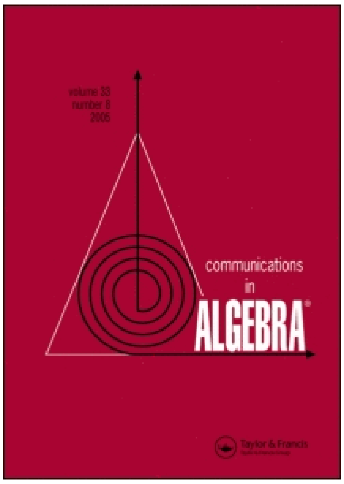


Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597239

Quasi-hereditary algebras which are dual extensions of algebras

Bangming Deng [a]; Changchang Xi [a]
[a] Department of Mathematics, Beijing Normal University, Beijing, P.R.China

Online Publication Date: 01 January 1994

To cite this Article Deng, Bangming and Xi, Changchang(1994)'Quasi-hereditary algebras which are dual extensions of algebras',Communications in Algebra,22:12,4717 — 4735

To link to this Article: DOI: 10.1080/00927879408825097

URL: http://dx.doi.org/10.1080/00927879408825097

COMMUNICATIONS IN ALGEBRA, 22(12), 4717-4735 (1994)

QUASI-HEREDITARY ALGEBRAS WHICH ARE DUAL EXTENSIONS OF ALGEBRAS

BANGMING DENG AND CHANGCHANG XI *

Department of Mathematics, Beijing Normal University, 100875 Beijing, P.R.China

June 28, 1993

Dedicated to Professor Xue-Fu Duan on his 80^{th} birthday

Introduction.

In connection with the study of highest weight categories arising in the representation theory of semisimple complex Lie algebras and algebraic groups Cline, Parshall and Scott introduced in [CPS] the notion of quasi-hereditary algebras. Many important algebras such as hereditary algebras, Schur algebras and algebras to blocks of category $\mathcal{O}$ which is studied in [BGG] are special classes of quasi-hereditary algebras. Quasi-hereditary algebras can be defined in ring-theoretic terms by the existence of a special sequence of ideals. These algebras have many applications and appear quite common.

Suppose A is a quasi-hereditary algebra, then there is a partial order $\leq$ on the set Λ of simple modules, and one studies the standard modules $\Delta = \{\Delta(\lambda)|\lambda \in \Lambda\}$. Of main interest is the category $\mathcal{F}(\Delta)$ of A-modules which have a Δ-filtration. C.M.Ringel proved in [R1] that $\mathcal{F}(\Delta)$ has relative almost split sequences. As a notable example, one considers the Schur algebra ([G]), in this case $\mathcal{F}(\Delta)$ becomes just the category consisting of all modules which have Weyl module filtration and is studied by many authors (see [D], [E], and others).

One of the interesting questions on $\mathcal{F}(\Delta)$ of a quasi-hereditary algebra is when the category $\mathcal{F}(\Delta)$ is finite (namely, there are only finitely many non-isomorphic

* Supported partially by the National Education Committee of China and the National Natural Science Foundation of China under Grant 19201003 .

indecomposable modules). The present paper provides an attempt to investigate $\mathcal{F}(\Delta)$ for a class of quasi-hereditary algebras which are defined as dual extensions (see below) of algebras without oriented cycle. These quasi-hereditary algebras are indeed BGG-algebras and have exact Borel subalgebras (see [X]). Our main result provides a useful reduction for these algebras with which one can decide whether $\mathcal{F}(\Delta)$ is finite even if the algebras themselves may be complicated (of wild type). Moreover, if a quasi-hereditary algebra is the dual extension of a hereditary algebra with radical square zero then we show that the relative Auslander-Reiten quiver of $\mathcal{F}(\Delta)$ is almost the same as the one of the given hereditary algebra.

The paper is organised as follows. In section 1 we recall some definitions and include some basic results needed in the sequel. From section 2 to 4 we prove the main results and give an explanation of our method. The last section contains results on the quadratic dual of the quadratic BGG-algebra which is the dual extension of a quadratic algebra without oriented cycle in its quiver.

Throughout the paper, algebras always mean finite-dimensional algebras over an algebraically closed field k and modules always mean finitely generated left modules, and notation on quivers is taken from [R2].

1. Preliminaries.

1.1 Let A be a finite-dimensional k-algebra. We denote by A-mod the category of all finite-dimensional left A-modules. Given a class Θ of A-modules, we denote by $\mathcal{F}(\Theta)$ the full subcategory of A-mod defined to be the class of all A-modules which have a Θ-filtration, that is, a filtration

$$0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M$$

such that each $M_{i-1}/M_i, 1 \leq i \leq t$, is isomorphic to an objects in Θ.

Let $E(1), \cdots, E(n)$ be the simple A-modules, one from each isomorphism class. Note that here we have fixed an ordering of simple modules. Let $P(i)$ be the projective cover of $E(i)$ and $Q(i)$ the injective hull of $E(i)$. We define $\Delta(i)$ to be the largest factor module of $P(i)$ with all composition factors of the form $E(j)$ for $j \leq i$ and call it a standard module. Dually, the so-called costandard module $\nabla(i)$ is defined to be the largest submodule of $Q(i)$ with composition factors $E(j)$ with $j \leq i$. Let Δ be the full subcategory formed by all $\Delta(i), 1 \leq i \leq n$ and ∇ the full subcategory formed by all $\nabla(i), 1 \leq i \leq n$. (Sometimes we denote Δ (respectively, ∇) by Δ_A (respectively, ∇_A) in order to indicate the considered algebra A).

The algebra A (or better the pair (A, Λ)) is called quasi-hereditary if

(i) $\mathrm{End}_A(\Delta(i)) \cong k$ for all i, and

(ii) Every projective module belongs to $\mathcal{F}(\Delta)$.

The algebra A is called a BGG-algebra ([I]) if (i),(ii) and the following condition (iii) are satisfied:

(iii) There is a duality $\delta : A$-mod $\longrightarrow A$-mod which fixes simple modules.

1.2 Suppose A is a quasi-hereditary algebra. We will collect now some results which will be needed in this context, proofs may be found in [R1].

Lemma. (1) $\mathrm{Ext}_A^t(X,Y)=0$ for all $X\in\mathcal{F}(\Delta)$ and $Y\in\mathcal{F}(\nabla)$ and all $t\geq 1$.

(2) For each i, there is a unique (up to isomorphism) indecomposable module $T(i)\in\mathcal{F}(\Delta)\cap\mathcal{F}(\nabla)$ with the folowing exact sequences

$$0\longrightarrow\Delta(i)\longrightarrow T(i)\longrightarrow X(i)\longrightarrow 0$$

$$0\longrightarrow Y(i)\longrightarrow T(i)\longrightarrow\nabla(i)\longrightarrow 0$$

such that $X(i)\in\mathcal{F}(\Delta(1),\cdots,\Delta(i-1))$, $Y\in\mathcal{F}(\nabla(1),\cdots,\nabla(i-1))$ and $\mathcal{F}(\Delta)\cap\mathcal{F}(\nabla)=\mathrm{add}(\bigoplus_{i=1}^n T(i))$.

(3) Put $T=\bigoplus T(i)$, we call T the characteristic module. Then for every A-module $M\in\mathcal{F}(\Delta)$, there is an exact sequence

$$0\longrightarrow M\longrightarrow T_0\longrightarrow T_1\longrightarrow\cdots\longrightarrow T_d\longrightarrow 0$$

with $T_i\in\mathrm{add}T$.

(4) $\mathrm{Ext}_A^1(\Delta(i),\Delta(j))=0$ for all $i\geq j$ and $\mathrm{Hom}_A(\Delta(i),\Delta(j))=0$ for all $i>j$. In particular, $\mathrm{Hom}_A(\mathcal{F}(\Delta(i),\cdots,\Delta(n)),\mathcal{F}(\Delta(1),\cdots,\Delta(i-1)))=0$.

(5) For each i and for each module $M\in\mathcal{F}(\Delta)$, there is a submodule M' of M such that $M'\in\mathcal{F}(\Delta(i),\cdots,\Delta(n))$ and $M/M'\in\mathcal{F}(\Delta(1),\cdots,\Delta(i-1))$.

1.3 A special class of quasi-hereditary algebras is constructed in [X]. Let us now recall the construction.

Let B be a finite-dimensional basic algebra over k. As usual we say that B is given by a quiver $Q_B=(Q_0,Q_1)$ with relations $\{\rho_i|\ i\in I_B\}$, that is, we consider the algebra $kQ^*/<\{\rho_i^*\ |\ i\in I_B\}>$, where Q^* is the opposite quiver of Q and the multiplication $\alpha\beta$ of two arrows α and β means that α comes first and then β follows (for the notation see [R2, 2] for details). For each α from i to j in Q_1, let α' be an arrow from j to i. We denote by Q_1' the set of all such α' with $\alpha\in Q_1$. For a path $\alpha_1\cdots\alpha_m$ we denote by $(\alpha_1\cdots\alpha_m)'$ the path $\alpha_m'\cdots\alpha_1'$ in (Q_0,Q_1'). With this notation we may define a BGG-algebra.

Definition. Suppose that B is an algebra given by the quiver $Q_B=(Q_0,Q_1)$ with relations $\{\rho_i|\ i\in I_B\}$. Let A be the algebra given by the quiver $(Q_0,Q_1\cup Q_1')$ with relations $\{\rho_i|\ i\in I_B\}\cup\{\rho_i'|\ i\in I_B\}\cup\{\alpha\beta'|\ \alpha,\beta\in Q_1\}$. Then it is a finite-dimensional algebra over k.

If B has no oriented cycle in its quiver, we may assume that $Q_0=\{1,\cdots,n\}$ such that $\mathrm{Hom}_B(P_B(i),P_B(j))=0$ for $i>j$, then A is quasi-hereditary. Furthermore, the standard A-modules are $\Delta_A(i)=P_B(i)$ for $i\in\{1,\cdots,n\}$. We say that A is the dual extension of B, denoted by $\mathcal{A}(B)$.

From now on, we suppose that B is a basic algebra without oriented cycle.

Since for this algebra $A=\mathcal{A}(B)$ one has a duality which fixes all simple modules, it is in fact a BGG-algebra in the sense of [I]. For this BGG-algebra A obtained from B we have the following properties.

1.4 Lemma. (1) B is a subalgebra of A with the same maximal semisimple subalgebra and B is also a factor algebra of A.

(2) Let B' be given by (Q_0, Q_1') with relations $\{\rho_i' \mid i \in I_B\}$. Then B' is a subalgebra of A with the same maximal semisimple subalgebra and B' is also the factor algebra of A modulo the ideal generated by $\{\alpha^* \mid \alpha \in Q_1\}$. Moreover. $\nabla_A(i) = Q_{B'}(i)$ for all i.

(3) The module $A_{B'}$ is projective.

(4) $\dim_k A = \sum_i (\dim_k P_B(i))^2$.

(5) Let $\mu : B \to A$ be the inclusion in (1). Then for each algebra-homomorphism $f : B \to R$, there is a homomorphism $g : A \to R$ such that $f = \mu g$.

Proof. By the construction of A, the statements (1) to (3) and (5) are trivial. The statement (4) follows from the following fact which can be proved by using the BGG-reciprocity (see [CPS] or [I]):

Let A be a BGG-algebra with standard modules $\Delta(1), \cdots, \Delta(n)$. Then $\dim_k A = \sum_i (\dim_k \Delta(i))^2$.

To see whether a module M lies in $\mathcal{F}(\Delta)$ we have the following

1.5 Lemma. Let M be an A-module. Then $M \in \mathcal{F}(\Delta)$ if and only if ${}_BM$ is projective.

Proof. Let M be in $\mathcal{F}(\Delta)$. Since $\Delta_A(i), 1 \le i \le n$, are projective B-modules, one knows that M as a B-module is a successive extension of projective B-modules and hence is a projective B-module.

Now assume that ${}_BM$ is a projective B-module. We shall show that M belongs to $\mathcal{F}(\Delta)$. Let ${}_BM'$ be a submodule of ${}_BM$ such that ${}_BM'$ is a direct sum of all direct summands of ${}_BM$ isomorphic to Be_n. Then M' is also an A-module since $Be_n = Ae_n = \Delta_A(n)$. We may write ${}_BM = {}_B M' \oplus_B M''$. Thus there is an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0$$

in A-mod. Since $M/M' \cong M''$ as B-modules, by induction, the A/Ae_nA-module $M/M' \in \mathcal{F}(\Delta_{\bar{A}}(1), \cdots, \Delta_{\bar{A}}(n-1))$, where $\bar{A} = A/Ae_nA$ can be considered as the dual extension of $\bar{B} = B/Be_nB$ by 1.3. Hence $M \in \mathcal{F}(\Delta)$.

1.6 To compare two categories $\mathcal{F}(\Delta_A)$ and B'-mod, we consider the functors $F = {}_AA \otimes_{B'} - : B'\text{-mod} \to A\text{-mod}$ and $G = B'_A \otimes - : A\text{-mod} \to B'\text{-mod}$. The basic properties of these functors are formulated in the following lemma.

Lemma. (1) F is an exact functor.

(2) $FE(i) \cong \Delta_A(i), FP_{B'}(i) \cong P_A(i)$ for all i.

(3) $G(FM) \cong {}_{B'}M$ for each B'-module M.

Proof. (1) follows from 1.4.

(2) It is easy to see that B' is an exact Borel subalgebra of A in the sense of [K]. Thus (2) follows from the definition of an exact Borel subalgebra.

(3) Since $G(FM) = {}_{B'}B'_A \otimes (A \otimes_{B'} M) \cong (B' \otimes_A A) \otimes_{B'} M \cong B' \otimes_{B'} M \cong M$, the statement follows.

1.7 Remark. From the definition 1.3 we know that the dual extension $A = \mathcal{A}(B)$ of an algebra B without oriented cycle is always a lean quasi-hereditary algebra in the sense of [ADL] since $C_t = \varepsilon_t A \varepsilon_t$ can be obtained from the quiver of A by restricting it to the vertices $\{t, t+1, \cdots, n\}$, where $\varepsilon_t = e_t + e_{t+1} + \cdots + e_n$.

2. Reduction.

In this section we will use the theory of vector space categories to give a method to determine whether $\mathcal{F}(\Delta)$ is finite. In this way the question is reduced from a bigger algebra to a smaller one. It turns out that this reduction is powerful when one deals with the dual extension algebra A of an algebra having no oriented cycle as in section 1.

Let us first recall some definitions.

2.1 Definition ([R2]). Let $\mathcal{K}$ be a Krull-Schmidt k-category and $|\cdot| : \mathcal{K} \to k-\text{mod}$ an additive functor. The pair $(\mathcal{K}, |\cdot|)$ is called a vector space category. We denote by $\check{\mathcal{U}}(\mathcal{K}, |\cdot|)$, called subspace category of $(\mathcal{K}, |\cdot|)$, the category of all triples $V = (V_0, V_\omega, \gamma_V : V_\omega \to |V_0|)$, where $V_\omega \in k-\text{mod}$, $V_0 \in \mathcal{K}$ and γ_V is a k-linear map. A morphism from $V \to V'$ by definition is a pair (f_0, f_ω), where $f_0 : V_0 \to V_0'$ and $f_\omega : V_\omega \to V_\omega'$ such that $f_\omega \gamma_{V'} = \gamma_V |f_0|$.

An additive k-category is called finite if there are finitely many isomorphism classes of indecomposable objects.

If A_0 is an algebra over k, and R is an A_0-module, one may form the one-point extension

$$A = \begin{bmatrix} A_0 & R \\ 0 & k \end{bmatrix}$$

This algebra is denoted by $A = A_0[R]$. We denote by ω the extension vertex of A. Clearly, $A-\text{mod} \simeq \check{\mathcal{U}}(A_0-\text{mod}, \text{Hom}_{A_0}(R,-))$. Dually, one has one-point coextension $[R]A_0$ which is defined to be the following matrix algebra

$$\begin{bmatrix} k & \text{Hom}_k(R,k) \\ 0 & A_0 \end{bmatrix},$$

Suppose A_0 is a quasi-hereditary algebra and $R \in \mathcal{F}(\Delta_{A_0})$. Then, by setting $\Delta_A(\omega) = (0, k, 0)$, $\Delta_A(1) = \Delta_{A_0}(1), \cdots, \Delta_A(n) = \Delta_{A_0}(n)$, the algebra $A = A_0[R]$ becomes a quasi-hereditary algebra. Moreover, we have

2.2 Lemma. $\mathcal{F}(\Delta_A) = \check{\mathcal{U}}(\mathcal{F}(\Delta_{A_0}), \text{Hom}_{A_0}(R,-))$.

Proof. Since $R \in \mathcal{F}(\Delta_{A_0})$ we see that $P(\omega) \in \mathcal{F}(\Delta_A)$. For any object $V = (V_0, V_\omega, \gamma_V) \in A-\text{mod}$ there is the following exact sequence

$$0 \longrightarrow (V_0, 0, 0) \longrightarrow (V_0, V_\omega, \gamma) \longrightarrow (0, V_\omega, 0) \longrightarrow 0$$

which shows that $V_0 \in \mathcal{F}(\Delta_{A_0})$ if $V \in \mathcal{F}(\Delta_A)$. Conversely, if $V = (V_0, V_\omega, \gamma_V) \in \check{\mathcal{U}}(\mathcal{F}(\Delta_{A_0}), \text{Hom}_{A_0}(R,-))$, one can again form the above sequence and knows from $(0, V_\omega, 0) \cong \Delta_A(\omega)^{\dim_k V_\omega}$ that V is an extension of $(V_0, 0, 0) \in \mathcal{F}(\Delta_{A_0})$ and a module in $\text{add}\Delta_A(\omega)$. Hence $V \in \mathcal{F}(\Delta_A)$.

2.3 Let D be a quasi-hereditary algebra with standard modules $\Delta_D(2), \cdots, \Delta_D(n)$. Suppose that there is a quasi-hereditary algebra C which is a coextension of D by a D-module such that $\Delta_D(i) \cong \Delta_C(i)/Ce_1\Delta_C(i)$ for $i \geq 2$, where 1 is the coextension vertex of C. (Note that the simple C-modules have the ordering $E_C(1), \cdots, E_C(n)$). Now we want to build a relationship between $\mathcal{F}(\Delta_C)$ and $\mathcal{F}(\Delta_D)$ and compare these two categories.

Proposition. There is an exact functor $\eta : \mathcal{F}(\Delta_C) \longrightarrow \mathcal{F}(\Delta_D)$ such that

(1) η is dense and full.

(2) If $f : M \to N$ is a homomorphism, then $\eta(f) = 0$ if and only if f factors through $\operatorname{add}(E_C(1))$.

(3) If $\eta(M) = 0$ for some indecomposable module $M \in \mathcal{F}(\Delta_C)$, then $M \cong \Delta_C(1) = E_C(1)$.

(4) For any $M \in \mathcal{F}(\Delta_C)$, if M has no direct summand isomorphic to $E_C(1)$, then the module $\eta(M)$ is indecomposable if and only if M is indecomposable.

(5) Given two indecomposable modules M_1 and M_2 in $\mathcal{F}(\Delta_C)$, they are isomorphic if and only if $\eta(M_1)$ and $\eta(M_2)$ are isomorphic.

(6) If R is a module in $\mathcal{F}(\Delta_C)$ and contains no direct summand isomorphic to $E_C(1)$, then $\operatorname{Hom}_C(R, M) \cong \operatorname{Hom}_D(\eta(R), \eta(M))$ for all $M \in \mathcal{F}(\Delta_C)$.

Proof. Note that D is the factor algebra of C modulo the ideal Ce_1C and we can identify D with C/Ce_1C. Put $\eta(M) = M/Ce_1M$ for all $M \in C-\text{mod}$. Since Ce_1 is a simple projective module, Ce_1M is just the maximal semisimple submodule of M with each direct summand isomorphic to $E_C(1)$. This implies that for each $f : M \to N$ one can define $\eta(f)$ to be the induced map from $\eta(M)$ to $\eta(N)$ by f:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Ce_1M & \longrightarrow & M & \xrightarrow{\pi_M} & \eta(M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow f & & \downarrow \eta(f) & & \\ 0 & \longrightarrow & Ce_1N & \longrightarrow & N & \xrightarrow{\pi_N} & \eta(N) & \longrightarrow & 0 \end{array}$$

Clearly, η is a well-defined functor from C-mod to D-mod.

Since Ce_1 is a projective C-module and η is naturally equivalent to the functor $(C/Ce_1C) \otimes_C -$, we see that η is an exact functor which sends $\Delta_C(i)$ to $\Delta_D(i)$ for all $i \geq 2$. Hence if $M \in \mathcal{F}(\Delta_C)$ then $\eta(M) \in \mathcal{F}(\Delta_D)$.

(1) η is dense. Indeed, let ${}_DX \in \mathcal{F}(\Delta_D)$. If ${}_DX \cong \Delta_D(n)$ then there is nothing to prove. So we assume that if ${}_DX$ has Δ_D-composition factors of the form $\Delta_D(j)$ with $i \leq j \leq n$ then there is a module ${}_CM \in \mathcal{F}(\Delta_C)$ such that $\eta(M) \cong X$.

Using induction, we shall prove that if ${}_DX \in \mathcal{F}(\Delta_D)$ with Δ_D- composition factors of the form $\Delta_D(j)$, $i-1 \leq j \leq n$ then we can find a module $M \in \mathcal{F}(\Delta_C)$ with $\eta(M) \cong X$.

Let X be such a module in $\mathcal{F}(\Delta_D)$. Then by Lemma 1.2 (5), there exists a submodule X_0 of X with $X_0 \in \mathcal{F}(\Delta_D(i), \cdots, \Delta_D(n))$ such that $X/X_0 \in \mathcal{F}(\Delta_D(i-1))$. Hence we have the following exact sequence

$$0 \longrightarrow X_0 \xrightarrow{\beta'} X \xrightarrow{\beta} \Delta_D(i-1)^l \longrightarrow 0$$

with $l = [M : \Delta_D(i-1)]$, the number of factors $\Delta_D(i-1)$ in a Δ_D- filtration of M.

By induction, there is a module $M_0 \in \mathcal{F}(\Delta_C)$ such that $\eta(M_0) = X_0$. Since $\mathrm{Ext}^2_C(\Delta_C(i-1), E_C(1)) = 0$ by 1.2 (1), we may form the following diagram in C-mod:

$$\begin{array}{ccccccccc}
 & & Ce_1M_0 & = & Ce_1M_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M_0 & \xrightarrow{x} & M & \xrightarrow{y} & \Delta_C(i-1)^l & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_{M_0}} & & \downarrow & & \| & & \\
0 & \longrightarrow & X_0 & \xrightarrow{\mu} & X' & \xrightarrow{\delta} & \Delta_C(i-1)^l & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & X_0 & \xrightarrow{\beta'} & {}_DX & \xrightarrow{\beta} & \Delta_D(i-1)^l & \longrightarrow & 0
\end{array}$$

where γ is the canonical projection, and where X' is the pullback of β and γ and also the pushout of π and x. (Note that for $M_0 \in C$–mod there holds $Ce_1M_0 \cong E_C(1)^l = \ker(\gamma)$). Then we may form the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Ce_1M_0 & \longrightarrow & M' & \longrightarrow & Ce_1X' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Ce_1M_0 & \longrightarrow & M & \longrightarrow & X' & \longrightarrow & 0
\end{array}$$

with M' a pullback and $M' \cong E_C(1)^{l+m}$. Since $M/M' \cong {}_DX$, we get that $M' = Ce_1M$ and $\eta(M) \cong {}_DX$. This shows that η is dense.

Suppose $M_1, M_2 \in \mathcal{F}(\Delta_C)$ and $f : \eta(M_1) \to \eta(M_2)$. Applying $\mathrm{Hom}_C(M_1, -)$ to the exact sequence (in C–mod)

$$0 \longrightarrow Ce_1M_2 \longrightarrow M_2 \longrightarrow \eta(M_2) \longrightarrow 0,$$

one obtains the following exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_C(M_1, M_2) \longrightarrow \mathrm{Hom}_C(M_1, \eta(M_2)) \longrightarrow \mathrm{Ext}^1_C(M_1, Ce_1M_2) \longrightarrow \cdots$$

Since $Ce_1M_1 \cong E_C(1)^m \in \mathcal{F}(\nabla_C)$ and $\mathrm{Ext}^1_C(\mathcal{F}(\Delta_C), \mathcal{F}(\nabla_C)) = 0$, one has $\mathrm{Ext}^1_C(M_1, Ce_1M_2) = 0$ and the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Ce_1M_1 & \longrightarrow & M_1 & \longrightarrow & \eta(M_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & Ce_1M_2 & \longrightarrow & M_2 & \longrightarrow & \eta(M_2) & \longrightarrow & 0
\end{array}$$

This shows that there exists a homomorphism $f' \in \mathrm{Hom}_C(M_1, M_2)$ with $\eta(f') = f$ by the definition of η. Hence η is full.

(2) This is trivial.

(3) Suppose M is indecomposable and $\eta(M) = 0$. Then $M = Ce_1M \cong E_C(1)^l$ and (3) follows.

(4) If M is indecomposable with $\eta(M) \neq 0$ then $\eta(M)$ is indecomposable since η is full. Now let M be a module in $\mathcal{F}(\Delta_C)$ which has no direct summand isomorphic to $E_C(1)$. Suppose that $\eta(M)$ is indecomposable. We shall prove that M is also indecomposable. Taking an $f \in \mathrm{End}_C(M)$, then $\eta(f)$ is invertible or nilpotent. If $\eta(f)$ is invertible then there exists a homomorphism $g' \in \mathrm{End}_C(\eta(M))$ such that $\eta(f)g' = 1$. By (1), there is an endomorphism g of M such that $\eta(g) = g'$ and $\eta(fg) = \eta(f)g'$. Hence $\eta(fg-1) = 0$ and $fg-1$ factors through Ce_1M, say $fg - 1 = h\mu$, where $h : M \to Ce_1M$ and $\mu : Ce_1M \to M$. If $fg - 1 \neq 0$ then the image of h is nonzero and M contains a direct summand isomorphic to $\mathrm{Im}(h)$ since Ce_1M is projective semisimple module. This means that M has a direct summand isomorphic to $E_C(1)$, a contradiction. Hence $fg = 1$. If $\eta(f)$ is nilpotent then there is a natural number m such that f^m factors over Ce_1M. Similar to the above discussion we can show that $f^m = 0$. Hence M is indecomposable.

(5) Similar argument as in the proof of (4) gives the assertion.

(6) follows from (1) and (2).

Now the functor η induces a functor $\Phi : \check{\mathcal{U}}(\mathcal{F}(\Delta_C), \mathrm{Hom}_C(R,-)) \to \check{\mathcal{U}}(\mathcal{F}(\Delta_D),$ $\mathrm{Hom}_D(\eta(R),-))$, where R is a module in $\mathcal{F}(\Delta_C)$.

2.4 Lemma. Suppose R is a module in $\mathcal{F}(\Delta_C)$ and has no direct summand isomorphic to $E_C(1)$. Then the functor $\eta : \mathcal{F}(\Delta_C) \to \mathcal{F}(\Delta_D)$ induces a canonical functor $\Phi : \check{\mathcal{U}}(\mathcal{F}(\Delta_C), \mathrm{Hom}_C(R,-)) \to \check{\mathcal{U}}(\mathcal{F}(\Delta_D), \mathrm{Hom}_D(\eta(R),-))$ such that

(1) Φ is exact, full and dense.

(2) If $V \in \check{\mathcal{U}}(\mathcal{F}(\Delta_C), \mathrm{Hom}_C(R,-))$ is indecomposable with $\Phi(V) = 0$, then $V \cong (E_C(1), 0, 0)$.

Proof. For each object $V = (V_0, V_\omega, \gamma_V) \in \check{\mathcal{U}}(\mathcal{F}(\Delta_C), \mathrm{Hom}_C(R,-))$ define $\Phi(V) = (\eta(V_0), V_\omega, \bar{\gamma}_V)$, where $\bar{\gamma}_V$ is the composition of γ_V and $\eta_{R,V_0} : \mathrm{Hom}_C(R, V_0) \simeq \mathrm{Hom}_D(\eta(R), \eta(V_0))$. To each map (f_0, f_ω) define $\Phi(f_0, f_\omega) = (\eta(f_0), f_\omega)$. It is easy to see that Φ is a well-defined functor.

(1) Since η is exact and dense, Φ is exact and dense too. Now we show that Φ is even full. Since η is full, given any homomorphism $g : \eta(V_0) \to \eta(V_0')$, we can find a homomorphism $f_0 : V_0 \to V_0'$ such that $\eta(f_0) = g$. Let $(\eta(f_0), f_\omega)$ be a homomorphism from $(\eta(V_0), V_\omega, \bar{\gamma}_V)$ to $(\eta(V_0'), V_\omega', \bar{\gamma}_{V'})$. Then we consider the

following diagram:

$$\begin{array}{ccccc} V_\omega & \xrightarrow{\gamma_V} & \mathrm{Hom}_C(R,V_0) & \xrightarrow{\eta_{R,V_0}} & \mathrm{Hom}_D(\eta(R),\eta(V_0)) \\ \downarrow{\scriptstyle f_\omega} & & \downarrow{\scriptstyle \mathrm{Hom}_C(R,f_0)} & & \downarrow{\scriptstyle \mathrm{Hom}_D(\eta(R),\eta(f_0))} \\ V'_\omega & \xrightarrow[\gamma_{V'}]{} & \mathrm{Hom}_C(R,V'_0) & \xrightarrow[\eta_{R,V'_0}]{} & \mathrm{Hom}_D(\eta(R),\eta(V'_0)) \end{array}$$

Put $x = \gamma_V \mathrm{Hom}_C(R,f_0) - f_\omega \gamma_{V'}$, then $x\eta_{R,V'_0} = 0$ by definition. According to 2.3 (6), η_{R,V'_0} is an isomorphism, thus $x = 0$ and the first square of the above diagram is commutative. This implies that Φ is full.

(2) If $(\eta(V_0), V_\omega, \bar{\gamma}_V) = 0$ then $V_\omega = 0$ and $\eta(V_0) = 0$. This means $V_0 \cong E_C(1)^l$. Since V is indecomposable, $V_0 \cong E_C(1)$ and $V \cong (E_C(1), 0, 0)$.

Combining 2.3 and 2.4 we have

2.5 Theorem. Let C be a quasi-hereditary algebra with standard modules $\Delta_C(1), \cdots, \Delta_C(n)$ such that

(1) C is the coextension of a quasi-hereditary algebra D with standard modules $\Delta_D(2), \cdots, \Delta_D(n)$, and

(2) For $2 \leq j$ there holds $\Delta_D(j) \cong \Delta_C(j)/Ce_1\Delta_C(j)$. Then

(i) $\mathcal{F}(\Delta_C)$ is finite if and only if $\mathcal{F}(\Delta_D)$ is finite.

(ii) If R is a module in $\mathcal{F}(\Delta_C)$ which does not contain a direct summand isomorphic to $E_C(1)$, then $\check{\mathcal{U}}(\mathcal{F}(\Delta_C), \mathrm{Hom}_C(R,-))$ is finite if and only if $\check{\mathcal{U}}(\mathcal{F}(\Delta_D), \mathrm{Hom}_D(R/Ce_1R,-))$ is finite.

Proof. (i) follows from 2.3 and (ii) follows from 2.3 and 2.4.

This result shows that in order to know whether $\mathcal{F}(\Delta_C)$ is finite one may verify if $\mathcal{F}(\Delta_D)$ is finite. In the next section we apply this result to give a reduction to determine the finiteness of $\mathcal{F}(\Delta)$ for a class of quasi-hereditary algebras.

3. Application.

As an application of the results in section 2 we study in this section the class of quasi-hereditary algebras which are the dual extensions of algebras whose quivers have no oriented cycle. We reduce the determination of $\mathcal{F}(\Delta_A)$ to that of $\mathcal{F}(\Delta_D)$ with D the dual extension of a factor algebra of B. In such a way one can determine whether $\mathcal{F}(\Delta_A)$ is finite.

Let us fix some notation. Let B again be a basic algebra without oriented cycle in its quiver. Since for the dual extension algebra A of B the standard modules $\{\Delta_A(i) | \ 1 \leq i \leq n\}$ are just the indecomposable projective B-modules, we may always assume that the vertices of the quiver of B are ordered in such a way that $\mathrm{Hom}_B(P_B(i), P_B(j)) = 0$ for $i > j$. Hence the vertex 1 is always a sink in the quiver of the algebra B. Let $\alpha_1, \cdots, \alpha_m$ be all the arrows in the quiver of B ending at the vertex 1. We denote by D the dual extension of $\overline{B} = B/Be_1B$.

Now we try to use 2.2 to describe our category $\mathcal{F}(\Delta_A)$. From this point of view we introduce the following new algebra A_0.

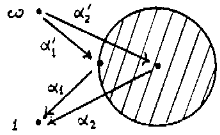
By the construction of A, an A-module M can be always regarded as a module over A_0, where A_0 is obtained from A by decomposing the vertex 1 in the quiver of A into 1 and $1' = \omega$ such that

(1) α_i' starts at ω and there is no any other arrow between ω and $j \in Q_0$, and

(2) all other arrows in the quiver of A remain in that of A_0, and

(3) all relations of A are just the relations for A_0.

The quiver of A_0 looks like the following



Conversely, any module over A_0 can be regarded as a module over A.

Denote by C the full subalgebra of A_0 with the vertex set $\{1, \cdots, n\}$. Since $\text{rad}P_{A_0}(\omega) = \text{rad}P_A(1)$, we see that A_0 is the one-point extension of the algebra C by the C-module $R = \text{rad}P_A(1)$ and C is a quasi-hereditary algebra with standard modules $\Delta_C(1) = E_C(1) = \Delta_A(1), \cdots, \Delta_C(n) = \Delta_A(n)$. (Note that $P_C(1) = E_C(1), P_C(j) = P_A(j)$ for $j \neq 1$). Also, A_0 is a quasi-hereditary algebra with standard modules $\Delta_{A_0}(\omega) = E_{A_0}(\omega), \Delta_{A_0}(i) = \Delta_A(i)$ for $i \neq \omega$. Furthermore, we have the following observation.

3.1 Lemma. $\mathcal{F}(\Delta_{A_0}) = \mathcal{F}(\Delta_A) \vee E_{A_0}(\omega)$.

From this lemma we know that the main question is to determine $\mathcal{F}(\Delta_{A_0})$. By 2.2, this is equivalent to the investigation of $\check{\mathcal{U}}(\mathcal{F}(\Delta_C), \text{Hom}_C(R, -))$. Since the conditions in 2.3 are satisfied by our algebras C and D, we can apply the results in section 2 to these algebras to reduce the investigation of $\mathcal{F}(\Delta_A)$ to $\mathcal{F}(\Delta_D)$, namely, we have the following result:

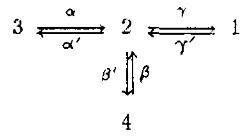
3.2 Theorem. $\mathcal{F}(\Delta_A)$ is finite if and only if $\check{\mathcal{U}}(\mathcal{F}(\Delta_D), \text{Hom}_D(R/Ce_1R, -))$ is finite.

Thus one can use the well-known results on the vector space category or representation theory of finite partially ordered sets as a tool to study the subcategories $\mathcal{F}(\Delta_A)$. In this case the following lemma may be useful.

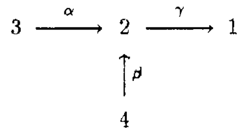
Lemma ([GR], sect. 4.7). If the category $\check{\mathcal{U}}(\mathcal{K}, |\cdot|)$ is finite, then for each indecomposable object $X \in \mathcal{K}$, the right module $|X|$ over $\text{End}_{\mathcal{K}}(X)$ is uniserial.

To explain our reduction let us consider the following examples.

3.3 Examples (1) Let A be given by the quiver

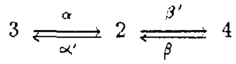


with $\alpha\alpha' = \beta\beta' = \gamma\gamma' = \alpha\gamma = \alpha\beta' = \beta\alpha' = \gamma'\alpha' = \beta\gamma = \gamma'\beta' = 0$ which is the dual extension of B given by



$$\alpha\gamma = \beta\gamma = 0.$$

By reduction, the quiver of D is



with $\quad \alpha\alpha' = \beta\beta' = 0$

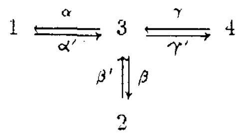
and the corresponding subspace category is $\check{\mathcal{U}}(\mathcal{F}(\Delta_D), \mathrm{Hom}_D(E_D(2), -))$, where $E_D(2)$ denotes the simple module corresponding to the vertex 2.

Consider $\mathrm{Hom}_D(E_D(2), P_D(2)) \cong k^2$ as right $\mathrm{End}_D(P_D(2))$-module which is annihilated by the radical of $\mathrm{End}_D(P_D(2)))$, thus it is a semisimple right $\mathrm{End}_D(P_D(2))$-module and is not uniserial. By the above lemma, $\check{\mathcal{U}}(\mathcal{F}(\Delta_D), \mathrm{Hom}_D(E_D(2), -))$ is infinite. It follows from 3.2 that $\mathcal{F}(\Delta_A)$ is infinite. In fact, the modules $M_\lambda = (M_1, M_2, M_3, M_4;\ \alpha, \alpha', \beta, \beta', \gamma, \gamma'_\lambda)$ for $\lambda \in k$ given by

$$\left(k^2, k^3, k, k; [1\ 0\ 0],\ \begin{bmatrix}0\\1\\0\end{bmatrix},\ [0\ 0\ 1],\ \begin{bmatrix}0\\1\\0\end{bmatrix},\ \begin{bmatrix}0&0\\0&1\\0&0\end{bmatrix},\ \begin{bmatrix}1&0&0\\0&0&\lambda\end{bmatrix}\right)$$

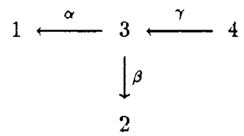
provide a family of infinitely many non-isomorphic indecomposable modules in $\mathcal{F}(\Delta_A)$. (One can use 1.5 to decide that M_λ, $\lambda \in k$, have really a Δ_A-filtration).

(2) Let A be given by the quiver



$$\alpha\alpha' = \beta\beta' = \gamma\gamma' = 0$$
$$\gamma\alpha = \gamma\beta = \alpha'\gamma' = \beta'\gamma' = 0$$

It is obtained from the algebra B:



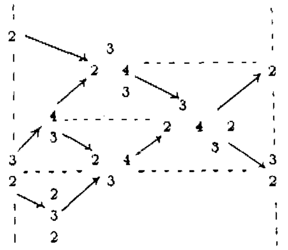
$$\gamma\alpha = \gamma\beta = 0$$

Then the quiver of D looks like

$$2 \underset{\beta}{\overset{\beta'}{\rightleftarrows}} 3 \underset{\gamma}{\overset{\gamma'}{\rightleftarrows}} 4 \qquad \beta\beta' = \gamma\gamma' = \gamma\beta = \beta'\gamma' = 0$$

and $R/Ce_1R = \Delta_D(3)$.

The Auslander–Reiten quiver of $\mathcal{F}(\Delta_D)$ has the form



where the indecomposable modules are displayed by their Loewy factors and the dotted vertical lines should be identified.

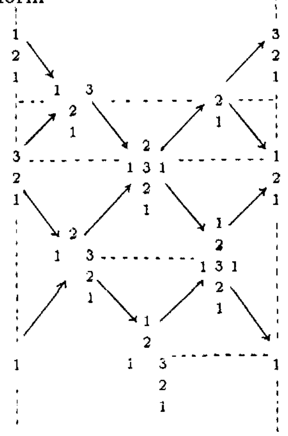
It is easy to show that $\mathcal{U}(\mathcal{F}(\Delta_D), \mathrm{Hom}_D(\Delta_D(3), -))$ is finite. Indeed, the investigation of this subspace category can be easily converted into that of a poset of finite type ([GR], sect. 4.1). As a consequence of 3.2, $\mathcal{F}(\Delta_D)$ is finite. In fact, there are 17 isomorphism classes of indecomposables in $\mathcal{F}(\Delta_A)$.

3.4 Proposition. Let $\mathcal{A}_n$ be the algebra given by the quiver

$$1 \underset{\alpha'_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\alpha'_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \cdots n-1 \underset{\alpha'_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} n$$

with $\alpha_i\alpha'_i = 0, 1 \leq i \leq n-1$, which is the dual extension of the Dynkin diagram A_n. Then $\mathcal{F}(\Delta_A)$ is finite if $n \leq 3$, and infinite if $n > 3$.

Proof. The cases $n = 1, 2$ are trivial. In case $n = 3$, the Auslander–Reiten quiver of $\mathcal{F}(\Delta_{\mathcal{A}_3})$ has the form



where the dotted vertical lines are identified. Then $\mathcal{F}(\Delta_{A_3})$ is finite.

In case $n = 4$, by deleting the vertex 1, we obtain D given by

$$2 \underset{\alpha_2'}{\overset{\alpha_2}{\rightleftarrows}} 3 \underset{\alpha_3'}{\overset{\alpha_3}{\rightleftarrows}} 4 \qquad \alpha_2\alpha_2' = \alpha_3\alpha_3' = 0.$$

The subspace category $\check{\mathcal{U}}(\mathcal{F}(\Delta_D), \mathrm{Hom}_D(P_D(2), -))$ is infinite since the space $\mathrm{Hom}_D(P_D(2), P_D(2)) \cong k^3$ considered as right $\mathrm{End}_D(P_D(2))$–module is not uniserial, for $\dim_k \mathrm{rad}(\mathrm{End}_D(P_D(2))) = 2$ and $\mathrm{rad}^2(\mathrm{End}_D(P_D(2))) = 0$. Hence $\mathcal{F}(\Delta_{A_4})$ is infinite.

The general case $n \geq 4$ follows directly from the case $n = 4$. The proof is finished.

4. A special case.

In this section we investigate the full subcategory $\mathcal{F}(\Delta)$ of the dual extension A of B in the special case where B is a hereditary algebra with $\mathrm{rad}^2(B) = 0$. We shall see that in this case $\mathcal{F}(\Delta)$ behaves as the module category of the algebra B. Namely, we prove the following

4.1 Theorem. Let B be a hereditary algebra with radical square zero and A the quasi–hereditary algebra which is the dual extension of B. Then the Auslander–Reiten quiver of $\mathcal{F}(\Delta_A)$ has the same number of vertices as that of the algebra B, and every irreducible map in B–mod induces an irreducible map in $\mathcal{F}(\Delta_A)$.

Proof. Since B' as well as B is hereditary, the projective dimension of a simple B'–module is smaller than 2. Hence proj.dim $\Delta_A(i) \leq 1$ for all i by 1.6. It follows from [DR] that $\mathcal{F}(\Delta_A)$ is closed under submodules. Let Q be the ideal of A such that $B' \cong A/Q$. Then $\mathrm{rad}(A) \cdot Q = 0$ since $\mathrm{rad}^2(B) = 0$. To prove the Theorem 4.1, we demonstrate that the functor $A \otimes_{B'} -$ is dense and preserves irreducible maps.

4.2 Lemma. The functor $F : B'\text{-mod} \longrightarrow \mathcal{F}(\Delta_A)$ is dense.

Proof. Clearly, the functor maps B'–modules into $\mathcal{F}(\Delta_A)$ according to 1.6. Suppose $M \in \mathcal{F}(\Delta_A)$. We will show that there is a B'–module X such that $F(X) \cong M$. Since $\mathcal{F}(\Delta_A)$ is closed under submodules, QM lies in $\mathcal{F}(\Delta_A)$. Note that the module QM is a semisimple A–module. Thus each simple direct summand of QM belongs to $\mathcal{F}(\Delta_A)$. It follows from A being a BGG-algebra that the simple direct summands of QM lie also in $\mathcal{F}(\nabla_A)$. Hence $\mathrm{Ext}^1_A(\mathcal{F}(\Delta_A), QM) = 0$. From the construction of A we know that $A = B' \oplus Q$ and there is an exact sequence

$$0 \longrightarrow Q_{B'} \longrightarrow A_{B'} \longrightarrow B' \longrightarrow 0$$

in B'–mod. Applying $- \otimes_{B'} (M/QM)$ to this sequence, one gets an exact sequence

$$Tor_1^{B'}(B', M/QM) \longrightarrow Q \otimes_{B'} (M/QM) \\ \longrightarrow A \otimes_{B'} (M/QM) \longrightarrow B' \otimes_{B'} (M/QM) \longrightarrow 0.$$

This gives a short exact sequence in A-mod:

$$0 \longrightarrow Q \otimes_{B'} (M/QM) \longrightarrow A \otimes_{B'} (M/QM) \xrightarrow{\alpha} M/QM \longrightarrow 0$$

since $\mathrm{Tor}_1^{B'}(B', M/QM) = 0$. If we apply $\mathrm{Hom}_A(Y, -)$ with $Y \in \mathcal{F}(\Delta_A)$ to the above sequence then we get

$$\begin{aligned} 0 \longrightarrow &\mathrm{Hom}_A(Y, Q \otimes_{B'} (M/QM)) \longrightarrow \mathrm{Hom}_A(Y, A \otimes_{B'} (M/QM)) \\ &\longrightarrow \mathrm{Hom}_A(Y, M/QM) \longrightarrow \mathrm{Ext}_A^1(Y, Q \otimes_{B'} (M/QM)). \end{aligned}$$

Note that with $A \otimes_{B'} (M/QM))$ also $Q \otimes_{B'} (M/QM)$ is in $\mathcal{F}(\Delta_A)$, and hence $Q \otimes_{B'} (M/QM)$ is a semisimple module and belongs to $\mathcal{F}(\nabla_A)$. Thus $\mathrm{Ext}_A^1(Y, Q \otimes_{B'} (M/QM)) = 0$ by 1.2 (1) and $\mathrm{Hom}_A(Y, \alpha)$ is surjective. This yields that α is an $\mathcal{F}(\Delta_A)$-approximation of M/QM. (Recall that a morphism $f : Y \longrightarrow M$ with Y in a full subcategory $\mathcal{C}$ of A-mod is called a right $\mathcal{C}$-approximation if $\mathrm{Hom}_A(Y', f)$ is surjective for all $Y' \in \mathcal{C}$). On the other hand, we have the natural exact sequence in A-mod:

$$0 \longrightarrow QM \longrightarrow M \xrightarrow{\beta} M/QM \longrightarrow 0$$

Without loss of generality, we may assume that M is indecomposable. Then β is right minimal. (A homomorphism $f : Y \longrightarrow M$ in A-mod is said to be right minimal if an endomorphism $g : Y \longrightarrow Y$ is an automorphism whenever $gf = f$). Similarly, one can see that β is also a right $\mathcal{F}(\Delta_A)$-approximation. Hence, β is a minimal right $\mathcal{F}(\Delta_A)$-approximation for M/QM and we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & QM & \longrightarrow & M & \xrightarrow{\beta} & M/QM & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle f} & & \| & & \\ 0 & \longrightarrow & Q \otimes_{B'} (M/QM) & \longrightarrow & A \otimes_{B'} (M/QM) & \xrightarrow{\alpha} & M/QM & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle g} & & \| & & \\ 0 & \longrightarrow & QM & \longrightarrow & M & \xrightarrow{\beta} & M/QM & \longrightarrow & 0 \end{array}$$

Since $\beta = (fg)\beta$, we know that fg is an automorphism of M. Hence M is a direct summand of $A \otimes_{B'} (M/QM)$. Let $M/QM = \bigoplus_j X_j$, where X_j are indecomposable B'-modules. Then $A \otimes_{B'} (M/QM) \cong \bigoplus_j A \otimes_{B'} X_j$ with $A \otimes_{B'} X_j$ indecomposable by 1.6 (3) and there is an X_j such that $F(X_j) \cong M$, and therefore the functor is dense.

4.3 Lemma. If $f : M \longrightarrow N$ is an irreducible map between indecomposable B'-modules, then $A \otimes_{B'} f$ is an irreducible map in $\mathcal{F}(\Delta_A)$.

Proof. If $A \otimes_{B'} f$ factors through a module $X' \in \mathcal{F}(\Delta_A)$ then, by Lemma 4.2, we may write $X' = A \otimes_{B'} X$ with X a B'-module:

$$\begin{array}{ccc} A \otimes_{B'} M & \xrightarrow{A\otimes f} & A \otimes_{B'} N \\ & {}_{h}\searrow \quad \nearrow_{g} & \\ & A \otimes_{B'} X & \end{array}$$

Applying the functor $G = B' \otimes_A -$ to this commutative diagram we have

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & {}_{B' \otimes_A h}\searrow \quad \nearrow_{B' \otimes_A g} & \\ & X & \end{array}$$

Hence $B' \otimes_A h$ is split mono, or $B' \otimes_A g$ is split epi. And therefore $h = A \otimes_{B'} (B' \otimes_A h)$ is split mono, or $g = A \otimes_{B'} (B' \otimes_A g)$ is split epi. Thus the lemma follows.

Now the Theorem 4.1 follows immediately from 4.2, 4.3 and 1.6.

4.4 Remark. The condition that B has radical-square-zero in the theorem is necessary. Let B be the algebra given by the quiver $1 \longleftarrow 2 \longleftarrow 3$. Then the corresponding BGG-algebra A has $\mathcal{F}(\Delta_A)$ with 9 indecomposable modules while the algebra B has only 6 indecomposable modules (cf. 3.4). Also the example 3.3 (1) shows that the heredity of B is necessary.

5. Quadratic duality.

In this section we discuss quadratic algebras and their dual quadratic algebras. Especially, we prove that the dual quadratic algebras of the dual extensions of quadratic algebras B without oriented cycle in their quivers are quasi-hereditary.

5.1 Definition. The algebra $A = kQ/ < R >$ is called quadratic if R is a subset of the space spanned by all paths of length 2.

Each quadratic algebra $A = kQ/ < R >$ has a natural Z-grading

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

where A_i is the space generated by the residual classes of paths of length i. We simply identify R with a subset of $A_1 \otimes_{A_0} A_1$, and the space spanned by such subset is denoted by $R(A)$.

For convenience, we write A as

$$A \leftrightarrow \{A_1, R(A) \subset A_1 \otimes_{A_0} A_1\}$$

Then $A = T(A_1)/ < R(A) >$, where $T(A_1)$ is the tensor algebra of A_1 over A_0.

A morphism $f : A \to B$ of quadratic algebras is an algebra homomorphism preserving gradings.

Let $A = kQ/ < R >= \bigoplus_{i\geq 0} A_i$ be a quadratic algebra. Regard the k-dual space A_1^* of A_1 as A_0-bimodule with the action $(afb)(v) = f(bva)$, where $a, b \in A_0$ and $f \in A_1^*$. The dual quadratic algebra $A^!$ is defined by

$$A^! \leftrightarrow \{A_1^*, R(A)^\perp \subset (A_1 \otimes_{A_0} A_1)^* = A_1^* \otimes_{A_0} A_1^*\}$$

here we identify $(A_1 \otimes_{A_0} A_1)^*$ with $A_1^* \otimes_{A_0} A_1^*$ by sending $(f \otimes g)^*$ to $g^* \otimes f^*$. More precisely,

$$A^! = T(A_1^*)/I$$

where I is the ideal of $T(A_1^*)$ generated by $R(A)^\perp$, in other words,

$$A^! = kQ^*/ < R(A)^\perp >$$

where $Q_0^* = Q_0$ and $Q_1^* = \{\alpha^* | \alpha \in Q_1\}$. The construction of dual quadratic algebras yields a functor

$$! : \mathcal{QA} \longrightarrow \mathcal{QA}^{op}, \qquad A \longmapsto A^!$$

which maps a morphism $f : A \longrightarrow B$ to a morphism $f^! : B^! \longrightarrow A^!$ induced by $f_1^* : B_1^* \longrightarrow A_1^*$ (see [M]).

5.2 Lemma([M]). The functor $!$ is an equivalence and $!^2 \simeq id_{\mathcal{QA}}$, where $id_{\mathcal{QA}}$ denotes the identity functor.

Let $A = kQ/ < R >= \bigoplus_{i\geq 0} A_i$ be a quadratic algebra. Recall that by $\mathcal{A}(A)$ we denote the dual extension of A (see section 1). By construction,

$$\mathcal{A}(A) \leftrightarrow \{A_1 \oplus A_1', R(\mathcal{A}(A)) \subset (A_1 \oplus A_1') \otimes_{A_0} (A_1 \oplus A_1')\}$$

where $A_1' = \oplus_{\alpha\in Q_1} k\alpha'$ and $R(\mathcal{A}(A))$ is the space spanned by $R(A) \cup R(A)' \cup \{\alpha\beta' | \alpha, \beta \in Q_1\}$.

The correspondence $A \longmapsto \mathcal{A}(A)$ gives rise to a functor

$$\mathcal{A} : \mathcal{QA} \longrightarrow \mathcal{QA}, \qquad A \longmapsto \mathcal{A}(A).$$

Then we have the following

5.3 Theorem. For each quadratic algebra $A = kQ/ < R >= \oplus_{i\geq 0} A_i$, there is an isomorphism $\eta(A) : \mathcal{A}(A)^! \to \mathcal{A}(A^!)$.

Proof. By construction, the algebra $\mathcal{A}(A)$ is given as follows:

$$\mathcal{A}(A) \leftrightarrow \{A_1 \oplus A_1', R(\mathcal{A}(A)) = R(A) \oplus R(A)' \oplus A_1 \otimes_{A_0} A_1' \subset (A_1 \oplus A_1') \otimes_{A_0} (A_1 \oplus A_1')\}$$

where $R(A)' = \{\sum_i \lambda_i \alpha_i' \beta_i' \mid \sum_i \lambda_i \beta_i \alpha_i \in R(A)\}$. Therefore,

$\mathcal{A}(A)^! \leftrightarrow \{(A_1 \oplus A_1')^* = A_1^* \oplus A_1'^*, R(\mathcal{A}(A)^!) = R(\mathcal{A}(A))^\perp \subset ((A_1 \oplus A_1') \otimes (A_1 \oplus A_1'))^*\}$.

Similarly, $A^! \leftrightarrow \{A_1^*, R(A)^\perp \subset (A_1 \otimes A_1)^*\}$ and

$$\begin{aligned} \mathcal{A}(A^!) \leftrightarrow \{A_1^* \oplus A_1^{*\prime}, R(\mathcal{A}(A^!)) &= R(A)^\perp \oplus (R(A)^\perp)' \oplus A_1^* \otimes A_1^{*\prime} \\ &\subset (A_1^* \oplus A_1^{*\prime}) \otimes (A_1^* \oplus A_1^{*\prime})\} \end{aligned}$$

where $A_1^{*\prime} \otimes A_1^{*\prime} \supset (R(A)^\perp)' = \{\sum_j \mu_j u_j^{*\prime} v_j^{*\prime} \mid \sum_j \mu_j v_j^* u_j^* \in R(A)^\perp, \mu_j \in k\}$.

Set

$$f : A_1^* \oplus A_1'^* \longrightarrow A_1^* \oplus A_1^{*\prime}$$

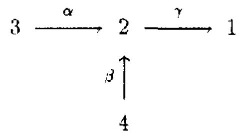
$$\Big(\sum_i \lambda_i \alpha_i^*, \sum_j \mu_j \beta_j'^*\Big) \longmapsto \Big(\sum_i \lambda_i \alpha_i^*, \sum_j \mu_j \beta_j^{*\prime}\Big)$$

Then f is an A_0-linear map such that $f \otimes f(R(\mathcal{A}(A)^!)) = R(\mathcal{A}(A^!))$ and it induces the wanted isomorphism $\eta(A)$.

Indeed, the isomorphisms $\eta(A)$ give rise to a natural isomorphism $\eta : {}^!\mathcal{A} \to \mathcal{A}^!$.

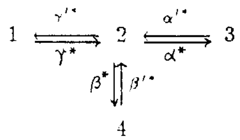
Let us now give an example to explain 5.3 before we go further.

5.4 Example. Let A be the dual extension of the path algebra of the quiver $Q = (Q_0, Q_1)$:

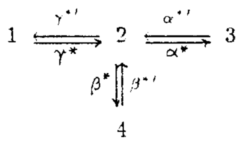


modulo the ideal generated by $R = \{\alpha\gamma, \beta\gamma\}$.

Then $\mathcal{A}(A)^!$ and $\mathcal{A}(A^!)$ are, respectively, the path algebras of the following quivers with the relations



and



$$\alpha\alpha'^* = \beta^*\beta'^* = \gamma^*\gamma'^* = 0 \qquad\qquad \alpha^*\alpha^{*\prime} = \beta^*\beta^{*\prime} = \gamma^*\gamma^{*\prime} = 0$$

Obviously, they are isomorphic.

Now we return to arbitrary algebras defined by quivers and relations.

5.5 Theorem. Let $A = kQ/ < R >$ and $B = kS/ < T >$, and Q, S have no oriented cycle. Then $\mathcal{A}(A) \otimes_k \mathcal{A}(B) \simeq \mathcal{A}(A \otimes_k B)$.

Proof. Note that the quiver of $A \otimes_k B$ consists of the vertices $e_i \otimes f_j$ and arrows $\alpha \otimes e_i$, $f_j \otimes \beta$, where e_i, f_j are the idempotents of A and B corresponding to vertices $i \in Q$, $j \in S$, and α, β are arrows in Q and S, respectively.

By construction, the quiver of $\mathcal{A}(A \otimes_k B)$ has the same set of vertices as that of $A \otimes_k B$, and it consists of arrows $\alpha \otimes e_i, f_j \otimes \beta$, $(\alpha \otimes e_i)', (f_j \otimes \beta)'$, $i \in Q_0, j \in S_0, \alpha \in Q_1$ and $\beta \in S_1$.

The embeddings $i_A : \mathcal{A}(A) \to \mathcal{A}(A \otimes_k B)$ defined by $a \longmapsto a \otimes 1$ and $i_B : \mathcal{A}(B) \to \mathcal{A}(A \otimes_k B)$ defined by $b \longmapsto 1 \otimes b$, where we identify $\alpha' \otimes e_i$ with $(\alpha \otimes e_i)'$ and $f_j \otimes \beta'$ with $(f_j \otimes \beta)'$, induce a k-linear map

$$\Psi : \mathcal{A}(A) \times \mathcal{A}(B) \longrightarrow \mathcal{A}(A \otimes_k B)$$

$$(a, b) \longmapsto i_A(a) \cdot i_B(b)$$

which is balanced. It then induces an algebra homomorphism

$$\Phi : \mathcal{A}(A) \otimes_k \mathcal{A}(B) \longrightarrow \mathcal{A}(A \otimes_k B)$$

$$a \otimes b \longmapsto \Psi(a, b) = i_A(a) \cdot i_B(b)$$

which provides a quiver isomorphism from the quiver of $\mathcal{A}(A) \otimes_k \mathcal{A}(B)$ to that of $\mathcal{A}(A \otimes_k B)$. Therefore, Φ is surjective.

On the other hand, using Lemma 1.4 (4) and comparing the dimensions of $\mathcal{A}(A) \otimes_k \mathcal{A}(B)$ and $\mathcal{A}(A \otimes_k B)$, we obtain

$$\begin{aligned}
\dim_k \mathcal{A}(A) \otimes_k \mathcal{A}(B) &= \dim_k \mathcal{A}(A) \cdot \dim_k \mathcal{A}(B) = \Big(\sum_i \dim_k^2 P_A(i)\Big)\Big(\sum_j \dim_k^2 P_B(j)\Big) \\
&= \sum_{i,j} (\dim_k P_A(i) \dim_k P_B(j))^2 = \sum_{i,j} (\dim_k (P_A(i) \otimes_k P_B(j)))^2 \\
&= \dim_k \mathcal{A}(A \otimes_k B)
\end{aligned}$$

since ${}_{A \otimes_k B} A \otimes_k B = \bigoplus_{i,j} P_A(i) \otimes_k P_B(j)$.

As a result, Φ is an isomorphism.

5.6 Corollary. Let A and B be quadratic algebras without oriented cycle in their quivers. Then $(\mathcal{A}(A) \otimes_k \mathcal{A}(B))^!$ is quasi-hereditary.

Proof. Since A and B have no oriented cycle, $A \otimes_k B$ has no oriented cycle either. Then so does $(A \otimes_k B)^!$.

From 5.3 and 5.5 it follows that the algebra

$$(\mathcal{A}(A) \otimes_k \mathcal{A}(B))^! \simeq (\mathcal{A}(A \otimes_k B))^! \simeq \mathcal{A}((A \otimes_k B)^!)$$

is quasi-hereditary.

Acknowledgement. The authors would like to thank Prof. Shaoxue Liu for help and encouragement.

References

[ADL] I.Ágoston, V.Dlab and E.Lukács, Lean quasi-hereditary algebras, Preprint.

[BGG] I.N.Bernstein, I.M.Gelfand and S.I.Gelfand, A category of $\mathcal{G}$-modules, Funct. Anal. Appl., 10(1976), 67-92.

[CPS] E.Cline, B.Parshall and L.Scott, Finite dimensional algebras and highest weight categories, J. Reine Angew. Math., 391(1988), 85-99.

[D] S.Donkin, On tilting modules for algebraic groups, Math. Z., 212(1993), 39-60.

[DR] V.Dlab and C.M.Ringel, The module theoretical approach to quasi- hereditary algebras, LMS Lecture Notes Series 168, "Representations of algebras and related topics", ed. H. Tachikawa and S.Brenner, 1992.

[E] K.Erdmann, Symmetric groups and quasi-hereditary algebras, Preprint.

[G] J.A.Green, Polynomial representations of GL_n, Springer LNM 830, 1980.

[GR] P.Gabriel and A.V.Roiter, The representation theory of finite dimensional algebras, Encyclopaedia of Math. Sci., Vol. 73, Algebra VIII (1992).

[I] R.S.Irving, BGG-algebras and the BGG reciprocity principle, J. Algebra, 135 (1990), 363-380.

[K] S.König, A guide to exact Borel subalgebras of quasi-hereditary algebras, Proceeding of ICRA, Carleton-Ottawa Math. LNS 14(1992).

[M] Yu.I.Manin, Some remarks on Koszul algebras and quantum groups, Ann. Inst. Fourier, Grenoble, 37(1987), no.4, 191-205.

[R1] C.M.Ringel, The category of modules with good filtrations over quasi-hereditary algebras has almost split sequences, Math.Zeit., 208(1991), 209-223.

[R2] C.M.Ringel, Tame algebras and integral quadratic forms, Springer LNM 1099, 1984.

[X] C.C.Xi, Quasi-hereditary algebras with a duality, to appear in J. Reine Angew. Math..

Received: July 1993

Revised: February 1994