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On the finitistic dimension conjecture, III: Related to the pair $eAe \subseteq A$

Changchang Xi

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, 100875 Beijing, People's Republic of China

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Abstract

Let A be an Artin algebra and e an idempotent element in A. In this paper, we use co-homological conditions on A to control the finitistic dimension of eAe. Such a consideration is of particular interest for understanding the finitistic dimension conjecture. Let us denote the finitistic dimension and global dimension of A by fin.dim(A) and gl.dim(A), respectively. Suppose gl.dim$(A) \leqslant 4$. Then fin.dim$(eAe) < \infty$ if one of the following conditions holds: (1) A/AeA has representation dimension at most 3; (2) $\Omega_A^{-3}(A)$ is an A/AeA-module; (3) proj.dim$({}_AS) \leqslant 3$ for all simple A/AeA-modules S. This result can be considered as a first step to the question of whether gl.dim$(A) \leqslant 4$ implies fin.dim$(eAe) < \infty$. Moreover, we show the following: Let A be an arbitrary Artin algebra and e an idempotent element of A such that the $*$-syzygy dimension or the Gorenstein dimension of the eAe-module Ae is finite. If fin.dim$(A) < \infty$, then fin.dim$(eAe) < \infty$.

Keywords: Finitistic dimension; Global dimension; Gorenstein dimension; Idempotent element; $*$-Syzygy module

1. Introduction

Let A be an Artin algebra. The famous finitistic dimension conjecture says that there exists a uniform bound for the finite projective dimensions of all finitely generated (left) A-modules of finite projective dimension. This conjecture is related to many other homological conjectures and attracts many algebraists, for example, Maurice Auslander, one of the founders of the mod-

E-mail address: xicc@bnu.edu.cn.

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ern aspects of the representation theory of Artin algebras [3, p. 501]. As we know, “one of his main interests in the theory of artin algebras was the finitistic dimension conjecture and related homological conjectures” [3, p. 815]. Now, this conjecture is more than 45 years old (see [9]) and, to the best knowledge of the author, still remains open. To attack this conjecture, a new idea was introduced in [29,30], that is, the comparison of the finitistic dimensions of the pairs $B \subseteq A$ of algebras such that B is a subalgebra of A with the same identity and that the radical of B is a left ideal in A. In this note, we discuss another type of pair of algebras: $B = eAe \subseteq A$ with e an idempotent element in A, and compare their finitistic dimensions. In this case, the algebras A and B usually have different identities.

Contrary to the usual consideration in the literature, where one often uses information on eAe to get information on A (see [13] for a discussion), we may consider the following inverse question:

> Suppose the global dimension of A is finite. Is it possible to show that the finitistic dimension of eAe is finite?

Apparently, this question is of particular interest for solving the finitistic dimension conjecture. The reason is that an affirmative answer to the question will lead to a positive solution to the finitistic dimension conjecture. This can be seen from a result of Auslander, which states that every Artin algebra B is of the form eAe with A being of finite global dimension (see [4]). In fact, Dlab and Ringel proved that A even can be assumed to be quasi-hereditary (see [11] and [24]). Another motivation for our consideration is an effort to generalize a recent result of Igusa and Todorov, which says that if the global dimension of A is at most three, then the finitistic dimension of eAe is finite (see [20]). Therefore, as a first step in this direction, we consider in this note the following special question:

Question. If the global dimension of A is at most 4, is it possible to show that the finitistic dimension of eAe is finite?

Even in this case the question seems to be hard. Though we cannot answer our question in general at present, we have, as a main result of this note, the following partial answer.

Theorem 1.1. *Let A be an Artin algebra and e an idempotent element in A. Suppose* $\mathrm{gl.dim}(A) \leqslant 4$. *Then*:

(1) *if A/AeA has representation dimension at most* 3, *then* $\mathrm{fin.dim}(eAe) < \infty$;
(2) *if* $\mathrm{inj.dim}_{eAe}(e\Omega_A^{-3}(A)) \leqslant 1$, *then* $\mathrm{fin.dim}(eAe) < \infty$;
(3) *if* $\mathrm{proj.dim}({}_AS) \leqslant 3$ *for all simple A/AeA-modules S, then* $\mathrm{fin.dim}(eAe) < \infty$.

If we relax the condition on the global dimension of A by restriction of the Gorenstein dimension of the right eAe-module Ae, then we have the following result.

Theorem 1.2. *Suppose that A is an Artin algebra and e is an idempotent element of A such that the $*$-syzygy dimension or the Gorenstein dimension of the right eAe-module Ae is finite. If* $\mathrm{fin.dim}(A) < \infty$, *then* $\mathrm{fin.dim}(eAe) < \infty$.

The paper is organized as follows: In Section 2 we recall some definitions and basic facts. The proofs of the results will be given in Sections 3 and 4, where we also show that if the full subcategory of A-modules with finite Gorenstein dimensions is contravariantly finite in A-mod, then $\text{fin.dim}(A) < \infty$. In the last section, we present some examples to illustrate the main results.

2. Preliminaries

In this section, we recall some definitions and basic results required in the paper.

Let A be an Artin algebra, that is, A is a finitely generated module over its center which is assumed to be a commutative Artin ring. We denote by A-mod the category of all finitely generated left A-modules and by $\text{rad}(A)$ the Jacobson radical of A. Given an A-module M, we denote by $\text{proj.dim}_A(M)$ the projective dimension of M, and by $\text{add}(M)$ the additive subcategory of A-mod generated by the module M. The nth syzygy of the A-module M is denoted by $\Omega_A(M)$ or simply by $\Omega(M)$ if there is no danger of confusion. Symmetrically, we denote the nth cosyzygy operator by Ω_A^{-n} or simply by Ω^{-n}. The left global dimension of A is denoted by $\text{gl.dim}(A)$ in this paper.

Let $K(A)$ be the quotient of the free abelian group generated by the isomorphism classes $[M]$ of modules M in A-mod modulo the relations:

(1) $[Y] = [X] + [Z]$ if $Y \simeq X \oplus Z$; and
(2) $[P] = 0$ if P is projective.

Thus $K(A)$ is a free abelian group with the basis of non-isomorphism classes of non-projective indecomposable A-modules in A-mod. Igusa and Todorov in [20] use the noetherian property of the ring of integers, and define a function Ψ on this abelian group, which depends on the algebra A and takes values of non-negative integers.

The following result is due to Igusa and Todorov [20].

Lemma 2.1. *For any Artin algebra A, there is a function Ψ which is defined on the objects of* A-mod *and takes non-negative integers as values, such that*

(1) $\Psi(M) = \text{proj.dim}_A(M)$ *if* ${}_AM$ *has finite projective dimension.*
(2) *For any A-modules X and Y, we have $\Psi(X) \leqslant \Psi(Y)$ if* $\text{add}(X) \subseteq \text{add}(Y)$. *In case* $\text{add}(X) = \text{add}(Y)$*, the equality holds.*
(3) *If $0 \to X \to Y \to Z \to 0$ is an exact sequence in* A-mod *with* $\text{proj.dim}(Z) < \infty$*, then* $\text{proj.dim}(Z) \leqslant \Psi(X \oplus Y) + 1$.

Note that given an exact sequence $0 \to X \to Y \to Z \to 0$ in A-mod, there are three relevant exact sequences

$$0 \longrightarrow \Omega^i(X) \longrightarrow \Omega^i(Y) \oplus P'' \longrightarrow \Omega^i(Z) \longrightarrow 0,$$

$$0 \longrightarrow \Omega(Z) \longrightarrow X \oplus P' \longrightarrow Y \longrightarrow 0,$$

$$0 \longrightarrow \Omega(Y) \longrightarrow \Omega(Z) \oplus P \longrightarrow X \longrightarrow 0,$$

where Ω^i is the ith syzygy operator, and where P, P' and P'' are projective modules. We call the above sequences *the syzygy shifted sequences*. So the following result is a direct consequence of Lemma 2.1.

Lemma 2.2. *If* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* A-mod*, then*

(1) $\operatorname{proj.dim}(Y) \leqslant \Psi(X \oplus \Omega(Z)) + 1$ *if* $\operatorname{proj.dim}(Y) < \infty$,
(2) $\operatorname{proj.dim}(X) \leqslant \Psi(\Omega(Y \oplus Z)) + 1$ *if* $\operatorname{proj.dim}(X) < \infty$.

The following is a standard result in homological algebra.

Lemma 2.3. *If* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in* A-mod*, then*

(1) $\operatorname{proj.dim}(X) < \operatorname{proj.dim}(Y)$ *implies* $\operatorname{proj.dim}(Z) = \operatorname{proj.dim}(Y)$;
(2) $\operatorname{proj.dim}(X) > \operatorname{proj.dim}(Y)$ *implies* $\operatorname{proj.dim}(Z) = \operatorname{proj.dim}(X) + 1$;
(3) $\operatorname{proj.dim}(X) = \operatorname{proj.dim}(Y)$ *implies* $\operatorname{proj.dim}(Z) \leqslant \operatorname{proj.dim}(X) + 1$;
(4) $\operatorname{proj.dim}(Z) \leqslant \max\{\operatorname{proj.dim}(X), \operatorname{proj.dim}(Y)\} + 1$.

The following lemma is a consequence of Lemma 2.3(4) by induction.

Lemma 2.4. *If* $X : 0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \xrightarrow{d_0} M \to 0$ *is a complex in* A-mod *with* d_0 *surjective, then*

$$\operatorname{proj.dim}(M) \leqslant n + \max\bigl\{\operatorname{proj.dim}(X_i), \operatorname{proj.dim}\bigl(H_i(X)\bigr) \bigm| i = 0, 1, \ldots, n\bigr\},$$

where $H_i(X)$ *denotes the* i*th homology of the complex* X.

Now, let us recall a few definitions. An Artin algebra A is called *representation-finite* if there is only finitely many non-isomorphic indecomposable A-modules in A-mod.

Given an Artin algebra A, the *finitistic dimension* of A, denoted by $\operatorname{fin.dim}(A)$, is defined as

$$\operatorname{fin.dim}(A) = \sup\bigl\{\operatorname{proj.dim}({}_AM) \bigm| M \in A\text{-mod and } \operatorname{proj.dim}({}_AM) < \infty\bigr\}.$$

The *representation dimension* of A, denoted by $\operatorname{rep.dim}(A)$, was defined by Auslander in [4] as follows:

$$\operatorname{rep.dim}(A) = \inf\bigl\{\operatorname{gl.dim}(\Lambda) \bigm| \Lambda \text{ is an Artin algebra with } \operatorname{dom.dim}(\Lambda) \geqslant 2 \text{ and } \operatorname{End}({}_\Lambda T) \text{ is Morita equivalent to } A, \text{ where } T \text{ is the injective envelope of } \Lambda\bigr\}.$$

Auslander also proved in [4] that the above definition is equivalent to the following definition:

$$\operatorname{rep.dim}(A) = \inf\bigl\{\operatorname{gl.dim}\bigl(\operatorname{End}({}_AM)\bigr) \bigm| M \text{ is a generator-cogenerator for } A\text{-mod}\bigr\},$$

where M is called a *generator* for A-mod if every indecomposable projective A-module is isomorphic to a direct summand of M; and a *cogenerator* for A-mod if every indecomposable injective A-module is isomorphic to a direct summand of M.

Similarly, one defines the *weak representation dimension* of A to be

$$\operatorname{wrep.dim}(A) = \inf\bigl\{\operatorname{gl.dim}\bigl(\operatorname{End}\bigl({}_AM\bigr)\bigr) \bigm| M \text{ is a generator for } A\text{-mod}\bigr\},$$

and the weak co-representation dimension of A to be

$$\operatorname{wco-rep.dim}(A) = \inf\bigl\{\operatorname{gl.dim}\bigl(\operatorname{End}\bigl({}_AM\bigr)\bigr) \bigm| M \text{ is a cogenerator for } A\text{-mod}\bigr\}.$$

Clearly, $\max\{\operatorname{wrep.dim}(A), \operatorname{wco-rep.dim}(A)\} \leqslant \operatorname{rep.dim}(A)$. A connection of the representation dimension with finitistic dimension is a result of Igusa and Todorov in [20], which says that if the weak representation dimension of A is at most three, then the finitistic dimension of A is finite.

Finally, let us recall the following result which is implied in [4].

Lemma 2.5. *Let A be an Artin algebra, and m a non-negative integer.*

If M is a generator-cogenerator for A-mod, then $\operatorname{gl.dim}(\operatorname{End}({}_AM)) \leqslant m$ if and only if, for each A-module Y, there is an exact sequence

$$0 \longrightarrow M_{m-2} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow Y \longrightarrow 0,$$

with $M_j \in \operatorname{add}({}_AM)$ for $j = 0, \ldots, m-2$, such that

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_A(X, M_{m-2}) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_A(X, M_1) \longrightarrow \operatorname{Hom}_A(X, M_0) \\ \longrightarrow \operatorname{Hom}_A(X, Y) \longrightarrow 0 \end{aligned}$$

is exact for all $X \in \operatorname{add}({}_AM)$.

The finitistic dimension conjecture says that $\operatorname{fin.dim}(A) < \infty$ for all Artin algebras A (see [9]). It was proved that the conjecture is true for algebras with radical-cube-zero in [16], and for monomial algebras in [15]. For further information about advances on the finitistic dimension conjecture as well as its relationship with other homological conjectures in the representation theory of Artin algebras, we may refer to [7,29,30,33,34] and the references therein. For representation dimension, we refer to [2,4,10,28,31,32] and the references therein.

3. Proof of Theorem 1.1

Throughout this section, we suppose that A is an Artin algebra, and let e be an idempotent element in A. We denote the algebra eAe by B. Then Ae is an A-B-bimodule in a natural way.

To prove our results in Theorem 1.1, we first establish several facts.

Lemma 3.1. *If M is a B-module with a projective cover $f : P \to M$, then the induced map $1 \otimes_B f : Ae \otimes_B P \to Ae \otimes_B M$ is a projective cover of the A-module $Ae \otimes_B M$*

Proof. This follows from the fact that B-mod is equivalent to the category $Ae \otimes_B (B\text{-mod})$, which is the image of the functor $Ae \otimes_B -$ from B-mod to A-mod. □

With this lemma in hand, we have the following result.

Corollary 3.2.

(1) *For any B-module M, we have an exact sequence*

$$0 \longrightarrow \operatorname{Tor}_1^B(Ae, M) \longrightarrow Ae \otimes_B \Omega_B(M) \longrightarrow \Omega_A(Ae \otimes_B M) \longrightarrow 0.$$

(2) *For any B-module M, we have $\operatorname{Hom}_A(Ae, \operatorname{Tor}_i^B(Ae, M)) = 0$ for all $i \geqslant 1$, that is, $\operatorname{Tor}_i^B(Ae, M)$ is an A/AeA-module.*

Proof. (1) Let $P \longrightarrow M \longrightarrow 0$ be a projective cover of the B-module M. Then we have an exact sequence

$$0 \longrightarrow \Omega_B(M) \longrightarrow P \longrightarrow M \longrightarrow 0.$$

This yields the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Tor}_1^B(Ae, M) \arrow[r] & Ae \otimes_B \Omega_B(M) \arrow[r] \arrow[d] & Ae \otimes_B P \arrow[r] \arrow[d, equal] & Ae \otimes_B M \arrow[r] \arrow[d, equal] & 0 \\
 & 0 \arrow[r] & \Omega_A(Ae \otimes_B M) \arrow[r] & Ae \otimes_B P \arrow[r] & Ae \otimes_B M \arrow[r] & 0,
\end{tikzcd}$$

which gives the desired exact sequence in A-mod:

$$0 \longrightarrow \operatorname{Tor}_1^B(Ae, M) \longrightarrow Ae \otimes_B \Omega_B(M) \longrightarrow \Omega_A(Ae \otimes_B M) \longrightarrow 0.$$

(2) If we apply $\operatorname{Hom}_A(Ae, -)$ to the top row of the commutative diagram, then we get an exact sequence

$$0 \longrightarrow e\operatorname{Tor}_1^B(Ae, M) \longrightarrow \Omega_B(M) \longrightarrow P \longrightarrow M \longrightarrow 0$$

which shows that $e\operatorname{Tor}_1^B(Ae, M) = 0$ for any B-module M. This implies that $e\operatorname{Tor}_i^B(Ae, M) = 0$ for all $i \geqslant 1$. $\square$

Lemma 3.3. *Suppose M is an arbitrary B-module. Then, for any $i \geqslant 0$,*

$$\Omega_B^{i+2}(M) \simeq e\Omega_A\big(Ae \otimes_B \Omega_B^{i+1}(M)\big) \simeq e\Omega_A^2\big(Ae \otimes_B \Omega_B^{i}(M)\big) \oplus eP,$$

where P is a projective A-module depending on M.

Proof. It follows from Corollary 3.2 that

$$\Omega_B(M) \simeq e\Omega_A(Ae \otimes_B M).$$

If we replace M by $\Omega_B^{i+1}(M)$ in this isomorphism formula, then we get the first isomorphism in Lemma 3.3. To obtain the second isomorphism, we first replace M by $\Omega_B^i(M)$ in the exact sequence in Corollary 3.2 and get the following exact sequence

$$0 \longrightarrow \operatorname{Tor}_{i+1}^B(Ae, M) \longrightarrow Ae \otimes_B \Omega_B^{i+1}(M) \longrightarrow \Omega_A\big(Ae \otimes_B \Omega_B^{i}(M)\big) \longrightarrow 0.$$

We then obtain the following syzygy shifted sequence

$$0 \longrightarrow \Omega_A\big(Ae \otimes_B \Omega^{i+1}(M)\big) \longrightarrow \Omega_A^2\big(Ae \otimes_B \Omega_B^i(M)\big) \oplus P \longrightarrow \mathrm{Tor}_{i+1}^B(Ae, M) \longrightarrow 0,$$

where P is a projective A-module. By Corollary 3.2(2), $e\,\mathrm{Tor}_{i+1}^B(Ae, M) = 0$. Thus, by multiplying with the idempotent element e, we get the second isomorphism in Lemma 3.3. $\square$

The next sequence is a generalization of the sequence in Corollary 3.2.

Lemma 3.4. *For any $n \geqslant 0$ and any B-module X, there is an exact sequence in A-*mod:

$$0 \longrightarrow \Omega_A^n\big(\mathrm{Tor}_1^B(Ae_B, X)\big) \longrightarrow \Omega_A^n\big(Ae \otimes_B \Omega_B(X)\big) \oplus P(X) \longrightarrow \Omega_A^{n+1}(Ae \otimes_B X) \longrightarrow 0,$$

with $P(X)$ a projective A-module depending on X.

Proof. The proof is a direct consequence of the following fact in homological algebra. If

$$0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow 0$$

is an exact sequence in A-mod, then there is an exact sequence for the corresponding syzygy modules:

$$0 \longrightarrow \Omega_A^m(X_n) \longrightarrow \Omega_A^m(X_{n-1}) \oplus P_{n-1} \longrightarrow \cdots \longrightarrow \Omega_A^m(X_2) \oplus P_2 \longrightarrow \Omega_A^m(X_1) \longrightarrow 0,$$

where P_j is a projective module for all $2 \leqslant j \leqslant n-1$. $\square$

As a consequence of Lemma 3.3, we obtain the following known result of Igusa and Todorov [20].

Corollary 3.5. *If* $\mathrm{gl.dim}(A) \leqslant 3$*, then* $\mathrm{fin.dim}(B) < \infty$.

Proof. Suppose M is a B-module with finite projective dimension. Since $\mathrm{gl.dim}(A) \leqslant 3$, the module $\Omega_A^2(Ae \otimes_B M)$ has projective dimension at most 1. Suppose the sequence

$$0 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow \Omega_A^2(Ae \otimes_B M) \longrightarrow 0$$

is a minimal projective resolution of the module $\Omega_A^2(Ae \otimes_B M)$. Then we get an exact sequence

$$(*) \quad 0 \longrightarrow eQ_1 \longrightarrow eQ_0 \longrightarrow e\Omega_A^2(Ae \otimes_B M) \longrightarrow 0.$$

By Lemma 3.3, there is a projective A-module P such that $e\Omega_A^2(Ae \otimes_B M) \oplus eP \simeq \Omega_B^2(M)$. So we may change the sequence $(*)$ into

$$0 \longrightarrow eQ_1 \longrightarrow eQ_0 \oplus eP \longrightarrow e\Omega_A^2(Ae \otimes_B M) \oplus eP = \Omega_B^2(M) \longrightarrow 0.$$

Now it follows from Lemma 2.1 that $\mathrm{proj.dim}({}_BM) \leqslant \mathrm{proj.dim}_B(\Omega_B^2(M)) + 2 \leqslant \Psi(eQ_0 \oplus eQ_1 \oplus eP) + 1 + 2 \leqslant \Psi(eA) + 3$. Thus the finitistic dimension of B is bounded above by $\Psi(eA) + 3$. $\square$

Proof of Theorem 1.1(1). Suppose M is a B-module of finite projective dimension. It follows from the exact sequence in Corollary 3.2 that we have the following shift of syzygy modules:

$$0 \longrightarrow \Omega_A\big(Ae \otimes_B \Omega_B(M)\big) \longrightarrow \Omega_A^2(Ae \otimes_B M) \oplus P_1 \longrightarrow \mathrm{Tor}_1^B(Ae, M) \longrightarrow 0$$

with P_1 a projective A-module. Now we apply the syzygy operator to the above sequence and get the following exact sequence

$$0 \longrightarrow \Omega_A^2\big(Ae \otimes_B \Omega_B(M)\big) \longrightarrow \Omega_A^3(Ae \otimes_B M) \oplus P' \longrightarrow \Omega_A\big(\mathrm{Tor}_1^B(Ae, M)\big) \longrightarrow 0,$$

where P' is a projective module. Again we apply syzygy shift once and reach the following exact sequence

$$0 \longrightarrow \Omega_A^4(Ae \otimes_B M) \longrightarrow \Omega_A^2\big(\mathrm{Tor}_1^B(Ae, M)\big) \oplus Q \longrightarrow \Omega_A^2\big(Ae \otimes_B \Omega_B(M)\big) \longrightarrow 0,$$

where Q is a projective A-module. By assumption, $\mathrm{rep.dim}(A/AeA) \leqslant 3$. Let V be an A/AeA-module such that $\mathrm{rep.dim}(A/AeA) = \mathrm{gl.dim}(\mathrm{End}(_{A/AeA}V))$. Since $\mathrm{Tor}_1^B(Ae, M)$ is an A/AeA-module by Corollary 3.2, it follows from Lemma 2.5 that there is an exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow \mathrm{Tor}_1^B(Ae, M) \longrightarrow 0$$

in (A/AeA)-mod, where V_0 and V_1 are A/AeA-modules in $\mathrm{add}(V)$. Now we consider this sequence in A-mod. By applying the syzygy operator to the above sequence, we get the following exact sequence

$$0 \longrightarrow \Omega_A^2(V_1) \longrightarrow \Omega_A^2(V_0) \oplus P'' \longrightarrow \Omega_A^2\big(\mathrm{Tor}_1^B(Ae, M)\big) \longrightarrow 0$$

in A-mod, where P'' is a projective A-module. Now we may form the following commutative exact diagram in A-mod:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \Omega_A^2(V_1) \arrow[r, equal] \arrow[d] & \Omega_A^2(V_1) \arrow[d] & & \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & \Omega_A^2(V_0) \oplus P'' \oplus Q \arrow[r] \arrow[d] & \Omega_A^2(Ae \otimes_B \Omega_B(M)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega_A^4(Ae \otimes_B M) \arrow[r] \arrow[d] & \Omega_A^2(\mathrm{Tor}_1^B(Ae, M)) \oplus Q \arrow[r] \arrow[d] & \Omega_A^2(Ae \otimes_B \Omega_B(M)) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Note that the first term in the lower sequence is projective since $\mathrm{gl.dim}(A) \leqslant 4$. Thus $Y \simeq \Omega_A^4(Ae \otimes_B M) \oplus \Omega_A^2(V_1)$, and we have an exact sequence

$$0 \longrightarrow \Omega_A^4(Ae \otimes_B M) \oplus \Omega_A^2(V_1) \longrightarrow \Omega_A^2(V_0) \oplus P'' \oplus Q \longrightarrow \Omega_A^2\big(Ae \otimes_B \Omega_B(M)\big) \longrightarrow 0.$$

From this sequence we get the following exact sequence in B-mod:

$$\begin{aligned}0 &\longrightarrow e\Omega_A^4(Ae\otimes_B M)\oplus e\Omega_A^2(V_1)\longrightarrow e\Omega_A^2(V_0)\oplus eP''\oplus eQ\\ &\longrightarrow e\Omega_A^2\big(Ae\otimes_B \Omega_B(M)\big)\longrightarrow 0.\end{aligned}$$

Note that there is a projective A-module P such that $e\Omega_A^2(Ae\otimes_B\Omega_B(M))\oplus eP$ is isomorphic to $\Omega_B^3(M)$ by Lemma 3.3. Now we can estimate the projective dimension of the B-module M. By Lemma 2.1, we have

$$\begin{aligned}\text{proj.dim}({}_BM) &\leqslant \text{proj.dim}\big(\Omega_B^3(M)\big)+3\\ &=\Psi\big(\Omega_B^3(M)\big)+3\\ &=\Psi\big(e\Omega_A^2\big(Ae\otimes_B\Omega_B(M)\big)\oplus eP\big)+3\\ &\leqslant \Psi\big(e\Omega_A^4(Ae\otimes_B M)\oplus e\Omega_A^2(V_1)\oplus\Omega_A^2(V_0)\oplus eP''\oplus eQ\oplus eP\big)+1+3\\ &\leqslant \Psi\big(e\Omega_A^4(Ae\otimes_B M)\oplus e\Omega_A^2(V)\oplus eP''\oplus eQ\big)+4\\ &\leqslant \Psi\big(eA\oplus e\Omega_A^2(V)\big)+4.\end{aligned}$$

Note that the B-module $eA\oplus e\Omega_A^2(V)$ does not depend on the choice of the module M. So the finitistic dimension of B is bounded above by $\Psi(eA\oplus e\Omega_A^2(V))+4$. This finishes the proof of Theorem 1.1(1). □

As a consequence of Theorem 1.1(1), we have the following result.

Corollary 3.6. *Let A be an Artin algebra. Suppose there is an A-module M such that it is a generator for A-mod and that the representation dimension of $\underline{\text{End}}({}_AM)$ is at most 3, where $\underline{\text{End}}({}_AM)$ is the quotient algebra of $\text{End}(M)$ modulo the morphisms that factor through a projective A-module. If $\text{gl.dim}(\text{End}({}_AM))\leqslant 4$, then $\text{fin.dim}(A)<\infty$.*

Proof. Since the finitistic dimension of an Artin algebra is invariant under Morita equivalences, we may assume that A is basic and M is of the form $A\oplus M'$, where M' has no projective summand. Let e be the canonical projection from M to M with the image of e being A. Then e is an idempotent in $\Gamma:=\text{End}(M)$. We may identify A with $e\Gamma e$, and the ideal in Γ generated by e with the set of all endomorphisms of M which factor through a projective A-module. Then $\Gamma/\Gamma e\Gamma\simeq\underline{\text{End}}({}_AM)$ has representation dimension bounded above by 3. Thus our corollary follows immediately from Theorem 1.1(1). □

We know that $\text{fin.dim}(A)<\infty$ if $\text{rad}^3(A)=0$ (see [16]). The following result is a partial answer to the question of whether $\text{rad}^4(A)=0$ implies that $\text{fin.dim}(A)<\infty$, and also a special case of Corollary 3.6.

Corollary 3.7. *Let A be an Artin algebra with $\text{rad}^4(A)=0$, and $M=\bigoplus_{i=1}^4 A/\text{rad}^i(A)$. Suppose $\underline{\text{End}}({}_AM)$ is of representation dimension at most three, then $\text{fin.dim}(A)<\infty$.*

Proof. By a result of Auslander, $\mathrm{gl.dim}(\mathrm{End}(_A M)) \leqslant 4$. Now the corollary follows immediately from Corollary 3.6. □

As another corollary we have the following result. For the notion of a stably hereditary algebra and a laura algebra, we refer to [31] and [2], respectively. For the definition of glued algebras, we refer to [1].

Corollary 3.8. *Let A be an Artin algebra with* $\mathrm{gl.dim}(A) \leqslant 4$ *and e an idempotent element in A. Then* $\mathrm{fin.dim}(eAe) < \infty$ *if one of the following holds*:

(1) *A/AeA is stably hereditary*;
(2) *A/AeA is a special biserial algebra*;
(3) *A/AeA is representation-finite*;
(4) *A/AeA is a tilted algebra, or a laura algebra*;
(5) *A/AeA is a glued algebra*;
(6) *A/AeA is a tame Schur algebra.*

Proof. All algebras displayed in Corollary 3.8 have representation dimensions at most three. For a proof of this fact, we refer to [2,10,12,31] and [19]. Thus the result follows from Theorem 1.1. □

Note that we may replace A/AeA in the statements of Corollary 3.8 by any algebra C such that there is a stable equivalence of Morita type between A/AeA and C (see [31]). For further information on stable equivalences of Morita type, we refer the reader to [25] and the references therein.

Let us mention the following special case of Corollary 3.8.

Corollary 3.9. *Let A be a hereditary algebra with radical of nilpotency index at most* 4. *Let Λ be the endomorphism algebra of the module $M = \bigoplus_{i=1}^{4} A/\mathrm{rad}^i(A)$. Suppose $M = M_0 \oplus M_1$ with M_0 projective, and with M_1 having no projective summands. Let e be the idempotent in Λ corresponding to the projection onto M_1. Then* $\mathrm{fin.dim}(e\Lambda e) < \infty$.

Proof. By a result of Auslander, we know that $\mathrm{gl.dim}(\Lambda) \leqslant 4$. We claim that no non-zero homomorphism $g : M_0 \longrightarrow M_0$ can factor through a module in $\mathrm{add}(M_1)$. Otherwise, we suppose $g = g_1 g_2$ with $g_1 : M_0 \longrightarrow M'$, $g_2 : M' \longrightarrow M_0$ and $M' \in \mathrm{add}(M_1)$. The image of g_2 is a submodule in M_0, and therefore projective. But this shows that M' contains a projective module as a direct summand, thus M_1 has a projective summand. This is a contradiction. So our claim follows. Then it is clear that $\Lambda/\Lambda e\Lambda$ is isomorphic to $\mathrm{End}(M_0)$ which is Morita equivalent to A, thus $\Lambda/\Lambda e\Lambda$ is hereditary. Note that a stably hereditary algebra has representation dimension at most 3. Now the corollary follows immediately from Corollary 3.8(1) since hereditary algebras have representation dimension bounded by 3. □

Proof of Theorem 1.1(2) and (3). Let $B = eAe$. Suppose M is a B-module with finite projective dimension. As in the proof of Theorem 1.1(1), we have an exact sequence:

$$0 \longrightarrow \Omega_A^4(Ae \otimes_B M) \longrightarrow \Omega_A^2\bigl(\mathrm{Tor}_1^B(Ae, M)\bigr) \oplus Q \longrightarrow \Omega_A^2\bigl(Ae \otimes_B \Omega_B(M)\bigr) \longrightarrow 0,$$

where Q is a projective A-module.

Let $K = \mathrm{Tor}_1^B(Ae, M)$. Suppose $f : P \longrightarrow \Omega_A^2(K)$ is a projective cover of $\Omega_A^2(K)$. We then have an exact sequence

$$0 \longrightarrow \Omega_A^3(K) \longrightarrow P \longrightarrow \Omega_A^2(K) \longrightarrow 0.$$

From this exact sequence we form the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \Omega_A^3(K) \arrow[r, equal] \arrow[d] & \Omega_A^3(K) \arrow[d] & & \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & P \oplus Q \arrow[r] \arrow[d] & \Omega_A^2(Ae \otimes_B \Omega_B(M)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega_A^4(Ae \otimes_B M) \arrow[r] \arrow[d] & \Omega_A^2(K) \oplus Q \arrow[r] \arrow[d] & \Omega_A^2(Ae \otimes_B \Omega_B(M)) \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

Since $\mathrm{gl.dim}(A) \leqslant 4$, the module $\Omega_A^4(Ae \otimes_B M)$ is projective and the first column of this diagram splits. Thus we get an exact sequence

$$0 \longrightarrow \Omega_A^3(K) \oplus \Omega_A^4(Ae \otimes_B M) \longrightarrow P \oplus Q \longrightarrow \Omega_A^2\big(Ae \otimes_B \Omega_B(M)\big) \longrightarrow 0.$$

Let $P' \longrightarrow \Omega_A^3(K)$ be a projective cover of the A-module $\Omega_A^3(K)$. Then we have two canonical exact sequences $0 \longrightarrow \Omega_A^4(K) \longrightarrow P' \longrightarrow \Omega_A^3(K) \longrightarrow 0$ and

$$(*) \quad 0 \longrightarrow \Omega_A^4(K) \longrightarrow P' \oplus \Omega_A^4(Ae \otimes_B M) \longrightarrow \Omega_A^3(K) \oplus \Omega_A^4(Ae \otimes_B M) \longrightarrow 0.$$

Now we apply $\mathrm{Hom}_A(\Omega_A^2(Ae \otimes_B \Omega_B(M)), -)$ to the sequence $(*)$ and obtain the following exact sequence:

$$\begin{aligned}
&\mathrm{Ext}_A^1\big(\Omega_A^2\big(Ae \otimes_B \Omega_B(M)\big), \Omega_A^4(K)\big) \\
&\quad\longrightarrow \mathrm{Ext}_A^1\big(\Omega_A^2\big(Ae \otimes_B \Omega_B(M)\big), Q_3\big) \\
&\quad\longrightarrow \mathrm{Ext}_A^1\big(\Omega_A^2\big(Ae \otimes_B \Omega_B(M)\big), \Omega_A^3(K) \oplus \Omega_A^4(Ae \otimes_B M)\big) \\
&\quad\longrightarrow \mathrm{Ext}_A^2\big(\Omega_A^2\big(Ae \otimes_B \Omega_B(M)\big), \Omega_A^4(K)\big)
\end{aligned}$$

with $Q_3 = P' \oplus \Omega_A^4(Ae \otimes_B M)$.

In the following, we show that $\mathrm{Ext}_A^2(\Omega_A^2(Ae \otimes_B \Omega_B(M)), \Omega_A^4(K)) = 0$ in the cases (2) and (3) of Theorem 1.1. Clearly, in case (3), we have $\Omega_A^4(K) = 0$ since K is an A/AeA-module and $\mathrm{proj.dim}({}_AK) \leqslant 3$.

Now we consider the case (2). Since $\mathrm{gl.dim}(A) \leqslant 4$, we know that $\mathrm{Ext}_A^2(\Omega_A^2(Ae \otimes_B \Omega_B(M)), \Omega_A^4(K))$ vanishes if $\mathrm{Ext}_A^2(\Omega_A^2(Ae \otimes_B \Omega_B(M)), {}_AA) = 0$. To prove the latter, we first show that for any A-module Y and B-module X, there is an embedding from $\mathrm{Ext}^1_A(Ae\otimes_B X, Y)$ into $\mathrm{Ext}^1_B(X, eY)$.

Let $0\longrightarrow Y\longrightarrow I_0\longrightarrow Y'\longrightarrow 0$ be an exact sequence with I_0 the injective hull of Y. Then we have an exact sequence in B-mod:

$$0\longrightarrow eY\longrightarrow eI_0\longrightarrow eY'\longrightarrow 0.$$

Now we may establish the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_A(Ae\otimes_B X, I_0) \arrow[r] \arrow[d, "f_0"'] & \mathrm{Hom}_A(Ae\otimes_B X, Y') \arrow[r] \arrow[d, "f_1"'] & \mathrm{Ext}^1_A(Ae\otimes_B X, Y) \arrow[r] \arrow[d, "f'"'] & 0 \\
\mathrm{Hom}_B(X, eI_0) \arrow[r] & \mathrm{Hom}_B(X, eY') \arrow[r] & \mathrm{Ext}^1_B(X, eY), &
\end{tikzcd}$$

where f_0, f_1 are the adjunction isomorphisms, and thus the morphism f' exists and is injective. This has shown that $\mathrm{Ext}^1_A(Ae\otimes_B X, Y)$ is embedded into $\mathrm{Ext}^1_B(X, eY)$. Thus we see that $\mathrm{Ext}^1_A(Ae\otimes_B \Omega_B(M), \Omega_A^{-3}(A))$ is embedded in $\mathrm{Ext}^1_B(\Omega_B(M), e\Omega_A^{-3}({}_AA)) = \mathrm{Ext}^2_B(M, e\Omega_A^{-3}({}_AA)) = 0$ since $\mathrm{inj.dim}_B(e\Omega_A^{-3}({}_AA)) \leqslant 1$. This implies that $\mathrm{Ext}^2_A(\Omega^2_A(Ae\otimes_B \Omega_B(M)), {}_AA) = \mathrm{Ext}^4_A(Ae\otimes_B \Omega_B(M), {}_AA) = \mathrm{Ext}^1_A(Ae\otimes_B \Omega_B(M), \Omega_A^{-3}({}_AA)) = 0$; and therefore $\mathrm{Ext}^2_A(\Omega^2_A(Ae\otimes_B \Omega_B(M)), \Omega^4_A(K)) = 0$.

Thus, by lifting elements in $\mathrm{Ext}^1_A(\Omega^2_A(Ae\otimes_B \Omega_B(M)), \Omega^3_A(K)\oplus\Omega^4_A(Ae\otimes_B M))$, we obtain the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \Omega^4_A(K) \arrow[r, equal] \arrow[d] & \Omega^4_A(K) \arrow[d] & & \\
0 \arrow[r] & P'\oplus\Omega^4_A(Ae\otimes_B M) \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & \Omega^2_A(Ae\otimes_B \Omega_B(M)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^3_A(K)\oplus\Omega^4_A(Ae\otimes_B M) \arrow[r] \arrow[d] & P\oplus Q \arrow[r] \arrow[d] & \Omega^2_A(Ae\otimes_B \Omega_B(M)) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where P' is a projective cover of $\Omega^3_A(K)$. This diagram provides us with an exact sequence

$$0\longrightarrow P'\oplus\Omega^4_A(Ae\otimes_B M)\longrightarrow Y\longrightarrow \Omega^2_A\big(Ae\otimes_B \Omega_B(M)\big)\longrightarrow 0$$

in A-mod with $Y = P\oplus Q\oplus\Omega^4_A(K)$ a projective A-module, and this yields an exact sequence in B-mod:

$$0\longrightarrow eP'\oplus e\Omega^4_A(Ae\otimes_B M)\longrightarrow eY\longrightarrow e\Omega^2_A\big(Ae\otimes_B \Omega_B(M)\big)\longrightarrow 0.$$

Note that $\Omega_B^3(M) \simeq e\Omega_A^2(Ae \otimes_B \Omega_B(M)) \oplus eT$ for some projective A-module T by Lemma 3.3. Thus we can use Lemma 2.1 to proceed with the following estimation:

$$\begin{aligned}
\operatorname{proj.dim}({}_B M) &\leqslant \operatorname{proj.dim}\bigl(\Omega_B^3(M)\bigr) + 3 \\
&= \Psi\bigl(\Omega_B^3(M)\bigr) + 3 \\
&\leqslant \Psi\bigl(eP' \oplus e\Omega_A^4(Ae \otimes_B M) \oplus eY \oplus eT\bigr) + 1 + 3 \\
&\leqslant \Psi(eA) + 4.
\end{aligned}$$

Note that the B-module eA does not depend on the choice of the module M. Thus we know that $\operatorname{fin.dim}(B)$ is finite. $\square$

Proposition 3.10. *Let A be an Artin algebra of global dimension at most 5. If, for any A-module X, we have $\operatorname{inj.dim}(\Omega_A^4(X)) \leqslant 3$, then $\operatorname{fin.dim}(eAe) < \infty$ for every idempotent element e in A.*

Proof. Let M be a B-module with $B = eAe$. Then we have a minimal projective resolution of $Ae \otimes_B M$:

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow Ae \otimes_B M \longrightarrow 0$$

with P_i projective. Put $X = Ae \otimes_B M$. We get the following exact sequence

$$0 \longrightarrow \Omega_A^4(X) \longrightarrow Q_3 \longrightarrow \Omega_A^3(X) \longrightarrow 0.$$

Since $\operatorname{inj.dim}(\Omega_A^4(X)) \leqslant 3$, the last term in the following induced sequence

$$\begin{aligned}
\cdots \longrightarrow \operatorname{Hom}_A\bigl(\Omega_A^2(X), \Omega_A^3(X)\bigr) &\longrightarrow \operatorname{Ext}_A^1\bigl(\Omega_A^2(X), \Omega_A^4(X)\bigr) \longrightarrow \operatorname{Ext}_A^1\bigl(\Omega_A^2(X), Q_3\bigr) \\
\longrightarrow \operatorname{Ext}_A^1\bigl(\Omega_A^2(X), \Omega_A^3(X)\bigr) &\longrightarrow \operatorname{Ext}_A^2\bigl(\Omega_A^2(X), \Omega_A^4(X)\bigr)
\end{aligned}$$

vanishes. This shows that we may form the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \Omega_A^4(X) \arrow[r, equal] \arrow[d] & \Omega_A^4(X) \arrow[d] & & \\
0 \arrow[r] & Q_3 \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & \Omega_A^2(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_A^3(X) \arrow[r] \arrow[d] & Q_2 \arrow[r] \arrow[d] & \Omega_A^2(X) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

This diagram shows that $Y \simeq Q_2 \oplus \Omega_A^4(X)$ since Q_2 is projective. Thus we have the following exact sequence

$$0 \longrightarrow Q_3 \longrightarrow Q_2 \oplus \Omega_A^4(X) \longrightarrow \Omega_A^2(X) \longrightarrow 0.$$

Since $\operatorname{gl.dim}(A) \leqslant 5$, $\operatorname{proj.dim}(\Omega_A^4(X)) \leqslant 1$. Hence $\operatorname{proj.dim}({}_A\Omega_A^2(X)) \leqslant 1$ by Lemma 2.3(1). Let $0 \longrightarrow P_3 \longrightarrow P_2 \longrightarrow \Omega_A^2(X) \longrightarrow 0$ be a minimal projective resolution of $\Omega_A^2(X)$. Then we multiply this resolution by e and obtain the following exact sequence

$$0 \longrightarrow eP_3 \longrightarrow eP_2 \longrightarrow e\Omega_A^2(X) \longrightarrow 0.$$

Note that $\Omega_B^2(M) \simeq e\Omega_A^2(X) \oplus eP$ for some projective A-module P by Lemma 3.3. Now we can use Lemma 2.1 to proceed the following estimation:

$$\begin{aligned} \operatorname{proj.dim}({}_BM) &\leqslant \operatorname{proj.dim}\big(\Omega_B^2(M)\big) + 2 \\ &= \Psi\big(\Omega_B^2(M)\big) + 2 \\ &= \Psi\big(e\Omega_B^2(X) \oplus eP\big) + 2 \\ &\leqslant \Psi(eP_3 \oplus eP_2 \oplus eP) + 1 + 2 \\ &\leqslant \Psi(eA) + 3. \end{aligned}$$

Note that the B-module eA does not depend on the choice of the module M. So the finitistic dimension of B is bounded above by $\Psi(eA) + 3$. This finishes the proof of the proposition. □

Next, we point out the following known result in the literature (see [22, Proposition 10]). For convenience of the reader, we include here a different proof which will be used later.

Proposition 3.11. *Suppose that A is an Artin algebra and e is an idempotent element of A with* $\operatorname{proj.dim}(Ae_{eAe}) < \infty$. *If* $\operatorname{fin.dim}(A) < \infty$, *then* $\operatorname{fin.dim}(eAe) < \infty$.

Proof. First, we have the following fact.

Lemma 3.12. *Suppose we are given an exact sequence of chain complexes*:

$$0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow Y \longrightarrow 0.$$

If there is an integer t such that $H_j(X_i) = 0$ for all i and for all $j \geqslant t$, then $H_j(Y) = 0$ for all $j \geqslant n + t - 1$. Here H_j denotes the j-th homology operator.

Proof. Let K_j be the image of the chain map $X_j \longrightarrow X_{j-1}$ for $j = 1, 2, \ldots, n$ with $K_1 = Y$ and $K_n = X_n$. Then there is an exact sequence of chain complexes

$$(*) \quad 0 \longrightarrow K_j \longrightarrow X_{j-1} \longrightarrow K_{j-1} \longrightarrow 0$$

for each $1 \leqslant j \leqslant n$.

Now, it follows from the exact sequence $(*)$ with $j = n$ that the following long exact sequence of homologies holds:

$$\cdots \longrightarrow H_j(K_n) \longrightarrow H_j(X_{n-1}) \longrightarrow H_j(K_{n-1}) \longrightarrow H_{j-1}(K_n) \longrightarrow \cdots$$

Since $H_j(X_i) = 0$ for all $j \geqslant t$, we have that $H_j(K_{n-1}) = 0$ for all $j \geqslant t+1$. Similarly, we can prove that $H_j(K_{n-2}) = 0$ for all $j \geqslant t+2$. In general, we can show that $H_j(K_i) = 0$ for all $j \geqslant n+t-i$. In particular, we have that $H_j(Y) = 0$ for all $j \geqslant n+t-1$. This finishes the proof of Lemma 3.12. □

Now we use Lemma 3.12 to show Proposition 3.11: Let $m = \mathrm{fin.dim}(A)$ and $n = \mathrm{proj.dim}(Ae_{eAe})$. We shall prove that $\mathrm{fin.dim}(eAe) \leqslant m+n$.

We fix a minimal projective resolution of the right eAe-module Ae:

$$0 \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow Ae \longrightarrow 0.$$

Note that every projective eAe-module is of the form eP with $P \in \mathrm{add}(Ae)$. Now, let X be an eAe-module of finite projective dimension, and let

$$0 \longrightarrow eP_s \longrightarrow \cdots \longrightarrow eP_1 \longrightarrow eP_0 \longrightarrow X \longrightarrow 0$$

be a minimal projective resolution of X, where P_j lies in $\mathrm{add}(Ae)$. We may form the following chain complex:

$$0 \longrightarrow Q_j \otimes_{eAe} eP_s \longrightarrow \cdots \longrightarrow Q_j \otimes_{eAe} eP_1 \longrightarrow Q_j \otimes_{eAe} P_0 \longrightarrow 0$$

for each $j = 0, 1, \ldots, n$. We denote this chain complex by X_j. Then we have an exact sequence of chain complexes:

$$0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow Y \longrightarrow 0,$$

where Y is the chain complex

$$0 \longrightarrow Ae \otimes_{eAe} eP_s \longrightarrow \cdots \longrightarrow Ae \otimes_{eAe} eP_1 \longrightarrow Ae \otimes_{eAe} P_0 \longrightarrow 0.$$

Since for all i we have $H_j(X_i) = 0$ for $j \geqslant 1$, we may use Lemma 3.12 to conclude that $H_j(Y) = 0$ for all $j \geqslant n+1$.

If $s < n$, then $s \leqslant m+n$. So we assume that $s \geqslant n+1$. In this case we have the following exact sequence

$$0 \longrightarrow Ae \otimes_{eAe} eP_s \longrightarrow \cdots \longrightarrow Ae \otimes_{eAe} eP_{n+1} \longrightarrow Ae \otimes_{eAe} P_n \longrightarrow M \longrightarrow 0,$$

where M is the cokernel of the last map. Thus the projective dimension of M is finite. Since the finitistic dimension of A is m, we have $\mathrm{proj.dim}(M) \leqslant m$.

If $s > m+n$, then it follows from $\mathrm{proj.dim}(M) \leqslant m$ that the kernel Q of the map $Ae \otimes_{eAe} eP_{n+m-1} \longrightarrow Ae \otimes_{eAe} eP_{n+m-2}$ is a projective A-module and generated by Ae. Thus we have an exact sequence

$$0 \longrightarrow Q \longrightarrow Ae \otimes_{eAe} eP_{n+m-1} \longrightarrow Ae \otimes_{eAe} eP_{n+m-2} \longrightarrow \cdots \longrightarrow Ae \otimes_{eAe} eP_n \longrightarrow M \longrightarrow 0.$$

It follows from this exact sequence that the sequence

$$0 \longrightarrow eQ \longrightarrow eP_{n+m-1} \longrightarrow eP_{n+m-2} \longrightarrow \cdots \longrightarrow eP_n$$

is exact. Note that $Q \in \operatorname{add}(Ae)$. Hence the projective resolution of X can be written as

$$0 \to eQ \to eP_{n+m-1} \to eP_{n+m-2} \to \cdots \to eP_{n+1} \to eP_n \to \cdots \to eP_1 \to eP_0 \to X \to 0.$$

This implies that $s \leqslant n+m$, a contradiction to the assumption that $s > m+n$. So we must have $s \leqslant m+n$. Thus the finitistic dimension of eAe is finite. $\square$

Now, we mention a condition for Ae to have finite projective dimension as a right eAe-module. Recall that an ideal I in an Artin algebra A is called a *homological ideal* if $\operatorname{Tor}_i^A(A/I, A/I) = 0$ for all $i \geqslant 1$. Observe that such an ideal was called a *strong idempotent ideal* in [6], or a stratified ideal in other papers. For further characterizations of homological ideals we refer to [6,14,26].

Proposition 3.13. *(See [6].) Let A be an Artin algebra with* $\operatorname{gl.dim}(A) < \infty$, *and let e be an idempotent in A. Then* $\operatorname{proj.dim}(Ae_{eAe}) < \infty$ *if the ideal AeA is a homological ideal in A.*

Let us remark that in the sense of recollements of derived categories one may use a result of Happel in [17] together with a result of König in [23] to judge some special cases for eAe to have finite finitistic dimension. However, our results in this section seem not to be covered by their results because in our situation the existence of a recollement between the three algebras A, eAe and A/AeA implies that the global dimension of eAe is finite. This usually does not happen in our situation, as will be shown by examples in the last section.

4. Gorenstein dimensions and finitistic dimensions

In this section, we shall generalize Proposition 3.11 by considering the so-called Gorenstein dimension of Ae_{eAe}, and prove Theorem 1.2. First, let us recall some definitions and facts (see [5] and [18] for more information).

Let A be an Artin algebra. An A-module M in A-mod is said to be *Gorenstein-projective* if there is an exact sequence of projective modules in A-mod,

$$P^{\bullet}\colon \quad \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots,$$

such that $M \simeq \operatorname{Im}(P_0 \longrightarrow P^0)$ and $\operatorname{Hom}(P^{\bullet}, Q)$ is exact for every projective module $Q \in A$-mod.

For finitely generated modules over Artin algebras, there is another description of Gorenstein-projective modules. Recall from [5] that a module X in A-mod is of *G-dimension zero*, denoted by G-$\dim(X) = 0$, if $\operatorname{Ext}_A^i(X, {}_AA) = 0 = \operatorname{Ext}_A^i(\operatorname{Tr}(X), A_A)$ for all $i > 0$, where Tr is the transpose operator. It was pointed out that a finitely generated A-module X has G-dimension zero if and only if X is Gorenstein-projective (see [18] and [27]).

We observe that if X is Gorenstein-projective defined by the complex $P^\bullet$, then all the images, kernels and cokernels of morphisms in the complex are Gorenstein-projective. Furthermore, every projective A-module in A-mod is Gorenstein-projective.

We say that an A-module $M \in A$-mod has *Gorenstein-dimension n* if there is an exact sequence $0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$ such that all X_i are Gorenstein-projective and that n is minimal among the lengths of such exact sequences. In this case, we write $G\text{-dim}(M) = n$.

Clearly, $G\text{-dim}(X) \leqslant \text{proj.dim}(X)$ for all A-modules X. Furthermore, if $\text{proj.dim}(X) < \infty$, then $G\text{-dim}(X) = \text{proj.dim}(X)$. We remark that there are modules with both finite Gorenstein dimension and infinite projective dimension.

Let us introduce some more notations. Let A be an Artin algebra. We denote by $\mathcal{G}^n(A)$ the full subcategory of A-mod consisting of all modules with G-dimension at most n, and by $\mathcal{G}^\infty(A)$ the union of all $\mathcal{G}^n(A)$ for $i \geqslant 0$. As usual, let $\mathcal{P}^\infty(A)$ stand for the full subcategory of A-mod with objects of finite projective dimensions. Clearly, $\mathcal{P}^\infty(A) \subseteq \mathcal{G}^\infty(A)$.

Let $\mathcal{C}$ be a full subcategory of A-mod and X an arbitrary A-module. Recall that a morphism $f : C \longrightarrow X$ is called a *right $\mathcal{C}$-approximation* of X if $C \in \mathcal{C}$ and the induced map $\text{Hom}_A(-, f)$: $\text{Hom}_A(C', C) \longrightarrow \text{Hom}_A(C', X)$ is surjective for all $C' \in \mathcal{C}$. The category $\mathcal{C}$ is said to be *contravariantly finite* in A-mod if every A-module in A-mod has a right $\mathcal{C}$-approximation.

Lemma 4.1. *If M is a Gorenstein-projective right A-module and if X is an A-module of finite projective dimension, then* $\text{Tor}_i^A(M, X) = 0$ *for all $i > 0$. In general, if G-dim$(M) = n$, then* $\text{Tor}_i^A(M, X) = 0$ *for all $i > n$ and all A-modules X of finite projective dimension.*

Proof. Let X be an A-module with $\text{proj.dim}({}_AX) \leqslant 1$. Then we pick a minimal projective resolution of X:

$$0 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow X \longrightarrow 0$$

with Q_0 and Q_1 projective. From this exact sequence we obtain the following exact sequence:

$$0 \longrightarrow \text{Tor}_1^A(M, X) \longrightarrow M \otimes_A Q_1 \xrightarrow{\alpha} M \otimes_A Q_0 \longrightarrow M \otimes_A X \longrightarrow 0.$$

To see $\text{Tor}_1^A(M, X) = 0$, we shall show that α is injective. However, this follows from the following exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & M \otimes_A Q_1 \arrow[r, "\alpha"] \arrow[d] & M \otimes_A Q_0 \arrow[r] \arrow[d] & M \otimes_A X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P^0 \otimes_A Q_1 \arrow[r] & P^0 \otimes_A Q_0 \arrow[r] & P^0 \otimes_A X \arrow[r] & 0.
\end{tikzcd}$$

Thus we have shown that if $\text{proj.dim}({}_AX) \leqslant 1$, then $\text{Tor}_i^A(M_A, X) = 0$ for all $i > 0$ and every Gorenstein-projective right A-module M.

Now suppose $\mathrm{proj.dim}(X) = n > 1$. Let K_i be the image of the map $P^i \longrightarrow P^{i+1}$, and $K_{-1} = M$. Then we have an exact sequence

$$0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots \longrightarrow P^i \longrightarrow K_i \longrightarrow 0.$$

Clearly, $M \simeq \Omega^{i+1}(K_i) \oplus U_i$ for some projective A-module U_i. Note that all K_i are Gorenstein-projective.

Given a positive integer j with $n \geqslant j \geqslant 1$, we define $i = n - j - 1$. Then $\mathrm{proj.dim}(\Omega^{j+i}(X)) \leqslant 1$. Thus $\mathrm{Tor}_1^A(N, \Omega^{j+i}(X)) = 0$ for all Gorenstein-projective modules N. From this we have

$$\mathrm{Tor}_j^A(M, X) = \mathrm{Tor}_{j+i+1}^A(K_i, X) = \mathrm{Tor}_1^A\big(K_i, \Omega^{j+i}(X)\big) = 0.$$

This implies that $\mathrm{Tor}_j^A(M, X) = 0$ for all $j > 0$, as desired.

The last statement follows by induction on the Gorenstein dimension of M. $\square$

As an immediate consequence we have the following result.

Corollary 4.2. *Let A be an Artin algebra and X an A-module.*

(1) *If there is an exact sequence*

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

with all P_i projective, then, for every Gorenstein-projective right module M_A, the sequence

$$0 \longrightarrow M \otimes_A P_n \longrightarrow \cdots \longrightarrow M \otimes_A P_1 \longrightarrow M \otimes_A P_0 \longrightarrow M \otimes_A X \longrightarrow 0$$

is exact.

(2) *If* $\mathrm{gl.dim}(A) < \infty$*, then X is Gorenstein-projective if and only if X is projective.*

The following result is true (see [18]).

Lemma 4.3. *Let X be an A-module in* A-mod *with finite Gorenstein dimension, say $G\text{-}dim(X) = n$. Then there is an exact sequence $0 \longrightarrow K \longrightarrow M \xrightarrow{f} X \longrightarrow 0$ such that $M \in \mathcal{G}^0(A)$, f is a right $\mathcal{G}^0(A)$-approximation of X and* $\mathrm{proj.dim}(K) = n - 1$. *For $n = 0$, we understand that $K = 0$.*

The next result tells us that G-dimensions and projective dimensions of modules are closely related to each other (see also [18]).

Lemma 4.4. *If $M \in A\text{-}mod$ with $G\text{-}dim(M) = m < \infty$, then there is an A-module X with* $\mathrm{proj.dim}(X) = m$. *In particular,* $\mathrm{fin.dim}(A) = \sup\{G\text{-}dim(X) \mid X \in \mathcal{G}^\infty(A)\}$.

Proof. We may assume that $m > 0$. By Lemma 4.3, there is an exact sequence $0 \longrightarrow K \longrightarrow U \longrightarrow M \longrightarrow 0$ with $U \in \mathcal{G}^0(A)$ and $\mathrm{proj.dim}(K) = m - 1$. Since U is Gorenstein-projective, the U can be embedded into a projective A-module $P \in A$-mod. Thus we get an exact commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
 & & C \arrow[r, equal] \arrow[d] & C \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Suppose that X is Gorenstein-projective. Then, since M is not Gorenstein-projective, we know that $G\text{-dim}(C) = G\text{-dim}({}_AM) + 1$. By Lemma 4.3, we can get a module X' with $\text{proj.dim}(X') = m$. Now we assume that X is not Gorenstein-projective. So the sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow X \longrightarrow 0$$

does not split. This shows that $\text{proj.dim}(X) = \text{proj.dim}(K) + 1 = m$. Hence we have shown the first statement in Lemma 4.4. This yields that $\text{fin.dim}(A) \geqslant \sup\{G\text{-dim}(X) \mid X \in \mathcal{G}^{\infty}(A)\}$. Clearly, it follows from $\mathcal{P}^{\infty}(A) \subseteq \mathcal{G}^{\infty}(A)$ that $\text{fin.dim}(A) \leqslant \sup\{G\text{-dim}(X) \mid X \in \mathcal{G}^{\infty}(A)\}$. Thus the last statement follows. $\square$

The following result is the promised generalization of Proposition 3.11.

Theorem 4.5. *Suppose that A is an Artin algebra and e is an idempotent element of A with G-dim$(Ae_{eAe}) < \infty$. If* $\text{fin.dim}(A) < \infty$*, then* $\text{fin.dim}(eAe) < \infty$.

Proof. The idea of the proof of this result is similar to the one of Proposition 3.11. Here we just outline the main argument.

Since Ae has finite Gorenstein dimension as a right eAe-module, by Lemma 4.3, we have an exact sequence in mod-eAe:

$$0 \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow Ae \longrightarrow 0,$$

with all Q_i Gorenstein-projective, where n is the Gorenstein dimension of the right eAe-module Ae. Suppose X is an eAe-module of projective dimension $s < \infty$, and has the following minimal exact sequence:

$$0 \longrightarrow eP_s \longrightarrow \cdots \longrightarrow eP_1 \longrightarrow eP_0 \longrightarrow X \longrightarrow 0$$

with $P_i \in \text{add}(Ae)$. Let $X_j^{\bullet}$ be the complex

$$0 \longrightarrow Q_j \otimes_{eAe} eP_s \longrightarrow \cdots \longrightarrow Q_j \otimes_{eAe} eP_1 \longrightarrow Q_j \otimes_{eAe} eP_0 \longrightarrow 0$$

for each $j = 0, 1, \dots, n$. By Corollary 4.2, the sequence $X_j^\bullet$ is exact everywhere except at $Q_j \otimes_{eAe} eP_0$. Let $Y^\bullet$ be the complex

$$0 \longrightarrow Ae \otimes_{eAe} eP_s \longrightarrow \cdots \longrightarrow Ae \otimes_{eAe} eP_1 \longrightarrow Ae \otimes_{eAe} eP_0 \longrightarrow 0.$$

Then we have an exact sequence of chain complexes

$$0 \longrightarrow X_n^\bullet \longrightarrow \cdots \longrightarrow X_1^\bullet \longrightarrow X_0^\bullet \longrightarrow Y^\bullet \longrightarrow 0.$$

Therefore the rest of the proof of Theorem 4.5 will be the same as the one of Proposition 3.11, that is, if we put $m = \text{fin.dim}(A)$ and $n = G\text{-dim}(Ae_{eAe})$, then we can show that the finitistic dimension of eAe is bounded above by $m + n$. □

We may state Theorem 4.5 in a slightly more general form. Recall that for an integer $d \geqslant 1$ an A-module M in A-mod is called a *d-syzygy module* if M is projective, or there is an A-module N such that $M \simeq \Omega^d(N)$. We denote by $\Omega^d(A\text{-mod})$ the full subcategory of A-mod consisting of all d-syzygy modules. An A-module M is called a *$*$-syzygy module* if $M \in \text{add}(\Omega^d(A\text{-mod}))$ for all $d \geqslant 1$. We say that an A-module M has *finite $*$-syzygy dimension* if there is an exact sequence $0 \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$ in A-mod for some integer $n \geqslant 0$ such that all Q_i are $*$-syzygy modules. Clearly, every Gorenstein-projective module is a $*$-syzygy module, and modules of finite Gorenstein dimensions have finite $*$-syzygy dimensions. With these notion in mind, we see from the proofs of Lemma 4.1 and Theorem 4.5 that the following weak version of Theorem 4.5 holds:

Theorem 4.6. *Suppose that A is an Artin algebra and e is an idempotent element of A such that the right eAe-module Ae has a finite $*$-syzygy dimension. If* $\text{fin.dim}(A) < \infty$, *then* $\text{fin.dim}(eAe) < \infty$.

Proof. For a $*$-syzygy right A-module M and an A-module X of finite projective dimension, we can show that $\text{Tor}_i^A(M, X) = 0$ for all $i > 0$. With this fact in hand, we can proceed with the proof of Theorem 4.6 smoothly as was done in Theorem 4.5. □

A trivial consequence of Theorem 4.5 is the following corollary.

Corollary 4.7. *Let A be an arbitrary Artin algebra with n the nilpotency index of* $\text{rad}(A)$, *and let Λ be the endomorphism algebra of the module $M := \bigoplus_{i=1}^n A/\text{rad}^i(A)$. If the right Λ-module* $\text{Hom}_A(M, A)$ *has finite Gorenstein dimension, then* $\text{fin.dim}(\Lambda)$ *is finite.*

Finally, let us mention the following fact concerning the modules of finite Gorenstein dimensions.

Proposition 4.8. *If $\mathcal{G}^\infty(A)$ is contravariantly finite in A-mod, then* $\text{fin.dim}(A) < \infty$.

Proof. By [18], $\mathcal{G}^\infty(A)$ is a resolving subcategory of A-mod. Note also that if $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is an exact sequence in A-mod and if the Gorenstein dimensions of both X and Z are at most n then the Gorenstein dimension of Y is at most n (see [5, Theorem 3.13]). Thus, with the same argument as in [8], one can prove that $\max\{G\text{-dim}(X) \mid X \in \mathcal{G}^\infty(A)\}$ is finite since $\mathcal{G}^{\infty}(A)$ is contravariantly finite in A-mod. By Lemma 4.4, we have that $\text{fin.dim}(A) = \max\{G\text{-dim}(X) \mid X \in \mathcal{G}^{\infty}(A)\}$. Thus the finitistic dimension of A is finite. $\square$

Remark. It can be shown that $\mathcal{G}^{\infty}(A)$ in general is not contravariantly finite in A-mod. An easy example is the algebra given in [21] where the same algebra was used to demonstrate that $\mathcal{P}^{\infty}(A)$ may not be contravariantly finite in A-mod. Examples for the subcategory $\mathcal{G}^{0}(A)$ not to be contravariantly finite can be found in [27]. In fact, the results in [27] show also that even for a finite-dimensional local commutative algebra over a field, $\mathcal{G}^{0}(A)$ may not be contravariantly finite in A-mod; for instance, the algebra $A := k[x, y, z]/(x^2, y^2, yz, z^2)$ over a field k is a desired example (I thank R. Takahashi for explaining me this example). Furthermore, we can show that $\mathcal{G}^{\infty}(A) = \mathcal{G}^{0}(A)$ for this algebra. Note that our consideration in this paper is concentrated on A-mod, not on the category A-Mod of *all* A-modules. As to A-Mod, things may differ greatly from what we mentioned here.

5. Examples

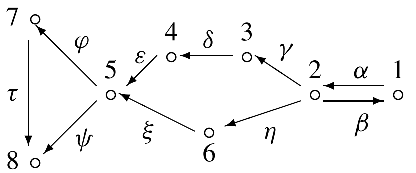
Now we illustrate the main results in the paper by two simple examples.

Example 1. In general, under the conditions of Corollary 3.8, one cannot get the finiteness of global dimension of eAe. For example, let A be the algebra given by quiver and relations:

$$1 \circ \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \circ\, 2, \qquad \beta\alpha = 0.$$

If e is the idempotent corresponding to the vertex 1, then eAe is isomorphic to $k[x]/(x^2)$, thus has infinite global dimension, but we know that $\text{gl.dim}(A) = 2$ and $\text{gl.dim}(A/AeA) = 0$. For a discussion of some general situation concerning idempotent ideals as well as when $\text{gl.dim}(eAe)$ is finite for an idempotent element e in an Artin algebra A, one may refer to [6].

Example 2. Let A be the algebra given by the following quiver and relations:



$$\alpha\beta = \alpha\eta = \alpha\gamma\delta = \gamma\delta\varepsilon - \eta\xi = \beta\alpha\gamma = 0.$$

This algebra has global dimension equal to 4. Let e_i be the primitive idempotent element corresponding to the vertex i. If we put $e = e_2 + e_3 + e_5 + e_6 + e_8$, then eAe is given by the following quiver with relations:

$$\begin{tikzcd}
& & 3 \arrow[dl, "\delta"'] & \\
8 & 5 \arrow[l, shift right, "\varphi"'] \arrow[l, shift left, "\psi"] & & 2 \arrow[ul, "\gamma"'] \arrow[dl, "\eta"] \arrow[loop right, "\alpha"] \\
& & 6 \arrow[ul, "\xi"] &
\end{tikzcd}$$

$$\alpha^2 = \alpha\eta = \alpha\gamma = \gamma\delta - \eta\xi = 0.$$

Clearly, the quotient algebra A/AeA is representation-finite, thus the finitistic dimension of eAe is finite by Theorem 1.1(1). Note that the algebra eAe is neither monomial, nor representation-finite, nor of finite global dimension, and that there is an alternative proof to the finiteness of the finitistic dimension of eAe by applying [30, Theorem 4.2].

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