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Gorenstein projective modules over rings of Morita contexts

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Abstract Under semi-weak and weak compatibility conditions of bimodules, we establish necessary and sufficient conditions of Gorenstein-projective modules over rings of Morita contexts with one bimodule homomorphism zero. This extends greatly the results on triangular matrix Artin algebras and on Artin algebras of Morita contexts with two bimodule homomorphisms zero in the literature, where only sufficient conditions are given under a strong assumption of compatibility of bimodules. An application is provided to describe Gorenstein-projective modules over noncommutative tensor products arising from Morita contexts. Our results are proved under a general setting of noetherian rings and modules instead of Artin algebras and modules.

Keywords Gorenstein-projective module, noetherian ring, noncommutative tensor product, totally exact complex, weakly compatible bimodule

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1 Introduction

For finitely generated modules over noetherian rings, Auslander and Bridge [2] introduced Gorenstein-projective modules, i.e., modules of G-dimension zero, and this idea was generalized several decades later by Enochs et al. [8] and Enochs and Jenda [7] for arbitrary modules over arbitrary rings. Nowadays, the notion of Gorenstein-projective modules plays a very important role in the so-called Gorenstein homological algebra which has significant applications in commutative algebra, algebraic geometry and other fields. It is fundamental, but also difficult, to describe all the Gorenstein-projective modules over a given algebra or ring. Recently, there are many interesting works done in this direction. For example, in a series of articles [17,22,23], Gorenstein-projective modules over the triangular matrix Artin algebra $\left(\begin{smallmatrix} A & N \\ 0 & B \end{smallmatrix}\right)$ were determined under some assumptions on the bimodule ${}_AN_B$. A natural generalization of triangular matrix algebras is the Morita context rings $\Lambda_{(0,0)} = \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$ with two bimodule homomorphisms zero. A Morita context ring is generally the 2×2 matrix ring $\Lambda_{(\phi,\psi)} := \left(\begin{smallmatrix} A & N \\ M & B \end{smallmatrix}\right)$ associated with a Morita context

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$(A, B, {}_BM_A, {}_AN_B, \phi : M \otimes_A N \to B, \psi : N \otimes_B M \to A)$ with two bimodule homomorphisms ϕ and ψ, and $\Lambda_{(\phi,\psi)}$-modules are presented by quadruples (X, Y, f, g) (see Section 2 for details). In [10], Gao and Psaroudakis gave a set of concise sufficient conditions for $\Lambda_{(0,0)}$-modules to be Gorenstein-projective. To achieve their results, they required some compatibility conditions on the bimodules M and N, which were introduced in [23].

In this paper, we consider Morita context rings of the form $\Lambda_\psi := \Lambda_{(0,\psi)}$, which is more general than $\Lambda_{(0,0)}$, and characterize their Gorenstein-projective modules. To implement our characterization of Gorenstein-projective modules, we use weak versions of compatibility conditions (see Subsection 2.2 for definition). With an additional assumption on a couple of very special Λ_ψ-modules related to ingredients of the given Morita context, we even can show that the weak compatibility conditions are necessary and sufficient for a Λ_ψ-module to be Gorenstein-projective. Our discussion is in the frame of noetherian rings instead of Artin algebras. Our main results, Theorems 3.5 and 3.11, are summarized as follows.

Theorem 1.1. *Let A and B be noetherian rings, and $(A, B, {}_BM_A, {}_AN_B, 0, \psi)$ be a Morita context with the bimodules ${}_BM_A$ and ${}_AN_B$ finitely generated as one-sided modules. Furthermore, assume that A is the trivial extension of a subring Λ of A by the image I of ψ.*

(I) *The following two sets of conditions are equivalent for the Morita context ring $\Lambda_\psi := \Lambda_{(0,\psi)}$:*

(1) *${}_\Lambda N_B$, ${}_BM_\Lambda$ and ${}_\Lambda I_\Lambda$ are weakly compatible bimodules, $({}_AN, 0, 0, 0)$ and $({}_AI, 0, 0, 0)$ are semi-weakly compatible left Λ_ψ-modules, and $(M_A, 0, 0, 0)$ and $(I_A, 0, 0, 0)$ are semi-weakly compatible right Λ_ψ-modules.*

(2) *A finitely generated Λ_ψ-module (X, Y, f, g) is Gorenstein-projective if and only if*

(a) *${}_B\mathrm{Coker}(f)$ and ${}_\Lambda\mathrm{Coker}(g)$ are Gorenstein-projective, and*

(b) *${}_B\mathrm{Im}(f) \simeq {}_BM \otimes_A \mathrm{Coker}(g)$, ${}_A\mathrm{Im}(g)/IX \simeq {}_AN \otimes_B \mathrm{Coker}(f)$ and ${}_AIX \simeq {}_AI \otimes_A \mathrm{Coker}(g)$, where $\mathrm{Coker}(f)$ and $\mathrm{Im}(g)$ denote the cokernel of f and the image of g, respectively.*

(II) *Suppose that ${}_\Lambda N_B$, ${}_BM_\Lambda$ and ${}_\Lambda I_\Lambda$ are weakly compatible. If a finitely generated Λ_ψ-module (X, Y, f, g) satisfies the above conditions* (a) *and* (b) *in* (2)*, then it is Gorenstein-projective.*

Theorem 1.1(I) not only extends greatly the ones on triangular matrix algebras by Xiong and Zhang [22] and Zhang [23], and on Morita context algebras with two bimodule homomorphisms zero by Gao and Psaroudakis [10], respectively, to a large class of Morita context rings, but also can be applied to a class of noncommutative tensor products (see Corollary 4.2 for details). Notably, noncommutative tensor products generalize usual tensor products over commutative rings, capture many known constructions in ring theory, and are useful in constructing recollements of derived module categories (see [5, 6]).

The rest of this paper is organized as follows. In Section 2, we recall the definition of (weakly) compatible bimodules, a complete Horseshoe lemma and basic facts on Morita context rings. In Section 3, we prove the main result, Theorem 1.1, and then formulate it for the special Morita context rings $\Lambda_{(0,0)}$. In this case, the resulting statement appears in a quite simple form (see Proposition 3.14). Finally, in Section 4, we apply our result to noncommutative tensor products arising from Morita contexts with two bimodule homomorphisms zero. This provides in fact a corresponding result for Morita context rings $\Lambda_{(\phi,0)}$, as indicated by Corollary 4.2.

2 Preliminaries

In this section, we recall basic definitions and facts for later proofs.

Let A be a unitary (associative) ring. We denote by A-Mod (resp. A-mod) the category of all the (resp. finitely generated) left A-modules. As usual, A-Proj and A-proj are the full subcategories of A-Mod consisting of all the projective modules and finitely generated projective A-modules, respectively. Similarly, we have the notations A-Inj and A-inj for the full subcategories of all the injective A-modules and finitely generated injective A-modules, respectively. For a full subcategory $\mathcal{X}$ of A-Mod, we denote by $\mathscr{C}(\mathcal{X})$ the category of complexes over $\mathcal{X}$, and write $\mathscr{C}(A)$ for $\mathscr{C}(A\text{-Mod})$.

The composite of two homomorphisms $f : X \to Y$ and $g : Y \to Z$ will be denoted by fg instead of gf. Thus the image of $x \in X$ under f is written as $(x)f$ or xf, and the image of f is denoted by $\mathrm{Im}(f)$.

A complex $X^\bullet = (X^i, d_X^i) \in \mathscr{C}(A)$ is *exact* if the cohomology group $H^i(X^\bullet) = 0$ for all i, and *totally exact* if it is exact and the complex $\mathrm{Hom}_A(X^\bullet, P)$ is exact for all projective A-modules P. Let X be an A-module. An exact complex $P^\bullet \in \mathscr{C}(A\text{-Proj})$ is called a *complete projective resolution* of X if $\mathrm{Ker}(d_P^0) = X$. By a *total projective resolution* of X, we mean a totally exact, complete projective resolution of X. Following [7], the module ${}_AX$ is *Gorenstein-projective* if X has a total projective resolution. Dually, an A-module Y is *Gorenstein-injective* if there is a complete injective resolution $I^\bullet \in \mathscr{C}(A\text{-Inj})$ such that $\mathrm{Ker}(d_I^0) = Y$ and $\mathrm{Hom}_A(E, I^\bullet)$ is exact for all $E \in A$-Inj. In A-mod, Gorenstein-projective modules are nothing else than modules of G-dimension 0 in the sense of Auslander and Bridge [2]. We denote by A-GProj (resp. A-Gproj) the category of all the (resp. finitely generated) Gorenstein-projective A-modules, and by A-GInj (resp. A-Ginj) the category of all the (resp. finitely generated) Gorenstein-injective A-modules. Note that A-Gproj contains A-proj and is closed under direct summands, extensions and kernels of surjective homomorphisms (see [13]).

Since Gorenstein-projective modules involve complete projective resolutions, a complete Horseshoe lemma is needed. For the convenience of the readers, we state it here for module categories and still refer it to the Horseshoe lemma. For other versions, see [10, 13, 23].

Lemma 2.1 (Horseshoe lemma)**.** *Given a short exact sequence* $0 \to U \to W \to V \to 0$ *of* A*-modules and two exact complexes* $X^\bullet = (X^i, d_X^i)$ *and* $Y^\bullet = (Y^i, d_Y^i)$ *of* A*-modules with* $\mathrm{Ker}(d_X^0) = U$ *and* $\mathrm{Ker}(d_Y^0) = V$*, if* $\mathrm{Ext}_A^1(\mathrm{Ker}(d_Y^i), X^i) = 0$ *for all* $i \geqslant 0$ *and if* $\mathrm{Ext}_A^1(Y^{-i}, \mathrm{Im}(d_X^{-i})) = 0$ *for all* $i \geqslant 1$*, then there are an exact complex* $Z^\bullet = (Z^i, d_Z^i)$ *and an exact sequence of complexes*

$$0 \longrightarrow X^\bullet \longrightarrow Z^\bullet \longrightarrow Y^\bullet \longrightarrow 0,$$

where

$$Z^i = X^i \oplus Y^i, \quad d_Z^i = \begin{pmatrix} d_X^i & 0 \\ \rho^i & d_Y^i \end{pmatrix},$$

and $\rho^i : Y^i \to X^{i+1}$ *is a homomorphism of* A*-modules such that the induced exact sequence* $0 \to \mathrm{Ker}(d_X^0) \to \mathrm{Ker}(d_Z^0) \to \mathrm{Ker}(d_Y^0) \to 0$ *coincides with the given short exact sequence.*

Furthermore, if $X^i = X^{i+1}$*,* $Y^i = Y^{i+1}$*,* $d_X^i = d_X^{i+1}$ *and* $d_Y^i = d_Y^{i+1}$ *for all* i*, then* $Z^i = Z^{i+1}$ *and* $d_Z^i = d_Z^{i+1}$ *for all* i*.*

The following easy lemma is often used.

Lemma 2.2. (1) *If*

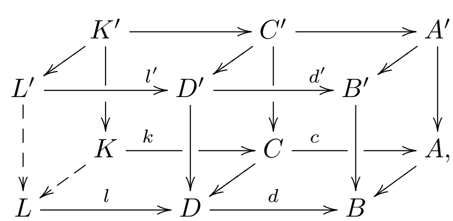
$$0 \to C^\bullet \xrightarrow{c^\bullet} E^\bullet \xrightarrow{d^\bullet} G^\bullet \to 0$$

is an exact sequence of complexes of A*-modules, then there is an induced exact sequence*

$$0 \to \mathrm{Ker}(d_C^i) \xrightarrow{\bar{c}^i} \mathrm{Ker}(d_E^i) \xrightarrow{\bar{d}^i} \mathrm{Ker}(d_G^i)$$

in A-Mod *for any* $i \in \mathbb{Z}$*. In particular, if* $C^\bullet$ *and* $E^\bullet$ *are exact, then* $\bar{d}^i$ *is surjective.*

(2) *Given a 3-dimensional diagram of* A*-modules with the squares consisting of solid arrows commutative*



if $l'd' = 0 = kc$ *and* l *is the kernel of* d*, then the dashed arrows exist and every new square commutes.*

The next lemma is well known.

Lemma 2.3. *Let* R *be a left noetherian ring, and* M *and* N *be finitely generated* R*-modules.*

(1) *Every surjective homomorphism* $f : {}_RM \to {}_RM$ *is an automorphism.*

(2) *If* $M \simeq N$ *and* $f : M \to N$ *is a surjective homomorphism of* R*-modules, then* f *is an isomorphism.*

Finally, we recall the definition of approximations. Let $\mathcal{D}$ be a full additive subcategory of an additive category $\mathcal{C}$ and X an object in $\mathcal{C}$. A morphism $f: X \to D$ in $\mathcal{C}$ is called a *left $\mathcal{D}$-approximation* of X if $D \in \mathcal{D}$ and $\mathrm{Hom}_{\mathcal{C}}(f, D'): \mathrm{Hom}_{\mathcal{C}}(D, D') \to \mathrm{Hom}_{\mathcal{C}}(X, D')$ is surjective for any object $D' \in \mathcal{D}$. Dually, a morphism $f: D \to X$ in $\mathcal{C}$ is called a *right $\mathcal{D}$-approximation* of X if $D \in \mathcal{D}$ and $\mathrm{Hom}_{\mathcal{C}}(D', f):$ $\mathrm{Hom}_{\mathcal{C}}(D', D) \to \mathrm{Hom}_{\mathcal{C}}(D', X)$ is surjective for any object $D' \in \mathcal{D}$. Left and right approximations are also termed as preenvelopes and precovers in ring theory, respectively.

2.1 Morita context rings and their modules

Morita context rings stemmed from a description of Morita equivalences of rings (see [3,19]), and now appear in many situations (see, for example, [12] for some cases). There is a large variety of literature on Morita contexts, duality and equivalences (see, for example, [3,4,10,11,16,18,19]). Here, we briefly recall Morita context rings and their modules.

Let A and B be unitary rings, ${}_AN_B$ be an A-B-bimodule, ${}_BM_A$ be a B-A-bimodule, $\phi: M \otimes_A N \to B$ be a homomorphism of B-B-bimodules and $\psi: N \otimes_B M \to A$ be a homomorphism of A-A-bimodules. Furthermore, let $I := \mathrm{Im}(\psi)$ and $J := \mathrm{Im}(\phi)$. The sextuple (A, B, M, N, ϕ, ψ) is called a *Morita context* (see [19]) if the two diagrams are commutative, i.e.,

$$\begin{array}{ccc} N \otimes_B M \otimes_A N & \xrightarrow{1_N \otimes \phi} & N \otimes_B B \\ \downarrow{\scriptstyle \psi \otimes 1_N} & & \downarrow{\scriptstyle \mathrm{mlt}} \\ A \otimes_A N & \xrightarrow{\mathrm{mlt}} & N, \end{array} \qquad \begin{array}{ccc} M \otimes_A N \otimes_B M & \xrightarrow{\phi \otimes 1_M} & B \otimes_B M \\ \downarrow{\scriptstyle 1_M \otimes \psi} & & \downarrow{\scriptstyle \mathrm{mlt}} \\ M \otimes_A A & \xrightarrow{\mathrm{mlt}} & M, \end{array}$$

where mlt stands for the multiplication map universally.

Associated with a Morita context (A, B, M, N, ϕ, ψ), we can define a *Morita context ring* (see [3,19]), denoted by $\Lambda_{(\phi,\psi)}$, which has the underlying abelian group of the matrix form with the multiplication induced by ϕ and ψ:

$$\Lambda_{(\phi,\psi)} := \begin{pmatrix} A & N \\ M & B \end{pmatrix} = \left\{ \begin{pmatrix} a & n \\ m & b \end{pmatrix} \;\middle|\; a \in A,\, b \in B,\, n \in N,\, m \in M \right\},$$

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} \begin{pmatrix} a' & n' \\ m' & b' \end{pmatrix} = \begin{pmatrix} aa' + (n \otimes m')\psi & an' + nb' \\ ma' + bm' & (m \otimes n')\phi + bb' \end{pmatrix}.$$

In the following, we write Λ_ψ for $\Lambda_{(0,\psi)}$. To avoid confusion with Morita algebras in [15], we adopt here the terminology of Morita context rings instead of Morita rings. For simplicity, we write Λ_ψ for $\Lambda_{(0,\psi)}$.

The description of modules over $\Lambda_{(\phi,\psi)}$ was well known (see, for example, [11,16]). Every $\Lambda_{(\phi,\psi)}$-module is determined by a quadruple (X, Y, f, g), where X and Y are modules over A and B, respectively, $f \in \mathrm{Hom}_B(M \otimes_A X, Y)$ and $g \in \mathrm{Hom}_A(N \otimes_B Y, X)$ such that the following two diagrams are commutative:

$$\begin{array}{ccc} N \otimes_B M \otimes_A X & \xrightarrow{1_N \otimes f} & N \otimes_B Y \\ {\scriptstyle \psi \otimes 1_X}\downarrow & & \downarrow{\scriptstyle g} \\ A \otimes_A X & \xrightarrow{\simeq} & X, \end{array} \qquad \begin{array}{ccc} M \otimes_A N \otimes_B Y & \xrightarrow{1_M \otimes g} & M \otimes_A X \\ {\scriptstyle \phi \otimes 1_Y}\downarrow & & \downarrow{\scriptstyle f} \\ B \otimes_B Y & \xrightarrow{\simeq} & Y, \end{array} \tag{2.1}$$

where the two isomorphisms are the multiplication maps.

If $({}_AX, {}_BY, f, g)$ is a $\Lambda_{(\phi,\psi)}$-module, then $I\mathrm{Coker}(g) = 0$ and $J\mathrm{Coker}(f) = 0$. This follows from (2.1) since $IX \subseteq \mathrm{Im}(g)$ and $JY \subseteq \mathrm{Im}(f)$. Thus $\mathrm{Coker}(g)$ is an A/I-module and $\mathrm{Coker}(f)$ is a B/J-module. Since $\mathrm{Hom}_B(M \otimes_A X, Y) \simeq \mathrm{Hom}_A(X, \mathrm{Hom}_B(M, Y))$, we denote by $\tilde{f}: X \to \mathrm{Hom}_B(M, Y)$ the image of f under this adjunction. Similarly, we define $\tilde{g}: Y \to \mathrm{Hom}_A(N, X)$.

A homomorphism from a $\Lambda_{(\phi,\psi)}$-module (X, Y, f, g) to another $\Lambda_{(\phi,\psi)}$-module (X', Y', f', g') is a pair (α, β) with $\alpha \in \mathrm{Hom}_A(X, X')$ and $\beta \in \mathrm{Hom}_B(Y, Y')$ such that the following two diagrams are commutative:

$$\begin{array}{ccc} M\otimes_A X & \xrightarrow{f} & Y \\ {\scriptstyle 1_M\otimes\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ M\otimes_A X' & \xrightarrow{f'} & Y', \end{array} \qquad \begin{array}{ccc} N\otimes_B Y & \xrightarrow{g} & X \\ {\scriptstyle 1_N\otimes\beta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ N\otimes_B Y' & \xrightarrow{g'} & X'. \end{array} \tag{2.2}$$

Clearly, for a homomorphism $(\alpha,\beta):(X,Y,f,g)\to(X',Y',f',g')$ of $\Lambda_{(\phi,\psi)}$-modules, its kernel $\mathrm{Ker}(\alpha,\beta)$ is $(\mathrm{Ker}(\alpha),\mathrm{Ker}(\beta),h,j)$, where h and j are uniquely given by the commutative diagrams, respectively. It holds that

$$\begin{array}{ccccc} M\otimes_A \mathrm{Ker}(\alpha) & \xrightarrow{1_M\otimes i_X} & M\otimes_A X & \xrightarrow{1_M\otimes\alpha} & M\otimes_A X' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \mathrm{Ker}(\beta) & \xhookrightarrow{i_Y} & Y & \xrightarrow{\beta} & Y', \end{array} \quad \begin{array}{ccccc} N\otimes_B \mathrm{Ker}(\beta) & \xrightarrow{1_N\otimes i_Y} & N\otimes_B Y & \xrightarrow{1_N\otimes\beta} & N\otimes_B Y' \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ \mathrm{Ker}(\alpha) & \xhookrightarrow{i_X} & X & \xrightarrow{\alpha} & X', \end{array} \tag{2.3}$$

where $i_X:\mathrm{Ker}(\alpha)\to X$ and $i_Y:\mathrm{Ker}(\beta)\to Y$ are the inclusions. Dually, one describes the cokernel of (α,β).

Let

$$0\to(X_1,Y_1,f_1,g_1)\xrightarrow{(\alpha_1,\beta_1)}(X_2,Y_2,f_2,g_2)\xrightarrow{(\alpha_2,\beta_2)}(X_3,Y_3,f_3,g_3)\to 0$$

be a sequence of $\Lambda_{(\phi,\psi)}$-modules. This sequence is exact if and only if the induced sequences $0\to X_1\xrightarrow{\alpha_1}X_2\xrightarrow{\alpha_2}X_3\to 0$ and $0\to Y_1\xrightarrow{\beta_1}Y_2\xrightarrow{\beta_2}Y_3\to 0$ are exact in A-Mod and B-Mod, respectively.

Given $\Lambda_{(\phi,\psi)}$-modules (X,Y,f,g) and (X',Y',f',g'), their direct sum is given by $(X\oplus X',Y\oplus Y', f\oplus f',g\oplus g')$, where

$$f\oplus f'=\begin{pmatrix} f & 0\\ 0 & f'\end{pmatrix}:M\otimes_A X\oplus M\otimes_A X'\to Y\oplus Y'$$

is defined to be the diagonal homomorphism of B-modules.

For $X\in A$-Mod and $Y\in B$-Mod, we denote by Ψ_X and Φ_Y the composites of the maps, respectively,

$$\Psi_X: N\otimes_B M\otimes_A X\xrightarrow{\psi\otimes 1_X}A\otimes_A X\xrightarrow{\simeq}X,\qquad \Phi_Y: M\otimes_A N\otimes_B Y\xrightarrow{\phi\otimes 1_Y}B\otimes_B Y\xrightarrow{\simeq}Y.$$

The bimodules ${}_BM_A$ and ${}_AN_B$ define two natural transformations ζ and ξ between tensor functors and hom-functors, i.e.,

$$\begin{aligned} &M\otimes_A -\xrightarrow{\zeta}\mathrm{Hom}_A(N,-):A\text{-Mod}\longrightarrow B\text{-Mod},\\ &\zeta_X:M\otimes_A X\longrightarrow \mathrm{Hom}_A(N,X),\quad m\otimes x\mapsto[n\mapsto(n\otimes m)\psi\,x],\\ &N\otimes_B -\xrightarrow{\xi}\mathrm{Hom}_B(M,-):B\text{-Mod}\longrightarrow A\text{-Mod},\\ &\xi_Y:N\otimes_B Y\to\mathrm{Hom}_B(M,Y),\quad n\otimes y\mapsto[m\mapsto(m\otimes n)\phi\,y]. \end{aligned}$$

Following [12], we define functors related to Morita context rings as follows:

$$\begin{aligned} &\mathsf{T}_A:A\text{-Mod}\longrightarrow\Lambda_{(\phi,\psi)}\text{-Mod},\quad {}_AX\mapsto\mathsf{T}_A(X)=(X,M\otimes_A X,1_{M\otimes X},\Psi_X),\\ &\mathsf{H}_A:A\text{-Mod}\longrightarrow\Lambda_{(\phi,\psi)}\text{-Mod},\quad {}_AX\mapsto\mathsf{H}_A(X)=(X,\mathrm{Hom}_A(N,X),\zeta_X,\delta_X),\\ &\mathsf{T}_B:B\text{-Mod}\longrightarrow\Lambda_{(\phi,\psi)}\text{-Mod},\quad {}_BY\mapsto\mathsf{T}_B(Y)=(N\otimes_B Y,Y,\Phi_Y,1_{N\otimes Y}),\\ &\mathsf{H}_B:B\text{-Mod}\longrightarrow\Lambda_{(\phi,\psi)}\text{-Mod},\quad {}_BY\mapsto\mathsf{H}_B(Y)=(\mathrm{Hom}_B(M,Y),Y,\delta_Y,\xi_Y), \end{aligned}$$

where $\delta_X:N\otimes_B\mathrm{Hom}_A(N,X)\to X$ and $\delta_Y:M\otimes_A\mathrm{Hom}_B(M,Y)\to Y$ are evaluation maps. For an A-module X with $IX=0$ and a B-module Y with $JY=0$, we can get naturally $\Lambda_{(\phi,\psi)}$-modules $(X,0,0,0)$ and $(0,Y,0,0)$, respectively. This gives rise to the functors

$$\mathsf{Z}_{A/I}:(A/I)\text{-Mod}\longrightarrow\Lambda_{(\phi,\psi)}\text{-Mod},\quad {}_{A/I}U\mapsto\mathsf{Z}_{A/I}(U)=({}_AU,0,0,0),$$

$$\mathsf{Z}_{B/J} : (B/J)\text{-Mod} \longrightarrow \Lambda_{(\phi,\psi)}\text{-Mod}, \quad {}_{B/J}V \mapsto \mathsf{Z}_{B/J}(V) = (0, {}_BV, 0, 0).$$

The actions of the above functors on morphisms are defined naturally. The relation among these functors is given by the following lemma. Since our proofs only use adjoint pairs of functors in recollements, we will not recall here the definition of recollements of abelian categories, and just refer the readers to [9,21] for more details.

Lemma 2.4 (See [12])**.** *There are the following two recollements of module categories*:

$$\xymatrix{B/J\text{-Mod} \ar[rr]^{\mathsf{Z}_{B/J}} && \Lambda_{(\phi,\psi)}\text{-Mod} \ar@/_1.5pc/[ll]_{\mathsf{Q}_B} \ar@/^1.5pc/[ll]^{\mathsf{P}_B} \ar[rr]^{\mathsf{U}_A} && A\text{-Mod}, \ar@/_1.5pc/[ll]_{\mathsf{T}_A} \ar@/^1.5pc/[ll]^{\mathsf{H}_A}}$$

$$\xymatrix{A/I\text{-Mod} \ar[rr]^{\mathsf{Z}_{A/I}} && \Lambda_{(\phi,\psi)}\text{-Mod} \ar@/_1.5pc/[ll]_{\mathsf{Q}_A} \ar@/^1.5pc/[ll]^{\mathsf{P}_A} \ar[rr]^{\mathsf{U}_B} && B\text{-Mod}, \ar@/_1.5pc/[ll]_{\mathsf{T}_B} \ar@/^1.5pc/[ll]^{\mathsf{H}_B}}$$

where U_A *and* U_B *are the canonical projections to* $\Lambda_{(\phi,\psi)}$-Mod *and* B-Mod*, respectively,* $\mathsf{Q}_A = ((A/I) \otimes_A -)\mathsf{U_A}$ *and* $\mathsf{Q}_B = ((B/J) \otimes_B -)\mathsf{U_B}$*, and* P_A *and* P_B *are defined on objects* (X, Y, f, g) *by taking kernels of* $\tilde{f}$ *and* $\tilde{g}$*, respectively.*

Suppose that A and B are noetherian rings, and ${}_BM_A$ and ${}_AN_B$ are bimodules such that they are finitely generated as one-sided modules. Then it is known that $\Lambda_{(\phi,\psi)}$ is a noetherian ring (see, for example, [18, Proposition 1.7, p. 12]). For a noetherian ring, its identity has a complete decomposition of orthogonal primitive idempotent elements (see [1, Proposition 10.14, p. 128]). Thus the description of indecomposable projective modules over the Artin algebra $\Lambda_{(\phi,\psi)}$ in [12, Proposition 3.1] extends to the one over the noetherian ring $\Lambda_{(\phi,\psi)}$.

Lemma 2.5. *Suppose that* A *and* B *are noetherian rings, and* ${}_BM_A$ *and* ${}_AN_B$ *are bimodules such that they are finitely generated as one-sided modules.*

(1) (*See* [12, *Proposition* 3.1]) *An indecomposable finitely generated* $\Lambda_{(\phi,\psi)}$*-module is projective if and only if it is given by* $\mathsf{T}_A(P) = (P, M \otimes_A P, \mathrm{id}_{M\otimes_A P}, \Psi_P)$*, or* $\mathsf{T}_B(Q) = (N \otimes_B Q, Q, \Phi_Q, \mathrm{id}_{N\otimes_B Q})$*, where* P *and* Q *are finitely generated, indecomposable projective modules over* A *and* B*, respectively.*

(2) (*See* [20, *Corollary* 2.2]) *An indecomposable* $\Lambda_{(\phi,\psi)}$*-module is injective if and only if it is of the form* $\mathsf{H}_A(U) = (U, \mathrm{Hom}_A(N,U),\ \zeta_U, \delta_U)$*, or* $\mathsf{H}_B(V) = (\mathrm{Hom}_B(M,V), V, \delta_V, \xi_V)$*, where* U *and* V *are indecomposable injective modules over* A *and* B*, respectively.*

2.2 Weakly and semi-weakly compatible modules

Compatible modules were defined in [23] to describe a class of Gorenstein-projective modules for triangular matrix Artin algebras which are of course special Morita context rings. They were further pursued in [10] for the Morita context Artin algebras $\Lambda_{(0,0)}$. We use weakly and semi-weakly compatible modules to characterize Gorenstein-projective modules over the noetherian rings Λ_ψ that are more general than $\Lambda_{(0,0)}$.

Let A and B be unitary rings. First, we recall the definition of (weakly) compatible bimodules.

Definition 2.6 (See [23, Definition 1.1] and [14, Definition 4.1])**.** Let ${}_AN_B$ be a bimodule.

(1) ${}_AN_B$ is compatible if

(C1) $\mathrm{Hom}_A(P^\bullet, N)$ is exact for all totally exact complexes $P^\bullet \in \mathscr{C}(A\text{-proj})$, and

(C2) $N \otimes_B Q^\bullet$ is exact for all exact complexes $Q^\bullet \in \mathscr{C}(B\text{-proj})$.

(2) ${}_AN_B$ is *weakly compatible* if it satisfies (C1) and

(C3) $N \otimes_B Q^\bullet$ is exact for all totally exact complexes $Q^\bullet \in \mathscr{C}(B\text{-proj})$.

Weakly compatible bimodules require exactness only for totally exact complexes $Q^\bullet \in \mathscr{C}(B\text{-proj})$ in (C3), and the notion of weakly compatible bimodules is a proper generalization of the one of compatible bimodules (see [14, Example 4.3]).

Definition 2.7. A left A-module ${}_AX$ is *semi-weakly compatible* if it satisfies (C1), i.e., $\mathrm{Hom}_A(P^\bullet, X)$ is exact for all totally exact complexes $P^\bullet \in \mathscr{C}(A\text{-proj})$.

Lemma 2.8. *Let A, B and C be rings.*

(1) *A right B-module Y_B is* semi-weakly compatible *if and only if $Y \otimes_B Q^\bullet$ is exact for all totally exact complexes $Q^\bullet \in \mathscr{C}(B\text{-proj})$.*

(2) *Let ${}_AN_B$ be an A-B-bimodule.*

(i) *If the modules ${}_AN$ and N_B are of finite injective dimension, then ${}_AN_B$ is a weakly compatible A-B-bimodule.*

(ii) *If there is an exact complex $P^\bullet = (P^i, d_P^i) \in \mathscr{C}(B\text{-proj})$ such that $\mathrm{Tor}_1^B(N, \mathrm{Ker}(d_P^i)) \neq 0$ for an integer i, then ${}_AN_B$ is not compatible.*

(3) *If bimodules ${}_AX_B$ and ${}_{C^{\mathrm{op}}}Y_{B^{\mathrm{op}}}$ are compatible, then the bimodule ${}_AX \otimes_B Y_C$ is weakly compatible.*

Proof. (1) For a finitely generated projective module W_B, there is an isomorphism

$$V \otimes_A \mathrm{Hom}_{B^{\mathrm{op}}}(W, U) \simeq \mathrm{Hom}_{B^{\mathrm{op}}}(W, V \otimes_A U)$$

as abelian groups for any bimodule ${}_AU_B$ and A-module V_A. If $W^\bullet \in \mathscr{C}(B^{\mathrm{op}}\text{-proj})$ is totally exact, then so is $\mathrm{Hom}_{B^{\mathrm{op}}}(W^\bullet, B) \in \mathscr{C}(B\text{-proj})$. Hence, Y_B is semi-weakly compatible if and only if $\mathrm{Hom}_{B^{\mathrm{op}}}(W^\bullet, Y)$ is exact if and only if $Y \otimes_B \mathrm{Hom}_{B^{\mathrm{op}}}(W^\bullet, B)$ is exact for all totally exact complexes $W^\bullet$ in $\mathscr{C}(B^{\mathrm{op}}\text{-proj})$. Note that $\mathrm{Hom}_B(-, B)$ is a duality between B-proj and B^{op}-proj. Thus (1) holds.

(2) (i) If ${}_AN$ is injective, then it is semi-weakly compatible. Suppose that ${}_AN$ is of the injective dimension n and $0 \to N \to I \to X \to 0$ is an exact sequence with I injective and ${}_AX$ of the injective dimension $n-1$. Then ${}_AX$ is semi-weakly compatible by induction. For a totally exact complex $P^\bullet \in \mathscr{C}(A\text{-proj})$, since P^i is projective, we have an exact sequence of complexes, i.e.,

$$0 \to \mathrm{Hom}_A(P^\bullet, N) \to \mathrm{Hom}_A(P^\bullet, I) \to \mathrm{Hom}_A(P^\bullet, X) \to 0$$

with both $\mathrm{Hom}_A(P^\bullet, I)$ and $\mathrm{Hom}_A(P^\bullet, X)$ being exact. Thus $\mathrm{Hom}_A(P^\bullet, N)$ is exact, and therefore ${}_AN$ is semi-weakly compatible. Now, suppose that N_B has finite injective dimension. This means that the left B^{op}-module ${}_{B^{\mathrm{op}}}N$ has finite injective dimension. Thus ${}_{B^{\mathrm{op}}}N$ is semi-weakly compatible. By (1), the right B-module N_B is semi-weakly compatible. Hence (i) follows.

(*ii*) The exact sequence $0 \to \mathrm{Ker}(d_P^{i-1}) \to P^{i-1} \to \mathrm{Ker}(d_P^i) \to 0$ shows that $\mathrm{Tor}_1^B(N, \mathrm{Ker}(d_P^i)) \neq 0$ and $N \otimes_B \mathrm{Ker}(d_P^{i-1}) \to N \otimes_B P^{i-1}$ is not an injective homomorphism. This means that the complex $N \otimes_B P^\bullet$ is not exact in degree $i-1$, and therefore ${}_AN_B$ is not a compatible A-B-bimodule.

(3) Suppose that $P^\bullet \in \mathscr{C}(A\text{-proj})$ is a totally exact complex. Then each P^i is a finitely generated projective A-module, and there is an isomorphism $\mathrm{Hom}_A(P^i, X \otimes_B Y) \simeq \mathrm{Hom}_A(P^i, X) \otimes_B Y$. This yields an isomorphism of complexes, i.e.,

$$\mathrm{Hom}_A(P^\bullet, X \otimes_B Y) \simeq \mathrm{Hom}_A(P^\bullet, X) \otimes_B Y.$$

Since ${}_AX$ is semi-weakly compatible, the complex $\mathrm{Hom}_A(P^\bullet, X)$ is exact. Thus ${}_{C^{\mathrm{op}}}Y \otimes_{B^{\mathrm{op}}} \mathrm{Hom}_A(P^\bullet, X)$ is exact by the compatibility of ${}_{C^{\mathrm{op}}}Y_{B^{\mathrm{op}}}$, i.e., $\mathrm{Hom}_A(P^\bullet, X) \otimes_B Y$ is exact. Hence, ${}_AX \otimes_B Y$ is a semi-weakly compatible left A-module.

Let $Q^\bullet \in \mathscr{C}(C\text{-proj})$ be a totally exact complex. Since ${}_{C^{\mathrm{op}}}Y_{B^{\mathrm{op}}}$ is compatible, ${}_{C^{\mathrm{op}}}Y$ is semi-weakly compatible. By (1), Y_C is semi-weakly compatible. Thus $Y \otimes_C Q^\bullet$ is exact. Since ${}_AX_B$ is compatible, the complex $X \otimes_B (Y \otimes_C Q^\bullet)$ is exact. Thus, $X \otimes_B Y_C$ is a semi-weakly compatible right C-module. □

It is not hard to see that a bimodule ${}_AN_B$ is weakly compatible if and only if ${}_AN$ and N_B are semi-weakly compatible. For an Artin algebra A, there is a duality D: A-mod $\to A^{\mathrm{op}}$-mod. Thus a left A-module $X \in A$-mod is semi-weakly compatible if and only if the right A-module $D(X)_A$ is semi-weakly compatible. This follows from the isomorphism $D(X) \otimes_A P^\bullet \simeq D\mathrm{Hom}_A(P^\bullet, X)$ for all totally exact complexes $P^\bullet$ in $\mathscr{C}(A\text{-proj})$.

3 Proof of the main result

In the rest of this paper, we assume that all the rings considered are noetherian, that is both left and right noetherian, and all the modules are finitely generated.

Let $(A, B, {}_BM_A, {}_AN_B, \phi, \psi)$ be a Morita context with $\phi = 0$. We consider the Morita context ring

$$\Lambda_\psi := \begin{pmatrix} A & N \\ M & B \end{pmatrix}_{(0,\psi)}.$$

Let $I := \mathrm{Im}(\psi)$. Then $IN = MI = 0$ and $I^2 = 0$. Assume further that Λ is a subring of A with the same identity and A is the *trivial extension* of Λ by I, i.e., $A = \Lambda \ltimes I$ with the multiplication

$$(\lambda, x)(\lambda', x') = (\lambda\lambda', \lambda x' + x\lambda'), \quad \lambda, \lambda' \in \Lambda, \quad x, x' \in I.$$

Thus $\Lambda \simeq A/I$. Let $\pi : A \to \Lambda$ be the canonical surjection. The restriction of π on Λ is the identity id_Λ. Clearly, I is an ideal of A with $I^2 = 0$.

Every A-module restricts to a Λ-module via the inclusion of Λ into A. Conversely, every Λ-module X induces an A-module $A \otimes_\Lambda X = X \oplus I \otimes_\Lambda X$, and restricts to an A-module via π, i.e., by defining $IX = 0$.

For a Λ_ψ-module (X, Y, f, g), let $\lambda_X : X \to \mathrm{Coker}(g)$ and $\mu_Y : Y \to \mathrm{Coker}(f)$ be the canonical projections.

3.1 Sufficient conditions for Gorenstein-projective modules

We first prove the following lemma.

Lemma 3.1. *If* $({}_AX, {}_BY, f, g) \in \Lambda_\psi\text{-mod}$, *then*

(1) $I\mathrm{Coker}(g) = 0$ *and* $I\mathrm{Im}(g) = 0$;

(2) *there is a unique* B*-module homomorphism* $\eta_Y : M \otimes_A \mathrm{Coker}(g) \to Y$ *such that* $f = (1_M \otimes \lambda_X)\eta_Y$, *and thus* $\mathrm{Im}(f) = \mathrm{Im}(\eta_Y)$ *and* $\mathrm{Coker}(f) = \mathrm{Coker}(\eta_Y)$;

(3) *let* $p_X : X \to X/IX$ *be the canonical projection, and then there is a unique homomorphism* $\theta_X : N \otimes_B \mathrm{Coker}(f) \to X/IX$ *of* A*-modules such that* $gp_X = (1_N \otimes \mu_Y)\theta_X$; *thus* $\mathrm{Im}(\theta_X) = \mathrm{Im}(gp_X)$;

(4) *let* $\mathrm{mlt}_X : I \otimes_A X \to X$ *be the multiplication map, and then there is a unique homomorphism* $m_X : I \otimes_A \mathrm{Coker}(g) \to X$ *of* A*-modules such that* $\mathrm{Im}(m_X) = IX$ *and* $\mathrm{mlt}_X = (1_I \otimes \lambda_X)m_X$.

Proof. (1) Clearly, $I\mathrm{Coker}(g) = 0$ holds for any Morita context rings and their modules. It follows from $IN = 0$ that $I(N \otimes_B Y) = 0$. Thus, $I\mathrm{Im}(g) = I(N \otimes_B Y)g = (I(N \otimes_B Y))g = 0$.

(2) There is the exact commutative diagram of B-modules, i.e.,

$$\begin{array}{ccccccc}
{}_BM \otimes_A N \otimes_B Y & \xrightarrow{1_M \otimes g} & M \otimes_A X & \xrightarrow{1_M \otimes \lambda_X} & M \otimes_A \mathrm{Coker}(g) & \longrightarrow & 0. \\
\downarrow{\scriptstyle \phi \otimes 1_Y} & & \downarrow{\scriptstyle f} & \swarrow{\scriptstyle \exists!\, \eta_Y} & & & \\
B \otimes_B Y & \xrightarrow{\mathrm{mlt}} & Y & & & &
\end{array}$$

Since $\phi = 0$, there is a unique homomorphism η_Y of B-modules such that $f = (1_M \otimes \lambda_X)\eta_Y$. This implies $\mathrm{Im}(f) = \mathrm{Im}(\eta_Y)$ because $1_M \otimes \lambda_X$ is surjective. Thus it follows from Lemma 2.3 that η_Y is injective if and only if $M \otimes_A \mathrm{Coker}(g) \simeq \mathrm{Im}(f)$ as B-modules.

(3) There is the following commutative diagram of A-modules:

$$\begin{array}{ccccccc}
N \otimes_B M \otimes_A X & \xrightarrow{1_N \otimes f} & N \otimes_B Y & \xrightarrow{1_N \otimes \mu_Y} & N \otimes_B \mathrm{Coker}(f) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \psi \otimes 1_X} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \exists!\, \theta_X} & & \\
IX & \hookrightarrow & X & \xrightarrow{p_X} & X/IX & \longrightarrow & 0. \\
\downarrow & & \downarrow{\scriptstyle \lambda_X} & & \downarrow{\scriptstyle \lambda'_X} & & \\
0 & \longrightarrow & \mathrm{Coker}(g) & = & \mathrm{Coker}(g) & &
\end{array}$$

It follows from Lemma 2.3 that θ_X is injective if and only if $N \otimes_B \mathrm{Coker}(f) \simeq \mathrm{Im}(g)/IX$.

(4) Consider the following commutative diagram of A-modules:

$$\begin{array}{ccccccc} I \otimes_A \mathrm{Im}(g) & \longrightarrow & I \otimes_A X & \xrightarrow{1_I \otimes \lambda_X} & I \otimes_A \mathrm{Coker}(g) & \longrightarrow & 0. \\ & & \Big\downarrow {\scriptstyle \mathrm{mlt}_X} & \swarrow {\scriptstyle \exists!\, m_X} & & & \\ & & X & & & & \end{array}$$

Thanks to $I\mathrm{Im}(g) = 0$, there is a unique homomorphism m_X of A-modules such that $\mathrm{mlt}_X = (1_I \otimes \lambda_X) m_X$. Thus, m_X is injective if and only if $IX \simeq I \otimes_A \mathrm{Coker}(g)$ as A-modules by Lemma 2.3. □

Lemma 3.2. *If* $(X, Y, f, g) \in \Lambda_\psi\text{-mod}$ *satisfies*

$$_A N \otimes_B \mathrm{Coker}(f) \simeq {}_A\mathrm{Im}(g)/IX \quad \text{and} \quad {}_A IX \simeq {}_A I \otimes_A \mathrm{Coker}(g),$$

then $\mathrm{Im}(g)$ *is the pushout of* $1_N \otimes_B \eta_Y$ *and* $\psi \otimes_\Lambda 1_{\mathrm{Coker}(g)}$, *where* η_Y *is given in Lemma* 3.1(2).

Proof. Put $U := \mathrm{Coker}(g)$, $V := \mathrm{Coker}(f)$ and $H := \mathrm{Im}(g)$. Then one gets the canonical exact sequence

$$0 \to H \xrightarrow{\epsilon_X} X \xrightarrow{\lambda_X} {}_A U \to 0$$

of A-modules. By Lemma 3.1(4), $\mathrm{Im}(m_X) = IX$. Let i_X be the inclusion of IX into X and write $m_X = m'_X i_X$. Consider the diagram

$$\begin{array}{ccccccccc} & & & & I \otimes_A U & & & & \\ & & & & \Big\downarrow {\scriptstyle m'_X} & & & & \\ & & & & IX & & & & \\ & & & \swarrow {\scriptstyle m_H} & \Big\downarrow {\scriptstyle i_X} & & & & \\ 0 & \longrightarrow & H & \xrightarrow{\epsilon_X} & X & \xrightarrow{\lambda_X} & U & \longrightarrow & 0. \end{array}$$

As λ_X is a homomorphism of A-modules, it holds that

$$(IX)\lambda_X = I(\mathrm{Im}(\lambda_X)) = IU = I\mathrm{Coker}(g) = 0$$

by Lemma 3.1(1). Thus there is an injective homomorphism $m_H : IX \to H$ of A-modules such that $i_X = m_H \epsilon_X$. It follows from ${}_A IX \simeq {}_A I \otimes_A \mathrm{Coker}(g)$ that m'_X is an isomorphism by Lemma 2.3(2).

The isomorphism $N \otimes_B \mathrm{Coker}(f) \simeq \mathrm{Im}(g)/IX$ implies that θ_X is injective. Thus the proof of Lemma 3.1(3) implies that there is a homomorphism $q : H \to N \otimes_B V$ making the following diagram commutative:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \Big\downarrow & & \Big\downarrow & & & & \\ & & IX & = & IX & & & & \\ & & \Big\downarrow {\scriptstyle m_H} & & \Big\downarrow {\scriptstyle i_X} & & & & \\ 0 & \longrightarrow & H & \xrightarrow{\epsilon_X} & X & \xrightarrow{\lambda_X} & U & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle q} & & \Big\downarrow {\scriptstyle p_X} & & \Big\| & & \\ 0 & \longrightarrow & N \otimes_B V & \xrightarrow{\theta_X} & X/IX & \xrightarrow{\lambda'_X} & U & \longrightarrow & 0. \\ & & \Big\downarrow & & \Big\downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

This shows $\mathrm{Coker}(m_H) \simeq N \otimes_B V$ as A-modules. Now, we write $g = \sigma\epsilon_X$ with $\sigma : N \otimes_B Y \to H$ the canonical projection and $\epsilon_X : H \hookrightarrow X$ the inclusion, and consider the diagram

$$\begin{array}{ccccc} N\otimes_B M\otimes_A X & \xrightarrow{\psi\otimes 1_X} & & & I\otimes_A X \\ \downarrow{\scriptstyle 1_{N\otimes_B M}\otimes\lambda_X} & & & \swarrow{\scriptstyle 1_I\otimes\lambda_X} & \\ N\otimes_B M\otimes_A U & \xrightarrow{\psi\otimes 1_U} & I\otimes_A U & & \downarrow{\scriptstyle \mathrm{mlt}_X} \\ \downarrow{\scriptstyle 1_N\otimes\eta_Y} & & \downarrow{\scriptstyle m'_X m_H} & & \\ N\otimes_B Y & \xrightarrow{\sigma} & H & \xrightarrow{\epsilon_X} & X. \end{array} \tag{3.1}$$

Since (X, Y, f, g) is a Λ_ψ-module, the out-side square in (3.1) is commutative. From the definition of m_H, we know $\mathrm{mlt}_X = (1_I \otimes \lambda_X)(m'_X m_H)\epsilon_X$. From the natural homomorphism $\psi : {}_AN \otimes_B M_A \to {}_AI_A$, one sees that the upper square is commutative. As $1_{N\otimes_B M} \otimes \lambda_X$ is a surjective map and ϵ_X is an injective map, the down-left corner in (3.1) is commutative. This means that there is the following exact commutative diagram:

$$\begin{array}{ccccccccc} & & N\otimes_B M\otimes_A U & \xrightarrow{1_N\otimes\eta_Y} & N\otimes_B Y & \xrightarrow{1_N\otimes\mu_Y} & N\otimes_B V & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi\otimes 1_U} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle \exists\nu} & & \\ 0 & \longrightarrow & I\otimes_A U & \xrightarrow{m'_X m_H} & H & \xrightarrow{q} & N\otimes_B V & \longrightarrow & 0, \end{array}$$

where the top row is exact because of $N \otimes_B -$ acting on the exact sequence

$${}_BM \otimes_A U \xrightarrow{\eta_Y} {}_BY \xrightarrow{\mu_Y} {}_BV \longrightarrow 0.$$

By the assumption, A is left noetherian and $N \otimes_B V$ is a finitely generated A-module, whence every surjective endomorphism of ${}_AN \otimes_B V$ is an automorphism. Thus ν is an automorphism. This implies that H is the pushout of $1_N \otimes_B \eta_Y$ and $\psi \otimes_A 1_U$. Since $MI = IU = 0$, we have $I \otimes_A U \simeq I \otimes_\Lambda U$ and $N \otimes_B M \otimes_A U \simeq N \otimes_B M \otimes_\Lambda U$ as A-modules. Thus H is the pushout of $1_N \otimes_B \eta_Y$ and $\psi \otimes_\Lambda 1_U$, as desired. □

For ${}_\Lambda X \in \Lambda$-Mod, we define a quadruple

$$\mathsf{T}_\Lambda(X) := (A \otimes_\Lambda X, {}_BM \otimes_\Lambda X, \pi_X, \psi_X),$$

where $\pi_X : M \otimes_A A \otimes_\Lambda X \simeq M \otimes_\Lambda X$ is the canonical homomorphism of B-modules, and

$$\psi_X : N \otimes_B (M \otimes_\Lambda X) \longrightarrow A \otimes_\Lambda X, \quad n \otimes (m \otimes x) \mapsto (n \otimes m)\psi \otimes x \quad \text{for } n \in N,\ m \in M,\ x \in X$$

is a homomorphism of A-modules. Clearly, $\psi_X = \Psi_{A\otimes_\Lambda X}$, $\psi_\Lambda = \psi$ and $(A \otimes_\Lambda X, {}_BM \otimes_\Lambda X, \pi_X, \psi_X)$ is a Λ_ψ-module. Moreover, for $\alpha \in \mathrm{Hom}_\Lambda(X, X')$, the pair $(1_A \otimes \alpha, 1_M \otimes \alpha)$ is a homomorphism from the Λ_ψ-module $\mathsf{T}_\Lambda(X)$ to $\mathsf{T}_\Lambda(X')$. Thus, we get a functor

$$\mathsf{T}_\Lambda : \Lambda\text{-Mod} \longrightarrow \Lambda_\psi\text{-Mod}, \quad {}_\Lambda X \mapsto \mathsf{T}_\Lambda(X) := (A \otimes_\Lambda X, M \otimes_\Lambda X, \pi_X, \psi_X).$$

Moreover,

$$\mathsf{T}_A(A \otimes_\Lambda X) = (A \otimes_\Lambda X, M \otimes_A A \otimes_\Lambda X, 1_{M\otimes_A A\otimes_\Lambda X}, \psi \otimes 1_{A\otimes X}) \simeq \mathsf{T}_\Lambda(X)$$

as Λ_ψ-modules via the morphism $(1_{A\otimes_\Lambda X}, \pi_X)$. In particular, $\mathsf{T}_\Lambda(\Lambda) \simeq \Lambda_\psi e_1$, where

$$e_1 = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} \in \Lambda_\psi$$

with 1_A the identity of A.

To stress the Λ-decomposition of ${}_AA \otimes_\Lambda X$, we sometimes write ${}_AA \otimes_\Lambda X$ as $X(I) := X \oplus I \otimes_\Lambda X$. Clearly, the A-module structure on $X(I)$ is given by $(\lambda, i)(x, j \otimes x') = (\lambda x, i \otimes x + \lambda j \otimes x')$, $\lambda \in \Lambda$, $i, j \in I$ and $x, x' \in X$. Thus, $\mathsf{T}_\Lambda(X) = (X(I), M \otimes_\Lambda X, \pi_X, \psi_X)$.

Lemma 3.3. *Let X and X' be Λ-modules. Then there are the following homomorphisms (or isomorphisms) of abelian groups, which are natural in X and X'.*

(1)

$$\mathrm{Hom}_\Lambda(X, X') \simeq \mathrm{Hom}_A(X(I), X'), \quad f \mapsto \begin{pmatrix} f \\ 0 \end{pmatrix}$$

for $f \in \mathrm{Hom}_\Lambda(X, X')$.

(2)

$$\mathrm{Hom}_\Lambda(X, I \otimes_\Lambda X') \simeq \mathrm{Hom}_A(X, X'(I)), \quad g \mapsto (0, g)$$

for $g \in \mathrm{Hom}_\Lambda(X, I \otimes_\Lambda X')$.

(3)

$$\mathrm{Hom}_\Lambda(X, X' \oplus I \otimes_\Lambda X') \simeq \mathrm{Hom}_A(X(I), X'(I)), \quad (a, c) \mapsto \begin{pmatrix} a & c \\ 0 & 1_I \otimes a \end{pmatrix}$$

for $a \in \mathrm{Hom}_\Lambda(X, X')$ and $c \in \mathrm{Hom}_\Lambda(X, I \otimes_\Lambda X')$.

Proof. (1) There are isomorphisms

$$\mathrm{Hom}_\Lambda(X, X') \simeq \mathrm{Hom}_\Lambda(X, \mathrm{Hom}_A({}_A A_\Lambda, X')) \simeq \mathrm{Hom}_A(A \otimes_\Lambda X, X') \simeq \mathrm{Hom}_A(X(I), X'),$$

where the first isomorphism is induced from the isomorphism $X' \to \mathrm{Hom}_A({}_A A_\Lambda, X')$, $x' \mapsto \{1_A \mapsto x'\}$, $x' \in X'$, the second is the adjunction and the third is given by

$$X(I) \to A \otimes_\Lambda X, \quad (x, i \otimes y) \mapsto (1_\Lambda, 0) \otimes x + (0, i) \otimes y, \quad x, y \in X.$$

We can check that the composite of the above isomorphisms sends f to $\binom{f}{0}$ for $f \in \mathrm{Hom}_\Lambda(X, X')$, and is natural in each variable.

(2) For $g \in \mathrm{Hom}_\Lambda(X, I \otimes_\Lambda X')$, it suffices to prove that $(0, g) : X \to X'(I)$ is a homomorphism of A-modules. In fact, take $a = (\lambda, i) \in A$, $x \in X$ and consider the Λ-module X as A-module, that is $ix = 0$ and $ax = (\lambda, i)x = \lambda x$. Then $(ax)(0, g) = (0, (ax)g) = (0, (\lambda x)g)$. On the other hand,

$$a(0, (x)g) = (\lambda, i)(0, (x)g) = (0, \lambda(x)g) = (0, (\lambda x)g).$$

Thus, $(0, g)$ is a homomorphism of A-modules.

(3) The proof is similar to the one of (1). Since we have the isomorphisms

$$\begin{aligned} \mathrm{Hom}_\Lambda(X, X' \oplus I \otimes_\Lambda X') &\simeq \mathrm{Hom}_\Lambda(X, \mathrm{Hom}_A({}_A A_\Lambda, X'(I))) \simeq \mathrm{Hom}_A(A \otimes_\Lambda X, X'(I)) \\ &\simeq \mathrm{Hom}_A(X(I), X'(I)), \end{aligned}$$

their composite sends (a, c) to $\left(\begin{smallmatrix} a & c \\ 0 & 1_I \otimes a \end{smallmatrix}\right)$ for $a \in \mathrm{Hom}_\Lambda(X, X')$ and $c \in \mathrm{Hom}_\Lambda(X, I \otimes_\Lambda X')$. All the isomorphisms are natural in each variable. □

The next lemma for $\Lambda_{(0,0)}$ was indicated in [10]. We state it for Λ_ψ and include more details for our applications. Note that the functor $\mathsf{Z}_B : B\text{-Mod} \to \Lambda_\psi\text{-Mod}$, ${}_B Y \mapsto (0, Y, 0, 0)$ is well defined.

Lemma 3.4. *Let ${}_\Lambda X, X' \in \Lambda$-mod and ${}_B Y$, ${}_B Y' \in B$-mod. Then there are the following isomorphisms which are natural in each variable*:

(1) $\mathrm{Hom}_\Lambda(X, N \otimes_B Y) \simeq \mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_\Lambda(X), \mathsf{T}_B(Y))$, $f \mapsto ((\begin{smallmatrix} f \\ 0 \end{smallmatrix}), 0)$.

(2) $\mathrm{Hom}_\Lambda(X, X') \simeq \mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_\Lambda(X), \mathsf{Z}_\Lambda(X'))$, $g \mapsto ((\begin{smallmatrix} g \\ 0 \end{smallmatrix}), 0)$.

(3) $\mathrm{Hom}_\Lambda(X, X' \oplus I \otimes_\Lambda X') \simeq \mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_\Lambda(X), \mathsf{T}_\Lambda(X'))$, $(a, c) \mapsto ((\begin{smallmatrix} a & c \\ 0 & 1_I \otimes a \end{smallmatrix}), 1_M \otimes a)$ *for $a : X \to X'$ and $c : X \to I \otimes_\Lambda X'$.*

(4) $\mathrm{Hom}_B(Y, M \otimes_\Lambda X) \simeq \mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_B(Y), \mathsf{T}_\Lambda(X))$, $h \mapsto ((1_N \otimes h)\psi_X, h)$.

(5) $\mathrm{Hom}_B(Y, Y') \simeq \mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_B(Y), \mathsf{T}_B(Y'))$, $t \mapsto (1_N \otimes t, t)$.

(6) $\mathrm{Hom}_B(Y, Y') \simeq \mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_B(Y), \mathsf{Z}_B(Y'))$, $t \mapsto (0, t)$.

(7) $\mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_B(Y), \mathsf{Z}_\Lambda(X)) = \mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_\Lambda(X), \mathsf{Z}_B(Y)) = 0.$

Proof. (1) There are isomorphisms

$$\mathrm{Hom}_\Lambda(X, N\otimes_B Y)\simeq \mathrm{Hom}_A(X(I), N\otimes_B Y)\simeq \mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_A(X(I)),\mathsf{T}_B(Y))\simeq \mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_\Lambda(X),\mathsf{T}_B(Y)),$$

where the first isomorphism is given by Lemma 3.3(1), the second one follows from the adjoint pair $(\mathsf{T}_A,\mathsf{U}_A)$ of functors in Lemma 2.4 and $\mathsf{U}_A\mathsf{T}_B(Y)=N\otimes_B Y$, and the third is given by the isomorphism $\mathsf{T}_A(X(I))\simeq\mathsf{T}_\Lambda(X)$. Verifications show that the composite of theses isomorphisms sends $f\in\mathrm{Hom}_\Lambda(X,N\otimes_B Y)$ to $((\begin{smallmatrix} f\\ 0\end{smallmatrix}),0)$ and is natural in each variables.

(2) Its proof is similar to the one of (1).

(3) Due to Lemma 3.3(3), we have $\mathrm{Hom}_\Lambda(X, X'\oplus I\otimes_\Lambda X')=\mathrm{Hom}_A(X(I),X'(I))$. Since T_A is fully faithful by Lemma 2.4, we have $\mathrm{Hom}_A(X(I),X'(I))\simeq\mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_A(X(I)),\mathsf{T}_A(X'(I)))$. Due to $\mathsf{T}_A(X(I))\simeq\mathsf{T}_\Lambda(X)$ for any X, we get $\mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_A(X(I)),\mathsf{T}_A(X'(I)))\simeq\mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_\Lambda(X),\mathsf{T}_\Lambda(X'))$. By verification, the composite of these isomorphisms sends $(a,c)\in\mathrm{Hom}_\Lambda(X,X'\oplus I\otimes_\Lambda X')$ to $((\begin{smallmatrix} a & c\\ 0 & 1_I\otimes a\end{smallmatrix}), 1_M\otimes a)$, and is natural in each variable.

(4) It follows from the adjoint pair $(\mathsf{T}_B,\mathsf{U}_B)$ in Lemma 2.4 and $\mathsf{U}_B\mathsf{T}_\Lambda(X)=M\otimes_\Lambda X$ that $\mathrm{Hom}_B(Y,M\otimes_\Lambda X)\simeq\mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_B(Y),\mathsf{T}_\Lambda(X))$, $h\mapsto((1_N\otimes h)\psi_X,h)$, which is natural in each variable.

(5) This is a consequence of the fully faithful functor T_B in Lemma 2.4.

(6) The proof is similar to the one of (4).

(7) By $\mathsf{U}_B\mathsf{Z}_\Lambda(X)=0$ and the adjoint pair $(\mathsf{T}_B,\mathsf{U}_B)$ of functors, we have $\mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_B(Y),\mathsf{Z}_\Lambda(X))\simeq\mathrm{Hom}_B(Y,\mathsf{U}_B\mathsf{Z}_\Lambda(X))=0$. Similarly, since $(\mathsf{T}_A,\mathsf{U}_A)$ is an adjoint pair of functors and $\mathsf{U}_A\mathsf{Z}_B(Y)=0$, we have $\mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_\Lambda(X),\mathsf{Z}_B(Y))\simeq\mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_A(X(I)),\mathsf{Z}_B(Y))\simeq\mathrm{Hom}_A(X(I),\mathsf{U}_A\mathsf{Z}_B(Y))=0$. □

Theorem 3.5. *Suppose that* ${}_\Lambda N_B$, ${}_B M_\Lambda$ *and* ${}_\Lambda I_\Lambda$ *are weakly compatible bimodules, and* (X,Y,f,g) *is a* Λ_ψ*-module. Then* (X,Y,f,g) *is Gorenstein-projective if the following hold*:

(a) *Both* ${}_\Lambda\mathrm{Coker}(g)$ *and* ${}_B\mathrm{Coker}(f)$ *are Gorenstein-projective*;

(b) ${}_BM\otimes_A\mathrm{Coker}(g)\simeq{}_B\mathrm{Im}(f)$, ${}_AN\otimes_B\mathrm{Coker}(f)\simeq{}_A\mathrm{Im}(g)/IX$ *and* ${}_AI\otimes_A\mathrm{Coker}(g)\simeq{}_AIX$, *where* $\mathrm{Coker}(f)$ *and* $\mathrm{Im}(g)$ *denote the cokernel of* f *and the image of* g, *respectively.*

Proof. Suppose that (a) and (b) hold true. Then ${}_\Lambda U:=\mathrm{Coker}(g)$ and ${}_BV:=\mathrm{Coker}(f)$ are Gorenstein-projective. By definition, there are two totally exact sequences of projective modules, i.e.,

$$P^\bullet:\cdots\longrightarrow P^{-1}\xrightarrow{d_P^{-1}}P^0\xrightarrow{d_P^0}P^1\longrightarrow\cdots\quad\text{and}\quad Q^\bullet:\cdots\longrightarrow Q^{-1}\xrightarrow{d_Q^{-1}}Q^0\xrightarrow{d_Q^0}Q^1\longrightarrow\cdots$$

over Λ and B, respectively, such that ${}_\Lambda U=\mathrm{Ker}(d_P^0)$ and ${}_BV=\mathrm{Ker}(d_Q^0)$. To prove (X,Y,f,g) is Gorenstein-projective, we construct a totally exact complex $T^\bullet=(T^i,d_T^i)_{i\in\mathbb{Z}}\in\mathscr{C}(\Lambda_\psi\text{-proj})$ such that $\mathrm{Ker}(d_T^0)\simeq(X,Y,f,g)$ as Λ_ψ-modules. We define $T^i:=\mathsf{T}_\Lambda(P^i)\oplus\mathsf{T}_B(Q^i)$ for all $i\in\mathbb{Z}$. Then $T^i\in\mathscr{C}(\Lambda_\psi\text{-proj})$ by Lemma 2.5(1). To define d_T^i, we have to define a few families of maps.

(1) A homomorphism $\rho^i:Q^i\to M\otimes_\Lambda P^{i+1}$ of B-modules for $i\in\mathbb{Z}$.

By assumptions, ${}_BM_\Lambda$ is weakly compatible, this implies that $M\otimes_\Lambda P^\bullet$ is exact by Definition 2.6(C3), and therefore $\mathrm{Ker}(1_M\otimes d_P^0)=M\otimes_\Lambda U$. Note that $M\otimes_\Lambda P^i\in\mathrm{add}({}_BM)$ for all i. Thus it follows from Definition 2.6(C1) that $\mathrm{Ext}_B^1(\mathrm{Ker}(d_Q^i),M\otimes_\Lambda P^i)=0$ for all $i\geqslant0$ and $\mathrm{Ext}_B^1(Q^{-i},\mathrm{Im}(1_M\otimes d_P^{-i}))=0$ for $i\geqslant1$. Thus, starting with the exact sequence

$$(*)\quad 0\longrightarrow{}_BM\otimes_\Lambda U\xrightarrow{\eta_Y}{}_BY\xrightarrow{\mu_Y}{}_BV\longrightarrow0$$

and applying Horseshoe Lemma 2.1, we get an exact sequence of complex

$$(**)\quad 0\longrightarrow M\otimes_\Lambda P^\bullet\xrightarrow{a^\bullet}Y^\bullet\xrightarrow{b^\bullet}Q^\bullet\longrightarrow0$$

of B-modules such that

$$Y^i:=M\otimes_\Lambda P^i\oplus Q^i\quad\text{and}\quad d_Y^i=\begin{pmatrix}1_M\otimes d_P^i & 0\\ \rho^i & d_Q^i\end{pmatrix},$$

where $\rho^i : Q^i \to M \otimes_\Lambda P^{i+1}$ is a homomorphism of B-modules, and $a^i = (1, 0)$ and $b^i = \binom{0}{1}$ are canonical maps for all i. Note that by taking kernels of $(**)$ at degree 0, we get back the exact sequence $(*)$.

(2) Two homomorphisms $\alpha^i : P^i \to I \otimes_\Lambda P^{i+1}$ and $\beta^i : P^i \to N \otimes_B Q^{i+1}$ of Λ-modules for $i \in \mathbb{Z}$.

In fact, we write $H := \mathrm{Im}(g)$ and get a canonical exact sequence $0 \to H \to X \to {}_AU \to 0$ of A-modules, which restricts to an exact sequence of Λ-modules

$$(\dagger) \quad 0 \longrightarrow {}_\Lambda H \xrightarrow{\epsilon_X} {}_\Lambda X \xrightarrow{\lambda_X} {}_\Lambda U \longrightarrow 0.$$

Now, we define

$$Z^i := (I \otimes_\Lambda P^i) \oplus (N \otimes_B Q^i), \quad d_Z^i = \begin{pmatrix} 1_I \otimes d_P^i & 0 \\ \tau^i & 1_N \otimes d_Q^i \end{pmatrix},$$

where $\tau^i = (1_N \otimes \rho^i)(\psi \otimes 1_{P^{i+1}})$ is the composite of the homomorphisms of Λ-modules, i.e.,

$${}_\Lambda N \otimes_B Q^i \xrightarrow{1_N \otimes \rho^i} {}_\Lambda N \otimes_B M \otimes_\Lambda P^{i+1} \xrightarrow{\psi \otimes 1_{P^{i+1}}} {}_\Lambda I \otimes_\Lambda P^{i+1}.$$

We show that $Z^\bullet = (Z^i, d_Z^i)_{i \in \mathbb{Z}}$ is an exact complex such that $H \simeq \mathrm{Ker}(d_Z^0)$. Indeed, it follows from the complex $Y^\bullet$ that

$$\begin{pmatrix} 1_M \otimes d_P^i & 0 \\ \rho^i & d_Q^i \end{pmatrix} \begin{pmatrix} 1_M \otimes d_P^{i+1} & 0 \\ \rho^{i+1} & d_Q^{i+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \rho^i(1_M \otimes d_P^{i+1}) + d_Q^i \rho^{i+1} & 0 \end{pmatrix} = 0.$$

This implies that $\rho^i(1_M \otimes d_P^{i+1}) + d_Q^i \rho^{i+1} = 0$ and $(1_N \otimes \rho^i)(1_N \otimes 1_M \otimes d_P^{i+1}) + (1_N \otimes d_Q^i)(1_N \otimes \rho^{i+1}) = 0$. By multiplying $\psi \otimes 1_{P^{i+2}}$, we further obtain

$$(1_N \otimes \rho^i)(1_N \otimes 1_M \otimes d_P^{i+1})(\psi \otimes 1_{P^{i+2}}) + (1_N \otimes d_Q^i)(1_N \otimes \rho^{i+1})(\psi \otimes 1_{P^{i+2}}) = 0.$$

Due to $(1_N \otimes 1_M \otimes d_P^{i+1})(\psi \otimes 1_{P^{i+2}}) = (\psi \otimes 1_{P^{i+1}})(1_I \otimes d_P^{i+1})$, we get $\tau^i(1_I \otimes d_P^{i+1}) + (1_N \otimes d_Q^i)\tau^{i+1} = 0$. Thus,

$$d_Z^i d_Z^{i+1} = \begin{pmatrix} 1_I \otimes d_P^i & 0 \\ \tau^i & 1_N \otimes d_Q^i \end{pmatrix} \begin{pmatrix} 1_I \otimes d_P^{i+1} & 0 \\ \tau^{i+1} & 1_N \otimes d_Q^{i+1} \end{pmatrix} = 0,$$

and $Z^\bullet$ is a complex. Moreover, there is an exact sequence of complexes of Λ-modules, i.e.,

$$0 \longrightarrow I \otimes_\Lambda P^\bullet \xrightarrow{c^\bullet} Z^\bullet \xrightarrow{d^\bullet} N \otimes_B Q^\bullet \longrightarrow 0,$$

where $c^\bullet$ and $d^\bullet$ are canonical inclusion and projection, respectively. By the weak compatibility of ${}_\Lambda I_\Lambda$ and ${}_\Lambda N_B$, both $I \otimes_\Lambda P^\bullet$ and $N \otimes_B Q^\bullet$ are exact complexes of Λ-modules. Thus $Z^\bullet$ is also an exact complex.

We now show $\mathrm{Ker}(d_Z^0) \simeq H$. Let

$$\sigma^i := \begin{pmatrix} \psi \otimes 1_{P^i} & 0 \\ 0 & 1_N \otimes 1_{Q^i} \end{pmatrix}_{i \in Z}$$

and $\sigma^\bullet = (\sigma^i)_{i \in \mathbb{Z}}$. Then $\sigma^\bullet : N \otimes_B Y^\bullet \to Z^\bullet$ is a chain map of complexes such that the following is an exact commutative diagram in $\mathscr{C}(A\text{-mod})$:

$$\begin{array}{ccccccccc}
 & & N \otimes_B M \otimes_\Lambda P^\bullet & \xrightarrow{1_N \otimes a^\bullet} & N \otimes_B Y^\bullet & \xrightarrow{1_N \otimes b^\bullet} & N \otimes_B Q^\bullet & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi \otimes 1_{P^\bullet}} & & \downarrow{\scriptstyle \sigma^\bullet} & & \| & & \\
0 & \longrightarrow & I \otimes_\Lambda P^\bullet & \xrightarrow{c^\bullet} & Z^\bullet & \xrightarrow{d^\bullet} & N \otimes_B Q^\bullet & \longrightarrow & 0.
\end{array}$$

Considering the differentials in degree 0, we have the following commutative diagram with exact rows:

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & N\otimes M\otimes P^1 \arrow[rr, "1_N\otimes a^1"] \arrow[dd, "\psi\otimes 1_{P^1}" near start] & & N\otimes Y^1 \arrow[rr, "1_N\otimes b^1"] \arrow[dd, "\sigma^1" near start] & & N\otimes_B Q^1 \arrow[dd, equal] \\
N\otimes M\otimes P^0 \arrow[ur, "1_N\otimes 1_M\otimes d_P^0"] \arrow[rr, "1_N\otimes a^0" near end] \arrow[dd, "\psi\otimes 1_{P^0}"] & & N\otimes Y^0 \arrow[ur, "1_N\otimes d_Y^0"] \arrow[rr, "1_N\otimes b^0" near end] \arrow[dd, "\sigma^0" near start] & & N\otimes_B Q^0 \arrow[ur, "1_N\otimes d_Q^0"] \arrow[dd, equal] & \\
 & I\otimes P^1 \arrow[rr, "c^1" near start] & & Z^1 \arrow[rr, "d^1" near start] & & N\otimes Q^1. \\
I\otimes P^0 \arrow[ur, "1_I\otimes d_P^0"] \arrow[rr, "c^0"] & & Z^0 \arrow[ur, "d_Z^0"] \arrow[rr, "d^0"] & & N\otimes_B Q^0 \arrow[ur, "1_N\otimes d_Q^0"] &
\end{tikzcd}
$$

By enlarging the diagram forward, we get the following exact commutative diagram of A-modules by Lemma 2.2(2):

$$
\begin{tikzcd}
 & N\otimes_B M\otimes_\Lambda U \arrow[r, "1_N\otimes\eta_Y"] \arrow[d, "\psi\otimes 1_U"] & N\otimes_B Y \arrow[r, "1_N\otimes\mu_Y"] \arrow[d, "\delta"] & N\otimes_B V \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I\otimes_\Lambda U \arrow[r, "c^0"] & \operatorname{Ker}(d_Z^0) \arrow[r, "d^0"] & N\otimes_B V \arrow[r] & 0.
\end{tikzcd}
$$

This shows that $\operatorname{Ker}(d_Z^0)$ is the pullback of $\psi\otimes_\Lambda 1_U$ and $1_N\otimes_B\eta_Y$. By Lemma 3.2, H is also the pushout of $\psi\otimes_\Lambda 1_U$ and $1_N\otimes_B\eta_Y$. Thus $H\simeq\operatorname{Ker}(d_Z^0)$ as A-modules. Thus there is the decomposition of d_Z^{-1}:

$$
\begin{tikzcd}[column sep=small]
Z^\bullet:\cdots \arrow[r, "d_Z^{-2}"] & (I\otimes_\Lambda P^{-1})\oplus(N\otimes_B Q^{-1}) \arrow[rr, "d_Z^{-1}"] \arrow[dr] & & (I\otimes_\Lambda P^0)\oplus(N\otimes_B Q^0) \arrow[r, "d_Z^0"] & \cdots \\
 & & H \arrow[ur, "d_H"'] & &
\end{tikzcd}
$$

such that the following two diagrams are commutative:

$$
\begin{tikzcd}
0 \arrow[r] & I\otimes_\Lambda U \arrow[r, "m'_X m_H"] \arrow[d, "1_I\otimes d_U"] & H \arrow[r, "q"] \arrow[d, "d_H"] & N\otimes_B V \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I\otimes_\Lambda P^0 \arrow[r, "{(1,0)}"] & I\otimes_\Lambda P^0\oplus N\otimes_B Q^0 \arrow[r, "\binom{0}{1}"] & N\otimes_B Q^0 \arrow[r] & 0,
\end{tikzcd}
\tag{3.2}
$$

$$
\begin{tikzcd}
 & N\otimes_B Y \arrow[r, "1_N\otimes d_Y"] \arrow[d, "\sigma"] & N\otimes_B Y^0 \arrow[r, "1_N\otimes d_Y^0"] \arrow[d, "\sigma^0"] & N\otimes_B Y^1 \arrow[r] \arrow[d, "\sigma^1"] & \cdots \\
0 \arrow[r] & H \arrow[r, "d_H"] & Z^0 \arrow[r, "d_Z^0"] & Z^1 \arrow[r] & \cdots.
\end{tikzcd}
$$

Finally, we give the definitions of α^i and β^i. In fact, there is the exact sequence

$$(\dagger)\quad 0\longrightarrow {}_\Lambda H\xrightarrow{\epsilon_X}{}_\Lambda X\xrightarrow{\lambda_X}{}_\Lambda U\longrightarrow 0.$$

Since ${}_\Lambda I_\Lambda$ is weakly compatible, $\operatorname{Hom}_\Lambda(P^\bullet,{}_\Lambda I)$ is exact. This yields $\operatorname{Ext}^1_\Lambda(\operatorname{Ker}(d_P^i),I)=0$ for all i. Due to $I\otimes_\Lambda P^i\in\operatorname{add}({}_\Lambda I)$, it holds that $\operatorname{Ext}^1_\Lambda(\operatorname{Ker}(d_P^i),I\otimes P^i)=0$ for all $i\geqslant 0$. Similarly, the weak compatibility of ${}_\Lambda N_B$ implies that $\operatorname{Ext}^1_\Lambda(\operatorname{Ker}(d_P^i),N\otimes_B Q^i)=0$ for all i. Thus $\operatorname{Ext}^1_\Lambda(\operatorname{Ker}(d_P^i),Z^i)=0$ for all $i\geqslant 0$. Clearly, $\operatorname{Ext}^1_\Lambda(P^{-i},\operatorname{Im}(d_Z^{-i}))=0$ for all $i\geqslant 1$. Hence, applying Lemma 2.1 to the exact sequence $(\dagger)$ and the exact complexes $Z^\bullet$ and $P^\bullet$, we get an exact sequence of exact complexes, i.e.,

$$(\dagger\dagger)\quad 0\longrightarrow Z^\bullet\xrightarrow{p^\bullet}E^\bullet\xrightarrow{q^\bullet}P^\bullet\longrightarrow 0$$

in $\mathscr{C}(\Lambda\text{-mod})$, where $p^\bullet$ and $q^\bullet$ are canonical inclusion and projection, respectively, and $E^i:=P^i\oplus(I\otimes_\Lambda P^i)\oplus(N\otimes_B Q^i)\in\Lambda$-mod,

$$d_E^i=\begin{pmatrix} d_P^i & \alpha^i & \beta^i\\ 0 & 1_I\otimes d_P^i & 0\\ 0 & \tau^i & 1_N\otimes d_Q^i\end{pmatrix}: P^i\oplus(I\otimes_\Lambda P^i)\oplus(N\otimes_B Q^i)\longrightarrow P^{i+1}\oplus(I\otimes_\Lambda P^{i+1})\oplus(N\otimes_B Q^{i+1}),$$

$\alpha^i : P^i \to I \otimes_\Lambda P^{i+1}$, and $\beta^i : P^i \to N \otimes_B Q^{i+1}$ are all the homomorphisms of Λ-modules. Recall that $\tau^i = (1_N \otimes \rho^i)(\psi \otimes 1_{P^{i+1}}) : N \otimes_B Q^i \to I \otimes_\Lambda P^{i+1}$ is a homomorphism of Λ-modules. Here, compared with Lemma 2.1, the order of direct summands of E^i is changed. Note that (†) is obtained by taking kernels at degree 0 in (††). Visually, the positive part of (††) looks as follows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H & \xrightarrow{d_H} & I \otimes_\Lambda P^0 \oplus N \otimes_B Q^0 & \xrightarrow{d_Z^0} & I \otimes_\Lambda P^1 \oplus N \otimes_B Q^1 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \epsilon_X} & \overset{(e_1,e_2)}{\nearrow} & \downarrow{\scriptstyle \left(\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}\right)} & & \\
0 & \dashrightarrow & {}_\Lambda X & \overset{d_{{}_\Lambda X}}{\dashrightarrow} & P^0 \oplus I \otimes_\Lambda P^0 \oplus N \otimes_B Q^0 & \overset{d_E^0}{\dashrightarrow} & P^1 \oplus I \otimes_\Lambda P^1 \oplus N \otimes_B Q^1 & \dashrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \lambda_X} & & \downarrow{\scriptstyle \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)} & & \\
0 & \longrightarrow & U & \xrightarrow{d_U} & P^0 & \xrightarrow{d_P^0} & P^1 & \longrightarrow & \cdots, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.3}$$

where $d_{{}_\Lambda X} = (\lambda_X d_U, e_1, e_2)$, and both $e_1 : {}_\Lambda X \to I \otimes_\Lambda P^0$ and $e_2 : {}_\Lambda X \to N \otimes_B Q^0$ are homomorphisms of Λ-modules.

Observe that $1_N \otimes d_Q^i$ and $(0, \tau^i)$ are homomorphisms of A-modules. The term $P^i \oplus I \otimes_\Lambda P^i$ has an A-module structure which is isomorphic to ${}_A A \otimes_\Lambda P^i$. By Lemma 3.3(3), the map

$$\begin{pmatrix} d_P^i & \alpha^i \\ 0 & 1_I \otimes d_P^i \end{pmatrix} : P^i \oplus I \otimes_\Lambda P^i \longrightarrow P^{i+1} \oplus I \otimes_\Lambda P^{i+1}$$

is the image of the homomorphism $(d_P^i, \alpha^i) : P^i \to P^{i+1} \oplus I \otimes_\Lambda P^{i+1}$ of Λ-modules, and thus it is a homomorphism of A-modules. Similarly, $\left(\begin{smallmatrix}\beta^i \\ 0\end{smallmatrix}\right) : P^i(I) \to N \otimes_B Q^{i+1}$ is an A-module homomorphism. Let

$$F^i := P^i(I) \oplus N \otimes_B Q^i, \quad d_F^i := \begin{pmatrix} d_P^i & \alpha^i & \beta^i \\ 0 & 1_I \otimes d_P^i & 0 \\ 0 & \tau^i & 1_N \otimes d_Q^i \end{pmatrix}.$$

Then $F^\bullet = (F^i, d_F^i)$ is a complex of A-modules, which is exact since the restriction of $F^\bullet$ to Λ-modules is the exact complex $E^\bullet$.

We show $\mathrm{Ker}(d_F^0) \simeq X$ as A-modules. First, the exact sequence $0 \to {}_\Lambda X \xrightarrow{d_{{}_\Lambda X}} E^0 \xrightarrow{d_E^0} E^1$ of Λ-modules, with $d_{{}_\Lambda X} = (\lambda_X d_U, e_1, e_2) : X \to P^0 \oplus I \otimes_\Lambda P^0 \oplus N \otimes_B Q^0$, can be regarded as an exact sequence of $\mathbb{Z}$-modules, i.e., $0 \longrightarrow X \xrightarrow{d_X} F^0 \xrightarrow{d_F^0} F^1$, where

$$d_X := ((\lambda_X d_U, e_1), e_2) : X \to F^0 = (P^0 \oplus (I \otimes_\Lambda P^0)) \oplus (N \otimes_B Q^0).$$

It suffices to show that d_X is a homomorphism of A-modules.

Let $x \in X$, $a = (\lambda, i) \in A = \Lambda \ltimes I$. Then $(ax)d_X = (((ax)\lambda_X d_U, (ax)e_1), (ax)e_2)$. Since λ_X is an A-homomorphism, it follows from Lemma 3.1 that $(IX)\lambda_X = 0$ and $(ax)\lambda_X d_U = (\lambda x)\lambda_X d_U = \lambda(x)\lambda_X d_U$. Note that $(ix)(e_1, e_2) = (i \otimes x)\mathrm{mlt}_X(e_1, e_2)$. We deduce from the diagrams (3.1)–(3.3) that

$$\begin{aligned} \mathrm{mlt}_X(e_1, e_2) &= (1_I \otimes \lambda_X)m'_X m_H \epsilon_X(e_1, e_2) = (1_I \otimes \lambda_X)m'_X m_H d_H \\ &= (1_I \otimes \lambda_X)(1_I \otimes d_U)(1, 0) = (1_I \otimes \lambda_X d_U, 0). \end{aligned}$$

Thus, $(ix)(e_1, e_2) = (i \otimes (x)\lambda_X d_U, 0)$. As e_i is a Λ-homomorphism, it holds that $(\lambda x)e_i = \lambda(x)e_i$ for $i = 1, 2$. Hence,

$$(ax)d_X = ((\lambda(x)\lambda_X d_U, \lambda(x)e_1 + i \otimes (x)\lambda_X d_U), \lambda(x)e_2).$$

It then follows from the module structures of $P^0(I)$ and $N\otimes_B Q^0$ that

$$(\lambda,i)((x)\lambda_X d_U,(x)e_1)=(\lambda(x)\lambda_X d_U,\lambda(x)e_1+i\otimes(x)\lambda_X d_U)$$

and $(\lambda,i)(x)e_2=\lambda(x)e_2$. This shows

$$\begin{aligned}a(x)d_X&=(\lambda,i)(x)d_X=(\lambda,i)(((x)\lambda_X d_U,(x)e_1),(x)e_2)\\&=((\lambda(x)\lambda_X d_U,\lambda(x)e_1+i\otimes(x)\lambda_X d_U),\lambda(x)e_2)=(ax)d_X,\end{aligned}$$

i.e., d_X is an A-homomorphism. Thus, ${}_AX$ has a complete projective resolution $F^\bullet$.

At this stage, we define the complex $T^\bullet=(T^i,d_T^i)$ of Λ_ψ-modules as follows. Let

$$T^i:=\mathsf{T}_\Lambda(P^i)\oplus\mathsf{T}_B(Q^i)=\left(F^i,Y^i,\begin{pmatrix}\pi_{P^i}&0\\0&0\end{pmatrix},\begin{pmatrix}\psi_{P^i}&0\\0&1_{N\otimes_B Q^i}\end{pmatrix}\right)$$

and

$$d_T^i:=(d_F^i,d_Y^i)=\left(\begin{pmatrix}d_P^i&\alpha^i&\beta^i\\0&1_I\otimes d_P^i&0\\0&\tau^i&1_N\otimes d_Q^i\end{pmatrix},\begin{pmatrix}1_M\otimes d_P^i&0\\\rho^i&d_Q^i\end{pmatrix}\right).$$

Note that $\Phi_{Q^i}=0$ since we assume $\phi=0$ in the Morita context ring Λ_ψ.

We have to check that (d_F^i,d_Y^i) is a homomorphism of Λ_ψ-modules. In fact, (d_F^i,d_Y^i) can be written as $\left(\begin{smallmatrix}t_{11}^i&t_{12}^i\\t_{21}^i&t_{22}^i\end{smallmatrix}\right)$ with

$$t_{11}^i=\left(\begin{pmatrix}d_P^i&\alpha^i\\0&1_I\otimes d_P^i\end{pmatrix},1_M\otimes d_P^i\right):\mathsf{T}_\Lambda(P^i)\longrightarrow\mathsf{T}_\Lambda(P^{i+1}),\quad t_{12}^i=\left(\begin{pmatrix}\beta^i\\0\end{pmatrix},0\right):\mathsf{T}_\Lambda(P^i)\longrightarrow\mathsf{T}_B(Q^{i+1}),$$

$$t_{21}^i=((0,\tau^i),\rho^i):\mathsf{T}_B(Q^i)\longrightarrow\mathsf{T}_\Lambda(P^{i+1}),\quad t_{22}^i=(1_N\otimes d_Q^i,d_Q^i):\mathsf{T}_B(Q^i)\longrightarrow\mathsf{T}_B(Q^{i+1}).$$

To see that these t_{pq}^i's are homomorphisms of Λ_ψ-modules, we just note that t_{11}^i is the image of (d_P^i,α^i) under the isomorphism in Lemma 3.4(3). Similarly, it follows from (1), (4) and (5) in Lemma 3.4 that t_{12}^i,t_{21}^i and t_{22}^i are homomorphisms of Λ_ψ-modules.

We show that $T^\bullet$ is a total projective resolution of (X,Y,f,g). Actually, the complex $T^\bullet$ is exact because $F^\bullet$ and $Y^\bullet$ are exact. By Lemma 2.5, each term of $T^\bullet$ is a projective Λ_ψ-module. Thus, $T^\bullet$ is an exact complex in $\mathscr{C}(\Lambda_\psi\text{-proj})$.

Next, we show that the complex $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet,\Lambda_\psi)$ is exact. This is equivalent to saying that the complexes $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet,\mathsf{T}_\Lambda(\Lambda))$ and $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet,\mathsf{T}_B(B))$ are exact, due to the isomorphism ${}_{\Lambda_\psi}\Lambda_\psi\simeq\mathsf{T}_\Lambda(\Lambda)\oplus\mathsf{T}_B(B)$.

To show that $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet,\mathsf{T}_\Lambda(\Lambda))$ is exact, we consider the exact sequence

$$0\longrightarrow\mathsf{Z}_\Lambda(I)\overset{((0,1),0)}{\longrightarrow}\mathsf{T}_\Lambda(\Lambda)\longrightarrow\mathsf{T}'_\Lambda(\Lambda)\longrightarrow0$$

of Λ_ψ-modules, where $\mathsf{T}'_\Lambda(\Lambda):=(\Lambda,M,\mu,0)$ and $\mu:{}_BM\otimes_A\Lambda\to{}_BM$ is the multiplication map.

Since T^i is a projective Λ_ψ-module, the sequence of complexes

$$0\longrightarrow\mathrm{Hom}_{\Lambda_\psi}(T^\bullet,\mathsf{Z}_\Lambda(I))\longrightarrow\mathrm{Hom}_{\Lambda_\psi}(T^\bullet,\mathsf{T}_\Lambda(\Lambda))\longrightarrow\mathrm{Hom}_{\Lambda_\psi}(T^\bullet,\mathsf{T}'_\Lambda(\Lambda))\longrightarrow0$$

is exact. Thus, to show the exactness of $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet,\mathsf{T}_\Lambda(\Lambda))$, it is sufficient to prove the one of the complexes $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet,\mathsf{Z}_\Lambda(I))$ and $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet,\mathsf{T}'_\Lambda(\Lambda))$. However, it follows from (2) and (7) in Lemma 3.4 that $\mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_\Lambda(P^i)\oplus\mathsf{T}_B(Q^i),\mathsf{Z}_\Lambda(I))\simeq\mathrm{Hom}_\Lambda(P^i,I)$. Then we have the following commutative diagram for all i:

$$\begin{array}{ccc}\mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_\Lambda(P^i)\oplus\mathsf{T}_B(Q^i),\mathsf{Z}_\Lambda(I))&\xrightarrow{\mathrm{Hom}_{\Lambda_\psi}(d_T^i,\mathsf{Z}_\Lambda(I))}&\mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_\Lambda(P^{i+1})\oplus\mathsf{T}_B(Q^{i+1}),\mathsf{Z}_\Lambda(I))\\ \downarrow\simeq&&\downarrow\simeq\\ \mathrm{Hom}_\Lambda(P^i,I)&\xrightarrow{\mathrm{Hom}_\Lambda(d_P^i,I)}&\mathrm{Hom}_\Lambda(P^{i+1},I).\end{array}$$

To check the commutativity of this diagram, one only needs to notice the definition of $\mathrm{Hom}_{\Lambda_\psi}(d_T^i, \mathsf{Z}_\Lambda(I)) = \mathrm{Hom}_{\Lambda_\psi}(t_{11}^i, \mathsf{Z}_\Lambda(I))$ with

$$t_{11}^i = \left(\begin{pmatrix} d_P^i & \alpha^i \\ 0 & 1_I \otimes d_P^i \end{pmatrix}, 1_M \otimes d_P^i \right).$$

Thus, $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_\Lambda(I)) \simeq \mathrm{Hom}_\Lambda(P^\bullet, I)$, while the latter complex $\mathrm{Hom}_\Lambda(P^\bullet, I)$ is indeed exact, according to the weak compatibility of ${}_\Lambda I_\Lambda$. Thus, $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_\Lambda(I))$ is exact. Furthermore, it follows from the exact sequence

$$0 \longrightarrow \mathsf{Z}_B(M) \longrightarrow \mathsf{T}'_\Lambda(\Lambda) \longrightarrow \mathsf{Z}_\Lambda(\Lambda) \longrightarrow 0$$

of Λ_ψ-modules and the projectivity of T^i that the following sequence of complexes is exact:

$$0 \longrightarrow \mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_B(M)) \longrightarrow \mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{T}'_\Lambda(\Lambda)) \longrightarrow \mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_\Lambda(\Lambda)) \longrightarrow 0.$$

By (6) and (7) in Lemma 3.4, $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_B(M)) \simeq \mathrm{Hom}_B(Q^\bullet, M)$. Since ${}_B M_A$ is weakly compatible, the complex $\mathrm{Hom}_B(Q^\bullet, M)$ is exact, and therefore $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_B(M))$ is exact. Similarly, by (2) and (7) in Lemma 3.4, $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_\Lambda(\Lambda)) \simeq \mathrm{Hom}_\Lambda(P^\bullet, \Lambda)$ is exact. Thus, $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{T}'_\Lambda(\Lambda))$ is exact, so $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{T}_\Lambda(\Lambda))$ is exact.

Now, we show that $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{T}_B(B))$ is exact. Similarly, from the exact sequence $0 \to \mathsf{Z}_\Lambda(N) \to \mathsf{T}_B(B) \to \mathsf{Z}_B(B) \to 0$ of Λ_ψ-modules, we get the exact sequence of complexes, i.e.,

$$0 \longrightarrow \mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_\Lambda(N)) \longrightarrow \mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{T}_B(B)) \longrightarrow \mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_B(B)) \longrightarrow 0.$$

By (2) and (7) in Lemma 3.4, together with the weak compatibility of ${}_\Lambda N_B$, we can show that the complex $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_\Lambda(N))$ is exact. By (6) and (7) in Lemma 3.4, $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_B(B)) \simeq \mathrm{Hom}_B(Q^\bullet, B)$. Since $Q^\bullet$ is a totally exact complex, $\mathrm{Hom}_B(Q^\bullet, B)$ is exact. Hence, $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_B(B))$ is exact, and so is the complex $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{T}_B(B))$. Thus, the complex $T^\bullet \in \mathscr{C}(\Lambda_\psi\text{-proj})$ is totally exact.

Finally, we show $\mathrm{Ker}(d_T^0) \simeq (X, Y, f, g)$. Clearly, $\mathrm{Ker}(d_F^0) = X$ and $\mathrm{Ker}(d_Y^0) = Y$. Moreover, we can verify the following two exact commutative diagrams:

$$\begin{array}{ccccccc}
 & & M \otimes_A X & \xrightarrow{1_M \otimes d_X} & M \otimes_A F^0 & \xrightarrow{1_M \otimes d_F^0} & M \otimes_A F^1 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \begin{pmatrix} \pi_{P^0} & 0 \\ 0 & 0 \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} \pi_{P^1} & 0 \\ 0 & 0 \end{pmatrix}} \\
0 & \longrightarrow & Y & \longrightarrow & Y^0 & \xrightarrow{d_Y^0} & Y^1,
\end{array}
\qquad
\begin{array}{ccccccc}
 & & N \otimes_B Y & \xrightarrow{1_N \otimes d_Y} & N \otimes_B Y^0 & \xrightarrow{1_N \otimes d_Y^0} & N \otimes_B Y^1 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \begin{pmatrix} \psi_{P^0} & 0 \\ 0 & 1_{N \otimes_B Q^0} \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} \psi_{P^1} & 0 \\ 0 & 1_{N \otimes_B Q^1} \end{pmatrix}} \\
0 & \longrightarrow & X & \xrightarrow[d_X]{} & F^0 & \xrightarrow[d_F^0]{} & F^1.
\end{array}$$

This shows $\mathrm{Ker}(d_T^0) \simeq (X, Y, f, g)$. Thus, (X, Y, f, g) is a Gorenstein-projective Λ_ψ-module with a total projective resolution $T^\bullet$. □

3.2 Necessary conditions for Gorenstein-projective modules

In this subsection, we discuss the converse of Theorem 3.5. We start with the following lemma.

Lemma 3.6 (See [16, Proposition 6.1]). *Let $U := (C_A, D_B, h, k)$ be a right $\Lambda_{(\phi,\psi)}$-module with $h \in \mathrm{Hom}_{B^{\mathrm{op}}}(C \otimes_A N, D_B)$ and $k \in \mathrm{Hom}_{A^{\mathrm{op}}}(D \otimes_B M_A, C_A)$, and $V := (X, Y, f, g)$ be a left $\Lambda_{(\phi,\psi)}$-module. Then there is an isomorphism of abelian groups*

$$U \otimes_{\Lambda_{(\phi,\psi)}} V = (C \otimes_A X \oplus D \otimes_B Y)/H,$$

where H is a subgroup of $C \otimes_A X \oplus D \otimes_B Y$ generated by

$$\{c \otimes (n \otimes y)g - (c \otimes n)h \otimes y \mid c \in C, n \in N, y \in Y\} \cup \{d \otimes (m \otimes x)f - (d \otimes m)k \otimes x \mid d \in D, x \in X, m \in M\}.$$

Lemma 3.7. *Let $C \in \Lambda^{\mathrm{op}}$-mod, $X \in \Lambda$-mod, $D \in B^{\mathrm{op}}$-mod and $Y \in B$-mod. Then there are the following isomorphisms of abelian groups, which are natural in each variable*:

(1) $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(C) \otimes_{\Lambda_\psi} \mathsf{T}_\Lambda(X) \simeq C \otimes_\Lambda X$, $(c,0) \otimes ((x, i \otimes x'), m \otimes x'') \mapsto c \otimes x$, $x, x', x'' \in X$, $c \in C$, $i \in I$, $m \in M$.

(2) $\mathsf{Z}_{B^{\mathrm{op}}}(D) \otimes_{\Lambda_\psi} \mathsf{T}_B(Y) \simeq D \otimes_B Y$, $(0,d) \otimes (n \otimes y', y) \mapsto d \otimes y$, $y, y' \in Y$, $d \in D$, $n \in N$.

(3) $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(C) \otimes_{\Lambda_\psi} \mathsf{T}_B(Y) = 0$.

(4) $\mathsf{Z}_{B^{\mathrm{op}}}(D) \otimes_{\Lambda_\psi} \mathsf{T}_\Lambda(X) = 0$.

Proof. We only prove (1) and (3), while the rest can be proved similarly and is omitted.

(1) Since ${}_A A \otimes_\Lambda X \simeq X(I)$ as A-modules, we get $C \otimes_A X(I) \simeq C \otimes_A A \otimes_\Lambda X \simeq C \otimes_\Lambda X$. By definition, $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(C) = (C_A, 0, 0, 0)$ and $\mathsf{T}_\Lambda(X) = (X(I), M \otimes_\Lambda X, \pi_X, \psi_X)$. By Lemma 3.6, we have $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(C) \otimes_{\Lambda_\psi} \mathsf{T}_\Lambda(X) \simeq (C \otimes_\Lambda X)/H$, while the subgroup H is generated by

$$\{c \otimes (n \otimes m)\psi\, x \mid c \in C, n \in N, m \in M, x \in X\}.$$

Thanks to $IX = 0$, we get $H = 0$. Thus (1) holds. Precisely, we can define

$$\alpha : C \otimes_\Lambda X \to \mathsf{Z}_{\Lambda^{\mathrm{op}}}(C) \otimes_{\Lambda_\psi} \mathsf{T}_\Lambda(X)$$

by $c \otimes x \mapsto (c,0) \otimes ((x,0),0)$, and

$$\beta : \mathsf{Z}_{\Lambda^{\mathrm{op}}}(C) \otimes_{\Lambda_\psi} \mathsf{T}_\Lambda(X) \to C \otimes_\Lambda X, \quad (c,0) \otimes ((x, i \otimes x'), m \otimes x'') \mapsto c \otimes x, \quad x, x', x'' \in X$$

for $c \in C$, $i \in I$ and $m \in M$. One can check that they are homomorphisms of abelian groups satisfying $\alpha\beta = 1$ and $\beta\alpha = 1$. Clearly, the isomorphisms of α and β are natural in C and X.

(3) In this case, $H = C \otimes_A N \otimes_B Y$ in Lemma 3.6. Thus, $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(C) \otimes_{\Lambda_\psi} \mathsf{T}_B(Y) \simeq (C \otimes_A N \otimes_B Y)/H = 0$. □

In the rest of this section, we assume that $T^\bullet = (T^i, d_T^i)_{i \in \mathbb{Z}} \in \mathscr{C}(\Lambda_\psi\text{-proj})$ is a totally exact complex such that $\mathrm{Ker}(d_T^0) = (X, Y, f, g) \in \Lambda_\psi$-mod.

By Lemma 2.5, $T^i = \mathsf{T}_\Lambda(P^i) \oplus \mathsf{T}_B(Q^i)$ for some $P^i \in \Lambda$-proj and $Q^i \in B$-proj. Thus we may write precisely

$$d_T^i = \begin{pmatrix} t_{11}^i & t_{12}^i \\ t_{21}^i & t_{22}^i \end{pmatrix}$$

with

$$\begin{aligned} t_{11}^i &\in \mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_\Lambda(P^i), \mathsf{T}_\Lambda(P^{i+1})), \\ t_{12}^i &\in \mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_\Lambda(P^i), \mathsf{T}_B(Q^{i+1})), \\ t_{21}^i &\in \mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_B(Q^i), \mathsf{T}_\Lambda(P^{i+1})), \\ t_{22}^i &\in \mathrm{Hom}_{\Lambda_\psi}(\mathsf{T}_B(Q^i), \mathsf{T}_B(Q^{i+1})). \end{aligned}$$

By Lemma 3.4(3), there is a Λ-module homomorphism (d_P^i, α^i), where $d_P^i : P^i \to P^{i+1}$ and $\alpha^i : P^i \to I \otimes_\Lambda P^{i+1}$ are homomorphisms of Λ-modules such that

$$t_{11}^i = \left(\begin{pmatrix} d_P^i & \alpha^i \\ 0 & 1_I \otimes d_P^i \end{pmatrix}, 1_M \otimes d_P^i \right).$$

Similarly, by (1), (4) and (5) in Lemma 3.4, we have a Λ-module homomorphism $\beta^i : P^i \to N \otimes_B Q^{i+1}$ and two B-module homomorphisms $\rho^i : Q^i \to M \otimes_\Lambda P^{i+1}$ and $d_Q^i : Q^i \to Q^{i+1}$ such that

$$t_{12}^i = \left(\begin{pmatrix} \beta^i \\ 0 \end{pmatrix}, 0 \right), \quad t_{21}^i = ((0, \tau^i), \rho^i) \quad \text{with } \tau^i = (1_N \otimes \rho^i)\psi_{P^{i+1}},$$

and $t_{22}^i = (1_N \otimes d_Q^i, d_Q^i)$. Furthermore, the exact complex $T^\bullet$ provides two exact complexes $F^\bullet := (F^i, d_F^i) \in \mathscr{C}(A\text{-mod})$ and $Y^\bullet := (Y^i, d_Y^i) \in \mathscr{C}(B\text{-mod})$ by defining

$$(\ddagger) \quad F^i := P^i(I) \oplus (N \otimes_B Q^i), \quad d_F^i := \begin{pmatrix} \begin{pmatrix} d_P^i & \alpha^i \\ 0 & 1_I \otimes d_P^i \end{pmatrix} & \begin{pmatrix} \beta^i \\ 0 \end{pmatrix} \\ (0, \tau^i) & 1_N \otimes d_Q^i \end{pmatrix}$$

and

$$Y^i := M \otimes_\Lambda P^i \oplus Q^i, \quad d_Y^i := \begin{pmatrix} 1_M \otimes d_P^i & 0 \\ \rho^i & d_Q^i \end{pmatrix}.$$

Then $\mathrm{Ker}(d_F^0) = X$ and $\mathrm{Ker}(d_Y^0) = Y$.

Let $Q^\bullet := (Q^i, d_Q^i)$ and $P^\bullet := (P^i, d_P^i)$. Then it follows from $d_Y^i d_Y^{i+1} = 0$ and $d_F^i d_F^{i+1} = 0$ that $d_Q^i d_Q^{i+1} = 0$ and $d_P^i d_P^{i+1} = 0$, respectively. Thus $Q^\bullet \in \mathscr{C}(B\text{-proj})$ and $P^\bullet \in \mathscr{C}(\Lambda\text{-proj})$. Moreover, we define $Z^\bullet = (Z^i, d_Z^i)$ with

$$Z^i = (I \otimes_\Lambda P^i) \oplus (N \otimes_B Q^i) \in \Lambda\text{-mod}, \quad d_Z^i = \begin{pmatrix} 1_I \otimes d_P^i & 0 \\ \tau^i & 1_N \otimes d_Q^i \end{pmatrix}.$$

Then it follows again from $d_F^i d_F^{i+1} = 0$ that $d_Z^i d_Z^{i+1} = 0$, and therefore $Z^\bullet \in \mathscr{C}(\Lambda\text{-mod})$.

The complex $Y^\bullet$ gives rise to an exact sequence in $\mathscr{C}(B\text{-mod})$:

$$0 \longrightarrow M \otimes_\Lambda P^\bullet \xrightarrow{a^\bullet} Y^\bullet \xrightarrow{b^\bullet} Q^\bullet \longrightarrow 0, \tag{3.4}$$

where $a^\bullet$ and $b^\bullet$ are canonical inclusion and projection, respectively. Also, we have two exact sequences of complexes in $\mathscr{C}(\Lambda\text{-mod})$:

$$0 \longrightarrow I \otimes_\Lambda P^\bullet \xrightarrow{c^\bullet} Z^\bullet \xrightarrow{d^\bullet} N \otimes_B Q^\bullet \longrightarrow 0, \tag{3.5}$$

$$0 \longrightarrow Z^\bullet \xrightarrow{p^\bullet} F^\bullet \xrightarrow{q^\bullet} P^\bullet \longrightarrow 0, \tag{3.6}$$

where $c^\bullet$ and $p^\bullet$ are canonical inclusions, and $d^\bullet$ and $q^\bullet$ are canonical projections. Furthermore, there is a chain map $\sigma^\bullet$ in $\mathscr{C}(\Lambda\text{-mod})$:

$$\sigma^\bullet = (\sigma^i)_{i\in\mathbb{Z}} : N \otimes_B Y^\bullet \to Z^\bullet, \quad \sigma^i := \begin{pmatrix} \psi \otimes 1_{P^i} & 0 \\ 0 & 1_N \otimes 1_{Q^i} \end{pmatrix}, \quad i \in \mathbb{Z}$$

such that the following diagram of complexes of Λ-modules is commutative and exact:

$$\begin{array}{ccccccccc} & & N \otimes_B M \otimes_\Lambda P^\bullet & \xrightarrow{1_N \otimes a^\bullet} & N \otimes_B Y^\bullet & \xrightarrow{1_N \otimes b^\bullet} & N \otimes_B Q^\bullet & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \psi \otimes 1_{P^\bullet}} & & \big\downarrow{\scriptstyle \sigma^\bullet} & & \big\| & & \\ 0 & \longrightarrow & I \otimes_\Lambda P^\bullet & \xrightarrow{c^\bullet} & Z^\bullet & \xrightarrow{d^\bullet} & N \otimes_B Q^\bullet & \longrightarrow & 0. \end{array} \tag{3.7}$$

Lemma 3.8. (1) *If* $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(M)$ *and* $\mathsf{Z}_\Lambda(N)$ *are semi-weakly compatible* Λ_ψ*-modules, then* $Q^\bullet \in \mathscr{C}(B\text{-proj})$ *is totally exact.*

(2) *If* $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(I)$*,* $\mathsf{Z}_\Lambda(I)$*,* $\mathsf{Z}_{B^{\mathrm{op}}}(N)$ *and* $\mathsf{Z}_B(M)$ *are semi-weakly compatible* Λ_ψ*-modules, then* $P^\bullet \in \mathscr{C}(\Lambda\text{-proj})$ *is totally exact.*

Proof. (1) Since $T^\bullet \in \mathscr{C}(\Lambda_\psi\text{-proj})$ is a totally exact complex and $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(M)$ is semi-weakly compatible by assumption, the complex $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(M) \otimes_{\Lambda_\psi} T^\bullet$ is exact. By (1) and (3) in Lemma 3.7,

$$\mathsf{Z}_{\Lambda^{\mathrm{op}}}(M) \otimes_{\Lambda_\psi} (\mathsf{T}_\Lambda(P^i) \oplus \mathsf{T}_B(Q^i)) \simeq M \otimes_\Lambda P^i.$$

Then we have the commutative diagram for all i,

$$\begin{array}{ccc} \mathsf{Z}_{\Lambda^{\mathrm{op}}}(M) \otimes_{\Lambda_\psi} (\mathsf{T}_\Lambda(P^i) \oplus \mathsf{T}_B(Q^i)) & \xrightarrow{1 \otimes d_T^i} & \mathsf{Z}_{\Lambda^{\mathrm{op}}}(M) \otimes_{\Lambda_\psi} (\mathsf{T}_\Lambda(P^{i+1}) \oplus \mathsf{T}_B(Q^{i+1})) \\ \big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle \simeq} \\ M \otimes P^i & \xrightarrow{1_M \otimes d_P^i} & M \otimes P^{i+1}. \end{array}$$

To check the commutativity of this diagram, one only needs to note the definition of $1\otimes d_T^i = 1_{\mathsf{Z}_{\Lambda^{\mathrm{op}}}(M)}\otimes t_{11}^i$ with

$$t_{11}^i = \left(\begin{pmatrix} d_P^i & \alpha^i \\ 0 & 1_I \otimes d_P^i \end{pmatrix}, 1_M \otimes d_P^i\right).$$

Hence, $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(M) \otimes_{\Lambda_\psi} T^\bullet \simeq M \otimes_\Lambda P^\bullet$ as complexes, and this yields that $M \otimes_\Lambda P^\bullet$ is exact. It follows from the exact sequence (3.4) that $Q^\bullet$ is an exact complex. Since $\mathsf{Z}_\Lambda(N)$ is semi-weakly compatible, $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_\Lambda(N))$ is exact. As $T^\bullet$ is totally exact, the complex $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{T}_B(B))$ is exact. Applying $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, -)$ to the exact sequence $0 \to \mathsf{Z}_\Lambda(N) \to \mathsf{T}_B(B) \to \mathsf{Z}_B(B) \to 0$, we get the exact sequence of complexes of $\mathbb{Z}$-modules, i.e.,

$$0 \longrightarrow \mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_\Lambda(N)) \longrightarrow \mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{T}_B(B)) \longrightarrow \mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_B(B)) \longrightarrow 0.$$

It follows that $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_B(B))$ is exact. Similarly, by (6) and (7) in Lemma 3.4, the complex $\mathrm{Hom}_B(Q^\bullet, B) \simeq \mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_B(B))$ is exact. Hence, $Q^\bullet$ is a totally exact complex.

(2) Since $T^\bullet \in \mathscr{C}(\Lambda_\psi\text{-proj})$ is totally exact, it follows from (2) and (4) in Lemma 3.7, together with the semi-weak compatibility condition on $\mathsf{Z}_{B^{\mathrm{op}}}(N)$, that $\mathsf{Z}_{B^{\mathrm{op}}}(N) \otimes_{\Lambda_\psi} T^\bullet \simeq N \otimes_B Q^\bullet$ is exact. Similarly, by Lemma 3.7(1) and the assumption on $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(I)$, we know that $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(I) \otimes_{\Lambda_\psi} T^\bullet \simeq I \otimes_\Lambda P^\bullet$ is exact. It then follows from the exact sequence (3.5) that $Z^\bullet$ is exact. This implies, together with the exact sequence (3.6), that $P^\bullet$ is an exact complex. By the semi-weak compatibility of $\mathsf{Z}_\Lambda(I)$, we deduce that $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_\Lambda(I))$ is exact. From the exact sequence

$$0 \to \mathsf{Z}_\Lambda(I) \overset{((0,1),0)}{\longrightarrow} \mathsf{T}_\Lambda(\Lambda) \to \mathsf{T}'_\Lambda(\Lambda) \to 0$$

of Λ_ψ-modules, we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_\Lambda(I)) \longrightarrow \mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{T}_\Lambda(\Lambda)) \longrightarrow \mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{T}'_\Lambda(\Lambda)) \longrightarrow 0.$$

This shows that $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{T}'_\Lambda(\Lambda))$ is exact. Now, applying $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, -)$ to the exact sequence $0 \to \mathsf{Z}_B(M) \to \mathsf{T}'_\Lambda(\Lambda) \to \mathsf{Z}_\Lambda(\Lambda) \to 0$, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_B(M)) \longrightarrow \mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{T}'_\Lambda(\Lambda)) \longrightarrow \mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_\Lambda(\Lambda)) \longrightarrow 0.$$

As $\mathsf{Z}_B(M)$ is semi-weakly compatible, $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_B(M))$ is exact, and therefore so is $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_\Lambda(\Lambda))$. By (2) and (7) in Lemma 3.4, $\mathrm{Hom}_\Lambda(P^\bullet, \Lambda) \simeq \mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_\Lambda(\Lambda))$ is exact. Hence $P^\bullet$ is totally exact. □

Lemma 3.9. *Assume that ${}_\Lambda N_B$ and ${}_B M_\Lambda$ are weakly compatible. If $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(M)$ and $\mathsf{Z}_\Lambda(N)$ are semi-weakly compatible Λ_ψ-modules, then $\mathsf{Z}_{B^{\mathrm{op}}}(N)$ and $\mathsf{Z}_B(M)$ are semi-weakly compatible Λ_ψ-modules.*

Proof. Since $T^\bullet$ is a totally exact complex of projective Λ_ψ-modules, we deduce from Lemma 3.8(1) that $Q^\bullet$ is a totally exact complex. By the assumption, N_B is semi-weakly compatible. It follows from $\mathsf{Z}_{B^{\mathrm{op}}}(N) \otimes_{\Lambda_\psi} T^\bullet \simeq N \otimes_B Q^\bullet$ that $\mathsf{Z}_{B^{\mathrm{op}}}(N) \otimes_{\Lambda_\psi} T^\bullet$ is exact. Since ${}_B M$ is semi-weakly compatible and $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_B(M)) \simeq \mathrm{Hom}_B(Q^\bullet, M)$, we see that $\mathrm{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_B(M))$ is exact. □

Theorem 3.10. *Assume that ${}_\Lambda N_B$, ${}_B M_\Lambda$ and ${}_\Lambda I_\Lambda$ are weakly compatible bimodules and $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(M)$ and $\mathsf{Z}_\Lambda(N)$ are semi-weakly Λ_ψ-modules. Furthermore, assume that $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(I)_{\Lambda_\psi}$ and ${}_{\Lambda_\psi}\mathsf{Z}_\Lambda(I)$ are semi-weakly compatible. If a Λ_ψ-module(X, Y, f, g) is Gorenstein-projective, then*

(a) $\mathrm{Coker}(f) \in B$-Gproj *and* $\mathrm{Coker}(g) \in \Lambda$-Gproj*, and*

(b) $\mathrm{Im}(f) \simeq M \otimes_A \mathrm{Coker}(g)$*,* $\mathrm{Im}(g)/IX \simeq N \otimes_B \mathrm{Coker}(f)$ *and* $IX \simeq I \otimes_A \mathrm{Coker}(g)$*.*

Proof. Suppose that (X, Y, f, g) lies in Λ_ψ-Gproj. Then there is a totally exact complex $T^\bullet \in \mathscr{C}(\Lambda_\psi\text{-proj})$ such that $\mathrm{Ker}(d_T^0) = (X, Y, f, g)$. By Lemmas 3.9 and 3.8, the foregoing complexes $P^\bullet \in \mathscr{C}(\Lambda\text{-proj})$ and $Q^\bullet \in \mathscr{C}(B\text{-proj})$ are totally exact. Thus $U := \mathrm{Ker}(d_P^0) \in \Lambda$-Gproj and $V := \mathrm{Ker}(d_Q^0) \in B$-Gproj.

First, we show that $\mathrm{Coker}(g)$ is a Gorenstein-projective Λ-module.

Since $\mathrm{Ker}(d_F^0) = X$ (see notation in (‡)), we write $d_X : X \to P^0(I) \oplus (N \otimes_B Q^0)$ for the inclusion of A-modules. The restriction of d_X to Λ-modules will be denoted by $d_{{}_\Lambda X}$. Then

$$d_{{}_\Lambda X} = (e_0, e_1, e_2) : {}_\Lambda X \to P^0 \oplus ({}_\Lambda I \otimes_\Lambda P^0) \oplus ({}_\Lambda N \otimes_B Q^0).$$

We show that e_0, e_1 and e_2 have the properties $(ix)e_0 = 0$, $(ix)e_1 = i \otimes (x)e_0$ and $(ix)e_2 = 0$ for $i \in I$ and $x \in X$.

Indeed, the homomorphism d_X of A-modules shows that

$$(((ax)e_0, (ax)e_1), (ax)e_2) = a[(x)((e_0, e_1), e_2)] = a[((x)e_0, (x)e_1), (x)e_2]$$

for $x \in X$ and $a = (\lambda, i) \in A$. Furthermore, the Λ-homomorphisms e_i $(i = 0, 1, 2)$ show the equality

$$(((ax)e_0, (ax)e_1), (ax)e_2) = ((\lambda(x)e_0 + (ix)e_0, \lambda(x)e_1 + (ix)e_1), \lambda(x)e_2 + (ix)e_2).$$

By the A-module structure of $P^0(I)$ and $N \otimes_B Q^0$, one obtains immediately

$$a[((x)e_0, (x)e_1), (x)e_2] = (\lambda, i)(((x)e_0, (x)e_1), (x)e_2) = ((\lambda(x)e_0, \lambda(x)e_1 + i \otimes (x)e_0), \lambda(x)e_2),$$

i.e., $(ix)e_0 = 0$, $(ix)e_1 = i \otimes (x)e_0$ and $(ix)e_2 = 0$.

Let d_U be the inclusion of ${}_\Lambda U$ into ${}_\Lambda P^0$. It follows from the sequence (3.6) that there is an exact commutative diagram of Λ-modules, i.e.,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(d_Z^0) & \xrightarrow{\epsilon_X} & {}_\Lambda X & \xrightarrow{\lambda_X} & U & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_Z} & & \downarrow{\scriptstyle (e_0, e_1, e_2)} & & \downarrow{\scriptstyle d_U} & & \\
0 & \longrightarrow & I \otimes_\Lambda P^0 \oplus N \otimes_B Q^0 & \xrightarrow{\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)} & P^0 \oplus I \otimes_\Lambda P^0 \oplus N \otimes_B Q^0 & \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)} & P^0 & \longrightarrow & 0,
\end{array}$$

where all the vertical maps are injective and $e_0 = \lambda_X d_U$. Note that $(ix)\lambda_X d_U = (ix)e_0 = 0$ for $i \in I$ and $x \in X$. Since d_U is injective, one must have $(ix)\lambda_X = 0$. Now, if we consider ${}_\Lambda U$ as an A-module, i.e., $IU = 0$, then λ_X is a homomorphism of A-modules.

Furthermore, ϵ_X is a homomorphism of A-modules if ${}_\Lambda\mathrm{Ker}(d_Z^0)$ is regarded as an A-module. In fact, for $z \in \mathrm{Ker}(d_Z^0)$, let $x_z := (z)\epsilon_X$. Then it follows from $i \otimes (x_z)e_0 = i \otimes (x_z)\lambda_X d_U = 0$ that

$$(ix_z)(e_0, e_1, e_2) = (0, i \otimes (x_z)e_0, 0) = 0.$$

Since the map (e_0, e_1, e_2) is injective, we obtain $ix_z = 0$, i.e., $I\mathrm{Im}(\epsilon_X) = 0$. This implies that ϵ_X is a homomorphism of A-modules. Thus there is an exact sequence of A-modules, i.e.,

$$0 \longrightarrow \mathrm{Ker}(d_Z^0) \xrightarrow{\epsilon_X} X \xrightarrow{\lambda_X} U \longrightarrow 0, \tag{3.8}$$

which fits into the following exact commutative diagram of A-modules:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Ker}(d_Z^0) & \xrightarrow{d_Z} & (I \otimes_\Lambda P^0) \oplus (N \otimes_B Q^0) & \xrightarrow{d_Z^0} & (I \otimes_\Lambda P^1) \oplus (N \otimes_B Q^1) \\
 & & \downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle j^0} & & \downarrow{\scriptstyle j^1} \\
0 & \longrightarrow & X & \xrightarrow{d_X} & P^0(I) \oplus (N \otimes_B Q^0) & \xrightarrow{d_E^0} & P^1(I) \oplus (N \otimes_B Q^1) \\
 & & \downarrow{\scriptstyle \lambda_X} & & \downarrow{\scriptstyle k^0} & & \downarrow{\scriptstyle k^1} \\
0 & \longrightarrow & U & \xrightarrow{d_U} & P^0 & \xrightarrow{d_P^0} & P^1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

with

$$j^i = \begin{pmatrix} (0, 1) & 0 \\ 0 & 1_{N \otimes_B Q^i} \end{pmatrix}$$

in which $(0,1): I\otimes_\Lambda P^i \to P^i \oplus I\otimes_\Lambda P^i$ is the canonical inclusion, and with

$$k^i = \begin{pmatrix} \binom{1}{0} \\ 0 \end{pmatrix}$$

in which $\binom{1}{0}$ means the projection map $P^i \oplus I\otimes_\Lambda P^i \to P^i$. Note that all the homomorphisms of Λ-modules in the top and the bottom rows are regarded as homomorphisms of A-modules via the canonical map $A\to\Lambda$. According to (1) and (2) in Lemma 3.3, the vertical maps j^i and k^i are also homomorphisms of A-modules.

Since $\mathrm{Ker}(d_Y^0) = Y$, we have an inclusion $d_Y : Y \hookrightarrow M\otimes_\Lambda P^0 \oplus Q^0$. Let $\delta : N\otimes_B Y \to \mathrm{Ker}(d_Z^0)$ be the homomorphism of A-modules induced from the chain map $\sigma^\bullet$ in the diagram (3.7). Then δ is surjective. Actually, since I_Λ is semi-weakly compatible and we have shown that $P^\bullet$ is a totally exact complex of projective modules, the complex $I\otimes_\Lambda P^\bullet$ is exact. Similarly, $N\otimes_B Q^\bullet$ is exact. So the exact sequence (3.6) implies that the complex $Z^\bullet$ is exact. By the diagram (3.7), the following diagram of A-modules is exact and commutative:

$$\begin{array}{ccccccccc}
 & & N\otimes_B M\otimes_\Lambda U & \xrightarrow{1_N\otimes\eta_Y} & N\otimes_B Y & \xrightarrow{1_N\otimes\mu_Y} & N\otimes_B V & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \psi\otimes 1_U} & & \big\downarrow{\scriptstyle \delta} & & \big\| & & \\
0 & \longrightarrow & I\otimes_\Lambda U & \longrightarrow & \mathrm{Ker}(d_Z^0) & \longrightarrow & N\otimes_B V & \longrightarrow & 0.
\end{array}$$

Therefore, the Snake lemma shows that δ is surjective.

Moreover, the diagram (3.7) gives rise to the following one of A-modules:

$$\begin{array}{ccccccc}
 & & N\otimes_B Y & \xrightarrow{1_N\otimes d_Y} & (N\otimes M\otimes P^0)\oplus(N\otimes_B Q^0) & \xrightarrow{1_N\otimes d_Y^0} & (N\otimes M\otimes P^1)\oplus(N\otimes_B Q^1) \\
 & \swarrow{\scriptstyle\delta} & \big|{\scriptstyle g} & \swarrow{\scriptstyle\sigma^0} & \big|{\scriptstyle \begin{pmatrix}\psi_{P^0} & 0\\ 0 & 1_{N\otimes Q^0}\end{pmatrix}} & \swarrow{\scriptstyle\sigma^1} & \big\downarrow{\scriptstyle \begin{pmatrix}\psi_{P^1} & 0\\ 0 & 1_{N\otimes Q^1}\end{pmatrix}} \\
\mathrm{Ker}(d_Z^0) & \longrightarrow & (I\otimes P^0)\oplus(N\otimes_B Q^0) & \xrightarrow{d_Z^0} & (I\otimes P^1)\oplus(N\otimes_B Q^1) & & \\
 & \searrow{\scriptstyle\epsilon_X} & \big\downarrow & \searrow{\scriptstyle j^0} & \big\downarrow & \searrow{\scriptstyle j^1} & \big\downarrow \\
 & & X & \xrightarrow{d_X} & P^0(I)\oplus(N\otimes_B Q^0) & \xrightarrow{d_F^0} & P^1(I)\oplus(N\otimes_B Q^1).
\end{array}$$

By the definitions of g,δ and ϵ_X, the two top and two bottom squares are commutative. We can verify

$$\sigma^i j^i = as \begin{pmatrix}\psi_{P^i} & 0 \\ 0 & 1_{N\otimes Q^i}\end{pmatrix} \quad \text{for all } i.$$

Since d_X is injective, it holds that $\delta\epsilon_X = g$. Thus $\mathrm{Coker}(g) = \mathrm{Coker}(\epsilon_X) \simeq U \in \Lambda\text{-Gproj}$.

Next, we prove $\mathrm{Coker}(f) \in B\text{-Gproj}$. Observe that $M\otimes_A P^\bullet \simeq M\otimes_\Lambda P^\bullet$ as complexes of B-modules. So the exact sequence (3.4) may be rewritten as the following exact sequence of B-modules:

$$0 \longrightarrow M\otimes_A P^\bullet \xrightarrow{a'^\bullet} Y^\bullet \xrightarrow{b^\bullet} Q^\bullet \longrightarrow 0$$

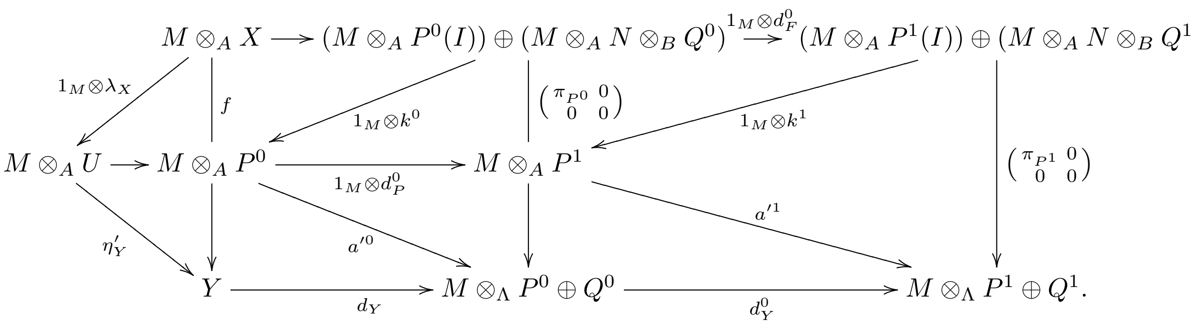
with

$$a'^i : M\otimes_A P^i \xrightarrow{\simeq} M\otimes_\Lambda P^i \xrightarrow{a^i} M\otimes_\Lambda P^i \oplus Q^i.$$

This gives rise to the exact sequence of B-modules, i.e.,

$$0 \longrightarrow M\otimes_A U \xrightarrow{\eta'_Y} Y \xrightarrow{\mu_Y} V \longrightarrow 0.$$

Now consider the following diagram of B-modules:



By the definition of λ_X as an A-module homomorphism, the upper two squares are commutative. By the definition of η'_Y, the lower two squares are commutative. Moreover,

$$(1_M \otimes k^i)a' = as \begin{pmatrix} \pi_{P^i} & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows from the injective map d_Y that $(1_M \otimes \lambda_X)\eta'_Y = f$. Therefore, $\mathrm{Coker}(f) = \mathrm{Coker}(\eta'_Y) = V \in B\text{-Gproj}$ and $\mathrm{Im}(f) \simeq M \otimes_A U = M \otimes_A \mathrm{Coker}(g)$. This completes the proof of (a).

Having proved that $\mathrm{Im}(f) \simeq M \otimes_A \mathrm{Coker}(g)$, we now prove $\mathrm{Im}(g)/IX \simeq N \otimes_B \mathrm{Coker}(f)$.

Recall that $F^\bullet \in \mathscr{C}(A\text{-mod})$ stands for the complex defined in (‡). Let

$$W^i := {}_\Lambda P^i \oplus {}_\Lambda N \otimes_B Q^i, \quad d_W^i := \begin{pmatrix} d_P^i & \beta^i \\ 0 & 1_N \otimes d_Q^i \end{pmatrix}.$$

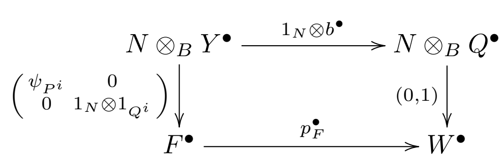
Due to $d_F^i d_F^{i+1} = 0$, we have $d_P^i \beta^{i+1} + \beta^i(1_N \otimes d_Q^i) = 0$ and $d_W^i d_W^{i+1} = 0$. So $W^\bullet \in \mathscr{C}(\Lambda\text{-mod})$. Regarding Λ-modules as A-modules, we have the exact sequence

$$(\#) \quad 0 \longrightarrow N \otimes_B Q^\bullet \xrightarrow{s^\bullet} W^\bullet \xrightarrow{t^\bullet} P^\bullet \longrightarrow 0$$

of complexes in $\mathscr{C}(A\text{-mod})$. Since ${}_\Lambda N_B$ is a weakly compatible bimodule and $Q^\bullet$ is a totaly exact complex in $\mathscr{C}(B\text{-proj})$, $N \otimes_B Q^\bullet$ is an exact complex. It then follows from the exactness of $P^\bullet$ that the complex $W^\bullet$ is exact. Now, since ${}_A\Lambda \otimes_A F^i = {}_A\Lambda \otimes_A (P^i(I) \oplus N \otimes_B Q^i) \simeq P^i \oplus (N \otimes_B Q^i)$ and $1_\Lambda \otimes_A d_F^i = d_W^i$, we have $W^\bullet \simeq \Lambda \otimes_A F^\bullet$ as complexes in $\mathscr{C}(A\text{-mod})$. Hence, ${}_A\Lambda \otimes_A F^\bullet$ is an exact complex of A-modules and ${}_A\mathrm{Ker}(d_W^0) \simeq {}_A\Lambda \otimes_A X \simeq X/IX$. Thus (#) induces the exact sequence of A-modules $0 \to {}_AN \otimes_B V \xrightarrow{s} X/IX \xrightarrow{t} {}_AU \to 0$. It follows from $\Lambda \otimes_A F^\bullet \simeq W^\bullet$ that there is a canonical chain map $p_F^\bullet : F^\bullet \to W^\bullet$ in $\mathscr{C}(A\text{-mod})$ with

$$p_F^i = \begin{pmatrix} \binom{1}{0} & 0 \\ 0 & 1_N \otimes 1_{Q^i} \end{pmatrix}.$$

Now, we take the kernels of $p_F^\bullet$ at degree 0 and get the canonical projection $p_X : X \to X/IX$. Considering the commutative diagram of complexes in $\mathscr{C}(A\text{-mod})$:



and the differentials in degree 0 in the diagram, we get the exact commutative diagram

$$\begin{array}{ccccccccc}
 & N\otimes_B Y & \longrightarrow & & N\otimes_B Y^0 & \xrightarrow{1_N\otimes_B d_Y^0} & & N\otimes_B Y^1 & \\
 {\scriptstyle g}\swarrow & \downarrow{\scriptstyle 1_N\otimes_B\mu_Y} & & \swarrow{\scriptstyle \begin{pmatrix}\psi_{P^0} & 0\\ 0 & 1_N\otimes 1_{Q^0}\end{pmatrix}} & \downarrow{\scriptstyle b^0} & & \swarrow{\scriptstyle \begin{pmatrix}\psi_{P^1} & 0\\ 0 & 1_N\otimes 1_{Q^1}\end{pmatrix}} & \downarrow{\scriptstyle b^1} & \\
X & \longrightarrow & P^0(I)\oplus(N\otimes_B Q^0) & & \xrightarrow{d_F^0} & P^1(I)\oplus(N\otimes_B Q^1) & & & \\
\downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_F^0} & & & \downarrow{\scriptstyle p_F^1} & & & \\
 & N\otimes_B V & \longrightarrow & & N\otimes_B Q^0 & \xrightarrow{1_N\otimes_B d_Q^0} & & N\otimes_B Q^1 & \\
 {\scriptstyle s}\swarrow & & & \swarrow{\scriptstyle (0,1)} & & & \swarrow{\scriptstyle (0,1)} & & \\
X/IX & \xrightarrow{d_W} & P^0\oplus(N\otimes_B Q^0) & & \xrightarrow{d_W^0} & P^1\oplus(N\otimes_B Q^1) & & &
\end{array}$$

in $\mathscr{C}(A\text{-mod})$ by Lemma 2.2(2). Thus $gp_X = (1_N\otimes\mu_Y)s$. Since s is injective and $1_N\otimes\mu_Y$ is surjective, we obtain $\mathrm{Im}(g)/IX \simeq N\otimes_B V = N\otimes_B \mathrm{Coker}(f)$.

Finally, we show $IX \simeq I\otimes_A \mathrm{Coker}(g)$. Actually, it follows from the chain map $c^\bullet : I\otimes_\Lambda P^\bullet \to Z^\bullet$ in (3.7) that the following exact commutative diagram exists:

$$\begin{array}{ccccccc}
 & & I\otimes_\Lambda U & \longrightarrow & I\otimes_\Lambda P^0 & \longrightarrow & I\otimes_\Lambda P^1\\
 & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c^0} & & \downarrow{\scriptstyle c^1}\\
0 & \longrightarrow & \mathrm{Ker}(d_Z^0) & \longrightarrow & Z^0 & \longrightarrow & Z^1.
\end{array}$$

Now, consider the diagram

$$\begin{array}{ccccc}
N\otimes_B M\otimes_A X & & \xrightarrow{\psi\otimes 1_X} & & I\otimes_A X\\
\downarrow{\scriptstyle 1_{N\otimes_B M}\otimes\lambda_X} & & & \swarrow{\scriptstyle 1_I\otimes\lambda_X} & \\
N\otimes_B M\otimes_A U & \xrightarrow{\psi\otimes 1_U} & I\otimes_A U & & \\
\downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle \mathrm{mlt}_X}\\
N\otimes_B M\otimes_\Lambda U & \xrightarrow{\psi\otimes 1_U} & I\otimes_\Lambda U & & \\
\downarrow{\scriptstyle 1_N\otimes\eta_Y} & & \downarrow{\scriptstyle c} & & \\
N\otimes_B Y & \xrightarrow{\delta} & \mathrm{Ker}(d_Z^0) & \xrightarrow{\epsilon_X} & X.
\end{array} \tag{3.9}$$

Note that the out-side square is commutative, due to $(X,Y,f,g)\in\Lambda_\psi$-mod. The down-left square commutes because of the commutative diagram (3.7), while the upper-left square commutes, due to the property of ψ. Thus it follows from the surjective map $\psi\otimes 1_X$ that the right-hand side of the square is commutative. Since c and ε_x are injective maps, $IX\simeq I\otimes_A U\simeq I\otimes_A \mathrm{Coker}(g)$. This completes the proof of (b). □

Theorem 3.11. *The following are equivalent for the Morita context ring* Λ_ψ:

(1) ${}_\Lambda N_B$, ${}_B M_\Lambda$ *and* ${}_\Lambda I_\Lambda$ *are weakly compatible bimodules, the left* Λ_ψ*-modules* $({}_A N,0,0,0)$ *and* $({}_A I,0,0,0)$ *and the right* Λ_ψ*-modules* $(M_A,0,0,0)$ *and* $(I_A,0,0,0)$ *are semi-weakly compatible.*

(2) *A* Λ_ψ*-module* (X,Y,f,g) *is Gorenstein-projective if and only if*

(a) ${}_B\mathrm{Coker}(f)$ *and* ${}_\Lambda\mathrm{Coker}(g)$ *are Gorenstein-projective, and*

(b) ${}_B\mathrm{Im}(f)\simeq {}_BM\otimes_A\mathrm{Coker}(g)$, ${}_A\mathrm{Im}(g)/IX\simeq {}_AN\otimes_B\mathrm{Coker}(f)$ *and* ${}_AIX\simeq {}_AI\otimes_A\mathrm{Coker}(g)$.

Proof. (1) ⇒ (2). This follows from Theorems 3.5 and 3.10.

(2) ⇒ (1). This will be done in the rest of this section. So in the following, we always assume (2).

Lemma 3.12. *If* $G\in\Lambda$-Gproj*, then* $\mathsf{T}_\Lambda(G)\in\Lambda_\psi$-Gproj*. Similarly, if* $Q\in B$-Gproj*, then* $\mathsf{T}_B(Q)\in\Lambda_\psi$-Gproj.

Proof. Since $\mathsf{T}_\Lambda(G) = (A\otimes_\Lambda G, M\otimes_\Lambda G, \pi_G, \psi_G)$ where $\pi_G : {}_BM\otimes_A A\otimes_\Lambda G\simeq {}_BM\otimes_\Lambda G$ and $\psi_G : {}_AN\otimes_B(M\otimes_\Lambda G)\to {}_AA\otimes_\Lambda G$ is given by $n\otimes(m\otimes x)\mapsto(n\otimes m)\psi\otimes x$ for $n\in N$, $m\in M$ and $x\in G$, it follows from $\mathrm{Coker}(\pi_G) = 0$ and ${}_A\mathrm{Im}(\psi_G) = I \otimes_\Lambda G$ that ${}_\Lambda\mathrm{Coker}(\psi_G) \simeq {}_\Lambda G$ and the condition (2)(a) is satisfied. On the other hand, $\mathrm{Im}(\pi_G) = M \otimes_\Lambda G \simeq M \otimes_\Lambda \mathrm{Coker}(\psi_G) \simeq M \otimes_A \mathrm{Coker}(\psi_G)$. We can show $I \otimes_A G = I(A \otimes_\Lambda G)$ in $A \otimes_\Lambda G$, and therefore $\mathrm{Im}(\psi_G)/I(A \otimes_\Lambda G) = 0$ and $I \otimes_A \mathrm{Coker}(\psi_G) = I \otimes_A G \simeq I(A \otimes_\Lambda G)$. This means that the condition (2)(b) is satisfied. Hence, $\mathsf{T}_\Lambda(G)$ is a Gorenstein-projective Λ_ψ-module by (2).

Let ${}_BQ$ be a Gorenstein-projective module. By definition, $\mathsf{T}_B(Q) = (N \otimes_B Q, Q, \Phi_Q, 1_{N\otimes Q})$, and we show that (a) and (b) in the conditions (2) hold for $\mathsf{T}_B(Q)$. However, this is easy to verify by the fact that $\Phi_Q : M \otimes_A N \otimes_B Q \to {}_BQ$ is a zero map. Thus, $\mathsf{T}_B(Q)$ is Gorenstein-projective by (2). □

Lemma 3.13. (i) *If* $G \in \Lambda$-Gproj, *then* $\mathrm{Tor}_1^\Lambda(I, G) = \mathrm{Tor}_1^\Lambda(M, G) = 0$. *Thus* $\mathrm{Tor}_i^\Lambda(I, G) = \mathrm{Tor}_i^\Lambda(M, G) = 0$ *for all* $i > 0$.

(ii) *If* ${}_BW \in B$-Gproj, *then* $\mathrm{Tor}_1^B(N, W) = 0$. *Thus* $\mathrm{Tor}_i^B(N, W) = 0$ *for all* $i > 0$.

Proof. Assume that $P^\bullet = (P^i, d_P^i)$ is a total projective resolution of G with $\mathrm{Ker}(d_P^0) = G$. Let $H := {}_\Lambda\mathrm{Ker}(d_P^{-1})$ and $b' : H \hookrightarrow P^{-1}$ be the inclusion. By Lemma 3.12,

$$\mathsf{T}_\Lambda(H) = (H(I), M \otimes_\Lambda H, \pi_H, \psi_H) \in \Lambda_\psi\text{-Gproj}.$$

Since $H(I)$ is a finitely generated A-module, $\mathrm{Hom}_A(H(I), {}_AA_A)$ is a finitely generated right A-module. Suppose that $f_1, f_2, \ldots, f_s$ form a set of generators for $\mathrm{Hom}_A(H(I), {}_AA_A)$. Then

$$\alpha : H(I) \xrightarrow{(f_1,\ldots,f_s)} ({}_AA)^s$$

is a left $\mathrm{add}({}_AA)$-approximation of $H(I)$. So we assume that $\alpha : H(I) \to Q$ is a left $\mathrm{add}({}_AA)$-approximation of $H(I)$ with $Q \in \mathrm{add}({}_AA)$. Since A is the trivial extension of Λ by I, we may further assume $Q = P(I)$ for some $P \in \mathrm{add}({}_\Lambda\Lambda)$. Then by Lemma 3.4(3),

$$\alpha = \begin{pmatrix} a & d \\ 0 & 1_I \otimes a \end{pmatrix} : H \oplus I \otimes_\Lambda H \longrightarrow P \oplus I \otimes_\Lambda P,$$

where $a : H \to P$ and $d : H \to I \otimes_\Lambda P$ are homomorphisms of Λ-modules. We show that $a : H \to P$ is an injective left $\mathrm{add}({}_\Lambda\Lambda)$-approximation of H. Actually, for $b : H \to P'$ with P' in Λ-proj, we have an A-module homomorphism

$$\bar{b} := \begin{pmatrix} b & 0 \\ 0 & 1_I \otimes b \end{pmatrix} : H(I) \to P'(I).$$

Since α is an approximation, there is a homomorphism

$$\bar{c} = \begin{pmatrix} c & e \\ 0 & 1_I \otimes c \end{pmatrix} : P(I) \to P'(I)$$

such that $\bar{b} = \alpha\bar{c}$, i.e.,

$$\begin{pmatrix} b & 0 \\ 0 & 1_I \otimes b \end{pmatrix} = \begin{pmatrix} a & d \\ 0 & 1_I \otimes a \end{pmatrix} \begin{pmatrix} c & e \\ 0 & 1_I \otimes c \end{pmatrix}.$$

This implies that $b = ac$ and a is a left $\mathrm{add}({}_\Lambda\Lambda)$-approximation of H. Taking $b = b'$, we see immediately that a is injective.

Since a is a left $\mathrm{add}({}_\Lambda\Lambda)$-approximation of H, there is a homomorphism $c : P \to P^{-1}$ of Λ-modules such that the following diagram is exact and commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow{a} & P & \longrightarrow & \mathrm{Coker}(a) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle c} & & \downarrow & & \\ 0 & \longrightarrow & H & \longrightarrow & P^{-1} & \longrightarrow & G & \longrightarrow & 0. \end{array}$$

Thus the right-hand side of the square is a pushout and pullback diagram. This induces an exact sequence of Λ-modules, i.e.,

$$0 \longrightarrow P \longrightarrow P^{-1} \oplus \mathrm{Coker}(a) \longrightarrow G \longrightarrow 0.$$

As ${}_\Lambda G$ is Gorenstein-projective, we always have $\mathrm{Ext}^i_\Lambda(G,X)=0$ for $i>0$ and any module ${}_\Lambda X$ of finite projective dimension. Thus the exact sequence splits and $\mathrm{Coker}(a)\oplus P^{-1}\simeq G\oplus P$ as Λ-modules. Therefore, to show Lemma 3.13, it is sufficient to show that

$$\mathrm{Tor}^\Lambda_1(I,\mathrm{Coker}(a))=\mathrm{Tor}^\Lambda_1(M,\mathrm{Coker}(a))=0,$$

i.e., the maps $1_I\otimes a$ and $1_M\otimes a$ are injective. This will be done by considering the left $\mathrm{add}({}_{\Lambda_\psi}\Lambda_\psi)$-approximation of $\mathsf{T}_\Lambda(H)$.

In fact, we have ${}_{\Lambda_\psi}\Lambda_\psi=\mathsf{T}_A(A)\oplus\mathsf{T}_B(B)$. Since T_A is a fully faithful additive functor (see Lemma 2.4), it follows from the left $\mathrm{add}({}_AA)$-approximation $\alpha: H(I)\to P(I)$ of $H(I)$ that

$$\mathsf{T}_A(\alpha):\mathsf{T}_A(H(I))\to\mathsf{T}_A(P(I))$$

is a left $\mathrm{add}(\mathsf{T}_A(A))$-approximation of $\mathsf{T}_A(H(I))$. This also implies that

$$\left(\begin{pmatrix} a & d\\ 0 & 1_I\otimes a\end{pmatrix}, 1_M\otimes a\right):\mathsf{T}_\Lambda(H)\to\mathsf{T}_\Lambda(P)$$

is a left $\mathrm{add}(\mathsf{T}_\Lambda(\Lambda))$-approximation of $\mathsf{T}_\Lambda(H)$. Let

$$\beta=\left(\begin{pmatrix} a & d\\ 0 & 1_I\otimes a\end{pmatrix}, 1_M\otimes a\right).$$

Take a left $\mathrm{add}(\mathsf{T}_B(B))$-approximation of $\mathsf{T}_\Lambda(H)$, i.e., $\theta:\mathsf{T}_\Lambda(H)\to\mathsf{T}_B(Q)$ for some $Q\in\mathrm{add}({}_BB)$. By Lemma 3.4(1), θ is of the form $(\binom{h}{0},0)$ with $h\in\mathrm{Hom}_\Lambda(H,N\otimes_B Q)$. Then we get a left $\mathrm{add}({}_{\Lambda_\psi}\Lambda_\psi)$-approximation $(\beta,\theta):\mathsf{T}_\Lambda(H)\to\mathsf{T}_\Lambda(P)\oplus\mathsf{T}_B(Q)$ of $\mathsf{T}_\Lambda(H)$. Since $\mathsf{T}_\Lambda(H)\in\Lambda_\psi$-Gproj by Lemma 3.12, there is an injective homomorphism from $\mathsf{T}_\Lambda(H)$ to a projective Λ_ψ-module, and therefore (β,θ) is injective. This shows that the homomorphisms

$$\begin{pmatrix} a & d & h\\ 0 & 1_I\otimes a & 0\end{pmatrix}: H\oplus I\otimes_\Lambda H\to P\oplus I\otimes_\Lambda P\oplus N\otimes_B Q$$

of Λ-modules and $(1_M\otimes a,0): M\otimes_\Lambda H\to M\otimes_\Lambda P\oplus Q$ of B-modules are injective. Therefore, $1_I\otimes a$ and $1_M\otimes a$ are injective. Thus, $\mathrm{Tor}^\Lambda_1(M,\mathrm{Coker}(a))=0$ and $\mathrm{Tor}^\Lambda_1(I,\mathrm{Coker}(a))=0$. Therefore, $\mathrm{Tor}^\Lambda_1(I,G)=0$. A dimension shift argument shows $\mathrm{Tor}^\Lambda_i(I,G)=0$ for $i>0$.

Now, let $W\in B$-Gproj with $Q^\bullet$ be a totally exact complex in $\mathscr{C}(B\text{-proj})$ such that $\mathrm{Ker}(d^0_Q)=W$. Then there is the short exact sequence $0\to V\xrightarrow{v}Q^{-1}\xrightarrow{w}W\to 0$ of B-modules with $V=\mathrm{Ker}(d^{-1}_Q)$. This yields an exact sequence of Λ_ψ-modules, i.e.,

$$0\longrightarrow (U,V,s,t)\xrightarrow{(i,v)}\mathsf{T}_B(Q^{-1})\xrightarrow{(1_N\otimes w,w)}\mathsf{T}_B(W)\longrightarrow 0$$

with $i: {}_AU\to{}_AN\otimes_B Q^{-1}$ the kernel of $1_N\otimes w$. By the diagram (2.2) (see Subsection 2.1), the homomorphisms s and t fit in the exact commutative diagrams, respectively,

$$\begin{array}{ccccc} & & M\otimes_A U & \xrightarrow{1_M\otimes i} & M\otimes_A N\otimes_B Q^{-1}\\ & & \downarrow s & & \downarrow 0\\ 0 & \longrightarrow & V & \xrightarrow{v} & Q^{-1},\end{array}\qquad \begin{array}{ccccccc} N\otimes_B V & \xrightarrow{1_N\otimes v} & N\otimes_B Q^{-1} & \xrightarrow{1_N\otimes w} & N\otimes_B W & \longrightarrow & 0\\ \downarrow t & & \| & & \| & & \\ 0\longrightarrow U & \xrightarrow{i} & N\otimes_B Q^{-1} & \xrightarrow{1_N\otimes w} & N\otimes_B W & \longrightarrow & 0.\end{array}$$

Since the homomorphism v is injective, we get $s=0$. By the Snake lemma, t is surjective. Therefore $\mathrm{Coker}(s)=V$ and $\mathrm{Coker}(t)=0$.

By Lemma 3.12, $\mathsf{T}_B(W) \in \Lambda_\psi$-Gproj. Since Λ_ψ-Gproj is closed under taking kernels of surjective homomorphisms (see [13, Theorem 2.7]), we have $(U, V, s, t) \in \Lambda_\psi$-Gproj. By the assumption (2), we have $IU \simeq I \otimes_\Lambda \operatorname{Coker}(t) = 0$ and $\operatorname{Im}(t)/IU \simeq N \otimes_B V$. Hence $\operatorname{Im}(t) = U \simeq N \otimes_B V$. This implies further that t is an isomorphism by Lemma 2.3(2) and $1_N \otimes v$ is injective. Thus, $\operatorname{Tor}_1^B(N, W) = 0$. □

(I) We show that ${}_BM_\Lambda$, ${}_\Lambda N_B$ and ${}_\Lambda I_\Lambda$ are weakly compatible.

Suppose that $P^\bullet$ is a totally exact complex in $\mathscr{C}(\Lambda\text{-proj})$. Then ${}_\Lambda\operatorname{Ker}(d_P^i)$ is Gorenstein-projective. It follows from Lemma 3.13 that $M \otimes_\Lambda P^\bullet$ and $I \otimes_\Lambda P^\bullet$ are exact. This further implies that the complex $T^\bullet := \mathsf{T}_\Lambda(P^\bullet) = (\mathsf{T}_\Lambda(P^i), d_T^i)$ with $d_T^i = (1_A \otimes d_P^i, 1_M \otimes d_P^i)$ of Λ_ψ-modules is exact and $\operatorname{Ker}(d_T^i) = \mathsf{T}_\Lambda(\operatorname{Ker}(d_P^i))$. By Lemma 3.12, $\mathsf{T}_\Lambda(\operatorname{Ker}(d_P^i)) \in \Lambda_\psi$-Gproj. Thus $\operatorname{Ker}(d_T^i) \in \Lambda_\psi$-Gproj for all i. This implies that $T^\bullet$ is a totally exact complex. It follows from Lemma 3.4(1) that $\operatorname{Hom}_\Lambda(P^\bullet, N) \simeq \operatorname{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{T}_B(B))$ is exact.

Next, we show that $\operatorname{Hom}_\Lambda(P^\bullet, I)$ is exact. Consider the exact sequence of Λ_ψ-modules:

$$0 \longrightarrow \mathsf{Z}_\Lambda(I) \stackrel{((0,1),0)}{\longrightarrow} \mathsf{T}_\Lambda(\Lambda) \longrightarrow \mathsf{T}'_\Lambda(\Lambda) \longrightarrow 0,$$

where $\mathsf{T}'_\Lambda(\Lambda) := ({}_A\Lambda, {}_BM, \mu, 0)$ with $\mu : M \otimes_A \Lambda \to M$ being the multiplication map, and $\mathsf{Z}_\Lambda(I) = ({}_AI, 0, 0, 0) \in \Lambda_\psi$-mod. As $T^i := \mathsf{T}_\Lambda(P^i)$ is a projective Λ_ψ-module (see Lemma 2.5(1) and $M \otimes_A P^i \simeq M \otimes_\Lambda P^i$), we have the exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_\Lambda(I)) \longrightarrow \operatorname{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{T}_\Lambda(\Lambda)) \longrightarrow \operatorname{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{T}'_\Lambda(\Lambda)) \longrightarrow 0.$$

Due to the total exactness of $T^\bullet$, the complex $\operatorname{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{T}_\Lambda(\Lambda))$ is exact. Then it follows from the exactness of $\operatorname{Hom}_\Lambda(P^\bullet, \Lambda)$ and the isomorphism

$$\operatorname{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{T}'_\Lambda(\Lambda)) = \operatorname{Hom}_{\Lambda_\psi}(\mathsf{T}_\Lambda(P^\bullet), \mathsf{T}'_\Lambda(\Lambda)) \simeq \operatorname{Hom}_\Lambda(P^\bullet, \Lambda)$$

as complexes of $\mathbb{Z}$-modules that $\operatorname{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_\Lambda(I))$ is exact. This implies that $\operatorname{Hom}_\Lambda(P^\bullet, I) = \operatorname{Hom}_A(P^\bullet, I)$ is exact. Hence, ${}_\Lambda I_\Lambda$ is a weakly compatible bimodule.

To complete the proof that ${}_BM_\Lambda$ and ${}_\Lambda N_B$ are weakly compatible, it remains to show that $\operatorname{Hom}_B(Q^\bullet, {}_BM)$ and $N \otimes_B Q^\bullet$ are exact for any totally exact complex $Q^\bullet \in \mathscr{C}(B\text{-proj})$. Actually, $\operatorname{Tor}_B^1(N, \operatorname{Ker}(d_Q^i)) = 0$ for all i by Lemma 3.13(ii). Thus $N \otimes_B Q^\bullet$ is exact, and therefore $E^\bullet := \mathsf{T}_B(Q^\bullet)$ is exact with $\operatorname{Ker}(d_E^i) = \mathsf{T}_B(\operatorname{Ker}(d_Q^i))$. It follows from Lemma 3.12 that $\operatorname{Ker}(d_E^i) \in \Lambda_\psi$-Gproj, whence $E^\bullet$ is a totally exact complex in $\mathscr{C}(\Lambda_\psi\text{-proj})$. Now, by Lemma 3.4(4), we know that $\operatorname{Hom}_B(Q^\bullet, M) \simeq \operatorname{Hom}_{\Lambda_\psi}(E^\bullet, \mathsf{T}_\Lambda(\Lambda))$ is exact.

(II) We prove that $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(M)$, $\mathsf{Z}_\Lambda(N)$, $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(I)$ and $\mathsf{Z}_\Lambda(I)$ are semi-weakly compatible Λ_ψ-modules.

Let $T^\bullet := (T^i, d_T^i)$ be a totally exact complex in $\mathscr{C}(\Lambda_\psi\text{-proj})$. By definition, we have to show that $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(M) \otimes_{\Lambda_\psi} T^\bullet$, $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(I) \otimes_{\Lambda_\psi} T^\bullet$, $\operatorname{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_\Lambda(N))$ and $\operatorname{Hom}_{\Lambda_\psi}(T^\bullet, \mathsf{Z}_\Lambda(I))$ are exact complexes.

By Lemma 2.5, $T^\bullet$ is of the form

$$T^\bullet \cdots \longrightarrow \mathsf{T}_\Lambda(P^{-1}) \oplus \mathsf{T}_B(Q^{-1}) \xrightarrow{d_T^{-1}} \mathsf{T}_\Lambda(P^0) \oplus \mathsf{T}_B(Q^0) \xrightarrow{d_T^0} \mathsf{T}_\Lambda(P^1) \oplus \mathsf{T}_B(Q^1) \longrightarrow \cdots,$$

and induces a complex $F^\bullet = (F^i, d_F^i) \in \mathscr{C}(A\text{-proj})$ and two complexes $P^\bullet = (P^i, d_P^i)$ and $Z^\bullet = (Z^i, d_Z^i) \in \mathscr{C}(\Lambda)$ as in (‡) and the sequence (3.6). By (1) and (3) in Lemma 3.7, to show that $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(M) \otimes_{\Lambda_\psi} T^\bullet$ and $\mathsf{Z}_{\Lambda^{\mathrm{op}}}(I) \otimes_{\Lambda_\psi} T^\bullet$ are exact, it is sufficient to prove that $M \otimes_\Lambda P^\bullet$ and $I \otimes_\Lambda P^\bullet$ are exact. Similarly, by (2) and (7) in Lemma 3.4, it is sufficient to show that $\operatorname{Hom}_\Lambda(P^\bullet, N)$ and $\operatorname{Hom}_\Lambda(P^\bullet, I)$ are exact complexes. Since we have shown that ${}_BM_\Lambda$, ${}_\Lambda N_B$ and ${}_\Lambda I_\Lambda$ are weakly compatible, it is enough to show that $P^\bullet$ is a totally exact complex. This is equivalent to saying that $P^\bullet$ is exact in degree i and $\operatorname{Ker}(d_P^i)$ is Gorenstein-projective for all i.

Now, we prove this statement for $i = 0$. Let $(X, Y, f, g) := \operatorname{Ker}(d_T^0)$, $V := \operatorname{Coker}(f)$, $U := \operatorname{Coker}(g)$, $(E, F, k, l) := \operatorname{Ker}(d_T^1)$, $T := \operatorname{Coker}(k)$ and $S := \operatorname{Coker}(l)$. Then (X, Y, f, g) and (E, F, k, l) are Gorenstein-projective Λ_ψ-modules. Consider the following canonical exact sequence of Λ_ψ-modules:

$$0 \longrightarrow (X, Y, f, g) \stackrel{(d_X, d_Y)}{\longrightarrow} T^0 \longrightarrow (E, F, k, l) \longrightarrow 0$$

with $d_X : {}_AX \to {}_AP^0(I) \oplus {}_AN \otimes_B Q^0$, $d_Y : {}_BY \to {}_BM \otimes_\Lambda P^0 \oplus Q^0$. Then (d_X, d_Y) is a left $\mathrm{add}({}_{\Lambda_\psi}\Lambda_\psi)$-approximation of (X, Y, f, g). By the diagram (2.2), there is the exact commutative diagram of A-modules, i.e.,

$$\begin{array}{ccccccccc}
 & & N \otimes_B Y & \longrightarrow & N \otimes_B M \otimes_\Lambda P^0 \oplus N \otimes_B Q^0 & \longrightarrow & N \otimes_B F & \longrightarrow & 0 \\
 & & \downarrow g & & \downarrow \begin{pmatrix} \psi_{P^0} & 0 \\ 0 & 1_{N \otimes_B Q^0} \end{pmatrix} & & \downarrow l & & \\
0 & \longrightarrow & X & \xrightarrow{d_X} & P^0(I) \oplus N \otimes_B Q^0 & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow \lambda_X & & \downarrow & & \downarrow & & \\
 & & U & \xrightarrow{a_U} & P^0 & \xrightarrow{b_S} & S & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.10}$$

The bottom row is exact by the Snake lemma, and thus an exact sequence of Λ-modules by Lemma 3.1(1). Let $d_{{}_\Lambda X} = (e_0, e_1, e_2) : {}_\Lambda X \to P^0 \oplus I \otimes_\Lambda P^0 \oplus N \otimes Q^0$ denote the restriction of d_X to Λ-modules. Then the above diagram shows $e_0 = \lambda_X a_U$.

Similarly, let $\mathrm{Ker}(d_T^{-1}) = (E', F', k', l')$, and $(t_X, t_Y) : T^{-1} \to \mathrm{Ker}(d_T^0) = (X, Y, f, g)$ be the canonical projection. Then there is a canonical exact sequence

$$0 \to (E', F', k', l') \to T^{-1} \overset{(t_X, t_Y)}{\longrightarrow} (X, Y, f, g) \to 0$$

of Λ_ψ-modules. This supplies us with the exact commutative diagram of A-modules, i.e.,

$$\begin{array}{ccccccccc}
 & & N \otimes_B F' & \longrightarrow & N \otimes_B M \otimes_\Lambda P^{-1} \oplus N \otimes_B Q^{-1} & \longrightarrow & N \otimes_B Y & \longrightarrow & 0 \\
 & & \downarrow l' & & \downarrow \begin{pmatrix} \psi_{P^{-1}} & 0 \\ 0 & 1_{N \otimes_B Q^{-1}} \end{pmatrix} & & \downarrow g & & \\
0 & \longrightarrow & E' & \longrightarrow & P^{-1}(I) \oplus N \otimes_B Q^{-1} & \xrightarrow{t_X} & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \lambda_X & & \\
 & & S' & \xrightarrow{a_{S'}} & P^{-1} & \xrightarrow{b_U} & U & \longrightarrow & 0.
\end{array} \tag{3.11}$$

Due to $d_T^{-1} = (t_X, t_Y)(d_X, d_Y)$, we have $t_X d_X = d_F^{-1}$. Thus the diagrams (3.10) and (3.11) provide the following commutative diagram of A-modules:

$$\begin{array}{ccccc}
P^{-1}(I) \oplus N \otimes_B Q^{-1} & \xrightarrow{t_X} & X & \xrightarrow{d_X} & P^0(I) \oplus N \otimes_B Q^0 \\
\downarrow & & \downarrow \lambda_X & & \downarrow \\
P^{-1} & \xrightarrow{b_U} & U & \xrightarrow{a_U} & P^0,
\end{array}$$

where the two unnamed vertical maps are natural projections. Since $d_F^{-1} = t_X d_X$, we get $b_U a_U = d_P^{-1}$.

We show that a_U is injective. Consider a_U as a homomorphism of Λ-modules. Since (X, Y, f, g) is Gorenstein-projective, we know $U \in \Lambda\text{-Gproj}$ by the assumption (2). Thus, for a_U to be injective, it suffices to show that a_U is a left $\mathrm{add}({}_\Lambda\Lambda)$-approximation of U. Actually, for $P \in \mathrm{add}({}_\Lambda\Lambda)$ and a homomorphism $a_0 : U \to P$ of Λ-module, we have to find a homomorphism $e : P^0 \to P$ of Λ-modules such that $a_0 = a_U e$. To define e, we construct a Λ-module homomorphism $a_1 : {}_\Lambda X \to I \otimes_\Lambda P$, where X is regarded as a Λ-module by restriction and a B-module homomorphism $h : Y \to M \otimes_\Lambda P$ such that $a := (\lambda_X a_0, a_1)$ is a homomorphism of A-modules, and $(a, h) : (X, Y, f, g) \to \mathsf{T}_\Lambda(P)$ is a homomorphism of Λ_ψ-modules.

Step 1. Construction of h. By Lemma 3.1(2), we have a commutative diagram

$$\begin{array}{ccc}
{}_BM \otimes_A X & \xrightarrow{\lambda'_X} & {}_BM \otimes_\Lambda U \longrightarrow 0, \\
\downarrow f & \swarrow \eta_Y & \\
{}_BY & &
\end{array}$$

where λ'_X is the composite of $1_M \otimes \lambda_X$ with the isomorphism $M \otimes_A U \to M \otimes_\Lambda U$ as B-modules. By the assumption, $\mathrm{Im}(f) \simeq M \otimes_\Lambda U$. This means that η_Y is injective and there is an exact sequence of B-modules, i.e.,

$$0 \longrightarrow M \otimes_\Lambda U \xrightarrow{\eta_Y} Y \xrightarrow{\mu_Y} V \longrightarrow 0.$$

Since ${}_BV \in B\text{-Gproj}$ and ${}_BM$ is semi-weakly compatible, it holds that $\mathrm{Ext}^1_B(V, M \otimes_\Lambda P) = 0$. This shows that $\mathrm{Hom}_B(\eta_Y, M \otimes_\Lambda P)$ is surjective, and therefore there is a homomorphism $h : Y \to M \otimes_\Lambda P$ of B-modules such that $\eta_Y h = 1_M \otimes a_0$.

Step 2. Construction of a_1. From $\eta_Y h = 1_M \otimes a_0$, one gets $(1_N \otimes_B \eta_Y)(1_N \otimes h) = 1_N \otimes_B 1_M \otimes a_0$. It follows from the natural property of ψ that the diagram of A-modules is commutative

$$\begin{tikzcd} N \otimes_B M \otimes_\Lambda U \arrow[r, "1_N \otimes \eta_Y"] \arrow[d, "\psi \otimes 1_U"] & N \otimes_B Y \arrow[d, "(1_N \otimes h)(\psi \otimes 1_P)"] \\ I \otimes_\Lambda U \arrow[r, "1_I \otimes a_0"] & I \otimes_\Lambda P. \end{tikzcd}$$

Now, let $H := \mathrm{Im}(g)$ and $g = \sigma\epsilon_X$ with $\sigma : N \otimes_B Y \to H$ the canonical projection and $\epsilon_X : H \hookrightarrow X$ the inclusion. According to Lemma 3.2 and its proof, there exists an injective homomorphism $m : {}_AI \otimes_\Lambda U \to {}_AH$, such that the following is a pushout diagram:

$$\begin{tikzcd} & N \otimes_B M \otimes_\Lambda U \arrow[r, "1_N \otimes \eta_Y"] \arrow[d, "\psi \otimes 1_U"] & N \otimes_B Y \arrow[d, "\sigma"] \\ 0 \arrow[r] & I \otimes_\Lambda U \arrow[r, "m"] & H. \end{tikzcd}$$

By a universal property of pushouts, there is a Λ-module homomorphism $t : H \to I \otimes_\Lambda P$ such that $1_I \otimes a_0 = mt$ and $(1_N \otimes h)(\psi \otimes 1_P) = \sigma t$.

The exact sequence $0 \to H \xrightarrow{\epsilon_X} X \xrightarrow{\lambda_X} {}_AU \to 0$ of A-modules restricts to an exact sequence of Λ-modules, i.e.,

$$0 \longrightarrow H \xrightarrow{\epsilon_X} {}_\Lambda X \xrightarrow{\lambda_X} U \longrightarrow 0.$$

It follows from ${}_\Lambda U \in \Lambda\text{-Gproj}$ and the semi-weak compatibility of ${}_\Lambda I$ that $\mathrm{Ext}^1_\Lambda(U, I \otimes_\Lambda P) = 0$, and therefore $\mathrm{Hom}_\Lambda(\epsilon_X, I \otimes_\Lambda P)$ is surjective. Hence, there is a homomorphism $a_1 : X \to I \otimes_\Lambda P$ of Λ-modules such that $\epsilon_X a_1 = t$.

Step 3. We show that $(\lambda_X a_0, a_1) : X \to P \oplus I \otimes_\Lambda P$ is a homomorphism of A-modules. We write a for $(\lambda_X a_0, a_1)$ for simplicity. On the one hand, $1_I \otimes a_0 = mt = m\epsilon_X a_1$. On the other hand, by the definition of m (see Lemma 3.2), the following diagram commutes:

$$\begin{tikzcd} I \otimes_A X \arrow[r, "\mathrm{mlt}_X"] \arrow[d, "1_I \otimes \lambda_X"'] & X \\ I \otimes_\Lambda U \arrow[r, "m"] & H, \arrow[u, "\epsilon_X"'] \end{tikzcd}$$

i.e., $\mathrm{mlt}_X = (1_I \otimes \lambda_X)\, m\, \epsilon_X$. Thus, for $i \in I$ and $x \in X$, it holds that

$$(ix)a_1 = ((i \otimes x)\mathrm{mlt}_X)a_1 = [(i \otimes x)((1_I \otimes \lambda_X)\, m\, \epsilon_X)]a_1 = (i \otimes x)[(1_I \otimes \lambda_X)(1_I \otimes a_0)] = i \otimes (x)\lambda_X a_0.$$

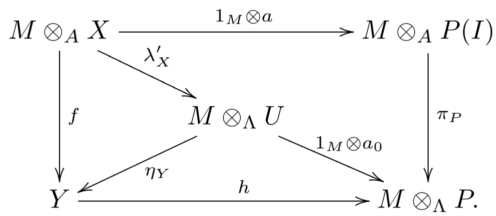
Clearly, $(ix)\lambda_X a_0 = 0$. Now it is easy to verify that a is a homomorphism of A-modules.

Step 4. We prove that (a, h) is a homomorphism of Λ_ψ-modules. First, we show that the out-side square of the following diagram of A-modules is commutative:

$$\begin{tikzcd} N \otimes_B Y \arrow[rr, "1_N \otimes h"] \arrow[dd, "g"] \arrow[dr, "\sigma"] & & N \otimes_B M \otimes_\Lambda P \arrow[dd, "\psi_P"] \\ & H \arrow[dl, "\epsilon_X"] \arrow[dr, "(0,t)"] & \\ X \arrow[rr, "a"] & & P(I). \end{tikzcd}$$

In fact, since $\epsilon_X \lambda_X a_0 = 0$, we have $\epsilon_X a = (0, \epsilon_X a_1) = (0, t)$ by Step 2. It follows from $(1_N \otimes h)(\psi \otimes 1_P) = \sigma t$ that $(1_N \otimes h)\psi_P = \sigma(0, t)$. Then $ga = (\sigma\epsilon_X)a = \sigma(0, t) = (1_N \otimes h)\psi_P$.

Second, we show that the out-side square of the following diagram of B-modules is commutative:



Note that $\lambda'_X \eta_Y = f$ and $\eta_Y h = 1_M \otimes a_0$. A straightforward verification shows $fh = (1_M \otimes a)\pi_P$. Thus, the pair (a, h) is a homomorphism of Λ_ψ-modules.

Step 5. Definition of e. Since (d_X, d_Y) is a left $\mathrm{add}({}_{\Lambda_\psi}\Lambda_\psi)$-approximation of (X, Y, f, g) and $\mathsf{T}_\Lambda(P) \in \mathrm{add}({}_{\Lambda_\psi}\Lambda_\psi)$, there are a homomorphism $u : \mathsf{T}_\Lambda(P^0) \to \mathsf{T}_\Lambda(P)$ and a homomorphism $v : \mathsf{T}_B(Q^0) \to \mathsf{T}_\Lambda(P)$ such that

$$(d_X, d_Y)\begin{pmatrix} u \\ v \end{pmatrix} = (a, h).$$

By (3) and (4) in Lemma 3.4, there exists a homomorphism $e : P \to P^0$ of Λ-modules satisfying $e_0 e = \lambda_X a_0$. Because $e_0 = \lambda_X a_U$ and λ_X is surjective, it holds that $a_U e = a_0$. Hence, a_U is a left $\mathrm{add}({}_\Lambda\Lambda)$-approximation of ${}_\Lambda U$. This completes the proof of a_U being injective.

Now we show that the complex $P^\bullet$ is totally exact. From the exact sequence

$$0 \longrightarrow U \xrightarrow{a_U} P^0 \xrightarrow{b_S} S \longrightarrow 0$$

of Λ-modules, we proceed with a similar proof of a_U being a left $\mathrm{add}({}_\Lambda\Lambda)$-approximation of U with $d_P^{-1} = b_U a_U$, and replace U with S to show that there is an injective homomorphism $a_S : S \to P^1$ such that $d_P^0 = b_S a_S$. This implies that $\mathrm{Ker}(d_P^0) = \mathrm{Ker}(b_S) = \mathrm{Im}(a_U) \simeq U \in \Lambda\text{-Gproj}$. Due to $\mathrm{Im}(d_P^{-1}) = \mathrm{Im}(a_U)$, we see that $\mathrm{Ker}(d_P^0) = \mathrm{Im}(d_P^{-1})$ and $P^\bullet$ is exact in degree 0 with $\mathrm{Ker}(d_P^0) \in \Lambda\text{-Gproj}$. Similarly, we can show that $P^\bullet$ is exact in any degree i with $\mathrm{Ker}(d_P^i) \in \Lambda\text{-Gproj}$. Thus, $P^\bullet$ is a totally exact complex. This finishes the proof of (2) implying (1). □

For the special Morita context ring $\Lambda_{(0,0)}$, it was shown in [10] that the compatibility conditions suffice a class of modules over $\Lambda_{(0,0)}$ to be Gorenstein-projective. Next, we point out that the weak compatibility conditions are both necessary and sufficient.

Proposition 3.14. *For the Morita context ring $\Lambda_{(0,0)}$, the following are equivalent*:

(1) ${}_A N_B$ *and* ${}_B M_A$ *are weakly compatible bimodules*; $({}_A N, 0, 0, 0)$ *and* $(M_A, 0, 0, 0)$ *are semi-weakly compatible left and right* $\Lambda_{(0,0)}$*-modules, respectively.*

(2) *A* $\Lambda_{(0,0)}$*-module* (X, Y, f, g) *is Gorenstein-projective if and only if*

(a) ${}_B\mathrm{Coker}(f)$ *and* ${}_A\mathrm{Coker}(g)$ *are Gorenstein-projective, and*

(b) ${}_B\mathrm{Im}(f) \simeq {}_B M \otimes_A \mathrm{Coker}(g)$ *and* ${}_A\mathrm{Im}(g) \simeq {}_A N \otimes_B \mathrm{Coker}(f)$*, where* $\mathrm{Coker}(f)$ *and* $\mathrm{Im}(g)$ *denote the cokernel of* f *and the image of* g*, respectively.*

Proof. This follows immediately from Theorem 3.11 because $I = 0$ in $\Lambda_{(0,0)}$. □

The following was proved in [10, Theorem A(i)]. Assume that both M and N are compatible bimodules over Artin algebras. If a $\Lambda_{(0,0)}$-module (X, Y, f, g) fulfills the conditions (a) and (b) in Proposition 3.14, then (X, Y, f, g) is Gorenstein-projective. It seems that the weak compatibility conditions are more suitable for describing Gorenstein-projective modules over $\Lambda_{(0,0)}$.

4 Applications to noncommutative tensor products

In this section, we describe Gorenstein-projective modules over the noncommutative tensor products of exact contexts arising from Morita contexts with two bimodule homomorphisms zero. This description is related to Gorenstein-projective modules over the Morita context rings $\Lambda_{(\phi,0)}$.

Definition 4.1 (See [5]). Let $\lambda : R \to S, \mu : R \to T$ be homomorphisms of unitary rings, and ${}_SW_T$ be an S-T-bimodule with $w \in W$. If the sequence

$$0 \longrightarrow R \xrightarrow{(\lambda,\mu)} S \oplus T \xrightarrow{\binom{\cdot w}{w\cdot}} W \longrightarrow 0$$

is exact of abelian groups, then (λ, μ, W, w) is called an exact context, where $\cdot w : S \to W$ is the right multiplication by w. The noncommutative tensor product of (λ, μ, W, w) is well defined.

Morita contexts provide prominent examples of exact contexts. For a Morita context $(A, \Gamma, {}_\Gamma M_A, {}_AN_\Gamma, \phi, \psi)$, let

$$R := \begin{pmatrix} A & 0 \\ 0 & \Gamma \end{pmatrix}, \quad S := \begin{pmatrix} A & N \\ 0 & \Gamma \end{pmatrix}, \quad T := \begin{pmatrix} A & 0 \\ M & \Gamma \end{pmatrix}, \quad W := \Lambda_{(\phi,\psi)}, \quad w := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If λ and μ are the inclusions, then (λ, μ, W, w) is an exact context. Its noncommutative tensor product, denoted by $C(A, \Gamma, M, N, \phi, \psi)$, can be described explicitly: $C(A, \Gamma, M, N, \phi, \psi)$ has the underlying abelian group of the matrix form

$$\begin{pmatrix} A & N \\ M & \Gamma \oplus (M \otimes_A N) \end{pmatrix}$$

with the multiplication $\circ$ defined by

$$\begin{pmatrix} a_1 & n_1 \\ m_1 & (b_1, m \otimes n) \end{pmatrix} \circ \begin{pmatrix} a_2 & n_2 \\ m_2 & (b_2, m' \otimes n') \end{pmatrix}$$
$$= \begin{pmatrix} a_1a_2 + (n_1 \otimes m_2)\psi & a_1n_2 + n_1b_2 + n_1(m' \otimes n')\phi \\ m_1a_2 + b_1m_2 + (m \otimes n)\phi m_2 & (b_1b_2,\ m_1 \otimes n_2 + (b_1m') \otimes n' + m \otimes (nb_2) + m \otimes (n \otimes m')\psi n') \end{pmatrix},$$

where $a_1, a_2 \in A$, $b_1, b_2 \in \Gamma$, $n_1, n_2, n, n' \in N$ and $m_1, m_2, m, m' \in M$. For details, we refer the readers to [5].

Let $C := C(A, \Gamma, M, N, 0, 0)$, and $B := \Gamma \ltimes (M \otimes_A N)$ be the trivial extension of Γ with the Γ-bimodule $M \otimes_A N$. We may regard M as a B-A-bimodule and N as an A-B-bimodule via the canonical surjective homomorphism $B \to \Gamma$. Thus we have a Morita context $(A, B, M, N, \phi, 0)$, where $\phi : M \otimes_A N \to B$, $m \otimes n \mapsto (0, m \otimes n)$ for $m \in M$ and $n \in N$, and the Morita context ring

$$\Lambda_{(\phi,0)} = \begin{pmatrix} A & N \\ M & B \end{pmatrix}_{(\phi,0)},$$

which is isomorphic to C. Thus, the dual versions of Theorems 3.5 and 3.10 also describe the Gorenstein-projective modules over the noncommutative tensor product C. For example, we have the following specifical corollary.

Corollary 4.2. *Suppose that the bimodules ${}_\Gamma M_A$, ${}_AN_\Gamma$ and ${}_\Gamma M \otimes_A N_\Gamma$ are weakly compatible. Let $B = \Gamma \ltimes J$ with $J := M \otimes_A N$. Then a C-module $({}_AX, {}_BY, f, g)$ is Gorenstein-projective if*

(i) ${}_\Gamma\mathrm{Coker}(f)$ *and* ${}_A\mathrm{Coker}(g)$ *are Gorenstein-projective, and*

(ii) ${}_AN \otimes_B \mathrm{Coker}(f) \simeq {}_A\mathrm{Im}(g)$, ${}_BM \otimes_A \mathrm{Coker}(g) \simeq {}_B\mathrm{Im}(f)/JY$ *and* ${}_BJ \otimes_B \mathrm{Coker}(f) \simeq {}_BJY$, *where* $\mathrm{Coker}(f)$ *and* $\mathrm{Im}(g)$ *denote the cokernel of f and the image of g, respectively.*

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