J. Noncommut. Geom. (Online first)
DOI 10.4171/JNCG/611

Derived equivalences constructed by Milnor patching

Wei Hu and Changchang Xi

Abstract. New derived equivalences of algebras are constructed from given ones by the pullbacks of tilting complexes. This leads to three explicit constructions of derived equivalences by gluing vertices, unifying arrows and identifying socle elements.

1. Introduction

Derived categories and equivalences between them were introduced by Grothendieck and Verdier (see [18]) and have played nowadays an important role in the representation theory of finite-dimensional algebras and finite groups (see [1, 4, 5, 17]). For example, many significant homological and numerical invariants of algebras are preserved by derived equivalences, such as Hochschild cohomology, Hochschild cyclic cohomology and the number of simple modules of algebras (see [10, 14]). Rickard's Morita theory on derived categories of rings provides a powerful tool to understand derived equivalences between algebras. Here, the crucial point is the notion of tilting complexes. They are by definition bounded complexes of finitely generated projective modules with self-orthogonality and generator property (see [14]). However, how to construct tilting complexes or derived equivalences is indeed a quite hard problem and seems still to be understood.

In 1971, Milnor proved that finitely generated projective modules over a pullback algebra can be constructed by patching the ones over its constituent algebras. In this way, Milnor established a well-known Mayer–Vietoris sequence for algebraic K-groups (see [13]). Motivated by this construction of projective modules, we consider in this paper naturally the question of constructing tilting complexes over pullback algebras by patching the ones over their constituent algebras. Here, one confronts immediately a great difficulty of proving the self-orthogonality and generator property of the constructed complexes since in general there are no expected homological relations on modules over different algebras in a pullback diagram. Though not solving this question completely in the paper, we do present the following partial answers when working with Artin algebras.

Mathematics Subject Classification 2020: 16G10 (primary); 18G80, 16D90, 16G20, 16E35 (secondary).
Keywords: derived category, derived equivalence, Milnor patching, pullback algebra, quiver, stable equivalence, tilting complex.

Suppose we are given a Milnor square of homomorphisms of Artin algebras, that is, a pullback diagram of Artin algebras

$$\begin{tikzcd} A \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"'] & A_1 \arrow[d, "\pi_1"] \\ A_2 \arrow[r, "\pi_2"] & A_0 \end{tikzcd}$$

with π_1 surjective, and a tilting complex $T_i^\bullet$ over A_i for $0 \leq i \leq 2$, such that $A_0 \otimes_{A_1} T_1^\bullet \simeq T_0^\bullet \simeq A_0 \otimes_{A_2} T_2^\bullet$. Let $B_i := \mathrm{End}_{A_i}(T_i^\bullet)$ for $i = 0, 1, 2$, and $h^\bullet$ an isomorphism from $A_0 \otimes_{A_1} T_1^\bullet$ to $A_0 \otimes_{A_2} T_2^\bullet$. The Milnor patching of the triple $(T_1^\bullet, T_2^\bullet, h^\bullet)$ is denoted by $M(T_1^\bullet, T_2^\bullet, h^\bullet)$ (see Section 2.4 for details).

In the following, we may assume that the complexes $T_i^\bullet$ are basic and radical. Now, our first main result can be stated as follows.

Theorem 1.1. *If $T_0^\bullet$ is a direct sum of shifts of projective A_0-modules, then*

(1) *$M(T_1^\bullet, T_2^\bullet, h^\bullet)$ is a tilting complex over A.*

(2) *There exist homomorphisms $\eta_1 : B_1 \to B_0$ and $\eta_2 : B_2 \to B_0$ of Artin algebras with η_1 surjective such that the pullback algebra B of η_1 and η_2 is isomorphic to the endomorphism algebra of $M(T_1^\bullet, T_2^\bullet, h^\bullet)$. Thus, the algebras A and B are derived equivalent.*

For general finite dimension algebras, derived equivalences of algebras need not to induce stable equivalences of algebras. However, almost ν-stable derived equivalences introduced in [7] always give rise to stable equivalences of Morita type [7, Theorem 1.1]. This generalizes a result by Rickard in [15, Corollary 5.5]. Moreover, almost ν-stable derived equivalences preserve dominant, finitistic and global dimensions. Also, Auslander–Reiten conjecture on stable equivalences, which says that stable equivalences preserve the number of non-projective simple modules, holds true for stable equivalences induced by almost ν-stable equivalences.

The following theorem shows that the Milnor patching construction in Theorem 1.1 is compatible with almost ν-stable derived equivalences.

Theorem 1.2. *Suppose that all A_i are finite-dimensional algebras over an algebraically closed field and $T_0^\bullet$ is a stalk complex at degree 0. If $T_i^\bullet$ defines an almost ν-stable derived equivalence between A_i and B_i for $i = 1, 2$, then the derived equivalence between A and B in Theorem 1.1 is almost ν-stable.*

Thus, by a result in [7], the algebras A and B in Theorem 1.2 are stably equivalent of Morita type. This implies that the Auslander–Reiten conjecture holds true for A and B.

As applications of Theorem 1.1, we construct new derived equivalences from given ones for algebras presented by quivers with relations. This is done by gluing vertices, unifying arrows and identifying socle elements; see Theorems 4.1, 4.5 and 4.8 for details. Remarkably, each of these operations can be iterated as many times as possible.

The paper is organized as follows: In Section 2, we fix notation and recall basic facts needed in later proofs. Also, we prove primary results on simple modules under derived equivalences, on tilting complexes and on their endomorphism rings. In Section 3, we show Theorems 1.1 and 1.2. In Section 4, we present details on constructions of derived equivalences by operations of gluing vertices, unifying arrows and identifying socle elements. These methods are illustrated by an example at the end of the section.

2. Preliminaries

In this section, we fix notation and recall some basic results on derived equivalences and on projective modules over pullback algebras. We then prove a few results concerning derived equivalences and tilting complexes. They will be used in the proof of Theorem 1.1.

2.1. Derived equivalences

Let $\mathcal{C}$ be an additive category.

Given two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, the composition of f with g is written as fg, which is a morphism from X to Z. But for two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ of categories, their composition is denoted by GF.

For an object X in $\mathcal{C}$, let $\mathrm{add}(X)$ be the full subcategory of $\mathcal{C}$ consisting of all direct summands of finite direct sums of copies of X. The object X is said to be *basic* if $X = \bigoplus_{i\in I} X_i$ with I an index set and X_i an indecomposable object for all $i \in I$ such that $X_i \not\simeq X_j$ for $i \neq j$.

A complex $X^\bullet = (X^i, d_X^i)$ over $\mathcal{C}$ is said to be *radical* if the differential $d_X^i : X^i \to X^{i+1}$ is a radical morphism for all $i \in \mathbb{Z}$. By $X^\bullet[n]$ we denote the n-th shift of $X^\bullet$; that is, $X^\bullet[n]$ has X^{i+n} as the i-th term and $(-1)^n d_X^{i+n}$ as the i-th differential.

Let $\mathscr{C}(\mathcal{C})$ and $\mathscr{K}(\mathcal{C})$ be the category of all complexes over $\mathcal{C}$ and the homotopy category of $\mathscr{C}(\mathcal{C})$, respectively. If $\mathcal{C}$ is an abelian category, we write $\mathscr{D}(\mathcal{C})$ for the (unbounded) derived category of $\mathcal{C}$. As usual, let $\mathscr{C}^{\mathrm{b}}(\mathcal{C})$, $\mathscr{K}^{\mathrm{b}}(\mathcal{C})$ and $\mathscr{D}^{\mathrm{b}}(\mathcal{C})$ denote the relevant full subcategories consisting of bounded complexes, respectively; and let $\mathscr{C}^{-}(\mathcal{C})$, $\mathscr{K}^{-}(\mathcal{C})$ and $\mathscr{D}^{-}(\mathcal{C})$ denote the corresponding full subcategories consisting of complexes bounded above. Analogously, $\mathscr{C}^{+}(\mathcal{C})$, $\mathscr{K}^{+}(\mathcal{C})$ and $\mathscr{D}^{+}(\mathcal{C})$ stand for the corresponding full subcategories consisting of complexes bounded below, respectively.

Let Λ be an Artin algebra over a commutative Artin ring. We denote by $\Lambda\text{-mod}$ the category of finitely generated left Λ-modules, and by $\Lambda\text{-proj}$ the full subcategory of $\Lambda\text{-mod}$ consisting of finitely generated projective Λ-modules. For simplicity, we write $\mathscr{C}(\Lambda)$, $\mathscr{K}(\Lambda)$ and $\mathscr{D}(\Lambda)$ for $\mathscr{C}(\Lambda\text{-mod})$, $\mathscr{K}(\Lambda\text{-mod})$ and $\mathscr{D}(\Lambda\text{-mod})$, respectively. Similarly, we have abbreviations $\mathscr{C}^{\mathrm{b}}(\Lambda)$, $\mathscr{K}^{\mathrm{b}}(\Lambda)$ and $\mathscr{D}^{\mathrm{b}}(\Lambda)$. In this paper, $\mathscr{D}^{\mathrm{b}}(\Lambda)$ is called the *derived category* of Λ.

Two Artin algebras Λ and Γ are said to be *derived equivalent* if their derived categories are equivalent as triangulated categories. It follows from Rickard's Morita theory for derived categories [14] that two Artin algebras Λ and Γ are derived equivalent if and only if there is a complex $T^\bullet$ in $\mathscr{K}^{\mathrm{b}}(\Lambda\text{-proj})$ satisfying the following:

(1) $T^\bullet$ is self-orthogonal, that is, $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\Lambda\text{-proj})}(T^\bullet, T^\bullet[n]) = 0$ for all integers $n \neq 0$;

(2) $\mathrm{add}(T^\bullet)$ generates $\mathscr{K}^{\mathrm{b}}(\Lambda\text{-proj})$ as a triangulated category; and

(3) $\Gamma \simeq \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(\Lambda\text{-proj})}(T^\bullet)$.

A complex $T^\bullet$ in $\mathscr{K}^{\mathrm{b}}(\Lambda\text{-proj})$ satisfying (1) and (2) is called a *tilting complex* over Λ. For a tilting complex $T^\bullet$, there exists always a basic, radical tilting complex $T_0^\bullet$ such that $\mathrm{add}(T_0^\bullet) = \mathrm{add}(T^\bullet)$ [7, Section 2, p. 112], and thus $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(\Lambda\text{-proj})}(T_0^\bullet)$ is Morita equivalent to $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(\Lambda\text{-proj})}(T^\bullet)$.

Given a derived equivalence $F : \mathscr{D}^{\mathrm{b}}(\Lambda) \to \mathscr{D}^{\mathrm{b}}(\Gamma)$, there is a unique (up to isomorphism) tilting complex $T^\bullet$ over Λ such that $F(T^\bullet)$ is isomorphic to Γ in $\mathscr{D}^{\mathrm{b}}(\Gamma)$. This complex $T^\bullet$ is called a tilting complex *associated with* F. For a survey of some constructions of derived equivalences, we may refer the reader to [20].

Finally, we recall two operations on complexes, used frequently in the sequel.

Let $X^\bullet = (X^i, d_X^i)_{i\in\mathbb{Z}}$ be a complex in $\mathscr{C}(\Lambda^{\mathrm{op}})$ and $Y^\bullet = (Y^i, d_Y^i)_{i\in\mathbb{Z}}$ a complex in $\mathscr{C}(\Lambda)$. By $X^\bullet \otimes_\Lambda^\bullet Y^\bullet$ we mean the total complex of the double complex with (i,j)-term $X^i \otimes_\Lambda Y^j$. That is, the n-th term of the complex $X^\bullet \otimes_\Lambda^\bullet Y^\bullet$ is

$$\bigoplus_{p+q=n} X^p \otimes_\Lambda Y^q = \bigoplus_{q\in\mathbb{Z}} X^{n-q} \otimes_\Lambda Y^q,$$

and the n-th differential is given by $x \otimes y \mapsto x \otimes (y)d_Y^q + (-1)^q (x)d_X^{n-q} \otimes y$ for $x \in X^{n-q}$ and $y \in Y^q$.

Let $X^\bullet$ and $Y^\bullet$ be two complexes in $\mathscr{C}(\Lambda)$. By $\mathrm{Hom}_\Lambda^\bullet(X^\bullet, Y^\bullet)$ we denote the total complex of the double complex with (i,j)-term $\mathrm{Hom}_\Lambda(X^{-i}, Y^j)$. Thus, the n-th term of the complex $\mathrm{Hom}_\Lambda^\bullet(X^\bullet, Y^\bullet)$ is $\prod_{p\in\mathbb{Z}} \mathrm{Hom}_\Lambda(X^p, Y^{n+p})$, and the n-th differential is given by $(\alpha^p)_{p\in\mathbb{Z}} \mapsto (\alpha^p d_Y^{n+p} - (-1)^n d_X^p \alpha^{p+1})_{p\in\mathbb{Z}}$ for $\alpha^p \in \mathrm{Hom}_\Lambda(X^p, Y^{n+p})$.

2.2. Complexes under change of rings

Let $f : \Lambda \to \Gamma$ be a homomorphism of R-algebras, where R is a commutative ring with identity. Then, the restriction functor ${}_\Lambda(-) : \Gamma\text{-mod} \to \Lambda\text{-mod}$ has a left adjoint functor $\Gamma \otimes_\Lambda - : \Lambda\text{-mod} \to \Gamma\text{-mod}$. The unit of this adjoint pair is the canonical homomorphism of Λ-modules X:

$$f^* : X \to {}_\Lambda\Gamma \otimes_\Lambda X, \quad x \mapsto 1 \otimes x \quad \text{for } x \in X.$$

Lemma 2.1. *Let $f : \Lambda \to \Gamma$ be a homomorphism of Artin algebras.*

(1) *If f is a surjective, then $\Gamma \otimes_\Lambda -$ gives a one-one correspondence between the set of isomorphism classes of indecomposable projective Λ-modules X with $\Gamma \otimes_\Lambda X \neq 0$ and the set of isomorphism classes of indecomposable projective Γ-modules.*

(2) *Let X be a Λ-module and U a Γ-module. Then, we have the following:*

   (i) *If f is surjective, then so is $f^* : X \to \Gamma \otimes_\Lambda X$.*

   (ii) *There is a natural isomorphism $\mathrm{Hom}_\Gamma(\Gamma \otimes_\Lambda X, U) \to \mathrm{Hom}_\Lambda(X, U)$ sending g to $f^* g$.*

Using Lemma 2.1 (ii), we can extend results on modules to complexes. The functor $\Gamma \otimes^{\bullet}_{\Lambda} - : \mathscr{C}(\Lambda) \to \mathscr{C}(\Gamma)$ has the restriction functor as its right adjoint functor. So the unit of this adjoint pair of functors provides a natural chain map $f^* : X^{\bullet} \to \Gamma \otimes^{\bullet}_{\Lambda} X^{\bullet}$ for $X^{\bullet} \in \mathscr{C}(\Lambda)$. More precisely, f^* is defined by $f^i : X^i \to \Gamma \otimes_{\Lambda} X^i$ for all integers i. Now, the following lemma is just a consequence of properties of units of adjoint functors.

Lemma 2.2. *Let $f : \Lambda \to \Gamma$ be a homomorphism of Artin algebras Λ and Γ. Then, for any $X^{\bullet} \in \mathscr{C}(\Lambda)$ and $U^{\bullet} \in \mathscr{C}(\Gamma)$,*

(1) $\mathrm{Hom}_{\mathscr{C}(\Gamma)}(\Gamma \otimes^{\bullet}_{\Lambda} X^{\bullet}, U^{\bullet}) \to \mathrm{Hom}_{\mathscr{C}(\Lambda)}(X^{\bullet}, U^{\bullet})$*, given by $h^{\bullet} \mapsto f^*h^{\bullet}$, is a natural isomorphism.*

(2) $\mathrm{Hom}_{\mathscr{K}(\Gamma)}(\Gamma \otimes^{\bullet}_{\Lambda} X^{\bullet}, U^{\bullet}) \to \mathrm{Hom}_{\mathscr{K}(\Lambda)}(X^{\bullet}, U^{\bullet})$*, given by $h^{\bullet} \mapsto f^*h^{\bullet}$, is a natural isomorphism.*

(3) *If $U^{\bullet} \simeq \Gamma \otimes^{\bullet}_{\Lambda} X^{\bullet}$ in $\mathscr{C}(\Gamma)$, then, for each epimorphism $g^{\bullet} : X^{\bullet} \to {}_{\Lambda}U^{\bullet}$ in $\mathscr{C}(\Lambda)$, there exists an isomorphism $h^{\bullet} : \Gamma \otimes^{\bullet}_{\Lambda} X^{\bullet} \to U^{\bullet}$ in $\mathscr{C}(\Gamma)$ such that $g^{\bullet} = f^*h^{\bullet}$.*

Let $g^{\bullet} : X^{\bullet} \to U^{\bullet}$ be a chain map from $X^{\bullet}$ to $U^{\bullet}$ in $\mathscr{C}(\Lambda)$. If, for each morphism $\alpha^{\bullet} : X^{\bullet} \to X^{\bullet}$ in $\mathscr{K}(\Lambda)$, there is a unique morphism $\beta^{\bullet} : U^{\bullet} \to U^{\bullet}$ in $\mathscr{K}(\Gamma)$ such that $g^{\bullet}\beta^{\bullet} = \alpha^{\bullet}g^{\bullet}$ in $\mathscr{K}(\Lambda)$, then the map

$$\mathrm{End}_{\mathscr{K}(\Lambda)}(X^{\bullet}) \to \mathrm{End}_{\mathscr{K}(\Gamma)}(U^{\bullet})$$

sending $\alpha^{\bullet}$ to $\beta^{\bullet}$ is a homomorphism of algebras, which is called the *algebra homomorphism determined by* $g^{\bullet}$. According to Lemma 2.2 (2), the morphism $f^* : X^{\bullet} \to \Gamma \otimes^{\bullet}_{\Lambda} X^{\bullet}$ determines a homomorphism of algebras:

$$\mathrm{End}_{\mathscr{K}(\Lambda)}(X^{\bullet}) \to \mathrm{End}_{\mathscr{K}(\Gamma)}(\Gamma \otimes^{\bullet}_{\Lambda} X^{\bullet}).$$

By the universal property of units of adjoint functors, the above homomorphism of algebras is actually given by $\alpha \mapsto \Gamma \otimes^{\bullet}_{\Lambda} \alpha$ for $\alpha \in \mathrm{End}_{\mathscr{K}(\Lambda)}(X^{\bullet})$.

2.3. Simple modules under derived equivalences

Let Λ be an Artin algebra and Y an indecomposable Λ-module. For each Λ-module X, we decompose X into a direct sum of indecomposable modules, say $X = \bigoplus_{i=1}^{n} X_i$, and let $[X : Y]$ be the multiplicity of Y as a direct summand of X, that is, the number of those X_j with $X_j \simeq Y$. Note that $[X : Y]$ is independent of the choice of the decomposition of X. For a bounded complex $X^{\bullet} \in \mathscr{C}(\Lambda)$, we define

$$[X^{\bullet} : Y] := \sum_{i \in \mathbb{Z}} [X^i : Y].$$

Note that $[X^{\bullet} : Y]$ is well defined in $\mathscr{C}^{\mathrm{b}}(\Lambda)$ by the Krull–Remak–Schmidt theorem. In the following, we denote by S_P the top of a projective module P.

Lemma 2.3. *If $T^{\bullet} = (T^i, d_T^i)$ is a tilting complex over Λ, then each indecomposable projective Λ-module P occurs as a direct summand of T^m for some integer m, that is, $[T^{\bullet} : P] > 0$.*

Proof. Let $F:\mathscr{D}^{\mathrm{b}}(\Lambda)\rightarrow\mathscr{D}^{\mathrm{b}}(\Gamma)$ be the derived equivalence induced by $T^{\bullet}$. If $[T^{\bullet}:P]=0$, then

$$\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Gamma)}\big(\Gamma, F(S_P)[i]\big)\simeq\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}\big(T^{\bullet}, S_P[i]\big)=0$$

for all $i\in\mathbb{Z}$. It follows that $F(S_P)$ is an acyclic complex and is isomorphic to the zero object in the derived category. This means that the equivalence functor F sends the non-zero object S_P to the zero object, which is impossible. ■

Lemma 2.4. *Let $T^{\bullet}$ be a basic, radical tilting complex over Λ, and let $\Gamma:=\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(T^{\bullet})$. Suppose that $F:\mathscr{D}^{\mathrm{b}}(\Lambda)\rightarrow\mathscr{D}^{\mathrm{b}}(\Gamma)$ is a derived equivalence induced by $T^{\bullet}$ and P is an indecomposable projective Λ-module. Let n be an integer. Then, $F(S_P)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(\Gamma)$ to $S[n]$ for a simple Γ-module S if and only if $[T^{\bullet}:P]=[T^n:P]=1$.*

Proof. Suppose $[T^{\bullet}:P]=[T^n:P]=1$. Then, $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^{\bullet}, S_P[i])=0$ for all $i\neq -n$. Hence, $F(S_P[-n])$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(\Gamma)$ to a Γ-module X. Now, we prove that X is simple. Since $[T^{\bullet}:P]=1$, there is only one indecomposable direct summand $T^{\bullet}_P$ of $T^{\bullet}$ such that P occurs in $T^{\bullet}_P$. Let $\bar{P}$ be the indecomposable projective Γ-module $F(T^{\bullet}_P)$. Then,

$$\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}\big(T^{\bullet}, S_P[-n]\big)\simeq\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}\big(T^{\bullet}_P, S_P[-n]\big),$$

or equivalently $\mathrm{Hom}_{\Gamma}(\Gamma, X)\simeq\mathrm{Hom}_{\Gamma}(\bar{P}, X)$. This means that X only contains composition factors isomorphic to $S_{\bar{P}}$. Moreover,

$$\mathrm{End}_{\Gamma}(X)\simeq\mathrm{End}_{\Lambda}(S_P)$$

is a division algebra. Hence, X must be simple and thus $F(S_P)\simeq X[n]$. Note that we only need $T^{\bullet}$ to be a tilting complex in the foregoing proof.

Conversely, suppose that $F(S_P)\simeq S[n]$ for simple Γ-module S. Then, by assumption, Γ is a basic algebra and S is a 1-dimensional module over $D:=\mathrm{End}_{\Gamma}(S)$. Thus, $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^{\bullet}, S_P[i])$ is zero for all $i\neq -n$, and 1-dimensional over D for $i=-n$. Since $T^{\bullet}$ is a radical complex,

$$\mathrm{Hom}_{\mathscr{K}(\Lambda)}\big(T^{\bullet}, S_P[i]\big)\simeq\mathrm{Hom}_{\Lambda}(T^{-i}, S_P)$$

for all integers i. This implies that P occurs in $T^{\bullet}$ only in degree n with the multiplicity 1. Hence, $[T^{\bullet}:P]=[T^n:P]=1$. ■

An immediate consequence of the proof of Lemma 2.4 is the corollary for tilting modules.

Corollary 2.5. *If $T=P\oplus P'$ is a basic titling Λ-module, where P is projective and P' has a minimal projective resolution $Q^{\bullet}=(Q^i,d^i)_{i\leq 0}$ such that each indecomposable direct summand of P does not appear in $\bigoplus_{i\leq 0}Q^i$, then there exists a derived equivalence $F:\mathscr{D}^{\mathrm{b}}(\Lambda)\rightarrow\mathscr{D}^{\mathrm{b}}(\mathrm{End}_{\Lambda}(T))$ such that $F(S)$ is isomorphic to a simple $\mathrm{End}_{\Lambda}(T)$-module for all simple modules $S\in\mathrm{add}(S_P)$.*

The following is an example of tilting modules satisfying the condition in Corollary 2.5.

Example 2.6. Let $n > 1$ and A be the path algebra of the following quiver:

$$\begin{tikzcd}[row sep=small] 1\bullet & \\ 2\bullet & \bullet n+1 \arrow[ul] \arrow[l] \arrow[ddl] \\ \vdots & \\ n\bullet & \end{tikzcd}$$

The indecomposable projective and simple modules corresponding to the vertex i are denoted by P_i and S_i, respectively. Fix an integer $1 \leq m < n$, set

$$P := P_{m+1} \oplus \cdots \oplus P_n \quad \text{and} \quad P' := P_{n+1} \oplus \bigoplus_{i=1}^{m} (P_{n+1}/S_i).$$

Then, it is easy to check that $T_m := P \oplus P'$ is a tilting module satisfying the assumptions of Corollary 2.5. Actually, for $1 \leq i \leq m$, the simple module $S_i = P_i$ is projective, and P_{n+1}/S_i admits a minimal projective resolution $0 \to P_i \to P_{n+1} \to P_{n+1}/S_i \to 0$. No direct summands of P are involved in the deleted minimal projective resolution of P_{n+1}/S_i.

The following lemma is often used in our later proofs.

Lemma 2.7. *Let $\{U_1, \ldots, U_s, V_1, \ldots, V_r\}$ be a complete set of pairwise non-isomorphic indecomposable projective Λ-modules and let $U := \bigoplus_{i=1}^{s} U_i$. Suppose that $T^\bullet$ is a basic, radical tilting complex over Λ with $[T^\bullet : V_i] = 1$ for all $1 \leq i \leq r$. Then, $T^\bullet \simeq U^\bullet \oplus V_1^\bullet \oplus \cdots \oplus V_r^\bullet$ in $\mathscr{K}^{\mathrm{b}}(\Lambda)$, where $U^\bullet$ and $V_i^\bullet$ are complexes, satisfying the properties:*

(a) *$[V_i^\bullet : V_j] = 1$ for $i = j$, and zero otherwise. Moreover, all $V_i^\bullet$ are indecomposable complexes.*

(b) *$U^\bullet \in \mathscr{K}^{\mathrm{b}}(\mathrm{add}(U))$, and $\mathrm{add}(U^\bullet)$ generates $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(U))$ as a triangulated category.*

Proof. Let $\Gamma := \mathrm{End}_{\mathscr{K}(\Lambda)}(T^\bullet)$ and $F : \mathscr{D}^{\mathrm{b}}(\Lambda) \to \mathscr{D}^{\mathrm{b}}(\Gamma)$ be a derived equivalence induced by the tilting complex $T^\bullet$. By Lemma 2.4, there are pairwise non-isomorphic indecomposable projective Γ-modules $\overline{V}_1, \ldots, \overline{V}_r$ such that $F(\mathrm{top}(V_i)) \simeq \mathrm{top}(\overline{V}_i)[n_i]$ for some n_i with $1 \leq i \leq r$. Let $\overline{U}_1, \ldots, \overline{U}_s$ be indecomposable projective Γ-modules such that

$$\{\overline{U}_1, \ldots, \overline{U}_s, \overline{V}_1, \ldots, \overline{V}_r\}$$

is a complete set of pairwise non-isomorphic indecomposable projective Γ-modules and set $\overline{U} := \bigoplus_{i=1}^{s} \overline{U}_i$. Since $T^\bullet$ is a basic tilting complex, Γ is a basic algebra, and therefore

$${}_\Gamma\Gamma \simeq \overline{U} \oplus \overline{V}_1 \oplus \cdots \oplus \overline{V}_r.$$

By definition, $F(T^\bullet) \simeq {}_\Gamma\Gamma$. Now, let $U^\bullet$ be a direct summand of $T^\bullet$ such that $F(U^\bullet) \simeq \overline{U}$, and let $V_i^\bullet$ be a direct summand of $T^\bullet$ such that $F(V_i^\bullet) \simeq \overline{V}_i$ for $1 \leq i \leq r$. Then, $F(U^\bullet \oplus V_1^\bullet \oplus \cdots \oplus V_r^\bullet) \simeq {}_\Gamma\Gamma \simeq F(T^\bullet)$, and consequently

$$T^\bullet \simeq U^\bullet \oplus V_1^\bullet \oplus \cdots \oplus V_r^\bullet \quad \text{in } \mathscr{D}^{\mathrm{b}}(\Lambda).$$

This implies $T^\bullet \simeq U^\bullet \oplus V_1^\bullet \oplus \cdots \oplus V_r^\bullet$ in $\mathscr{K}^{\rm b}(\Lambda)$. Note that

$$\begin{aligned}\mathrm{Hom}_{\mathscr{K}^{\rm b}(\Lambda)}\big(V_i^\bullet, \mathrm{top}(V_j)[k]\big) &\simeq \mathrm{Hom}_{\mathscr{D}^{\rm b}(\Lambda)}\big(V_i^\bullet, \mathrm{top}(V_j)[k]\big)\\ &\simeq \mathrm{Hom}_{\mathscr{D}^{\rm b}(\Gamma)}\big(\bar{V}_i, \mathrm{top}(\bar{V}_j)[k+n_j]\big) = 0,\end{aligned}$$

whenever $i \neq j$ or $k \neq -n_j$. By assumption, $[T^\bullet : V_i] = 1$ for $1 \le i \le r$. This implies that the projective module V_i only occurs in the $(-n_i)$-th degree of $V_i^\bullet$.

It is easy to see that all complexes $V_i^\bullet$ can be chosen to be indecomposable. This proves (a).

By (a) and $[T^\bullet : V_i] = 1$ for all i, the complex $U^\bullet$ is clearly in $\mathscr{K}^{\rm b}(\mathrm{add}(U))$. Now, we show that F induces a triangle equivalence between $\mathscr{K}^{\rm b}(\mathrm{add}(U))$ and $\mathscr{K}^{\rm b}(\mathrm{add}(\bar{U}))$. In fact, a complex $P^\bullet$ from $\mathscr{K}^{\rm b}(\Lambda\text{-proj})$ lies in $\mathscr{K}^{\rm b}(\mathrm{add}(U))$ if and only if

$$\mathrm{Hom}_{\mathscr{D}^{\rm b}(\Lambda)}\big(P^\bullet, \mathrm{top}(V_i)[k]\big) = 0$$

for all $1 \le i \le r$ and all $k \in \mathbb{Z}$. However, this is equivalent to

$$\mathrm{Hom}_{\mathscr{D}^{\rm b}(\Gamma)}\big(F(P^\bullet), \mathrm{top}(\bar{V}_i)[k+n_i]\big) = 0$$

for all $1 \le i \le r$ and all $k \in \mathbb{Z}$; that is, $F(P^\bullet)$ belongs to $\mathscr{K}^{\rm b}(\mathrm{add}(\bar{U}))$. Hence, F induces a triangle equivalence between $\mathscr{K}^{\rm b}(\mathrm{add}(U))$ and $\mathscr{K}^{\rm b}(\mathrm{add}(\bar{U}))$. Since $\mathrm{add}(\bar{U})$ generates $\mathscr{K}^{\rm b}(\mathrm{add}(\bar{U}))$ as a triangulated category, $\mathrm{add}(U^\bullet)$ generates $\mathscr{K}^{\rm b}(\mathrm{add}(U))$ as a triangulated category. This proves (b). ∎

2.4. Projective modules over Milnor squares of algebras

Let A_0, A_1 and A_2 be rings with identity. Given two homomorphisms $\pi_i : A_i \to A_0$ of rings, the *pullback ring* A of π_1 and π_2 is defined by

$$A := \big\{(x, y) \in A_1 \times A_2 \mid (x)\pi_1 = (y)\pi_2\big\}.$$

Transparently, there is a commutative diagram of ring homomorphisms

$$\begin{tikzcd} A \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"'] & A_1 \arrow[d, "\pi_1"] \\ A_2 \arrow[r, "\pi_2"] & A_0 \end{tikzcd}$$

where λ_i is the canonical projections from A to A_i for $i = 1, 2$. The pullback diagram has a universal property: For any ring homomorphisms $i_1 : B \to A_1$ and $i_2 : B \to A_2$ with $i_1\pi_1 = i_2\pi_2$, there is a unique ring homomorphism $\theta : B \to A$ such that $\theta\lambda_j = i_j$ for $j = 1, 2$. Note that if π_1 is surjective, then so is λ_2.

If one of π_1 and π_2 is surjective, then the pullback diagram is called a *Milnor square* of rings. For a Milnor square of rings, there is a nice description of projective A-modules via projective A_i-modules in [13], which was successfully used in algebraic K-theory, ring theory and representation theory (see [2, 13]) We recall this description right now.

Given a projective A_1-module X_1, a projective A_2-module X_2 and an isomorphism $h : A_0 \otimes_{A_1} X_1 \to A_0 \otimes_{A_2} X_2$ of A_0-modules, the *Milnor patching* of the triple (X_1, X_2, h) is defined by

$$\begin{aligned} M(X_1, X_2, h) &:= \{(x_1, x_2) \in X_1 \oplus X_2 \mid (x_1)\pi_1^* h = (x_2)\pi_2^*\} \\ &= \{(x_1, x_2) \in X_1 \oplus X_2 \mid (1 \otimes x_1)h = 1 \otimes x_2\} \end{aligned}$$

(see [2, Chapter 8 (C), p. 303] for terminology). Note that $M(X_1, X_2, h)$ has an A-module structure:

$$a \cdot (x_1, x_2) = \big((a)\lambda_1 \cdot x_1, (a)\lambda_2 \cdot x_2\big) \quad \text{for } a \in A,\ x_1 \in X_1,\ x_2 \in X_2.$$

Let $p_i : M(X_1, X_2, h) \to X_i$ be the canonical projection. The following description of projective A-modules was given in [13, Chapter 2].

Lemma 2.8. *Suppose that π_1 is surjective, X_i is a projective A_i-module for $i = 1, 2$, and $h : A_0 \otimes_{A_1} X_1 \to A_0 \otimes_{A_2} X_2$ is an isomorphism of A_0-modules. Then, we have the following:*

(1) *The module $M(X_1, X_2, h)$ is a projective A-module. Furthermore, if, in addition, X_1 and X_2 are finitely generated over A_1 and A_2, respectively, then $M(X_1, X_2, h)$ is finitely generated over A.*

(2) *Every projective A-module is isomorphic to $M(X_1, X_2, h)$ for some suitably chosen X_1, X_2 and h.*

(3) *For $i \in \{1, 2\}$, there is a natural isomorphism $\mu_i : A_i \otimes_A M(X_1, X_2, h) \to X_i$ sending $a_i \otimes (x_1, x_2)$ to $a_i x_i$, and the canonical projection $p_i : M(X_1, X_2, h) \to X_i$ is equal to $\lambda_i^* \mu_i$.*

(4) *There is an exact sequence of A-modules:*

$$0 \to M(X_1, X_2, h) \xrightarrow{[p_1, p_2]} X_1 \oplus X_2 \xrightarrow{\left[\begin{smallmatrix} \pi_1^* h \\ -\pi_2^* \end{smallmatrix}\right]} A_0 \otimes_{A_2} X_2 \to 0.$$

For the rest of this section, we shall assume that A_0, A_1 and A_2 are Artin algebras and π_1 is surjective. Thus, we have an exact sequence of A-bimodules:

$$0 \to A \xrightarrow{[\lambda_1, \lambda_2]} A_1 \oplus A_2 \xrightarrow{\left[\begin{smallmatrix} \pi_1 \\ -\pi_2 \end{smallmatrix}\right]} A_0 \to 0. \tag{$*$}$$

Let P_1 be the direct sum of all non-isomorphic indecomposable projective A_1-modules X such that $A_0 \otimes_{A_1} X = 0$, and let Q_1 be a direct sum of all non-isomorphic indecomposable projective A_1-modules Y such that $A_0 \otimes_{A_1} Y \neq 0$. Thus, $A_1\text{-proj} = \mathrm{add}(P_1 \oplus Q_1)$. Similarly, we define projective A_2-modules P_2 and Q_2, such that $A_2\text{-proj} = \mathrm{add}(P_2 \oplus Q_2)$.

Since π_1 is surjective, λ_2 is also surjective. Therefore, if X is an indecomposable projective A-module with $A_2 \otimes_A X \neq 0$, then $A_2 \otimes_A X$ is an indecomposable projective A_2-module by Lemma 2.1 (1). Hence, for an indecomposable projective A-module X, only the three cases occur:

- Case 1: $A_2 \otimes_A X = 0$.

- Case 2: $0 \neq A_2 \otimes_A X \in \operatorname{add}(P_2)$.
- Case 3: $0 \neq A_2 \otimes_A X \in \operatorname{add}(Q_2)$.

According to the three cases, we have a partition of indecomposable projective A-modules: For $1 \leq i \leq 3$, let F_i be the direct sum of all non-isomorphic indecomposable projective A-modules X corresponding to Case i. Then, A-proj $= \operatorname{add}(F_1 \oplus F_2 \oplus F_3)$.

Lemma 2.9. *The following properties hold for P_i:*

(1) *The functor $A_i \otimes_A -$ and the restriction functor ${}_A(-)$ induce mutually inverse equivalences between $\operatorname{add}(F_i)$ and $\operatorname{add}(P_i)$ for $i = 1, 2$.*

(2) *Let $i \in \{1, 2\}$ and $X \in \operatorname{add}(P_i)$. Then, the natural map*

$$\operatorname{Hom}_A({}_AX, A) \to \operatorname{Hom}_{A_i}(X, A_i),$$

sending α to $\alpha\lambda_i$, is an isomorphism of right A-modules.

(3) *Let $i \in \{1, 2\}$ and $X \in \operatorname{add}(P_i)$. If $\operatorname{add}({}_{A_i}X) = \operatorname{add}(\nu_{A_i}X)$, then $\operatorname{add}({}_AX) = \operatorname{add}(\nu_AX)$, where ν_A is the Nakayama functor $D\operatorname{Hom}_A(-, {}_AA)$ of A.*

Proof. (1) We prove the case $i = 1$. For X in $\operatorname{add}(F_1)$, we have $A_2 \otimes_A X = 0$, and therefore $A_0 \otimes_{A_1} A_1 \otimes_A X \simeq A_0 \otimes_{A_2} A_2 \otimes_A X = 0$ and $A_1 \otimes_A X \in \operatorname{add}(P_1)$. Thus, $X \simeq M(A_1 \otimes_A X, 0, 0)$ and the map $\lambda_1^* : X \to A_1 \otimes_A X$ is a bijection by the definition of $M(A_1 \otimes_A X, 0, 0)$. It follows from (1) and (3) in Lemma 2.8 that, for X and Y in $\operatorname{add}(F_1)$, the functor $A_1 \otimes_A -$ induces an isomorphism:

$$\operatorname{Hom}_A(X, Y) \simeq \operatorname{Hom}_{A_1}(A_1 \otimes_A X, A_1 \otimes_A Y).$$

Moreover, for $U \in \operatorname{add}(P_1)$, the module $M(U, 0, 0)$ lies in $\operatorname{add}(F_1)$ with $A_1 \otimes_A M(U, 0, 0) \simeq U$. Thus, $A_1 \otimes_A - : \operatorname{add}(F_1) \to \operatorname{add}(P_1)$ is an equivalence. Clearly, the restriction functor ${}_A(-)$ is right adjoint to $A_1 \otimes_A -$ by Lemma 2.1 (2), and therefore a quasi-inverse of $A_1 \otimes_A -$. This proves (1) for $i = 1$. Similarly, we prove the case $i = 2$.

(2) Assume $i = 1$ and $X \in \operatorname{add}(P_1)$. By Lemma 2.1,

$$\operatorname{Hom}_A({}_AX, A_i) \simeq \operatorname{Hom}_{A_i}(A_i \otimes_A X, A_i) \quad \text{for } 0 \leq i \leq 2.$$

It follows from $X \in \operatorname{add}(P_1)$ that ${}_AX \in \operatorname{add}(F_1)$ by (1). Consequently, $A_2 \otimes_A X = 0$ and $A_0 \otimes_A X \simeq A_0 \otimes_{A_2} A_2 \otimes_A X = 0$. Therefore, $\operatorname{Hom}_A({}_AX, A_0) = 0 = \operatorname{Hom}_A({}_AX, A_2)$. Applying $\operatorname{Hom}_A({}_AX, -)$ to $(*)$, we get an isomorphism of right A-modules:

$$\operatorname{Hom}_A({}_AX, A) \to \operatorname{Hom}_A({}_AX, {}_AA_1),$$

which sends α to $\alpha\lambda_1$. Similarly, we demonstrate the case $i = 2$.

(3) Without loss of generality, we assume that the module X is basic. Then, it follows from $\operatorname{add}({}_{A_i}X) = \operatorname{add}(\nu_{A_i}X)$ that $\nu_{A_i}X \simeq X$. This together with (2) implies the isomorphisms: $\nu_AX = D\operatorname{Hom}_A({}_AX, A) \simeq D(\operatorname{Hom}_{A_i}(X, A_i)_A) = {}_A(\nu_{A_i}X) \simeq {}_AX$. Thus, (3) follows. ∎

The next lemma describes indecomposable projective A-modules in $\mathrm{add}(F_3)$.

Lemma 2.10. *The following statements hold.*

(1) *For each indecomposable A_2-module V in $\mathrm{add}(Q_2)$, there is an A_1-module W (unique up to isomorphism) in $\mathrm{add}(Q_1)$ with an isomorphism $h : A_0 \otimes_{A_1} W \to A_0 \otimes_{A_2} V$ such that $M(W, V, h)$ is an indecomposable projective A-module in $\mathrm{add}(F_3)$.*

(2) *Let $\{V_1, \dots, V_s\}$ be a complete set of pairwise non-isomorphic indecomposable projective A_2-modules in $\mathrm{add}(Q_2)$, and let $W_i \in \mathrm{add}(Q_1)$ be the projective A_1-module determined by V_i in (1) for $1 \le i \le s$. Then, $\{M(W_i, V_i, h_i) \mid 1 \le i \le s\}$ is a complete set of pairwise non-isomorphic indecomposable projective A-modules in $\mathrm{add}(F_3)$.*

Proof. (1) Since π_1 is surjective, it follows from Lemma 2.1 (1) that there is an A_1-module W (unique up to isomorphism) and an isomorphism $h : A_0 \otimes_{A_1} W \to A_0 \otimes_{A_2} V$. We need to show that $M(W, V, h)$ is in $\mathrm{add}(F_3)$. Let X be an indecomposable direct summand of $M(W, V, h)$. Then, there are two possibilities: $A_2 \otimes_A X \neq 0$ or $A_2 \otimes_A X = 0$. If $A_2 \otimes_A X \neq 0$, then $A_2 \otimes_A X$ is a direct summand of V. Since V is indecomposable, $A_2 \otimes_A X \simeq V$. By definition, $X \in \mathrm{add}(F_3)$. Now, we exclude the case $A_2 \otimes_A X = 0$. If this happens, then $A_1 \otimes_A X \neq 0$. Otherwise, $X \simeq M(A_1 \otimes_A X, A_2 \otimes_A X, g) = 0$. So $A_1 \otimes_A X$ is a nonzero direct summand of W. However, $X \in \mathrm{add}(F_1)$ by definition. It follows from Lemma 2.9 (1) that $A_1 \otimes_A X$ lies in $\mathrm{add}(P_1)$. This is a contradiction. Thus, $M(W, V, h) \in \mathrm{add}(F_3)$. Since $A_2 \otimes_A M(W, V, h) \simeq V$ is indecomposable, the module $M(W, V, h)$ is indecomposable by Lemma 2.1 (1).

(2) It follows from (1) that $M(W_i, V_i, h_i) \in \mathrm{add}(F_3)$ is indecomposable for all $1 \le i \le s$. Now, let X be an indecomposable A-module in $\mathrm{add}(F_3)$. Then, the A_2-module $A_2 \otimes_A X$ is indecomposable since λ_2 is surjective. Thus, there is some V_i such that

$$A_2 \otimes_A X \simeq V_i \simeq A_2 \otimes_A M(W_i, V_i, h_i).$$

By Lemma 2.1 (1), $X \simeq M(W_i, V_i, h_i)$. ∎

Given a complex $X_1^\bullet$ in $\mathscr{C}^{\mathrm{b}}(A_1\text{-proj})$ and a complex $X_2^\bullet$ in $\mathscr{C}^{\mathrm{b}}(A_2\text{-proj})$ together with an isomorphism $h^\bullet : A_0 \otimes_{A_1}^\bullet X_1^\bullet \to A_0 \otimes_{A_2}^\bullet X_2^\bullet$ in $\mathscr{C}(A_0)$, we define a complex $M(X_1^\bullet, X_2^\bullet, h^\bullet) := (M(X_1^i, X_2^i, h^i), d^i)_{i \in \mathbb{Z}}$, where the differential is induced by the exact sequence given in Lemma 2.8 (4).

Lemma 2.11. *Suppose $X_1^\bullet \in \mathscr{C}^{\mathrm{b}}(A_1\text{-proj})$, $X_2^\bullet \in \mathscr{C}^{\mathrm{b}}(A_2\text{-proj})$ and $h^\bullet : A_0 \otimes_{A_1}^\bullet X_1^\bullet \to A_0 \otimes_{A_2}^\bullet X_2^\bullet$ is an isomorphism in $\mathscr{C}(A_0)$.*

(1) *$M(X_1^\bullet, X_2^\bullet, h^\bullet)$ is a bounded complex over A-proj.*

(2) *For $i \in \{1, 2\}$, there is a natural isomorphism $\mu_i^\bullet : A_i \otimes_A^\bullet M(X_1^\bullet, X_2^\bullet, h^\bullet) \to X_i^\bullet$ of complexes, sending $a_i \otimes (x_1^j, x_2^j)$ to $a_i x_i^j$, and the canonical projection $p_i^\bullet : M(X_1^\bullet, X_2^\bullet, h^\bullet) \to X_i^\bullet$ is equal to $\lambda_i^* \mu_i^\bullet$.*

(3) *There is an exact sequence of complexes of A-modules:*

$$0 \to M(X_1^\bullet, X_2^\bullet, h^\bullet) \xrightarrow{[p_1^\bullet, p_2^\bullet]} X_1^\bullet \oplus X_2^\bullet \xrightarrow{\left[\begin{smallmatrix} \pi_1^* h^\bullet \\ -\pi_2^* \end{smallmatrix}\right]} A_0 \otimes_{A_2}^\bullet X_2^\bullet \to 0,$$

where $p_i^\bullet$ is induced by the canonical projection p_i for $i = 1, 2$.

(4) *Set $X^\bullet := M(X_1^\bullet, X_2^\bullet, h^\bullet)$ and $X_0^\bullet := A_0 \otimes_{A_2}^\bullet X_2^\bullet$. If $\mathrm{Hom}_{\mathscr{K}(A_0)}(X_0^\bullet, X_0^\bullet[-1]) = 0$, then there exists a pullback diagram of algebras:*

$$\begin{tikzcd}
\mathrm{End}_{\mathscr{K}(A)}(X^\bullet) \arrow[r, "\varepsilon_1"] \arrow[d, "\varepsilon_2"'] & \mathrm{End}_{\mathscr{K}(A_1)}(X_1^\bullet) \arrow[d, "\eta_1"] \\
\mathrm{End}_{\mathscr{K}(A_2)}(X_2^\bullet) \arrow[r, "\eta_2"] & \mathrm{End}_{\mathscr{K}(A_0)}(X_0^\bullet),
\end{tikzcd}$$

where $\varepsilon_1, \varepsilon_2, \eta_1$ and η_2 are homomorphisms of algebras, determined by $p_1^\bullet$, $p_2^\bullet$, $\pi_1^* h^\bullet$ and π_2^*, respectively.

Proof. The statements (1)–(3) follow immediately from the definition of $M(X_1^\bullet, X_2^\bullet, h^\bullet)$ and Lemma 2.8 (1)–(3). Now, we prove (4). Since $X^\bullet \in \mathscr{K}^{\mathrm{b}}(A\text{-proj})$, it follows from the triangle

$$X^\bullet \xrightarrow{[p_1^\bullet, p_2^\bullet]} X_1^\bullet \oplus X_2^\bullet \xrightarrow{\left[\begin{smallmatrix} \pi_1^* h^\bullet \\ -\pi_2^* \end{smallmatrix}\right]} X_0^\bullet \to X^\bullet[1]$$

in $\mathscr{D}^{\mathrm{b}}(A)$ that the long sequence

$$\begin{aligned} \cdots &\to \mathrm{Hom}_{\mathscr{K}(A)}\left(X^\bullet, X_0^\bullet[-1]\right) \to \mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, X^\bullet) \\ &\to \mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, X_1^\bullet \oplus X_2^\bullet) \to \mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, X_0^\bullet) \to \cdots \end{aligned}$$

is exact. Since

$$\mathrm{Hom}_{\mathscr{K}(A)}\left(X^\bullet, X_i^\bullet[j]\right) \simeq \mathrm{Hom}_{\mathscr{K}(A_i)}\left(A_i \otimes_A^\bullet X^\bullet, X_i^\bullet[j]\right) \simeq \mathrm{Hom}_{\mathscr{K}(A_i)}\left(X_i^\bullet, X_i^\bullet[j]\right)$$

for $j \in \mathbb{Z}$, the assumption in (4) implies $\mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, X_0^\bullet[-1]) = 0$. Then, the above sequence is isomorphic to

$$0 \to \mathrm{End}_{\mathscr{K}(A)}(X^\bullet) \xrightarrow{[\varepsilon_1, \varepsilon_2]} \mathrm{End}_{\mathscr{K}(A_1)}(X_1^\bullet) \oplus \mathrm{End}_{\mathscr{K}(A_2)}(X_2^\bullet) \xrightarrow{\left[\begin{smallmatrix} \eta_1 \\ -\eta_2 \end{smallmatrix}\right]} \mathrm{End}_{\mathscr{K}(A_0)}(X_0^\bullet). \quad \blacksquare$$

3. Derived equivalences by Milnor patching

In this section, we prove all results in the introduction. We first establish the derived equivalence for pullback algebras in Theorem 1.1. Then, we consider when the derived equivalence in Theorem 1.1 is almost ν-stable and thus prove Theorem 1.2.

3.1. Proof of Theorem 1.1

We first show the lemma.

Lemma 3.1. *Let $f : \Lambda \to \Gamma$ be a surjective homomorphism between Artin algebras. If $T^\bullet$ is a basic, radical tilting complex over Λ such that $\Gamma \otimes^\bullet_\Lambda T^\bullet$ is a basic tilting complex over Γ of the form $\bigoplus_{i=1}^r X_i[n_i]$, where $\{X_1, \ldots, X_r\}$ is a complete set of non-isomorphic indecomposable projective Γ-modules, then the induced morphism $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, f^*)$: $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, T^\bullet) \to \mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, {}_\Lambda\Gamma \otimes^\bullet_\Lambda T^\bullet)$ is surjective.*

Proof. By Lemma 2.1 (1), we can assume that there is a complete set $\{V_1, \ldots, V_r, U_1, \ldots, U_s\}$ of pairwise non-isomorphic indecomposable projective Λ-modules such that $\Gamma \otimes_\Lambda V_i \simeq X_i$ for all $i = 1, \ldots, r$, and that $\Gamma \otimes_\Lambda U_i = 0$ for all $i = 1, \ldots, s$. Set $U := \bigoplus_{i=1}^s U_i$. By our assumption, $[\Gamma \otimes_\Lambda T^\bullet : X_i] = 1$ for all $1 \le i \le r$. This implies $[T^\bullet : V_i] = 1$ for $1 \le i \le r$. So, by Lemma 2.7, we can write $T^\bullet$ as

$$T^\bullet := U^\bullet \oplus V_1^\bullet \oplus \cdots \oplus V_r^\bullet$$

such that $U^\bullet \in \mathscr{K}^{\mathrm{b}}(\mathrm{add}(U))$, and $[V_i^\bullet : V_j] = 1$ for $i = j$ and 0 otherwise. Thus, $\Gamma \otimes_\Lambda U^\bullet = 0$ and $\Gamma \otimes^\bullet_\Lambda V_i^\bullet \simeq (\Gamma \otimes_\Lambda V_i)[n_i] \simeq X_i[n_i]$ for some integer n_i. To prove Lemma 3.1, it is sufficient to prove that

$$\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, f^*) : \mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, V_i^\bullet) \to \mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, {}_\Lambda\Gamma \otimes^\bullet_\Lambda V_i^\bullet)$$

is surjective for all $1 \le i \le r$.

We set $\Sigma := \mathrm{End}_{\mathscr{K}(\Lambda)}(T^\bullet)$. Since

$$\begin{aligned}
&\mathrm{Hom}_\Sigma\left(\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, U^\bullet), \mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, \Gamma \otimes^\bullet_\Lambda V_i^\bullet)\right)\\
&\quad\simeq \mathrm{Hom}_{\mathscr{K}(\Lambda)}\left(U^\bullet, \Gamma \otimes^\bullet_\Lambda V_i^\bullet\right)\\
&\quad\simeq \mathrm{Hom}_{\mathscr{K}(\Gamma)}(\Gamma \otimes^\bullet_\Lambda U^\bullet, \Gamma \otimes^\bullet_\Lambda V_i^\bullet) = 0,
\end{aligned}$$

we deduce that the Σ-module $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, \Gamma \otimes^\bullet_\Lambda V_i^\bullet)$ does not have composition factors in $\mathrm{add}(\mathrm{top}(\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, U^\bullet)))$.

Now, for $1 \le k \le r$, let S_k denote the top of V_k, and let $\overline{S}_k$ be the top of the Σ-module $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, V_k^\bullet)$. Let $G : \mathscr{D}^{\mathrm{b}}(\Lambda) \to \mathscr{D}^{\mathrm{b}}(\Sigma)$ be the derived equivalence induced by $T^\bullet$. Then, $G(S_k) \simeq \overline{S}_k[-n_k]$ for $1 \le k \le r$ by the proof of Lemma 2.4. Since $\Gamma \otimes^\bullet_\Lambda T^\bullet$ is a tilting complex over Γ, we have

$$\mathrm{Hom}_{\mathscr{K}(\Lambda)}\left(T^\bullet, (\Gamma \otimes^\bullet_\Lambda V_k^\bullet)[n]\right) \simeq \mathrm{Hom}_{\mathscr{K}(\Gamma)}\left(\Gamma \otimes^\bullet_\Lambda T^\bullet, (\Gamma \otimes^\bullet_\Lambda V_k^\bullet)[n]\right) = 0$$

for all $1 \le k \le r$ and $n \ne 0$, and $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, \Gamma \otimes^\bullet_\Lambda V_k^\bullet) \simeq G(\Gamma \otimes^\bullet_\Lambda V_k^\bullet)$ for all $1 \le k \le r$. Hence,

$$\begin{aligned}
\mathrm{Hom}_\Sigma\left(\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, \Gamma \otimes^\bullet_\Lambda V_i^\bullet), \overline{S}_k\right) &\simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Sigma)}\left(G(\Gamma \otimes^\bullet_\Lambda V_i^\bullet), G\left(S_k[n_k]\right)\right)\\
&\simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}\left(\Gamma \otimes^\bullet_\Lambda V_i^\bullet, S_k[n_k]\right)\\
&\simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}\left(X_i[n_i], S_k[n_k]\right)
\end{aligned}$$

is 0 for all $k \neq i$ and is 1-dimensional over $\mathrm{End}_\Lambda(S_k)$ for $k = i$. Thus, the top of the Σ-module $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, \Gamma \otimes^\bullet_\Lambda V_i^\bullet)$ is $\overline{S}_i$, and there is a projective cover

$$\varepsilon : \mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, V_i^\bullet) \to \mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, \Gamma \otimes^\bullet_\Lambda V_i^\bullet).$$

Clearly, this surjective homomorphism is given by $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, g^\bullet)$ for some morphism $g^\bullet : V_i^\bullet \to \Gamma \otimes^\bullet_\Lambda V_i^\bullet$. By Lemma 2.2 (2), there is a morphism $u^\bullet : \Gamma \otimes^\bullet_\Lambda V_i^\bullet \to \Gamma \otimes^\bullet_\Lambda V_i^\bullet$, such that $g^\bullet = f^* u^\bullet$. This yields

$$\varepsilon = \mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, f^*) \cdot \mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, u^\bullet).$$

Hence, the endomorphism $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, u^\bullet)$ of the Σ-module $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, \Gamma \otimes^\bullet_\Lambda V_i^\bullet)$ is surjective, and therefore an isomorphism. Consequently, $\mathrm{Hom}_{\mathscr{K}(\Lambda)}(T^\bullet, f^*)$ is surjective. ■

Lemma 3.2. *Keep the assumptions in Theorem 1.1. Then,* $\mathrm{add}(M(T_1^\bullet, T_2^\bullet, h^\bullet))$ *generates* $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ *as a triangulated category.*

Proof. By the assumptions of Theorem 1.1, the tilting complex $T_0^\bullet$ is of the form $T_0^\bullet = \bigoplus_{i=1}^m U_i[n_i]$ with U_i projective A_0-modules such that $n_i \neq n_j$ whenever $i \neq j$. Thus,

$$\mathrm{Hom}_{\mathscr{K}(A_0)}\big(U_i[n_i], U_j[n_j]\big) = 0$$

for all $i \neq j$ and $\bigoplus_{i=1}^m U_i$ is a basic, projective generator for $A_0\text{-mod}$.

Recall from Section 2.4 that

$$A_j\text{-proj} = \mathrm{add}(P_j \oplus Q_j) \quad \text{for } j = 1, 2,$$

where $A_0 \otimes_{A_j} P_j = 0$ and $A_0 \otimes_{A_j} Y \neq 0$ for each indecomposable direct summand Y of Q_j. Let $\{V_1, \ldots, V_r\}$ and $\{W_1, \ldots, W_s\}$ be complete sets of pairwise non-isomorphic indecomposable projective modules in $\mathrm{add}(Q_1)$ and $\mathrm{add}(Q_2)$, respectively. Since $A_0 \otimes^\bullet_{A_i} T_i^\bullet \simeq T_0^\bullet$ in $\mathscr{C}(A_0)$ for $i = 1, 2$, and since each indecomposable projective A_0-module occurs in $T_0^\bullet$ only once, there holds $[T_1^\bullet : V_i] = 1 = [T_2^\bullet : W_j]$ for all i, j. By Lemma 2.7, we can write

$$T_1^\bullet = P_1^\bullet \oplus V_1^\bullet \oplus \cdots \oplus V_r^\bullet \quad \text{and} \quad T_2^\bullet = P_2^\bullet \oplus W_1^\bullet \oplus \cdots \oplus W_s^\bullet,$$

such that

(1) $P_i^\bullet \in \mathscr{K}^{\mathrm{b}}(\mathrm{add}(P_i))$, and $\mathrm{add}(P_i^\bullet)$ generates $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(P_i))$ as a triangulated category for $i = 1, 2$, and

(2) $[V_i^\bullet : V_j] = \delta_{ij}$ and $[W_k^\bullet : W_l] = \delta_{kl}$, where δ_{ij} is the Kronecker symbol.

Note that $A_0 \otimes_{A_1} P_1 = 0$ and $A_0 \otimes^\bullet_{A_1} V_i^\bullet = (A_0 \otimes_{A_1} V_i)[n_{V_i}]$ for some integer n_{V_i} with $1 \leq i \leq r$. By assumption, we have an isomorphism of complexes:

$$\bigoplus_{i=1}^{r}(A_0 \otimes_{A_1} V_i)[n_{V_i}] \simeq \bigoplus_{i=1}^{m} U_i[n_i]. \qquad (**)$$

This gives rise to a partition $\sigma = \{\sigma_1, \dots, \sigma_m\}$ of $\{1, \dots, r\}$ with $\sigma_i := \{j \mid n_{V_j} = n_i\}$. Now we define

$$V_{\sigma_i} := \bigoplus_{j \in \sigma_i} V_j, \quad \text{and} \quad V^\bullet_{\sigma_i} := \bigoplus_{j \in \sigma_i} V^\bullet_j$$

for all $1 \le i \le m$. Then, $A_0 \otimes^\bullet_{A_1} V^\bullet_{\sigma_i} \simeq U_i[n_i]$ for $1 \le i \le m$. This partition means that we collect terms of the left-hand side of ($**$) according to the position n_i of terms in $T^\bullet_0$. Thus,

$$A_0 \otimes^\bullet_{A_1} T^\bullet_1 = (A_0 \otimes^\bullet_{A_1} P^\bullet_1) \oplus \Big(\bigoplus_{i=1}^r A_0 \otimes^\bullet_{A_1} V^\bullet_i\Big) = \bigoplus_{i=1}^m A_0 \otimes^\bullet_{A_1} V^\bullet_{\sigma_i} \simeq \bigoplus_{i=1}^m U_i[n_i].$$

By repeating the above procedure, we get a partition $\tau := \{\tau_1, \dots, \tau_m\}$ of $\{1, \dots, s\}$ with

$$\tau_i := \big\{k \in \{1, \dots, s\} \mid A_0 \otimes^\bullet_{A_2} W^\bullet_k \simeq (A_0 \otimes_{A_2} W_k)[n_{W_k}] \text{ and } n_{W_k} = n_i\big\}.$$

Define $W_{\tau_i} := \bigoplus_{k \in \tau_i} W_k$, and $W^\bullet_{\tau_i} := \bigoplus_{k \in \tau_i} W^\bullet_k$. Then, $A_0 \otimes^\bullet_{A_2} W^\bullet_{\tau_i} \simeq U_i[n_i]$ for $1 \le i \le m$, and

$$A_0 \otimes^\bullet_{A_2} T^\bullet_2 = (A_0 \otimes^\bullet_{A_2} P^\bullet_2) \oplus \Big(\bigoplus_{i=1}^s A_0 \otimes^\bullet_{A_2} W^\bullet_i\Big) = \bigoplus_{i=1}^m A_0 \otimes^\bullet_{A_2} W^\bullet_{\tau_i} \simeq \bigoplus_{i=1}^m U_i[n_i].$$

Since $\mathrm{Hom}(U_i[n_i], U_j[n_j]) = 0$ for all $i \ne j$, the isomorphism

$$h^\bullet : A_0 \otimes^\bullet_{A_1} T^\bullet_1 \to A_0 \otimes^\bullet_{A_2} T^\bullet_2$$

can be rewritten as

$$\mathrm{diag}[h^\bullet_1, \dots, h^\bullet_m] : \bigoplus_{i=1}^m (A_0 \otimes^\bullet_{A_1} V^\bullet_{\sigma_i}) \to \bigoplus_{i=1}^m A_0 \otimes^\bullet_{A_2} W^\bullet_{\tau_i},$$

where $h^\bullet_i : A_0 \otimes^\bullet_{A_1} V^\bullet_{\sigma_i} \to A_0 \otimes^\bullet_{A_2} W^\bullet_{\tau_i}$ is an isomorphism in $\mathscr{C}(A_0)$ for all i.

For simplicity, we write $T^\bullet$ for $M(T^\bullet_1, T^\bullet_2, h^\bullet)$, and write $Z^\bullet_i$ for $M(V^\bullet_{\sigma_i}, W^\bullet_{\tau_i}, h^\bullet_i)$ for $1 \le i \le m$. Thus, for each integer k, $Z^k_i = M(V^k_{\sigma_i}, W^k_{\tau_i}, h^k_i)$. For $k \ne -n_i$, the term $V^k_{\sigma_i}$ is in $\mathrm{add}(P_1)$, and the term $W^k_{\tau_i}$ is in $\mathrm{add}(P_2)$. Hence, $A_0 \otimes_{A_1} V^k_{\sigma_i} = 0 = A_0 \otimes_{A_2} W^k_{\tau_i}$, and $Z^k_i \simeq {}_A V^k_{\sigma_i} \oplus {}_A W^k_{\tau_i} \in \mathrm{add}(F_1 \oplus F_2)$ for all $k \ne -n_i$. Since V_{σ_i} is a direct summand of $V^{-n_i}_{\sigma_i}$ and since W_{τ_i} is a direct summand of $W^{-n_i}_{\tau_i}$, $M(V_{\sigma_i}, W_{\tau_i}, h^{-n_i}_i)$ is a direct summand of $Z^{-n_i}_i$. By Lemma 2.9 (1), the functor ${}_A(-) : \mathrm{add}(P_1) \to \mathrm{add}(F_1)$ is an equivalence and consequently induces a triangle equivalence $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(P_1)) \to \mathscr{K}^{\mathrm{b}}(\mathrm{add}(F_1))$. Since $\mathrm{add}(P^\bullet_1)$ generates $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(P_1))$ as a triangulated category, $\mathrm{add}(M(P^\bullet_1, 0, 0)) = \mathrm{add}({}_A P^\bullet_1)$ generates $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(F_1))$ as a triangulated category. Similarly, $\mathrm{add}(M(0, P^\bullet_2, 0))$ generates $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(F_2))$ as a triangulated category. Let $\mathscr{X}$ be the full subcategory of $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ generated by $\mathrm{add}(T^\bullet)$. Then, $F_1 \oplus F_2 \in \mathscr{X}$. As all terms Z^k_i with $k \ne -n_i$ are in $\mathrm{add}(F_1 \oplus F_2)$, the term $Z^{-n_i}_i \in \mathscr{X}$. Thus, the module $F_1 \oplus F_2 \oplus (\bigoplus_{i=1}^m Z^{-n_i}_i) \in \mathscr{X}$. By Lemma 2.10 (2), the direct sum

$$\bigoplus_{i=1}^m M(V_{\sigma_i}, W_{\tau_i}, h^{-n_i}_i)$$

is a basic, additive generator of F_3. Recall that $M(V_{\sigma_i}, W_{\tau_i}, h_i^{-n_i})$ is a direct summand of $Z_i^{-n_i}$ for all $1 \le i \le m$. It follows that $F_1 \oplus F_2 \oplus F_3 \in \mathscr{X}$. As $F_1 \oplus F_2 \oplus F_3$ is an additive generator of A-proj, $\mathrm{add}(T^\bullet)$ generates $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ as a triangulated category. ■

Proof of Theorem 1.1. As usual, we write $T^\bullet$ for $M(T_1^\bullet, T_2^\bullet, h^\bullet)$. We identify $A_0 \otimes^\bullet_{A_2} T_2^\bullet$ with $T_0^\bullet$. By Lemma 2.11 (3), there is a short exact sequence

$$0 \to T^\bullet \to T_1^\bullet \oplus T_2^\bullet \xrightarrow{\left[\begin{smallmatrix}\pi_1^* h^\bullet \\ -\pi_2^*\end{smallmatrix}\right]} T_0^\bullet \to 0,$$

which yields a triangle in $\mathscr{D}^{\mathrm{b}}(A)$. Applying $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^\bullet, -)$ to this triangle, we obtain the commutative diagram with exact rows for each integer i:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}\big(T^\bullet, T_0^\bullet[i-1]\big) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}\big(T^\bullet, T^\bullet[i]\big) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}\big(T^\bullet, \bigoplus_{k=1}^2 T_k^\bullet[i]\big) \arrow[d, "\simeq"] \\
\mathrm{Hom}_{\mathscr{K}(A)}\big(T^\bullet, T_0^\bullet[i-1]\big) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathscr{K}(A)}\big(T^\bullet, T^\bullet[i]\big) \arrow[r] \arrow[d, equal] & \bigoplus_{k=1}^2 \mathrm{Hom}_{\mathscr{K}(A)}\big(T^\bullet, T_k^\bullet[i]\big) \arrow[d, "\simeq"] \\
\mathrm{Hom}_{\mathscr{K}(A_0)}\big(T_0^\bullet, T_0^\bullet[i-1]\big) \arrow[r] & \mathrm{Hom}_{\mathscr{K}(A)}\big(T^\bullet, T^\bullet[i]\big) \arrow[r] & \bigoplus_{k=1}^2 \mathrm{Hom}_{\mathscr{K}(A_k)}\big(T_k^\bullet, T_k^\bullet[i]\big).
\end{tikzcd} \tag{$**$}$$

Here, we use the natural isomorphisms:

$$\mathrm{Hom}_{\mathscr{K}(A)}\big(T^\bullet, T_k^\bullet[i]\big) \simeq \mathrm{Hom}_{\mathscr{K}(A_k)}\big(A_k \otimes_A T^\bullet, T_k^\bullet[i]\big) \simeq \mathrm{Hom}_{\mathscr{K}(A_k)}\big(T_k^\bullet, T_k^\bullet[i]\big)$$

for $0 \le k \le 2$, where the last isomorphism is due to Lemma 2.11 (2). Since

$$\mathrm{Hom}_{\mathscr{K}(A_0)}\big(T_0^\bullet, T_0^\bullet[i-1]\big) = 0$$

for all $i \neq 1$ and since $\mathrm{Hom}_{\mathscr{K}(A_k)}(T_k^\bullet, T_k^\bullet[i]) = 0$ for all $i \neq 0$ and all $0 \le k \le 2$, $\mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet, T^\bullet[i]) = 0$ for all $i \neq 0, 1$. It follows from Lemma 3.1 that the morphism $\eta_1 : \mathrm{Hom}_{\mathscr{K}(A_1)}(T_1^\bullet, T_1^\bullet) \to \mathrm{Hom}_{\mathscr{K}(A_0)}(T_0^\bullet, T_0^\bullet)$ determined by $\pi_1^* h^\bullet$ is surjective. Consequently, from the long exact sequence $(**)$, we get $\mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet, T^\bullet[1]) = 0$. Thus, $T^\bullet$ is self-orthogonal. Together with Lemma 3.2, we have shown that $T^\bullet$ is a tilting complex over A.

By Lemma 2.11 (4), there exists a pullback diagram of homomorphisms of algebras:

$$\begin{tikzcd}
\mathrm{End}_{\mathscr{K}(A)}(T^\bullet) \arrow[r, "\varepsilon_1"] \arrow[d, "\varepsilon_2"] & \mathrm{End}_{\mathscr{K}(A_1)}(T_1^\bullet) \arrow[d, two heads, "\eta_1"] \\
\mathrm{End}_{\mathscr{K}(A_2)}(T_2^\bullet) \arrow[r, "\eta_2"] & \mathrm{End}_{\mathscr{K}(A_0)}(T_0^\bullet),
\end{tikzcd}$$

where η_1 and η_2 are determined by $\pi_1^* h^\bullet$ and π_2^*, respectively, and where ε_1 and ε_2 are determined by the projections from $T^\bullet$ to $T_1^\bullet$ and $T_2^\bullet$, respectively. This completes the proof of Theorem 1.1. ■

Now, we consider a special Milnor square induced by an ideal in an algebra.

Given an algebra A and an ideal I in A, let can. : $A \to A/I$ be the canonical surjective homomorphism. Then, we may form a natural Milnor square

$$\begin{tikzcd} \Lambda \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"'] & A \arrow[d, "\text{can.}"] \\ A \arrow[r, "\text{can.}"] & A/I. \end{tikzcd}$$

If $T^\bullet$ is a tilting complex over A with $B := \mathrm{End}_{\mathscr{D}^{\mathrm{b}}(A)}(T^\bullet)$, then A and B are derived equivalent. We may hope that the natural quotient complex $T^\bullet/IT^\bullet$ could be a tilting complex over A/I and give a derived equivalence between the quotients A/I and B/J for some ideal J in B. In this way, we get naturally a Milnor square

$$\begin{tikzcd} \Gamma \arrow[r, "\lambda_1'"] \arrow[d, "\lambda_2'"'] & B \arrow[d, "\text{can.}"] \\ B \arrow[r, "\text{can.}"] & B/J. \end{tikzcd}$$

So we might apply Theorem 1.1 and get a derived equivalence between Λ and Γ. In general, however, the complex $T^\bullet/IT^\bullet$ does not have to be a tilting complex over A/I, but a sufficient and necessary condition was provided for $T^\bullet/IT^\bullet$ to be a tilting complex in [8, Section 4].

An immediate consequence of Theorem 1.1 and [8, Theorem 4.2] is the following.

Corollary 3.3. *Let A be an Artin algebra and $T^\bullet$ a basic, radical tilting complex over A. Suppose that I is an ideal in A such that $\mathrm{rad}(A) \subseteq I$, $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, IT^\bullet[i]) = 0$ for all $i \neq 0$ and $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet/IT^\bullet, (T^\bullet/IT^\bullet)[-1]) = 0$. Let $B := \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet)$ and J be the ideal of B consisting of all those endomorphisms of $T^\bullet$ that factorize through the injection $IT^\bullet \to T^\bullet$. If $T^\bullet/IT^\bullet$ is a basic, radical complex over A/I, then the algebras*

$$\Lambda := \big\{(a,a') \in A \times A \mid a - a' \in I\big\} \quad \textit{and} \quad \Gamma := \big\{(b,b') \in B \times B \mid b - b' \in J\big\}$$

are derived equivalent.

3.2. Proof of Theorem 1.2

Almost ν-stable derived equivalences were introduced in [7] to get stable equivalences of Morita type. This generalized a result of Rickard [15] on self-injective algebras. As shown in [9, 17], stable equivalences of Morita type are extremely helpful to approaching Broué's abelian defect group conjecture.

Throughout this section, all algebras are finite-dimensional over a fixed field.

Let $F : \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$ be a derived equivalence of algebras A and B. Suppose that $Q^\bullet$ and $\bar{Q}^\bullet$ are radical tilting complexes associated with F and the quasi-inverse F^{-1} of F, respectively. By applying the shift functor if necessary, we may assume that $Q^\bullet$ and $\bar{Q}^\bullet$ are of the form

$$0 \to Q^{-n} \to \cdots \to Q^{-1} \to Q^0 \to 0, \quad 0 \to \bar{Q}^0 \to \bar{Q}^1 \to \cdots \to \bar{Q}^n \to 0,$$

respectively. Let $Q := \bigoplus_{i=1}^n Q^{-i}$ and $\bar{Q} := \bigoplus_{i=1}^n \bar{Q}^n$. The derived equivalence F is called *almost ν-stable* provided that $\mathrm{add}({}_AQ) = \mathrm{add}(\nu_A Q)$ and $\mathrm{add}({}_B\bar{Q}) = \mathrm{add}(\nu_B \bar{Q})$. The composition of finitely many almost ν-stable derived equivalences or their quasi-inverses is called an *iterated almost ν-stable derived equivalence*. Such a derived equivalence of finite-dimensional algebras over a field always induces a stable equivalence of Morita type (see [6, 7]).

A module $P \in A\text{-}\mathsf{mod}$ is said to be *ν-stably projective* if $\nu_A^i P$ is projective for all $i \geq 0$, where ν_A is the Nakayama functor $D\,\mathrm{Hom}_A(-, A) \simeq D(A) \otimes_A - : A\text{-}\mathsf{mod} \to A\text{-}\mathsf{mod}$. We denote by A-stp the full subcategory of A-proj consisting of all ν-stably projective A-modules and call it the *Nakayama-stable* category of A. For further information on this category, we refer to [9, 12].

To prove Theorem 1.2, we have to prepare a few lemmas. Recall that S_X denotes the top of an indecomposable projective module X.

Lemma 3.4. *If P is an indecomposable module in A-*stp*, then there exists an exact sequence of A-modules*

$$0 \to R_P \to \nu_A S_P \to S_{\nu P} \to 0 \qquad (\star)$$

*such that the composition factors of R_P are of the form S_X with X indecomposable projective modules not in A-*stp.

Proof. Since S_P is the top of P, the module $\nu_A S_P$ is a quotient of $\nu_A P \in A$-stp, while $\nu_A P$ is an indecomposable projective module in A-stp and has $S_{\nu_A P}$ as its top. Thus, $\nu_A S_P$ is an indecomposable module with a simple top $S_{\nu_A P}$. Hence, there is an exact sequence of A-modules:

$$0 \to R_P \to \nu_A S_P \to S_{\nu P} \to 0.$$

For each indecomposable module $Y \in A$-stp, the multiplicity of S_Y as a composition factor of $\nu_A S_P$ is the length of $\mathrm{Hom}_A(Y, \nu_A S_P)$ as an $\mathrm{End}_A(S_Y)$-module. However,

$$\begin{aligned}\mathrm{Hom}_A(Y, \nu_A S_P) &\simeq \mathrm{Hom}_A\big(Y, D(A) \otimes_A S_P\big) \simeq \mathrm{Hom}_A\big(Y, D(A)\big) \otimes_A S_P \\ &\simeq D(Y) \otimes_A S_P \simeq \mathrm{Hom}_A(\nu_A^{-1} Y, S_P)\end{aligned}$$

is zero if $Y \not\simeq \nu_A P$ and has length 1 if $Y \simeq \nu_A P$. Hence, $\nu_A S_P$ has the composition factor $S_{\nu P}$ at top with $[\nu_A S_P : S_{\nu P}] = 1$, and other composition factors of the form S_X with X an indecomposable projective module not in A-stp. ■

Lemma 3.5 ([6, Theorem 1.1]). *Let $F : \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(B)$ be a derived equivalence between algebras A and B over an algebraically closed field, and let $T^\bullet$ and $\bar{T}^\bullet$ be tilting complexes associated with F and F^{-1}, respectively. Set $T^{\pm} := \bigoplus_{i \neq 0} T^i$ and $\bar{T}^{\pm} := \bigoplus_{j \neq 0} \bar{T}^j$. Then, the following are equivalent:*

(1) *the functor F is an iterated almost ν-stable derived equivalence;*

(2) $\mathrm{add}(T^{\pm}) = \mathrm{add}(\nu_A T^{\pm})$ *and* $\mathrm{add}(\bar{T}^{\pm}) = \mathrm{add}(\nu_B \bar{T}^{\pm})$*;*

(3) $T^{\pm} \in A\text{-stp}$ *and* $\bar{T}^{\pm} \in B\text{-stp}$*;*

(4) *for an indecomposable projective A-module $X \notin A\text{-stp}$, $F(S_X)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(B)$ to a simple B-module;*

(5) *for each indecomposable projective A-module $X \notin A\text{-stp}$, there hold $X \notin \mathrm{add}(T^{\pm})$ and $[U^0 : X] = 1$, where $U^{\bullet} = (U^i, d_U)$ is the direct sum of all non-isomorphic indecomposable direct summands of $T^{\bullet}$.*

Moreover, if one of (1)–(5) is satisfied, then A and B are stably equivalent of Morita type.

Thus, a derived equivalence F is iterated almost ν-stable if and only if so is its quasi-inverse F^{-1} by (2).

Lemma 3.6. *Let Λ and Γ be algebras over an algebraically closed field and $F : \mathscr{D}^{\mathrm{b}}(\Lambda) \to \mathscr{D}^{\mathrm{b}}(\Gamma)$ an iterated almost ν-stable derived equivalence. Suppose that P is an indecomposable projective Λ-module in $\Lambda\text{-stp}$.*

(1) *If $F(S_P)$ is isomorphic to a simple Γ-module $S_{P'}$, then so is $F(S_{\nu_\Lambda P})$. Moreover, P' must be in $\Gamma\text{-stp}$.*

(2) *If $F(S_P)$ is not isomorphic to a simple Γ-module, then neither is $F(S_{\nu_\Lambda P})$.*

Proof. (1) We may assume that F is almost ν-stable with $Q^{\bullet}$ and $\bar{Q}^{\bullet}$ being radical, tilting complexes associated with F and F^{-1}, respectively. Let $Q := \bigoplus_{i>0} Q^{-i}$ and $\bar{Q} := \bigoplus_{i>0} \bar{Q}^{i}$. Then, $\mathrm{add}(\nu_\Lambda Q) = \mathrm{add}(Q)$ and $\mathrm{add}(\nu_\Gamma \bar{Q}) = \mathrm{add}(\bar{Q})$.

By [7, Lemma 5.2], there is a radical, two-sided tilting complex ${}_\Gamma\Delta^{\bullet}_\Lambda$:

$$0 \to \Delta^0 \to \Delta^1 \to \cdots \to \Delta^n \to 0$$

such that $F(X^{\bullet}) \simeq \Delta^{\bullet} \otimes^{\bullet}_\Lambda X^{\bullet}$ with $\Delta^i \in \mathrm{add}(\bar{Q} \otimes_k Q^*)$ for all $i > 0$. Here, $Q^* = \mathrm{Hom}_\Lambda(Q, \Lambda)$ is the Λ-duality of ${}_\Lambda Q$. Then, $\Theta^{\bullet} := \mathrm{Hom}^{\bullet}_\Gamma(\Delta^{\bullet}, \Gamma)$ is an inverse of $\Delta^{\bullet}$. The bimodules Δ^0 and Θ^0 define a stable equivalence of Morita type between Λ and Γ (see the proof of [7, Theorem 5.3]). Here, we stress that $\Delta^0 \otimes_\Lambda -$ is both a left and right adjoint to $\Theta^0 \otimes_\Gamma -$. Indeed, $\Theta^0 := \mathrm{Hom}_\Gamma(\Delta^0, \Gamma)$ implies that $\Delta^0 \otimes_\Lambda -$ is a left adjoint to $\Theta^0 \otimes_\Gamma -$. Note that there is an isomorphism $\Delta^{\bullet} \simeq \mathrm{Hom}^{\bullet}_\Lambda(\Theta^{\bullet}, \Lambda)$ in $\mathscr{D}^{\mathrm{b}}(\Gamma \otimes_k \Lambda^{\mathrm{op}})$, due to the fact that $\Delta^{\bullet}$ is an inverse of $\Theta^{\bullet}$. By the proof of [7, Theorem 5.3], the terms $\Theta^{-i} \in \mathrm{add}(Q \otimes_k \bar{Q}^*)$ for all $i > 0$, where $\bar{Q}^* = \mathrm{Hom}_\Gamma(\bar{Q}, \Gamma)$. The natural isomorphisms

$$\mathrm{Hom}_\Lambda(Q \otimes_k \bar{Q}^*, \Lambda) \simeq \mathrm{Hom}_k(\bar{Q}^*, Q^*) \simeq \nu_\Gamma \bar{Q} \otimes_k Q^*$$

imply that $\mathrm{Hom}_\Lambda(\Theta^{-i}, \Lambda) \in \mathrm{add}(\bar{Q} \otimes_k Q^*)$ for all $i > 0$. Thus, the terms of $\Delta^{\bullet}$ and $\mathrm{Hom}^{\bullet}_\Lambda(\Theta^{\bullet}, \Lambda)$ in positive degrees are all projective bimodules. By [7, Lemma 2.1], all morphisms between $\Delta^{\bullet}$ and $\mathrm{Hom}^{\bullet}_\Lambda(\Theta^{\bullet}, \Lambda)$ in $\mathscr{D}^{\mathrm{b}}(\Gamma \otimes_k \Lambda^{\mathrm{op}})$ are given by chain maps. Thus, there is a chain map $f^{\bullet} : \Delta^{\bullet} \to \mathrm{Hom}^{\bullet}_\Lambda(\Theta^{\bullet}, \Lambda)$ which induces an isomorphism in $\mathscr{D}^{\mathrm{b}}(\Gamma \otimes_k \Lambda^{\mathrm{op}})$. This means that the cone $\mathrm{con}(f^{\bullet})$ of $f^{\bullet}$ is an acyclic complex. However, except the terms in degrees $-1, 0$, all terms of $\mathrm{con}(f^\bullet)$ are projective bimodules. Hence, $\mathrm{con}(f^\bullet)$ splits; that is, $\mathrm{con}(f^\bullet)$ is a contractible complex. Thus, $f^\bullet$ is an isomorphism in $\mathscr{K}^{\mathrm{b}}(\Gamma\otimes_k\Lambda^{\mathrm{op}})$. Since both complexes $\Delta^\bullet$ and $\mathrm{Hom}^\bullet_\Lambda(\Theta^\bullet,\Lambda)$ are radical, they are even isomorphic in $\mathscr{C}^{\mathrm{b}}(\Gamma\otimes_k\Lambda^{\mathrm{op}})$ by [7, (b), p. 113]. It follows that $\Delta^0\simeq\mathrm{Hom}_\Lambda(\Theta^0,\Lambda)$ and $\Delta^0\otimes_\Lambda -$ is a right adjoint to $\Theta^0\otimes_\Gamma -$.

Suppose $F(S_P)\simeq S_{P'}$ in $\mathscr{D}^{\mathrm{b}}(\Gamma)$ for an indecomposable projective Γ-module P'. Then, $P'\in\Gamma$-stp. In fact, if $P'\notin\Gamma$-stp, then $\mathrm{Hom}_\Lambda(P,S_P)\simeq\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Gamma)}(F(P),S_{P'})$ would vanish since $F(P)$ is isomorphic to a complex in $\mathscr{K}^{\mathrm{b}}(\Gamma\text{-stp})$ by [7, Lemma 3.9]. This is a contradiction.

To prove (1), we show $F(S_{\nu P})\simeq S_{\nu P'}$.

Indeed, since $F(S_P)$ is simple,

$$\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}\big(T^\bullet, S_P[i]\big)\simeq\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Gamma)}\big(\Gamma, F(S_P)[i]\big)=0$$

for all $i\neq 0$. This implies $Q^*\otimes_\Lambda S_P\simeq\mathrm{Hom}_\Lambda(Q,S_P)=0$. Thus, $\Delta^i\otimes_\Lambda S_P=0$ for $i>0$ and $F(S_P)\simeq\Delta^\bullet\otimes^\bullet_\Lambda S_P=\Delta^0\otimes_\Lambda S_P\simeq S_{P'}$.

For $P\in\Lambda$-stp, there is the following exact sequence of Λ-modules by Lemma 3.4:

$$0\to R_P\to\nu_\Lambda S_P\to S_{\nu P}\to 0. \qquad (\star)$$

Now, applying $\Delta^0\otimes_\Lambda -$ to ($\star$), we get an exact sequence of Γ-modules

$$0\to\Delta^0\otimes_\Lambda R_P\to\Delta^0\otimes_\Lambda\nu_\Lambda S_P\to\Delta^0\otimes_\Lambda S_{\nu P}\to 0. \qquad (\star\star)$$

Note that $\Delta^0\otimes_\Lambda\nu_\Lambda S_P\simeq\nu_\Gamma(\Delta^0\otimes_\Lambda S_P)$ by a property of stable equivalences of Morita type (see (b) in the proof of [9, Lemma 3.1] and observe that (b) holds without any additional assumptions in [9, Lemma 3.1] because $\Delta^0\otimes_\Lambda -$ is both left and right adjoint to $\Theta^0\otimes_\Gamma -$). It follows from $F(S_P)\simeq\Delta^0\otimes_\Lambda S_P\simeq S_{P'}$ that $\nu_\Gamma(\Delta^0\otimes_\Lambda S_P)\simeq\nu_\Gamma(F(S_P))\simeq\nu_\Gamma(S_{P'})$. Hence, $\Delta^0\otimes_\Lambda\nu_\Lambda S_P\simeq\nu_\Gamma(S_{P'})$. Due to $\mathrm{Hom}_\Lambda(Q,S_P)=0$, we get $P\notin\mathrm{add}(Q)$ and $\nu_\Lambda P\notin\mathrm{add}(\nu_\Lambda Q)=\mathrm{add}(Q)$. This implies $\mathrm{Hom}_\Lambda(Q,S_{\nu P})=0$. Thus, $\Delta^i\otimes_\Lambda S_{\nu P}=0$ for $i\neq 0$ and $F(S_{\nu P})\simeq\Delta^\bullet\otimes_\Lambda S_{\nu P}\simeq\Delta^0\otimes_\Lambda S_{\nu P}$. So we assume $F(S_{\nu P})=\Delta^0\otimes_\Lambda S_{\nu P}\in\Gamma$-mod and rewrite ($\star\star$) as

$$0\to\Delta^0\otimes_\Lambda R_P\to\nu_\Gamma S_{P'}\to F(S_{\nu P})\to 0.$$

Both $\nu_\Gamma S_{P'}$ and $F(S_{\nu P})$ have a simple top isomorphic to $S_{\nu P'}$. Moreover, by Lemma 3.4, the other composition factors of $\nu_\Gamma S_{P'}$ are of the form $S_{X'}$ with X' indecomposable not in Γ-stp. So, to prove that $F(S_{\nu P})$ is simple, we only have to show that $F(S_{\nu P})$ does not have any submodule isomorphic to $S_{X'}$ for all indecomposable projective Γ-modules $X'\notin\Gamma$-stp. This is equivalent to saying that $\mathrm{Hom}_\Gamma(S_{X'},F(S_{\nu P}))=0$ for all indecomposable projective modules $X'\notin\Gamma$-stp. Indeed, F is iterated almost ν-stable if and only if F^{-1} is iterated almost ν-stable. Hence, by Lemma 3.5 (4), for each indecomposable projective Γ-module $X'\notin\Gamma$-stp, there is an indecomposable projective Λ-module $X\notin\Lambda$-stp such that $F(S_X)\simeq S_{X'}$. Thus, $\mathrm{Hom}_\Gamma(S_{X'},F(S_{\nu P}))\simeq\mathrm{Hom}_\Lambda(S_X,S_{\nu P})=0$. Consequently, $F(S_{\nu P})$ has a unique composition factor $S_{\nu P'}$, that is, $F(S_{\nu P})\simeq S_{\nu P'}$.

(2) follows from (1). ∎

Lemma 3.7. *Keep the assumptions in Theorem 1.2. For $i = 1, 2$, let Y_i be the direct sum of all indecomposable projective A_i-modules Y such that the image of* $\operatorname{top}(Y)$ *under the derived equivalence induced by $T_i^\bullet$ is not isomorphic to a simple module. Then, $Y_i \in$* $\operatorname{add}(P_i)$ *and* $\operatorname{add}(\nu_{A_i} Y_i) = \operatorname{add}(Y_i)$, *where P_i is the direct sum of all non-isomorphic indecomposable projective A_i-modules X such that $A_0 \otimes_{A_i} X = 0$ (see Section 2.4).*

Proof. Without loss of generality, we assume $i = 1$. Indeed, let Y be an indecomposable projective A_1-module such that

($*$) the image of $\operatorname{top}(Y)$ under the derived equivalence induced by $T_1^\bullet$ is not isomorphic to a simple module.

Then, we first show $Y \in \operatorname{add}(P_1)$.

By Lemma 2.3, Y must occur in a term T_1^i of $T_1^\bullet$ as a direct summand. Suppose $Y \notin \operatorname{add}(T_1^m)$ for all $m \neq 0$. Then, Y occurs in T_1^0 as a direct summand. Since the image of $\operatorname{top}(Y)$ under the derived equivalence induced by $T_1^\bullet$ is not isomorphic to a simple module, it follows from Lemma 2.4 that $[T_1^0 : Y] > 1$. As $A_0 \otimes_{A_1} Y \neq 0$, we have

$$[T_0^0 : A_0 \otimes_{A_1} Y] = [A_0 \otimes_{A_1} T_1^0 : A_0 \otimes_{A_1} Y] \geq [T_1^0 : Y] > 1.$$

This implies that the stalk complex $T_0^\bullet$ is not basic. This contradiction shows that Y must occur in some T_1^m with $m \neq 0$, that is, $Y \in \operatorname{add}(T_1^m)$. Since $T_0^\bullet$ is a stalk complex and $A_0 \otimes_{A_1} T_1^\bullet \simeq T_0^\bullet$ as complexes, we obtain $A_0 \otimes_{A_1} T_1^m \simeq A_0 \otimes_{A_1} T_0^m = A_0 \otimes_{A_1} 0 = 0$. Thus, $A_0 \otimes_{A_1} Y = 0$ and $Y \in \operatorname{add}(P_1)$. This shows $Y_1 \in \operatorname{add}(P_1)$.

Now, by ($*$) and Lemma 3.5 (4), $Y_1 \in A_1$-stp. It then follows from Lemma 3.6 (2) that, for each indecomposable $Y \in \operatorname{add}(Y_1)$, the module $\nu_{A_1} Y$ satisfies again the condition ($*$) and therefore is in $\operatorname{add}(P_1)$. Hence, $Y_1 \in \operatorname{add}(P_1) \cap A_1$-stp and $\nu_{A_1}(Y_1) \in \operatorname{add}(Y_1)$. This means $\operatorname{add}(\nu_{A_1} Y_1) = \operatorname{add}(Y_1)$. ∎

Proof of Theorem 1.2. We keep the notations in the proof of Theorem 1.1. The tilting complex $T^\bullet$ induces a derived equivalence between the pullback algebras. To prove that $T^\bullet$ induces an iterated almost ν-stable derived equivalence, we show the statements (a) and (b).

(a) $T^i \in A$-stp for all $i \neq 0$.

In fact, by assumption, $T_0^\bullet$ is a stalk complex concentrated in degree 0 and $A_0 \otimes_{A_i} T_i^\bullet \simeq T_0^\bullet$ for $i = 1, 2$. It follows that $T_i^m \in \operatorname{add}(P_i)$ for $i = 1, 2$ and $m \neq 0$, where P_i is as defined in Section 2.4. Thus, by the construction of $T^\bullet$, the term T^m is equal to $M(T_1^m, 0, 0) \oplus M(0, T_2^m, 0)$ for $m \neq 0$. By Lemma 3.5, for $i \in \{1, 2\}$, the A_i-module $T_i^\pm := \bigoplus_{m \neq 0} T_i^m$ satisfies $\operatorname{add}(\nu_{A_i} T_i^\pm) = \operatorname{add}(T_i^\pm)$. It follows from Lemma 2.9 (3) that $T^\pm := \bigoplus_{m \neq 0} T^m$ satisfies $\operatorname{add}(\nu_A T^\pm) = \operatorname{add}(T^\pm)$. Hence, $T^m \in A$-stp for all $m \neq 0$.

(b) $[T^0 : X] = 1$ for each indecomposable projective A-module $X \notin A$-stp.

Let X be an indecomposable projective A-module and $X \notin A$-stp. It follows from (a) that X has to be a direct summand of T^0. Suppose contrarily $[T^0 : X] = r > 1$. Clearly, from the construction of $T^\bullet$, we have

$$T^0 \simeq M(T_1^0, T_2^0, h^0)$$

with $h^0: A_0 \otimes_{A_1} T_1^0 \to A_0 \otimes_{A_2} T_2^0$ an isomorphism of A_0-modules. Moreover, $X \simeq M(X_1, X_2, h_X)$ for $X_1 = A_1 \otimes_A X$, $X_2 = A_2 \otimes_A X$ and an A_0-module isomorphism $h_X: A_0 \otimes_{A_1} X_1 \to A_0 \otimes_{A_2} X_2$. If $h_X \neq 0$, then $A_0 \otimes_{A_1} X_1 = A_0 \otimes_{A_1} A_1 \otimes_A X$ is a direct summand of $A_0 \otimes_{A_1} A_1 \otimes_A T^0 \simeq A_0 \otimes_{A_1} T_1^0 \simeq T_0^0$ with the multiplicity at least r. This contradicts the assumption that $T_0^\bullet$ is a basic projective generator of A_0-modules. Hence, $h_X = 0$, $A_0 \otimes_{A_i} X_i = 0$, for $i = 1, 2$, and $X \simeq M(X_1, 0, 0) \oplus M(0, X_2, 0) = X_1 \oplus X_2$. It follows that $X_i \in \mathrm{add}(P_i)$ for $i = 1, 2$, and either $X_1 = 0$ or $X_2 = 0$. Without loss of generality, we assume $X_1 \neq 0$. Then, $[T_1^0 : X_1] \geq r$, due to $A_1 \otimes_A T^0 \simeq T_1^0$. Consequently, $X_1 \in A_1$-stp by Lemma 3.5 (5). Thus, the image of $\mathrm{top}(X_1)$ of the indecomposable projective A_1-module X_1 under the derived equivalence induced by $T_1^\bullet$ is not isomorphic to a simple module by Lemma 2.4. It then follows from Lemmas 3.7 and 2.9 (3) that $X = X_1$ lies in A-stp. This contradicts the assumption that X does not belong to A-stp. Hence, $[T^0 : X] = 1$.

Altogether, we have shown that $\mathrm{add}(\nu_A T^\pm) = \mathrm{add}(T^\pm)$, $T^\pm \in A$-stp and $[T^0 : X] = 1$ for every indecomposable projective A-module $X \notin A$-stp. Observe that if $X \notin A$-stp, then $X \notin \mathrm{add}(T^\pm)$. Now, by Lemma 3.5 (5), $T^\bullet$ induces an iterated almost ν-stable derived equivalence. ∎

4. Derived equivalences from quiver operations

Based on Theorem 1.1, we consider three operations on derived equivalent algebras given by quivers with relations, namely gluing vertices, unifying arrows and identifying socle elements. The precise formulations are given by Theorems 4.1, 4.5 and 4.8, respectively. These operations can be applied repeatedly and combined with each other, thus producing a lot of new derived equivalences from given ones.

We start with a derived equivalence which sends some (usually not all) simple modules to simple modules. The collection of these simple modules does not have to be a simple-minded collection in general (for definition, see [11, 16]). Note that the set of the images of all simple modules under a derived equivalence forms a simple-minded collection. But this set may contain no simple modules. To get new derived equivalences from given ones by our operations on quivers with relations, we need to consider some invariant simple modules of derived equivalences.

Derived equivalences sending certain simple modules to simple modules occur very often and play actually an important role in representation theory (see Okuyama's study of Broué Conjecture [3]). The following is a general construction to get such derived equivalences.

Let A be a basic Artin algebra and e an idempotent in A such that

$$\mathrm{Hom}_A\big(A/AeA, A(1-e)\big) = 0.$$

Then, there is a tilting complex of the form

$$T^\bullet: \quad 0 \to Ae \oplus Q \xrightarrow{\left[\begin{smallmatrix} 0 \\ \phi \end{smallmatrix}\right]} A(1-e) \to 0,$$

where $A(1-e)$ is in degree zero and $\phi : Q \to A(1-e)$ is a right add(Ae)-approximation. By Lemma 2.4, the derived equivalence induced by $T^{\bullet}$ sends all simple A-modules corresponding to the indecomposable direct summands of $A(1-e)$ to simple modules.

Now, we recall terminology on quivers.

Let $Q = (Q_0, Q_1)$ be a quiver with Q_0 the set of vertices and Q_1 the set of arrows between vertices. For $m > 1$, let Q_m be the set of all paths in Q of length m. The starting and ending vertices of a path p are denoted by $s(p)$ and $e(p)$, respectively. As usual, the trivial path corresponding to a vertex $i \in Q_0$ is denoted by e_i.

Let k be a field and kQ the path algebra of Q over k. The composition of two paths p and q in kQ is written as pq if $e(p) = s(q)$, and 0 otherwise. A *relation* ω on Q is a k-linear combination of paths: $\omega = \lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_n p_n$ with $0 \neq \lambda_i \in k$, $e(p_1) = \cdots = e(p_n)$ and $s(p_1) = \cdots = s(p_n)$. Here, we assume that the length of each p_i is at least 2. If $n = 1$ in ω, then ω is called a *monomial* relation.

Let ρ be a set of relations in kQ and $\langle \rho \rangle$ the ideal of kQ generated by ρ. Then, an algebra of the form $kQ/\langle \rho \rangle$ is said to be presented by the quiver Q with relations ρ. Clearly, $\langle \rho \rangle \subseteq \langle Q_2 \rangle$. Note that for any ideal $I \subseteq \langle Q_2 \rangle$ of kQ such that kQ/I is finite-dimensional, there is a set ρ of relations such that $\langle \rho \rangle = I$.

4.1. Derived equivalences from gluing vertices

In this subsection, we shall construct derived equivalences from given ones by gluing vertices of quivers.

Let $A = kQ/\langle \rho \rangle$ be a finite-dimensional algebra over a field k presented by a quiver $Q = (Q_0, Q_1)$ with relations ρ. For $X \subseteq Q_0$, let $e_X = \sum_{i \in X} e_i \in A$. Suppose that $\sigma = \{\sigma_1, \ldots, \sigma_m\}$ is a partition of X; that is, X is a disjoint union of the subsets σ_j, $1 \le j \le m$. Let Q^σ be the quiver obtained from Q by gluing the vertices in σ_t into one vertex, also denoted by σ_t, for all t, and keeping all arrows. This means that an arrow $\alpha \in Q_1$ with the starting vertex in σ_i and the ending vertex in σ_j is an arrow α in Q^σ with the starting vertex σ_i and the ending vertex σ_j. Hence, the vertex set of Q^σ is the union of $\{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ and $Q_0 \setminus X$, and the arrow set of Q^σ is Q_1. Then, there is a natural homomorphism of algebras:

$$\lambda_\sigma : kQ^\sigma \to kQ/\langle \rho \rangle$$

which sends e_i to e_i for $i \notin X$ and e_{σ_t} to $\sum_{i \in \sigma_t} e_i$ for $1 \le t \le m$ and preserves all arrows. Clearly, the kernel of λ_σ is contained in $\langle Q_2^\sigma \rangle$ in kQ^σ. Let ρ^σ be a set of relations on Q^σ such that $\langle \rho^\sigma \rangle = \mathrm{Ker}(\lambda_\sigma)$. The relations ρ^σ can be obtained in the following way: For each t, let ρ^{σ_t} be the set of relations on Q^σ consisting of all $\alpha\beta$ with α, β being arrows such that $e(\alpha)$ and $s(\beta)$ are different vertices in σ_t. Then, $\rho^\sigma = \rho \cup \rho^{\sigma_1} \cup \cdots \cup \rho^{\sigma_m}$. The algebra $A^\sigma := kQ^\sigma/\langle \rho^\sigma \rangle$ is called the σ-*gluing algebra* of A. The above homomorphism λ_σ induces a homomorphism from A^σ to A, denoted again by λ_σ. Observe that $\lambda_\sigma : A^\sigma \to A$ is injective and the image of λ_σ is the subalgebra of A generated by the idempotents $\{e_i \mid i \in Q_0 \setminus X\}$, $\sum_{i \in \sigma_t} e_i$ for $1 \le t \le m$ and all arrows in Q. Thus, the Jacobson radicals of A^σ and A are equal. This construction has been used in the study of the finitistic dimension conjecture (for example, see [19]).

The procedure of σ-gluing algebras can be interpreted as pullbacks of algebras. We define $k^X := \bigoplus_{i\in X} k$ to be the path algebra of the quiver with isolated vertices indexed by X. Considering the set σ, we have the algebra k^σ which is just the σ-gluing algebra of k^X. There is an embedding $\lambda_\sigma : k^\sigma \to k^X$ sending e_{σ_i} to $\sum_{j\in\sigma_i} e_j$ for $1 \le i \le m$. Also, note that there is a canonical homomorphism of algebras

$$\pi : kQ/\langle\rho\rangle \to k^X$$

sending e_i to e_i for $i \in X$, and all other idempotents and all arrows to zero. Similarly, there is a canonical, surjective homomorphism $\pi : kQ^\sigma/\langle\rho^\sigma\rangle \to k^\sigma$ of algebras. Then, we have a commutative diagram of homomorphisms of algebras:

$$\begin{tikzcd} kQ^\sigma/\langle\rho^\sigma\rangle \arrow[r, "\lambda_\sigma"] \arrow[d, "\pi"'] & kQ/\langle\rho\rangle \arrow[d, "\pi"] \\ k^\sigma \arrow[r, "\lambda_\sigma"] & k^X \end{tikzcd}$$

Since $\dim_k A + \dim_k k^\sigma = \dim_k A^\sigma + \dim_k k^X$, the above commutative diagram is a Milnor square.

Theorem 4.1. *Suppose that F is a derived equivalence between algebras $A := kQ/\langle\rho\rangle$ and $A' := kQ'/\langle\rho'\rangle$. Let X be a subset of Q_0 such that the simple A-modules corresponding to the vertices in X are sent by F to simple A'-modules S'. Let X' be the set of vertices in Q'_0 corresponding to the A'-modules S'. Let σ be a partition of X and σ' the corresponding partition of X'. Then, A^σ and $A'^{\sigma'}$ are derived equivalent.*

Proof. By assumption, there is a basic, radical tilting complex $T^\bullet$ over A such that $F(T^\bullet) \simeq A'$ in $\mathscr{D}^{\mathrm{b}}(A')$. By Lemmas 2.4 and 2.7, we can rewrite $T^\bullet$ as $T^\bullet = U^\bullet \oplus \bigoplus_{i\in X} V_i^\bullet$ such that $U^\bullet \in \mathscr{K}^{\mathrm{b}}(\mathrm{add}(\bigoplus_{i\in Q_0\setminus X} Ae_i))$ and $V_i^\bullet$ is indecomposable with $[V_i^\bullet : Ae_j] = \delta_{ij}$ for all $i, j \in X$. Moreover, for each $i \in X$, the projective A-module Ae_i occurs as a direct summand of V_i^0 with the multiplicity 1 (see the proof of Lemma 2.4). By the definition of $\pi : A \to k^X$, we have $k^X \otimes_A Ae_i = 0$ for $i \notin X$ and $k^X \otimes_A Ae_i \simeq k^X e_i$ for $i \in X$. Thus, there is an isomorphism in $\mathscr{C}(k^X)$:

$$h^\bullet : k^X \otimes_A T^\bullet \to k^X.$$

Clearly, $k^X \otimes_{k^\sigma} k^\sigma \simeq k^X$. Let $\eta_1 : \mathrm{End}_{\mathscr{K}(A)}(T^\bullet) \to \mathrm{End}_{k^X}(k^X)$ be the algebra homomorphism determined by the composition $\pi^* h_1^\bullet : T^\bullet \to k^X \otimes_A T^\bullet \to k^X$, and let $\eta_2 : \mathrm{End}_{k^\sigma}(k^\sigma) \to \mathrm{End}_{k^X}(k^X)$ be the algebra homomorphism determined by λ_σ. By Theorem 1.1, the pullback algebra of η_1 and η_2 is derived equivalent to the pullback algebra A^σ of $\pi : A \to k^X$ and $\lambda_\sigma : k^\sigma \to k^X$. It remains to show that $A'^{\sigma'}$ is isomorphic to the pullback algebra of η_1 and η_2.

For each x in Q_0 (respectively, Q'_0), we denote by S_x (respectively, S'_x) the simple A-module (respectively, A'-module) corresponding to the vertex x. By relabeling the vertices if necessary, we may assume

$$X = \{1, \ldots, m\} = X' \quad \text{and} \quad F(S_i) \simeq S'_i \quad \text{for } 1 \le i \le m.$$

In this case, σ and σ' are the same partition of $\{1, \ldots, m\}$. For $i, j \in \{1, \ldots, m\}$, the Hom-space

$$\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A')}\big(F(V_i^\bullet), S'_j\big) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A')}\big(F(V_i^\bullet), F(S_j)\big) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(V_i^\bullet, S_j)$$

is 1-dimensional for $i = j$, and 0 for $i \neq j$. Thus, it follows from the indecomposability of $F(V_i^\bullet)$ that there exists an isomorphism $g_i : F(V_i^\bullet) \to A'e_i$ for $1 \le i \le m$. Let $f := \sum_{j \in Q'_0 \setminus X'} e_j \in A'$. Then, there is an isomorphism $g : F(U^\bullet) \to A'f$. Thus, we obtain an isomorphism

$$\mathrm{diag}[g, g_1, \ldots, g_m] : F(T^\bullet) \to A',$$

which induces an isomorphism $\tilde{g} : \mathrm{End}_{\mathscr{D}^{\mathrm{b}}(A')}(F(T^\bullet)) \to \mathrm{End}_{A'}(A')$. Let s be the composition of the maps

$$\mathrm{End}_{\mathscr{K}(A)}(T^\bullet) \simeq \mathrm{End}_{\mathscr{D}^{\mathrm{b}}(A)}(T^\bullet) \to \mathrm{End}_{\mathscr{D}^{\mathrm{b}}(A')}\big(F(T^\bullet)\big) \xrightarrow{\tilde{g}} \mathrm{End}_{A'}(A') \to A'.$$

Then, for each $i \in \{1, \ldots, m\}$, the map s sends the primitive idempotent corresponding to the direct summand $V_i^\bullet$ to e_i. According to this fact, we can check the following commutative diagram:

$$\begin{tikzcd}
\mathrm{End}_{\mathscr{K}(A)}(T^\bullet) \arrow[r, "\eta_1"] \arrow[d, "s"', "\simeq"] & \mathrm{End}_{k^X}(k^X) \arrow[d, "\simeq"] & \mathrm{End}_{k^\sigma}(k^\sigma) \arrow[l, "\eta_2"'] \arrow[d, "\simeq"] \\
A' \arrow[r, "\pi"] & k^{X'} & k^{\sigma'} \arrow[l, "\lambda_{\sigma'}"']
\end{tikzcd}$$

where the last two vertical isomorphisms are the canonical ones. This diagram shows that the pullback algebra $A'^{\sigma'}$ of π and $\lambda_{\sigma'}$ is isomorphic to the pullback algebra of η_1 and η_2. This finishes the proof. ■

Remark. In Theorem 4.1, the indecomposable projective A^σ-module corresponding to a part of σ occurs only once (in degree 0) in the tilting complex that induces a derived equivalence between A^σ and $A'^{\sigma'}$ (see the proof of Theorem 1.1). Therefore, by Lemma 2.4, this derived equivalence sends the simple modules corresponding to parts of σ to the simple modules corresponding to parts of σ'. Thus, Theorem 4.1 can be employed repeatedly as many times as possible.

Also, one can construct new derived equivalences from two given derived equivalences by Theorem 4.1.

Corollary 4.2. *Let F be a derived equivalence between two algebras $A := kQ/\langle\rho\rangle$ and $A' := kQ'/\langle\rho'\rangle$, and let G be a derived equivalence between $B := k\Gamma/\langle\phi\rangle$ and $B' := k\Gamma'/\langle\phi'\rangle$. Suppose that $\overline{Q}_0$ (respectively, $\overline{\Gamma}_0$) is a subset of Q_0 (respectively, Γ_0) such that* *the simple modules corresponding to the vertices in* $\bar{Q}_0$ *(respectively,* $\bar{\Gamma}_0$*) are sent by* F *(respectively,* G*) to simple modules corresponding to the vertices in* $\bar{Q}'_0$ *(respectively,* $\bar{\Gamma}'_0$*) and that* $|\bar{Q}_0| = |\bar{Q}'_0|$ *and* $|\bar{\Gamma}_0| = |\bar{\Gamma}'_0|$. *Let* σ *be a partition of the set* $\bar{Q}_0 \dot\cup \bar{\Gamma}_0$ *and* σ' *the corresponding partition of* $\bar{Q}'_0 \dot\cup \bar{\Gamma}'_0$. *Then,* $(A \times B)^\sigma$ *and* $(A' \times B')^{\sigma'}$ *are derived equivalent.*

Proof. Taking the products of two algebras, we get a derived equivalence between $A \times B$ and $A' \times B'$, which sends the simple modules corresponding to the vertices in $\bar{Q}_0 \cup \bar{\Gamma}_0$ to the simple modules corresponding to the vertices in $\bar{Q}'_0 \cup \bar{\Gamma}'_0$. Thus, the corollary follows immediately from Theorem 4.1. ∎

A special case of Corollary 4.2 is to attach an algebra simultaneously to derived equivalent algebras.

Corollary 4.3. *Let* F *be a derived equivalence between algebras* $A := kQ/\langle\rho\rangle$ *and* $A' := kQ'/\langle\rho'\rangle$ *such that* F *sends the simple* A*-modules corresponding to vertices in* $\bar{Q}_0$ *to the simple* A'*-modules corresponding to vertices in* $\bar{Q}'_0$ *and* $|\bar{Q}_0| = |\bar{Q}'_0|$. *Suppose that* $C := k\Gamma/\langle\rho''\rangle$ *is an arbitrary algebra. Let* σ *be a partition of* $\bar{Q}_0 \cup \Gamma_0$ *and* σ' *the corresponding partition of* $\bar{Q}'_0 \cup \Gamma_0$. *Then,* $(A \times C)^\sigma$ *and* $(A' \times C)^{\sigma'}$ *are derived equivalent.*

4.2. Derived equivalences from unifying arrows

In this section, we introduce the operation of unifying arrows for the first time and construct new derived equivalences from given ones by this operation.

Throughout this section, Δ is the quiver with the vertex set $\{x, 1, 2, \ldots, n\}$ and n arrows $\alpha_j : x \to j$, $1 \leq j \leq n$. Here, we understand that the arrows have pairwise distinct ending vertices. We define $E := \{1, \ldots, n\}$. It may happen that the vertex x falls into E. In this case, Δ has the vertex set E. Let σ be the partition of E with only one part, and let $\alpha := \{\alpha_1, \ldots, \alpha_n\}$ for simplicity.

Let $A = kQ/\langle\rho\rangle$ be a finite-dimensional k-algebra such that Δ is a subquiver of Q. By the discussion in the previous section, there is an algebra embedding

$$\lambda_\sigma : kQ^\sigma/\langle\rho^\sigma\rangle \to kQ/\langle\rho\rangle.$$

Let Q^α be the quiver obtained from Q^σ by unifying the arrows $\alpha_1, \ldots, \alpha_n$ into one arrow $\overline{\alpha}$ in Q^σ. Thus, Q^α has the vertex set $(Q^\sigma)_0$, while the set of arrows is $\{\overline{\alpha}\} \cup Q_1^\sigma \setminus \{\alpha_1, \ldots, \alpha_n\}$. Then, there is a canonical algebra homomorphism

$$\varphi : kQ^\alpha \to kQ^\sigma/\langle\rho^\sigma\rangle$$

sending $\overline{\alpha}$ to $\sum_{i=1}^n \alpha_i$, and preserving all other arrows and all vertices. It is easy to see that $\mathrm{Ker}(\varphi)$ is contained in $\langle Q_2^\alpha\rangle$. Let ρ^α be relations on Q^α such that $\langle\rho^\alpha\rangle = \mathrm{Ker}(\varphi)$. Then, we get a natural embedding

$$\lambda_\alpha : kQ^\alpha/\langle\rho^\alpha\rangle \to kQ^\sigma/\langle\rho^\sigma\rangle.$$

We define $A^\alpha := kQ^\alpha/\langle\rho^\alpha\rangle$. This is called the *unifying algebra* of A by α. The image of the composition $\lambda_\alpha\lambda_\sigma$ is the subalgebra of A generated by all the arrows $\beta \notin \alpha$, $\sum_{i=1}^n \alpha_i$ and idempotents $e_E, e_i, i \in Q_0\backslash E$.

Next, we shall interpret A^α as a pullback algebra. Actually, A^α fits into the following pullback diagram of algebra homomorphisms:

$$\begin{tikzcd}
A^\alpha \arrow[r, "\lambda_\alpha"] \arrow[d, "\pi"'] & A^\sigma \arrow[d, "\pi"] \\
k^{\Delta_0^\sigma} \arrow[r, "\lambda"] & (k\Delta)^\sigma/\langle\sum_{i=1}^n \alpha_i\rangle
\end{tikzcd}$$

The vertical homomorphisms in the above diagram are obviously defined.

Lemma 4.4. *The algebra $(k\Delta)^\sigma/\langle\sum_{i=1}^n \alpha_i\rangle$ is radical-square zero.*

Proof. If $x \notin \{1,\dots,n\}$, then $x \neq y$ and $(k\Delta)^\sigma/\langle\sum_{i=1}^n \alpha_i\rangle$ is radical-square zero. Without loss of generality, we now assume that α_1 is a loop in the quiver Δ. Then, none of $\alpha_2,\dots,\alpha_n$ is a loop by the assumption that the vertices $1,\dots,n$ are pairwise distinct. Thus,

$$\alpha_i\alpha_j = 0 \quad \text{for all } i \neq 1 \text{ and all } j \in \{1,\dots,n\}.$$

Further, for each $j \in \{1,\dots,n\}$, the path $\alpha_1\alpha_j = (\sum_{i=1}^n \alpha_i)\alpha_j$ is in $\langle\sum_{i=1}^n \alpha_i\rangle$. Altogether, we have shown that all paths in $(k\Delta)^\sigma$ of length 2 belong to $\langle\sum_{i=1}^n \alpha_i\rangle$. The lemma follows. ∎

Let $kQ/\langle\rho\rangle$ be a finite-dimensional algebra. Let i and j be vertices in Q_0, and let Q_{ij} be the k-vector space with all arrows from i to j as a basis. Then, every vector space automorphism $\chi : Q_{ij} \to Q_{ij}$ extends to an algebra automorphism $\phi_\chi : kQ \to kQ$ which sends $\alpha \in Q_{ij}$ to $(\alpha)\chi$ and preserves all other arrows and all vertices. If $(\langle\rho\rangle)\phi_\chi = \langle\rho\rangle$ for all such automorphisms χ on Q_{ij}, then ρ is said to be (i,j)*-invariant*. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a sub-quiver of Q. We say that ρ is Γ*-invariant* if ρ is (i,j)-invariant for all $i, j \in \Gamma_0$. For example, ρ is Γ-invariant if ρ consists only of monomial relations and there is at most 1 arrow from i to j in Q for any two vertices i, j in Γ_0. Note that ρ is Γ-invariant if and only if ρ^{op} in $kQ^{\mathrm{op}}/\langle\rho^{\mathrm{op}}\rangle$ is Γ^{op}-invariant.

Theorem 4.5. *Let $A := kQ/\langle\rho\rangle$ and $A' := kQ'/\langle\rho'\rangle$. Suppose that the quiver Δ is a sub-quiver of both Q and Q'. Assume that ρ or ρ' is Δ-invariant. If $F : \mathscr{D}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(A')$ is a derived equivalence such that $F(S_i) \simeq S_i'$ for all $i \in \Delta_0$, then A^α and A'^α are derived equivalent.*

Proof. Without loss of generality, we assume that ρ' is Δ-invariant. Further, we assume that the common starting vertex x of $\alpha_1,\dots,\alpha_n$ is not in E. The case that $x \in E$ can be proved similarly. Let $\widetilde{\Delta}$ be the full sub-quiver of Q defined by Δ_0. Then, Δ is a sub-quiver of $\widetilde{\Delta}$ with the same vertices and (possibly) less arrows. Let $B := k\widetilde{\Delta}/\langle\widetilde{\Delta}_2\rangle$ and $\Lambda := (k\Delta)^\sigma/\langle\sum_{i=1}^n \alpha_i\rangle$. Then, by Lemma 4.4, there is a canonical surjective homomorphism $\pi : B^\sigma \to \Lambda$ of algebras.

Let $T^\bullet$ be a basic, radical tilting complex associated with the derived equivalence F. Set $U := \bigoplus_{i\in Q_0\setminus\Delta_0} Ae_i$. Since $F(S_i)\simeq S_i'$ for all $i\in\Delta_0$, we can assume $T^\bullet = U^\bullet\oplus V_x^\bullet\oplus V_1^\bullet\oplus\cdots\oplus V_n^\bullet$ by Lemmas 2.4 and 2.7, where $V_i^\bullet$ is a complex in $\mathscr{K}^{\rm b}(A\text{-proj})$ such that, for each $i\in\Delta_0$, $V_i^0 = Ae_i\oplus U_i$ for some $U_i\in \mathrm{add}(U)$ and $V_i^j\in\mathrm{add}(U)$ for all $j\neq 0$. Note that there is a commutative diagram

$$\begin{tikzcd}
A^\sigma \arrow[r, "\pi"] \arrow[d, "\lambda_\sigma"] & B^\sigma \arrow[r, "\pi"] \arrow[d, "\lambda_\sigma"] & k^\sigma \arrow[d, "\lambda_\sigma"] \\
A \arrow[r, "\pi"] & B \arrow[r, "\pi"] & k^E
\end{tikzcd}$$

where the horizontal maps are the canonical maps. The right-hand square and the entire square are pullback diagrams of algebras. This implies that the left-hand square is also a pullback diagram. It is easy to see that $B\otimes_A U = 0$ and there is an isomorphism of stalk complexes in $\mathscr{C}(B)$:

$$h^\bullet : B\otimes_A T^\bullet = B\otimes_A\left(Ae_x\oplus\bigoplus_{i=1}^n Ae_i\right)\to B\simeq B\otimes_{B^\sigma} B^\sigma.$$

By the proof of Theorem 1.1, $T_\sigma^\bullet := M(T^\bullet, B^\sigma, h^\bullet)$ is a tilting complex over A^σ with $\mathrm{End}_{\mathscr{K}(A^\sigma)}(T_\sigma^\bullet)\simeq A'^\sigma$. Moreover, there is a pullback diagram

$$\begin{tikzcd}
\mathrm{End}_{\mathscr{K}(A^\sigma)}(T_\sigma^\bullet) \arrow[r, "\varepsilon_1"] \arrow[d, "\varepsilon_2\mu"] & \mathrm{End}_{\mathscr{K}(A)}(T^\bullet) \arrow[d, "\eta\mu"] \\
B^\sigma \arrow[r, "\lambda_\sigma"] & B,
\end{tikzcd}$$

where η is determined by $T^\bullet\to B$, ε_1 and ε_2 are determined by the projections from $T_\sigma^\bullet$ to $T^\bullet$ and B^σ, respectively, and μ is the canonical isomorphism from $\mathrm{End}({}_\Lambda\Lambda)$ to Λ.

By assumption, $F(S_i)\simeq S_i'$ for all $i\in\Delta_0$. It follows that $\mathrm{Ext}^1_A(S_i,S_j)\simeq\mathrm{Ext}^1_{A'}(S_i',S_j')$ for all $i,j\in\Delta_0$. This indicates that the number of arrows from i to j is equal in both Q and Q'. Hence, we can assume that $\widetilde{\Delta}$ is also a full sub-quiver of Q' with vertices Δ_0. As a consequence, there is a canonical, surjective homomorphism $\pi : A'\to B$ of algebras.

Let $\theta : A'\to \mathrm{End}_{\mathscr{K}(A)}(T^\bullet)$ be an isomorphism of algebras. Note that $\mathrm{End}_B(B)\simeq B$ is radical-square zero by definition. Thus, the map $\theta\eta\mu : A'\to B$ sends the kernel of $\pi : A'\to B$ to zero, and there is an algebra homomorphism $\chi : B\to B$, fixing all idempotents e_i, $i\in\Delta_0$, and satisfying $\theta\eta\mu = \pi\chi$. It follows that χ induces an automorphism of the vector space e_iBe_j which is isomorphic to the vector space Q'_{ij} for all $i,j\in\Delta_0$. Since ρ' is Δ-invariant, there is an automorphism $\phi_\chi : A'\to A'$ extending χ, that is, $\phi_\chi\pi = \pi\chi$. Thus, $\theta^{-1}\phi_\chi\pi = \eta\mu$; that is, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{End}_{\mathscr{K}(A)}(T^\bullet) \arrow[r, "\eta\mu"] \arrow[d, "\theta^{-1}\phi_\chi", "\simeq"'] & B \arrow[d, equal] & B^\sigma \arrow[l, "\lambda_\sigma"'] \arrow[d, equal] \\
A' \arrow[r, "\pi"] & B & B^\sigma \arrow[l, "\lambda_\sigma"']
\end{tikzcd}$$

It then follows that there is an isomorphism ψ from the pullback algebra $\mathrm{End}_{\mathscr{K}(A^\sigma)}(T_\sigma^\bullet)$ of $\eta\mu$ and λ_σ to the pullback algebra A'^σ of π and λ_σ such that the diagram

$$\begin{tikzcd}
\mathrm{End}_{\mathscr{K}(A^\sigma)}(T_\sigma^\bullet) \arrow[r, "\varepsilon_2"] \arrow[d, "\psi"] & \mathrm{End}_{B^\sigma}(B^\sigma) \arrow[d, "\mu"] \\
A'^\sigma \arrow[r, "\pi"] & B^\sigma
\end{tikzcd}$$

is commutative. This diagram can be extended to a commutative diagram

$$\begin{tikzcd}
\mathrm{End}_{\mathscr{K}(A^\sigma)}(T_\sigma^\bullet) \arrow[r, "\varepsilon_2"] \arrow[d, "\psi", "\simeq"'] & \mathrm{End}_{B^\sigma}(B^\sigma) \arrow[r, "p"] \arrow[d, "\mu", "\simeq"'] & \mathrm{End}_\Lambda(\Lambda) \arrow[d, "\mu", "\simeq"'] & \mathrm{End}_{k\Delta_0^\sigma}(k\Delta_0^\sigma) \arrow[l, "i"'] \arrow[d, "\mu", "\simeq"'] \\
A'^\sigma \arrow[r, "\pi"] & B^\sigma \arrow[r, "\pi"] & \Lambda & k\Delta_0^\sigma \arrow[l, "\lambda"']
\end{tikzcd}$$

where p and i are determined by π and λ, respectively. Hence, the pullback algebra A'^α of $\pi : A'^\sigma \to \Lambda$ and λ is isomorphic to the pullback algebra of $\varepsilon_2 p$ and i. Note that

$$\Lambda \otimes_{k\Delta_0^\sigma} k\Delta_0^\sigma \simeq \Lambda \simeq \Lambda \otimes_{B^\sigma} B^\sigma \simeq \Lambda \otimes_{B^\sigma} B^\sigma \otimes_{A^\sigma} T_\sigma^\bullet$$

in $\mathscr{C}(\Lambda)$. By the proof of Theorem 1.1, the pullback algebra of $\varepsilon_2 p$ and i is derived equivalent to the pullback algebra A^α of $\pi : A^\sigma \to \Lambda$ and $\lambda : k\Delta_0^\sigma \to \Lambda$. Consequently, A'^α is derived equivalent to A^α. ■

Remark 4.6. (1) The derived equivalence constructed in Theorem 4.5 sends the simple A^α-modules corresponding to x and y again to simple A'^α-modules. Thus, Theorem 4.5 can be applied repeatedly.

(2) Note that two algebras A and B are derived equivalent if and only if so are their opposite algebras A^{op} and B^{op}. This means that Theorem 4.5 holds true for the subquiver Δ^{op}.

4.3. Derived equivalences from identifying socle elements

In this section, we introduce the third operation, called identifying socle elements of algebras.

Let A be a basic Artin algebra with the Jacobson radical r_A, and let $1_A = e_1 + \cdots + e_n$ be a decomposition of 1_A into pairwise orthogonal primitive idempotents. Fix $i, j \in \{1, \ldots, n\}$. A *longest* (e_i, e_j)*-element* in A is a nonzero element $a \in e_i r_A e_j$ such that $r_A a = 0 = a r_A$, that is, $a \in \mathrm{soc}(r_A e_j) \cap \mathrm{soc}(e_i r_A)$. In this case, the ideal $\langle a \rangle$ of A generated by a is 1-dimensional and contained in $\mathrm{soc}({}_A A e_j) \cap \mathrm{soc}(e_i A_A)$. A longest (e_i, e_i)-element is called a *complete* e_i*-cycle*.

For the rest of this section, we fix two algebras $A := kQ/\langle \rho \rangle$ and $B := k\Gamma/\langle \omega \rangle$ given by quivers with relations. Suppose that a is a longest (e_i, e_j)-element in A and that b is a longest (e_s, e_t)-element in B, where $i, j \in Q_0$ and $s, t \in \Gamma_0$. We glue i and s into a new vertex u, and glue j and t into another new vertex v. Let σ be the corresponding partition of the set $\{i,j,s,t\}$. In case that $i=j$ or $s=t$, we actually glue all the vertices $\{i,j,s,t\}$ into one vertex, that is, $u=v$. Let $(A\times B)^{\sigma}$ be the σ-gluing algebra defined in Section 4.1. In case $i=j$ and $s=t$, we simply write $A\,{}_{e_i}\!\times_{e_s} B$ for $(A\times B)^{\sigma}$. Note that $a-b$ is a longest (e_u,e_v)-element in $(A\times B)^{\sigma}$ and $\langle a-b\rangle$ is a 1-dimensional ideal in $(A\times B)^{\sigma}$. So we can define a new algebra

$$A\,{}_a\!\diamond_b\, B := (A\times B)^{\sigma}/\langle a-b\rangle.$$

It is called the algebra of *identifying socle elements* in A and B.

Suppose that $A' := kQ'/\langle\rho'\rangle$ is another algebra and there is a derived equivalence $F:\mathscr{D}^{\rm b}(A)\to\mathscr{D}^{\rm b}(A')$ such that $F(S_i)\simeq S_{i'}$ and $F(S_j)\simeq S_{j'}$ for some $i',j'\in Q'_0$. Let $T^{\bullet}$ be a basic, radical tilting complex over A associated with F. We may identify A' with $\operatorname{End}_{\mathscr{K}^{\rm b}(A)}(T^{\bullet})$ via the isomorphism $\operatorname{End}_{\mathscr{K}^{\rm b}(A)}(T^{\bullet})\to A'$ induced by F. Further, by Lemma 2.4, both Ae_i and Ae_j only occur in degree 0 with the multiplicity 1 in $T^{\bullet}$. For $x\in\{i,j\}$, let $T_x^{\bullet}$ be the indecomposable direct summand of $T^{\bullet}$ such that Ae_x is a direct summand of T_x^0, namely, $T_x^0=Ae_x\oplus P_x$, and let $e_{x'}$ be the primitive idempotent element in A' corresponding to the summand $T_x^{\bullet}$. Let $m_a: T_i^{\bullet}\to T_j^{\bullet}$ be the (well-defined) particular morphism

$$\begin{tikzcd}
\cdots \arrow[r] & T_i^{-1} \arrow[r] \arrow[d, "0"] & Ae_i\oplus P_i \arrow[r] \arrow[d, "\left[\begin{smallmatrix} \cdot a & 0\\ 0 & 0\end{smallmatrix}\right]"] & T_i^{1} \arrow[r] \arrow[d, "0"] & \cdots \\
\cdots \arrow[r] & T_j^{-1} \arrow[r] & Ae_j\oplus P_j \arrow[r] & T_j^{1} \arrow[r] & \cdots
\end{tikzcd}$$

and let a' be the composition $T^{\bullet}\to T_i^{\bullet}\xrightarrow{m_a}T_j^{\bullet}\to T^{\bullet}$, where the first and last morphisms are the canonical projection and injection, respectively.

Lemma 4.7. *The element a' just defined is a longest $(e_{i'},e_{j'})$-element in A'.*

Proof. Since $a\in e_i r_A e_j$ is nilpotent, the element a' is nilpotent and lies in $e_{i'}r_{A'}e_{j'}$. It remains to show $r_{A'}a'=0$ and $a'r_{A'}=0$.

Since the complex $T^{\bullet}$ is basic, the algebra A' is basic. Let $g^{\bullet}:T^{\bullet}\to T^{\bullet}$ be an endomorphism in $\mathscr{K}^{\rm b}(A)$ such that $g^{\bullet}$ lies in $r_{A'}$. Then, $g^{\bullet}$ is nilpotent; that is, $(g^{\bullet})^m$ is null-homotopic for some integer $m\geq 1$. Particularly, $(g^0)^m=h^0d^{-1}+d^0h^1$ for some homomorphisms $h^0:T^0\to T^{-1}$ and $h^1:T^1\to T^0$ of A-modules. Since the differentials of $T^{\bullet}$ are radical by assumption, the homomorphism $(g^0)^m$ is radical. Note that T^0 can be written as $Ae_i\oplus U$ with $Ae_i\notin\operatorname{add}(U)$. Then, the homomorphism g^0 can be rewritten as

$$g^0=\begin{bmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{bmatrix}:\quad Ae_i\oplus U\to Ae_i\oplus U.$$

Since Ae_i does not lie in $\operatorname{add}(U)$, the homomorphisms ϕ_{12} and ϕ_{21} are radical. Thus, the $(1,1)$-component of $(g^0)^m$ as a 2×2 matrix is the sum of ϕ_{11}^m and a radical homomorphism. Hence, ϕ_{11}^m is radical, and therefore so is ϕ_{11} because Ae_i is indecomposable.

Hence, $g^0 p$ is radical, where $p : T^0 \to Ae_i$ is the canonical projection. Now, the fact $r_A a = 0$ indicates that the composition $Ae_l \xrightarrow{r} Ae_i \xrightarrow{\cdot a} Ae_j$ is zero for all $l \in Q_0$ and all radical homomorphisms r. It follows that $(g^\bullet \pi^\bullet m_a)^0 = [g^0 p(\cdot a), 0] = 0$, where $\pi^\bullet : T^\bullet \to T_i^\bullet$ is the canonical projection. Since m_a is zero for all non-zero degrees, the chain map $g^\bullet \pi^\bullet m_a$ is zero in all degrees, and consequently $g^\bullet a' = 0$. This shows $r_{A'} a' = 0$. Similarly, we can prove $a' r_{A'} = 0$ by the fact $a r_A = 0$. ■

The next theorem shows that the derived equivalence between A and A' can be extended by identifying socle elements.

Theorem 4.8. *The algebras $A\ {}_a\!\diamond_b\ B$ and $A'\ {}_{a'}\!\diamond_b\ B$ are derived equivalent.*

Proof. For simplicity, we write Λ for $(A \times B)^\sigma$. As explained in Section 4.1, Λ is the pullback algebra of the canonical surjective homomorphisms $B \to k^\sigma$ and $A \to k^\sigma$. Let $\sigma' = \{i', s\} \cup \{j', t\}$ be the corresponding partition of $\{i', j', s, t\}$. By the proof of Theorem 1.1, $\widetilde{T}^\bullet := M(T^\bullet, B, 1)$ is a tilting complex over Λ with the endomorphism algebra isomorphic to $(A' \times B)^{\sigma'}$. By definition, $\widetilde{T}_i^\bullet := M(T_i^\bullet, Be_s, 1)$ and $\widetilde{T}_j^\bullet := M(T_j^\bullet, B_t, 1)$ are indecomposable direct summands of $\widetilde{T}^\bullet$. Note that all other indecomposable direct summands of $\widetilde{T}^\bullet$ are of the form $M(P^\bullet, 0, 0)$ or $M(0, Q, 0)$, where $P^\bullet$ is an indecomposable direct summand of $T^\bullet$ and Q is an indecomposable projective B-module. Moreover, Λe_u and Λe_v, which are isomorphic to $M(Ae_i, Be_s, 1)$ and $M(Ae_j, Be_t, 1)$, respectively, only occur in degree 0 with the multiplicity 1 in $\widetilde{T}^\bullet$. Thus, $\widetilde{T}^\bullet$ is a basic, radical complex over Λ.

Let $I := \langle a - b \rangle$. Then, $I e_v = I = e_u I$ and $IX = 0$ for all indecomposable projective Λ-modules X not isomorphic to Λe_v. It follows that $I\widetilde{T}^\bullet = I\widetilde{T}^0 \simeq {}_\Lambda I$. As ${}_\Lambda I$ is a simple Λ-module with $e_u I \neq 0$, $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\widetilde{T}^\bullet, I\widetilde{T}^\bullet[l]) \simeq \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(\widetilde{T}^\bullet, I[l]) = 0$ for all $l \neq 0$. Now, the short exact sequence $0 \to I\widetilde{T}^\bullet \to \widetilde{T}^\bullet \to \widetilde{T}^\bullet / I\widetilde{T}^\bullet \to 0$ in $\mathscr{C}(\Lambda)$ gives rise to a triangle $I\widetilde{T}^\bullet \to \widetilde{T}^\bullet \to \widetilde{T}^\bullet/I\widetilde{T}^\bullet \to I\widetilde{T}^\bullet[1]$ in $\mathscr{D}^{\mathrm{b}}(\Lambda)$. Applying $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}(\widetilde{T}^\bullet, -)$ to this triangle, we get an exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}\big(\widetilde{T}^\bullet, \widetilde{T}^\bullet/I\widetilde{T}^\bullet[-1]\big) \to \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}(\widetilde{T}^\bullet, I\widetilde{T}^\bullet) \to \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(\Lambda)}(\widetilde{T}^\bullet, \widetilde{T}^\bullet),$$

which is isomorphic to

$$0 \to \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}\big(\widetilde{T}^\bullet, \widetilde{T}^\bullet/I\widetilde{T}^\bullet[-1]\big) \to \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\widetilde{T}^\bullet, I\widetilde{T}^\bullet) \xrightarrow{\theta} \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\widetilde{T}^\bullet, \widetilde{T}^\bullet). \quad (\sharp)$$

Observe that the map $\cdot(a-b) : \Lambda e_u \to {}_\Lambda I$ induces a morphism $g^\bullet$ in $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\Lambda)}(\widetilde{T}_i^\bullet, \widetilde{T}_j^\bullet)$:

$$\begin{tikzcd}
\widetilde{T}_i^\bullet \arrow[dd, "g^\bullet"] & \cdots \arrow[r] & T_i^{-1} \arrow[r, "d"] \arrow[d] & \Lambda e_u \oplus P_i \arrow[r] \arrow[d, "{\left[\begin{smallmatrix} \cdot(a-b) \\ 0 \end{smallmatrix}\right]}"] & T_i^1 \arrow[r] \arrow[d] & 0 \\
 & & 0 \arrow[r] \arrow[d] & I \arrow[r] \arrow[d, "{[1,0]}"] & 0 \arrow[d] & \\
\widetilde{T}_j^\bullet & 0 \arrow[r] & T_j^{-1} \arrow[r, "d"] & \Lambda e_v \oplus P_j \arrow[r] & T_j^1 \arrow[r] & 0.
\end{tikzcd}$$

The image of g^0 is $I\Lambda e_v = I$. It follows that $g^\bullet$ cannot be null-homotopic because the image of any morphism from T_j^{-1} or T_i^1 to Λe_v has image contained in Ae_j which intersects I trivially. Hence, $g^\bullet \neq 0$, and therefore

$$\tilde{g}^\bullet := \begin{bmatrix} g^\bullet & 0 \\ 0 & 0 \end{bmatrix}$$

is a nonzero endomorphism of $\widetilde{T}^\bullet$ and lies in $\mathrm{Im}(\theta)$ (see the sequence ($\sharp$)). Note that

$$\mathrm{Hom}_{\mathscr{K}^b(\Lambda)}(\widetilde{T}^\bullet, I\widetilde{T}^\bullet) \simeq \mathrm{Hom}_{\mathscr{K}^b(\Lambda)}(\widetilde{T}^\bullet, {}_\Lambda I) \simeq \mathrm{Hom}_\Lambda(\Lambda e_u, {}_\Lambda I) \simeq e_u I = I$$

and I is 1-dimensional. Hence, θ is an injective map and $\mathrm{Im}(\theta)$ is a 1-dimensional k-space with $\tilde{g}^\bullet$ as a basis. It follows from ($\sharp$) that $\mathrm{Hom}_{\mathscr{K}^b(\Lambda)}(\widetilde{T}^\bullet, \widetilde{T}^\bullet/I\widetilde{T}^\bullet[-1]) = 0$. Thus,

$$\mathrm{Hom}_{\mathscr{K}^b(\Lambda)}\left(\widetilde{T}^\bullet/I\widetilde{T}^\bullet, \widetilde{T}^\bullet/I\widetilde{T}^\bullet[-1]\right) \simeq \mathrm{Hom}_{\mathscr{K}^b(\Lambda)}\left(\widetilde{T}^\bullet, \widetilde{T}^\bullet/I\widetilde{T}^\bullet[-1]\right) = 0.$$

Now, by [8, Theorem 4.2], the algebras Λ/I and $\mathrm{End}_{\mathscr{K}^b(\Lambda)}(\widetilde{T}^\bullet)/\mathrm{Im}(\theta)$ are derived equivalent. It is easy to check that the isomorphism $\mathrm{End}_{\mathscr{K}^b(\Lambda)}(\widetilde{T}^\bullet) \simeq (A' \times B)^{\sigma'}$, induced by the projections $\Lambda \to A$ and $\Lambda \to B$, sends the element $\tilde{g}^\bullet$ in $\mathrm{End}_{\mathscr{K}^b(\Lambda)}(\widetilde{T}^\bullet)$ to $a' - b$ in $(A' \times B)^{\sigma'}$. As a result, $\mathrm{End}_{\mathscr{K}^b(\Lambda)}(\widetilde{T}^\bullet)/\mathrm{Im}(\theta) \simeq A'\,{}_{a'}\!\diamond_b\, B$. It follows from $\Lambda/I = A\,{}_a\!\diamond_b\, B$ that $A\,{}_a\!\diamond_b\, B$ is derived equivalent to $A'\,{}_{a'}\!\diamond_b\, B$. ■

Note that the derived equivalence in Theorem 4.8 sends the simple modules over $A\,{}_a\!\diamond_b\, B$ corresponding to u and v again to the simple modules over $A'\,{}_{a'}\!\diamond_b\, B$ corresponding to u' and v', respectively.

A special case of Theorem 4.8 is that complete cycles have the same starting and ending vertices.

Corollary 4.9. *Suppose that e and f are primitive idempotent elements in A and B, respectively, and that $a \in A$ is a complete e-cycle and $b \in B$ is a complete f-cycle. Let $T^\bullet$ be a basic, radical tilting complex over A with $[T^\bullet : Ae] = 1$, and let $A' = \mathrm{End}_{\mathscr{K}^b(A)}(T^\bullet)$. Then, $A\,{}_a\!\diamond_b\, B$ and $A'\,{}_{a'}\!\diamond_b\, B$ are derived equivalent.*

Finally, we illustrate our constructions by an example.

Example 4.10. Let A and A' be k-algebras given by the following quivers with relations, respectively:

$$A: \begin{tikzcd} \overset{1}{\bullet} \arrow[r, shift left, "\alpha"] & \overset{2}{\bullet} \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\beta"] & \overset{3}{\bullet} \arrow[l, shift left, "\gamma"] \end{tikzcd} \qquad A': \begin{tikzcd} \overset{1}{\bullet} \arrow[r, "\alpha'"] & \overset{2}{\bullet} \arrow[dl, "\beta'"] \\ \underset{3}{\bullet} \arrow[u, "\gamma'"] & \end{tikzcd}$$

$$\alpha\delta\alpha, \gamma\delta, \delta\alpha - \beta\gamma\ ; \qquad \alpha'\beta'\gamma'\alpha',\ \gamma'\alpha'\beta'\gamma'.$$

Let e_i be the primitive idempotent element of A corresponding to the vertex i and $e := e_1$. Then, there is a tilting complex $T^\bullet = T_1^\bullet \oplus Ae_2[1] \oplus Ae_3[1]$ over A, where $T_1^\bullet$ is the complex: $0 \to Ae_2 \xrightarrow{\cdot\delta} Ae \to 0$ with Ae in degree 0. Explicit calculations show that $\operatorname{End}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet)$ is the algebra A' given by the quiver with relations above. The vertices $1, 2$ and 3 in A' correspond to the indecomposable direct summands $T_1^\bullet$, $Ae_2[1]$ and $Ae_3[1]$, respectively.

Since Ae occurs in $T^\bullet$ only in degree zero with multiplicity 1, it follows from Lemma 2.4 that the derived equivalence induced by $T^\bullet$ sends the simple A-module S_1 corresponding to the vertex 1 to the simple A'-module S_1' corresponding to the vertex 1.

If we glue a vertex of an arbitrary algebra B with the vertex 1 in A and A', respectively, that is, we apply gluing vertex operation in $A \times B$ and $A' \times B$, respectively, then the resulting algebras Λ and Λ' are derived equivalent by Corollary 4.3. For instance, we take B to be the algebra

$$\begin{tikzcd} \overset{4}{\bullet} \arrow[r, "\eta"] & \overset{1}{\bullet} \arrow[loop right, "\varepsilon"] \end{tikzcd}, \qquad \varepsilon^n = 0,$$

and glue the vertex 1 in A with the vertex 1 in B as well as the vertex 1 in A' with 1 in B. Then, the gluing algebras Λ and Λ' are derived equivalent. Visually, they are given by the following quivers with relations, respectively:

$$\Lambda : \begin{tikzcd} \overset{4}{\bullet} \arrow[r, "\eta"] & \overset{1}{\bullet} \arrow[r, shift left, "\alpha"] \arrow[loop below, "\varepsilon"] & \overset{2}{\bullet} \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\beta"] & \overset{3}{\bullet} \arrow[l, shift left, "\gamma"] \end{tikzcd} \qquad \Lambda' : \begin{tikzcd} \overset{4}{\bullet} \arrow[r, "\eta"] & \overset{1}{\bullet} \arrow[r, "\alpha'"] \arrow[loop left, "\varepsilon"] & \overset{2}{\bullet} \arrow[dl, "\beta'"] \\ & 3\bullet \arrow[u, "\gamma'"] & \end{tikzcd}$$

$$\alpha\delta\alpha, \gamma\delta, \delta\alpha - \beta\gamma;\ \varepsilon^n, \eta\alpha, \varepsilon\alpha, \delta\varepsilon. \qquad \alpha'\beta'\gamma'\alpha',\ \gamma'\alpha'\beta'\gamma';\ \varepsilon^n, \eta\alpha', \varepsilon\alpha', \gamma'\varepsilon.$$

The above gluing operation of quivers of A and B is nothing else than forming a Milnor square which can be pictured as follows:

$$\begin{tikzcd} \overset{4}{\bullet} \arrow[r, "\eta"] & \overset{1}{\bullet} \arrow[r, shift left, "\alpha"] \arrow[loop below, "\varepsilon"] & \overset{2}{\bullet} \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\beta"] & \overset{3}{\bullet} \arrow[l, shift left, "\gamma"] & {} \arrow[r] & {} & \overset{4}{\bullet} \arrow[r, "\eta"] & \overset{1}{\bullet} \arrow[loop right, "\varepsilon"] \\ & & {} \arrow[d] & & & & & {} \arrow[d] \\ & \overset{1}{\bullet} \arrow[r, shift left, "\alpha"] & \overset{2}{\bullet} \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\beta"] & \overset{3}{\bullet} \arrow[l, shift left, "\gamma"] & {} \arrow[r] & {} & & \bullet 1 \end{tikzcd}$$

Note that the socle elements $\alpha\delta, \varepsilon^{n-1}$ in $A \times B$ and $\alpha'\beta'\gamma', \varepsilon^{n-1}$ in $A' \times B$ fulfill the conditions in Corollary 4.9. Hence, by identifying socle elements, we also have a derived equivalence between the quotient algebras $\Lambda/(\alpha\delta - \varepsilon^{n-1})$ and $\Lambda'/(\alpha'\beta'\gamma' - \varepsilon^{n-1})$.

Acknowledgments. Part of the primary manuscript was revised during a visit of the corresponding author CCX to the University of Stuttgart in 2018. He would like to thank Steffen Koenig for invitation and the Alexander von Humboldt-Stiftung for support.

Funding. The research work was partially supported by NSFC (12031014 and 12226314).

References

[1] M. Broué, Isométries de caractères et équivalences de Morita ou dérivées. *Publ. Math. Inst. Hautes Études Sci.* (1990), no. 71, 45–63 Zbl 0727.20005 MR 1079643

[2] W. Bruns and J. Gubeladze, *Polytopes, rings, and K-theory*. Springer Monogr. Math., Springer, Dordrecht, 2009 MR 2508056

[3] M. Cabanes, On Okuyama's theorems about Alvis-Curtis duality. 2008, arXiv:0810.0952v1

[4] J. Chuang and R. Rouquier, Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification. *Ann. of Math. (2)* **167** (2008), no. 1, 245–298 Zbl 1144.20001 MR 2373155

[5] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*. London Math. Soc. Lecture Note Ser. 119, Cambridge University Press, Cambridge, 1988 Zbl 0635.16017 MR 0935124

[6] W. Hu, On iterated almost ν-stable derived equivalences. *Comm. Algebra* **40** (2012), no. 10, 3920–3932 Zbl 1278.18024 MR 2982908

[7] W. Hu and C. Xi, Derived equivalences and stable equivalences of Morita type, I. *Nagoya Math. J.* **200** (2010), 107–152 Zbl 1275.16007 MR 2747880

[8] W. Hu and C. Xi, Derived equivalences for Φ-Auslander-Yoneda algebras. *Trans. Amer. Math. Soc.* **365** (2013), no. 11, 5681–5711 Zbl 1279.18008 MR 3091261

[9] W. Hu and C. Xi, Derived equivalences and stable equivalences of Morita type, II. *Rev. Mat. Iberoam.* **34** (2018), no. 1, 59–110 Zbl 1431.16012 MR 3763342

[10] B. Keller, Invariance and localization for cyclic homology of DG algebras. *J. Pure Appl. Algebra* **123** (1998), no. 1-3, 223–273 Zbl 0890.18007 MR 1492902

[11] S. Koenig and D. Yang, Silting objects, simple-minded collections, t-structures and co-t-structures for finite-dimensional algebras. *Doc. Math.* **19** (2014), 403–438 Zbl 1350.16010 MR 3178243

[12] R. Martínez-Villa, Algebras stably equivalent to l-hereditary. In *Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979)*, pp. 396–431, Lecture Notes in Math. 832, Springer, Berlin, 1980 Zbl 0443.16019 MR 0607166

[13] J. Milnor, *Introduction to algebraic K-theory*. Ann. of Math. Stud. 72, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1971 Zbl 0237.18005 MR 0349811

[14] J. Rickard, Morita theory for derived categories. *J. London Math. Soc. (2)* **39** (1989), no. 3, 436–456 Zbl 0642.16034 MR 1002456

[15] J. Rickard, Derived equivalences as derived functors. *J. London Math. Soc. (2)* **43** (1991), no. 1, 37–48 Zbl 0683.16030 MR 1099084

[16] J. Rickard, Equivalences of derived categories for symmetric algebras. *J. Algebra* **257** (2002), no. 2, 460–481 Zbl 1033.20005 MR 1947972

[17] R. Rouquier, Derived equivalences and finite dimensional algebras. In *International Congress of Mathematicians. Vol. II*, pp. 191–221, Eur. Math. Soc., Zürich, 2006 Zbl 1108.20006 MR 2275594

[18] J.-L. Verdier, Catégories dérivées: quelques résultats (état 0). In *Cohomologie étale*, pp. 262–311, Lecture Notes in Math. 569, Springer, Berlin, 1977 Zbl 0407.18008 MR 3727440

[19] C. Xi, On the finitistic dimension conjecture. II. Related to finite global dimension. *Adv. Math.* **201** (2006), no. 1, 116–142 Zbl 1103.18011 MR 2204752

[20] C. Xi, Derived equivalences of algebras. *Bull. Lond. Math. Soc.* **50** (2018), no. 6, 945–985 Zbl 1423.18046 MR 3891936

Received 27 August 2023; revised 9 September 2024.

Wei Hu
School of Mathematical Sciences, Beijing Normal University, 10875 Beijing, China;
huwei@bnu.edu.cn

Changchang Xi (corresponding author)
School of Mathematical Sciences, Capital Normal University, 100048 Beijing; School of Mathematics and Statistics, Shaanxi Normal University, 710119 Xi'an, China; xicc@cnu.edu.cn