H^0(\Phi(\psi_\lambda))} & H^0(W_0)
\end{array}$$

are commutative, where the isomorphisms are due to $\Theta, W_0 \in \mathscr{D}^{\leq 0}(B^{\mathrm{e}})$. Moreover, for the pair (B, f) and $\mu\mu_1 \in Z(\Gamma)$, we obtain similarly the following commutative diagrams, in which the second one is obtained from the first by the functor H^0:

$$(\natural_2) \qquad \begin{array}{ccc}
\Theta \otimes^{\mathbb{L}}_A \Theta & \xrightarrow{\mathbb{L}\omega_{\mu\mu_1}} & \Theta \\
\downarrow{\scriptstyle \theta_{\Theta,\Theta}} & & \downarrow{\scriptstyle \xi_\Theta} \\
\Theta_0 \otimes_A \Theta_0 & \xrightarrow{\omega_{\mu\mu_1\mu_2}} & \Theta_0\ ,
\end{array} \qquad\qquad
\begin{array}{ccc}
H^0(\Theta \otimes^{\mathbb{L}}_A \Theta) & \xrightarrow{H^0(\mathbb{L}\omega_{\mu\mu_1})} & \Theta_0 \\
\downarrow{\scriptstyle H^0(\theta_{\Theta,\Theta})} & & \downarrow{\scriptstyle H^0(\xi_\Theta)} \\
\Theta_0 \otimes_A \Theta_0 & \xrightarrow{\omega_{\mu\mu_1\mu_2}} & \Theta_0
\end{array}$$

where $\mu_2 \in Z(\Gamma)$ is invertible and $H^0(\xi_\Theta)$ is an automorphism by Lemma 3.2(1). Since the functor F_f induces an algebra isomorphism $Z(\Gamma) \to \mathrm{End}_{B^{\mathrm{e}}}(\Theta_0)$, there is an invertible element $\mu_3 \in Z(\Gamma)$ such that $H^0(\xi_\Theta) = F_f(\cdot\mu_3)$. This implies

$$(\natural_3) \quad \omega_{\mu\mu_1\mu_2} H^0(\xi_\Theta)^{-1} = \omega_{\mu\mu_1\mu_2} F_f(\cdot\mu_3^{-1}) = \omega_{\mu\mu_1\mu_2\mu_3^{-1}}.$$

Let $\mu' := \mu_1\mu_2\mu_3^{-1} \in Z(\Gamma)$. Then μ' is invertible. By $(\natural_1)$-$(\natural_3)$, we obtain the commutative diagram in (3).

Next, we apply Lemma 3.3(2) to show the algebra isomorphism in (3).

Let $A' := R(A, e, \lambda\lambda_1)$, $B' := H^0(\Phi(A'))$ and $\varphi : A \to A'$ be the canonical injection. By Lemma 3.3(2), B' is an algebra. Since $A' = A \oplus \Delta_0$ and $\Phi(A) \simeq B$, there holds $\Phi(A') \simeq B \oplus W_0$ in $\mathscr{D}(B^{\mathrm{e}})$. Now, we identity B' with $B \oplus H^0(W_0)$ as B^{e}-modules and describe the multiplication of B' in terms of the one of A' and the one in Lemma 3.3(2):

The multiplication of B with B' (or B' with B) is given by left (or right) multiplication since B' is a B-B-bimodule; while the multiplication on $H^0(W_0)$ is induced from the composition

$$\begin{array}{ccccc} H^0(W_0)\otimes_B H^0(W_0) & \xrightarrow{H^0(\theta_{W_0',W_0'})^{-1}} & H^0(W_0'\otimes_B^{\mathbb{L}} W_0') & \xrightarrow{H^0(\lambda^0_{W_0}\otimes_B^{\mathbb{L}}\lambda^0_{W_0})} & H^0(W_0\otimes_B^{\mathbb{L}} W_0) \\ & & & & \downarrow{\scriptstyle H^0(\phi_{\Delta_0,\Delta_0})} \\ H^0(W_0) & \xleftarrow{H^0(\Phi(\omega_{\lambda\lambda_1}))} & H^0\big(\Phi(\Delta_0\otimes_A\Delta_0)\big) & \xleftarrow{H^0(\Phi(\xi_{\Delta_0\otimes_A^{\mathbb{L}}\Delta_0}))} & H^0\big(\Phi(\Delta_0\otimes_A^{\mathbb{L}}\Delta_0)\big) \end{array}$$

where $W_0':=\tau^{\leq 0}W_0$ and the injection $\lambda^0_{W_0}: W_0'\to W_0$ is an isomorphism by $W_0\in\mathscr{D}^{\leq 0}(B^{\mathrm{e}})$. It then follows from

$$\theta_{W_0,W_0}=\big((\lambda^0_{W_0})^{-1}\otimes_B^{\mathbb{L}}(\lambda^0_{W_0})^{-1}\big)\theta_{W_0',W_0'}=\big(\lambda^0_{W_0}\otimes_B^{\mathbb{L}}\lambda^0_{W_0}\big)^{-1}\theta_{W_0',W_0'},$$

that $H^0(\theta_{W_0,W_0})^{-1}=H^0(\theta_{W_0',W_0'})^{-1}H^0(\lambda^0_{W_0}\otimes_B^{\mathbb{L}}\lambda^0_{W_0})$. Thus the multiplication of $H^0(W_0)$ with $H^0(W_0)$ in B' is induced from

$$H^0(\theta_{W_0,W_0})^{-1}H^0(\phi_{\Delta_0,\Delta_0})H^0(\Phi(\psi_\lambda)):\ H^0(W_0)\otimes_B H^0(W_0)\longrightarrow H^0(W_0).$$

Suppose that $H^0(\Phi(\xi_\Delta))$ is an isomorphism. Then $H^0(\tau_2)$ is an isomorphism and $B'\simeq B\oplus\Theta_0$ as B^{e}-modules. Moreover, the commutative diagram in (3) implies that $H^0(\tau_2)$ induces an algebra isomorphism $R(B,f,\mu\mu')\simeq B'$ which lifts the identity map of B. Since $\lambda_1\in Z(\Lambda)$ and $\mu'\in Z(\Gamma)$ are invertible, it follows from [4, Lemma 3.2(2)] that $A'\simeq R(A,e,\lambda)$ and $R(B,f,\mu\mu')\simeq R(B,f,\mu)$ as algebras. Thus there are algebra isomorphisms $H^0\big(\Phi(R(A,e,\lambda))\big)\simeq H^0\big(\Phi(A')\big)=B'\simeq R(B,f,\mu)$.

(4) Under the identifications $G_e(\Delta_0)=\Lambda$ and $\Delta=\mathbb{L}F_e(\Lambda)$, we see that $\xi_\Delta:\Delta=(\mathbb{L}F_e\circ G_e)(\Delta_0)\to\Delta_0$ is the counit adjunction morphism of Δ_0 associated with the adjoint pair $(\mathbb{L}F_e,G_e)$. Similarly, up to isomorphism, $\xi_\Theta:\Theta=(\mathbb{L}F_f\circ G_f)(\Theta_0)\to\Theta_0$ is the counit adjunction morphism of Θ_0 associated with the adjoint pair $(\mathbb{L}F_f,G_f)$. Now, recall that two morphisms $f_i:X_i\to Y_i$ for $i=1,2$ in an additive category are isomorphic if there are isomorphisms $\alpha_1:X_1\to X_2$ and $\alpha_2:Y_1\to Y_2$ such that $f_1\alpha_2=\alpha_1 f_2$. By the diagram ($\sharp$), the functor Φ is an equivalence and there is a natural isomorphism

$$\Phi\circ\mathbb{L}F_e\circ G_e\xrightarrow{\simeq}\mathbb{L}F_f\circ G_f\circ\Phi:\ \mathscr{D}(A^{\mathrm{e}})\longrightarrow\mathscr{D}(B^{\mathrm{e}}).$$

This implies that $\Phi(\xi_\Delta):\Phi(\Delta)\to W_0$ is isomorphic to the counit adjunction morphism of W_0 associated with $(\mathbb{L}F_f,G_f)$. If $W_0\simeq\Theta_0$ in $\mathscr{D}(B^{\mathrm{e}})$, then ξ_Θ and $\Phi(\xi_\Delta)$ are isomorphic as morphisms in $\mathscr{D}(B^{\mathrm{e}})$. Since $H^0(\xi_\Theta)$ is an isomorphism by Lemma 3.2(1), $H^0(\Phi(\xi_\Delta))$ is an isomorphism. This shows (4). $\square$

Proof of Theorem 3.1. For each $\lambda\in Z(\Lambda)$, let $A':=R(A,e,\lambda)$, $\varphi:A\to A'$ the canonical injection and $B':=H^0(\Phi(A'))$. Since $A'=A\oplus\Delta_0$ and $\Phi(A)\simeq B$, we have $\Phi(A')\simeq B\oplus\Phi(\Delta_0)$. By assumption, $\Phi(\Delta_0)\simeq\Theta_0$ in $\mathscr{D}(B^{\mathrm{e}})$. This implies $\Phi(A')\simeq B\oplus\Theta_0$ in $\mathscr{D}(B^{\mathrm{e}})$, and therefore $B'=B\oplus H^0(\Phi(\Delta_0))\simeq B\oplus\Theta_0$ and $H^n(\Phi(A'))=0$ for all $n\neq 0$. Now, let $\varphi':=H^0(\Phi(\varphi)):B\to B'$. By the multiplication of B' in Lemma 3.3(2), φ' is the canonical injection. Then $(e)\varphi=e\in A'$ and $(f)\varphi'=f\in B'$. By Proposition 3.4, (A',e) and (B',f) are derived equivalent. Since $\Phi(\Delta_0)\simeq\Theta_0$ in $\mathscr{D}(B^{\mathrm{e}})$, it follows from Lemma 3.6(3)(4) that there is an algebra isomorphism $B'\simeq R(B,f,(\lambda)\sigma)$ which lifts the identity map of B. Consequently, (A',e) and $\big(R(B,f,(\lambda)\sigma),f\big)$ are derived equivalent. Clearly, $(e)\sigma=f$ since e and f are identities of Λ and Γ, respectively. Thus $\big(R(A,e),e\big)$ and $\big(R(B,f),f\big)$ are derived equivalent. $\square$

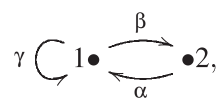
A sufficient condition for the isomorphism in Theorem 3.1 to hold true is the vanishing of positive Tor-groups over corner algebras.

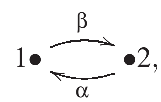
Proposition 3.7. *Suppose that there is a derived equivalence between (A,e) and (B,f) of algebras with idempotents, which is induced by a two-sided tilting complex ${}_AT_B$. If $\mathrm{Tor}_n^\Lambda(Ae,eA)=0=\mathrm{Tor}_n^\Gamma(Bf,fB)$ for all $n\geq 1$, then the derived equivalence $\Phi:\mathscr{D}(A^{\mathrm{e}})\to\mathscr{D}(B^{\mathrm{e}})$ associated with T between the enveloping algebras A^{e} and B^{e} satisfies $\Phi(Ae\otimes_\Lambda eA)\simeq Bf\otimes_\Gamma fB$ in $\mathscr{D}(B^{\mathrm{e}})$.*

Proof. Since $\mathrm{Tor}_n^{\Lambda}(Ae,eA)=0$ for all $n\geq 1$, we have $Ae\otimes^{\mathbb{L}}_{\Lambda}eA\simeq Ae\otimes_{\Lambda}eA$ in $\mathscr{D}(A^{\mathrm{e}})$. Similarly, $Bf\otimes^{\mathbb{L}}_{\Gamma}fB\simeq Bf\otimes_{\Gamma}fB$ in $\mathscr{D}(B^{\mathrm{e}})$. Moreover, since (A,e) and (B,f) are derived equivalent, it follows from Lemma 2.4(4) that $\Phi(Ae\otimes^{\mathbb{L}}_{\Lambda}eA)\simeq Bf\otimes^{\mathbb{L}}_{\Gamma}fB$ in $\mathscr{D}(B^{\mathrm{e}})$. Thus $\Phi(Ae\otimes_{\Lambda}eA)\simeq Bf\otimes_{\Gamma}fB$. $\square$

Proof of Theorem 1.1. Suppose that A and B are derived equivalent, gendo-symmetric algebras. Then the pair (A,e) and (B,f) are derived equivalent by Proposition 2.6. Without loss of generality, we assume that the derived equivalence between (A,e) and (B,f) is induced by a two-sided tilting complex $T\in\mathscr{D}(A\otimes_k B^{\mathrm{op}})$. This gives rise to a derived equivalence between A^{e} and B^{e}. Let $\Phi:=T^{\vee}\otimes^{\mathbb{L}}_{A}-\otimes^{\mathbb{L}}_{A}T:\mathscr{D}(A^{\mathrm{e}})\to\mathscr{D}(B^{\mathrm{e}})$ be the associated equivalence. Then Φ induces an algebra isomorphism $\sigma:Z(eAe)\to Z(fBf)$ (see the lines just before Theorem 3.1). Note that, for the gendo-symmetric algebra (A,e), there is an isomorphism ${}_AAe\otimes_{\Lambda}eA_A\simeq D(A)$ of A-A-bimodules by [6, Section 2.2] or [4, Lemma 4.1(2)]. Similarly, ${}_BBf\otimes_{\Gamma}fB_B\simeq D(B)$ as B-B-bimodules. Since $\Phi(D(A))\simeq D(B)$ in $\mathscr{D}(B^{\mathrm{e}})$, we have $\Phi(Ae\otimes_{\Lambda}eA)\simeq Bf\otimes_{\Gamma}fB$ in $\mathscr{D}(B^{\mathrm{e}})$. Now, Theorem 1.1 follows immediately from Theorem 3.1. $\square$

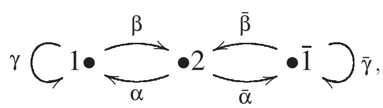
Finally, we present an example to illustrate the main result. We consider the truncated polynomial algebra $\Lambda:=k[x]/(x^3)$. Let X be the simple Λ-module and Y the indecomposable Λ-module of length 2. Then $A:=\mathrm{End}_{\Lambda}(\Lambda\oplus X)$ and $B:=\mathrm{End}_{\Lambda}(\Lambda\oplus Y)$ are derived equivalent, gendo-symmetric algebras. In this case, $Ae=\mathrm{Hom}_{\Lambda}(\Lambda\oplus X,\Lambda)$ and $Bf=\mathrm{Hom}_{\Lambda}(\Lambda\oplus Y,\Lambda)$. Clearly, $eAe\simeq eBe\simeq\Lambda$. Moreover, A and B are given by the following quivers with relations, respectively:

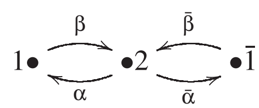
$A:$ 

$B:$ 

$$\alpha\beta=\alpha\gamma=0,\ \gamma^2=\beta\alpha. \qquad \alpha\beta\alpha\beta=0.$$

Further, e and f are corresponding to the vertex 1 in the quivers, respectively. $R(A,e)$ and $R(B,f)$ are presented by the following quivers with relations, respectively.

$R(A,e):$ 

$R(B,f):$ 

$$\beta\bar{\alpha}=\bar{\beta}\alpha=0,\ \alpha\gamma=\bar{\alpha}\bar{\gamma}=0, \qquad \beta\bar{\alpha}=\bar{\beta}\alpha=0,$$
$$\gamma^2=\beta\alpha,\ \bar{\gamma}^2=\bar{\beta}\bar{\alpha},\ \alpha\beta+\bar{\alpha}\bar{\beta}=0. \qquad \alpha\beta\alpha\beta+\bar{\alpha}\bar{\beta}\bar{\alpha}\bar{\beta}=0.$$

By Theorem 1.1, $R(A,e)$ and $R(B,f)$ are derived equivalent.

References

[1] M. AUSLANDER, I. REITEN and S. SMALØ, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics **36**. Cambridge University Press, Cambridge, 1995.

[2] H. X. CHEN and C. C. XI, Dominant dimensions, derived equivalences and tilting modules, *Isr. J. Math.* **215** (2016) 349-395.

[3] H. X. CHEN and C. C. XI, Homological theory of orthogonal modules, arXiv:2208.14712. Preprint, 1-40, 2022.

[4] H. X. CHEN, M. FANG and C. C. XI, Tachikawa's second conjecture, derived recollements, and gendo-symmetric algebras, Preprint, 2022, available at arXiv: 2211.08037.

[5] M. FANG and S. KOENIG, Endomorphism algebras of generators over symmetric algebras, *J. Algebra* **332** (2011) 428-433.

[6] M. FANG and S. KOENIG, Gendo-symmetric algebras, canonical comultiplication, bar cocomplexes and dominant dimension, *Trans. Amer. Math. Soc.* **368** (2016) 5037-5055.

[7] M. Fang and H. Miyachi, Hochschild cohomology and dominant dimension, *Trans. Amer. Math. Soc.* **371** (2019) 5267-5292.

[8] D. HAPPEL, On the derived category of a finite-dimensional algebra, *Comment. Math. Helv.* **62**(3) (1987) 339-389.

[9] M. HOSHINO, Modules without self-extensions and Nakayama's conjecture, *Arch. Math. (Basel)* **43** (1984) 493-500.

[10] M. HOSHINO, Y. KATO and J-I. MIYACHI, On t-structures and torsion theories induced by compact objects, *J. Pure Appl. Algebra* **167** (2002) 15-35.

[11] W. HU and C.C. XI, Derived equivalences and stable equivalences of Morita type, I. *Nagoya Math. J.* **200** (2010) 107-152.

[12] W. HU and C.C. XI, Derived equivalences for Φ-Auslander-Yoneda algebras, *Trans. Amer. Math. Soc.* **365** (2013) 5681-5711.

[13] B. KELLER, Deriving DG categories, *Ann. Sci. École Norm. Sup.* **74** (1994) 63-102.

[14] B. MÜLLER, The classification of algebras by dominant dimension, *Canad. J. Math.* **20** (1968) 398-409.

[15] J.-I. MIYACHI, Recollement and tilting complexes, *J. Pure Appl. Algebra* **183** (2003) 245-273.

[16] T. NAKAYAMA, On algebras with complete homology, *Abh. Math. Sem. Univ. Hamburg* **22** (1958) 300-307.

[17] A. NEEMAN, *Triangulated categories*, Annals of Mathematics Studies **148**, Princeton University Press, Princeton, NJ, 2001.

[18] J. RICKARD, Morita theory for derived categories, *J. London Math. Soc*. **39** (1989) 436-456.

[19] J. RICKARD, Derived categories and stable equivalences, *J. Pure Appl. Algebra* **61** (1989) 303-317.

[20] J. RICKARD, Derived equivalences as derived functors, *J. Lond. Math. Soc*. **43** (1991) 37-48.

[21] H. TACHIKAWA, Quasi-Frobenius rings and generalizations, *Lecture Notes in Mathematics* **351**, Berlin-Heidelberg New York 1973.

[22] CH. WEIBEL, *An introduction to homological algebra*, Cambridge University Press. Cambridge, 1994.

Hongxing Chen,
School of Mathematical Sciences & Academy for Multidisciplinary Studies, Capital Normal University, 100048 Beijing, P. R. China;
Email: chenhx@cnu.edu.cn (H.X.Chen)

Changchang Xi,
School of Mathematical Sciences, Capital Normal University, 100048 Beijing; & School of Mathematics and Statistics, Shaanxi Normal University, 710119 Xi'an, P. R. China
Email: xicc@cnu.edu.cn (C.C.Xi)