MINIMAL ELEMENTS OF THE POSET OF A HAMMOCK

CHANGCHANG XI

1. *Introduction*

Let k be an algebraically closed field and A a finite-dimensional k-algebra. As usual we assume that A is basic and connected. Thus A is a factor-algebra of the path algebra of a quiver $\Delta = (\Delta_0, \Delta_1)$ by an admissible ideal I. By A-mod we denote the category of all finitely generated left A-modules and by A-ind a full subcategory of A-mod consisting of the representatives of isomorphism classes of all indecomposable modules. Let $x \in \Delta_0$; we denote by $P(x)$, $Q(x)$ and $E(x)$ the indecomposable projective A-module, the indecomposable injective module and the simple module at the vertex x, respectively. Now we consider the set

$$S_x^A = \{M \in A-\mathrm{ind}\,|\, M \not\cong P(x), \mathrm{Hom}\,(P(x), M) \neq 0 \text{ and } \mathrm{Hom}\,(P(x), \tau M) = 0\},$$

where τ stands for the Auslander–Reiten translation, and we define on S_x^A the relation $X \leqslant Y$ if and only if there is a homomorphism $f\colon Y \to X$ such that $\mathrm{Hom}\,(P(x), f) \neq 0$. Let Γ_A denote the Auslander–Reiten quiver of A. Thus the vertices of Γ_A are isomorphism classes $[X]$ of A-modules X in A-ind. One defines a function $h_x\colon (\Gamma_A)_0 \to \mathbb{N}$ by $h_x([X]) = \dim \mathrm{Hom}\,(P(x), X)$ which is the number of times $E(x)$ occurs as a composition factor of the A-module X. Let A be representation-directed. Then we call the function h_x a hammock function and its support $H(x)$, as a full subquiver of Γ_A, a hammock which starts from $[P(x)]$ and ends at $[Q(x)]$. It was shown in [7] that $(S_x^A, \leqslant)$ is a poset and every hammock $H(x)$ is an Auslander–Reiten quiver of a poset which is isomorphic to $(S_x^A, \leqslant)$.

In the present paper we try to describe the minimal elements in S_x^A without using any knowledge of Γ_A. We approach this by two separate routes. One is a theoretical characterization in terms of minimal add S_x^A-approximation (see Definition 2.3). The other route shows how to construct the modules considered as minimal elements in S_x^A; there we use only the sink map ending at the indecomposable injective module $Q(x)$ corresponding to x.

The main result is the following.

THEOREM. *Suppose that A is a representation-directed algebra. Let*

$$\bigoplus_{i=1}^{m} X_i \longrightarrow Q(x) \quad \textit{and} \quad \bigoplus_{i=1}^{n} R_i \xrightarrow{(f_i)_i^t} Q(x)$$

be a minimal right add S_x^A*-approximation of $Q(x)$ and a sink map for $Q(x)$ respectively, where X_i and R_j are indecomposable. Then*

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(1) *$m = n$;*
(2) *$\mathrm{End}_A(\oplus X_i)$ is a semisimple finite-dimensional algebra;*
(3) *$\{X_i \mid i = 1, \ldots, m\}$ is the set of all minimal elements of $(S_x^A, \leqslant)$;*
(4) *for each $i \in \{1, 2, \ldots, m\}$ there is a number $j \in \{1, \ldots, m\}$ such that X_i is isomorphic to $\tau^- \mathrm{Ker}(f_j)$ if f_j is surjective, or to the projective cover of $\mathrm{Cok}(f_j)$ if f_j is injective.*

For further details about hammocks we refer to [4, 7].

2. *The minimal S_x^A-approximation of $Q(x)$*

Let A be an algebra and $x \in \Delta_0$. Put $\mathcal{S}_x^A := (\mathrm{add}\, S_x^A, \mathrm{Hom}(P(x), -))$. Let $\check{\mathcal{U}}(\mathcal{S}_x^A)$ be the subspace category of the vector space category $\mathcal{S}_x^A$, whose objects are of the form

$$V = (V_0, V_\omega, \gamma_V : P(x) \otimes_k V_\omega \longrightarrow V_0),$$

where V_ω is a finite-dimensional vector space over k and $V_0 \in \mathrm{add}\, S_x^A$. A morphism f from V to W is defined as a pair (f_0, f_ω), where $f_\omega : V_\omega \to W_\omega$ is a k-linear map and $f_0 \in \mathrm{Hom}_A(V_0, W_0)$, such that the diagram

$$\begin{tikzcd} P(x) \otimes V_\omega \arrow[r] \arrow[d, "1 \otimes f_\omega"'] & V_0 \arrow[d, "f_0"] \\ P(x) \otimes W_\omega \arrow[r] & W_0 \end{tikzcd}$$

is commutative. We recall the functor

$$\Sigma : \check{\mathcal{U}}(\mathcal{S}_x^A) \longrightarrow A\text{-mod}$$

defined in [8] (see also [5]). Let $V = (V_0, V_\omega, \gamma_V) \in \check{\mathcal{U}}(\mathcal{S}_x^A)$. Put $\Sigma V := \mathrm{Cok}(\gamma_V)$, the cokernel of γ_V. For $f = (f_0, f_\omega)$ define Σf to be the induced map

$$\begin{tikzcd} P(x) \otimes_k V_\omega \arrow[r] \arrow[d, "1 \otimes f_\omega"'] & V_0 \arrow[r] \arrow[d, "f_0"'] & \Sigma V \arrow[r] \arrow[d, "\Sigma f"'] & 0 \\ P(x) \otimes_k W_\omega \arrow[r] & W_0 \arrow[r] & \Sigma W \arrow[r] & 0 \end{tikzcd}$$

from ΣV to ΣW. We assume that there is an object $Q = (Q_0, Q_\omega, \gamma_Q)$ in $\check{\mathcal{U}}(\mathcal{S}_x^A)$ such that $\Sigma Q = Q(x)$, namely, the following exact sequence exists:

$$P(x) \otimes_k Q_\omega \xrightarrow{\gamma_Q} Q_0 = \bigoplus_{i=1}^{m} X_i \xrightarrow{\pi_Q} Q(x) \longrightarrow 0,$$

where $X_i \in S_x^A$ is indecomposable for $i = 1, \ldots, m$. This assumption is always satisfied if the algebra A is of finite representation type (see [5]). We may suppose that Q has no direct summand Q' with $\Sigma Q' = 0$. Now we fix such a choice of Q in $\check{\mathcal{U}}(\mathcal{S}_x^A)$ and put $K = \mathrm{Ker}(\gamma_Q)$ and $B := \mathrm{Im}(\gamma_Q)$. We want to determine the endomorphism ring of Q_0.

The set S_x^A is said to be scalar provided $\dim \mathrm{Hom}(P(x), M) = 1$ for all $M \in S_x^A$. Note that if S_x^A is scalar then $(S_x^A, \leqslant)$ is a poset.

2.1 Lemma. (1) *Q is an indecomposable object in $\check{\mathcal{U}}(\mathcal{S}_x^A)$.*

(2) *If N is a quotient of $P(x)^l$ for some l, and* $M \in \operatorname{add} S_x^A$, *then* $\operatorname{Ext}_A^1(M, N) = 0$.
(3) $\operatorname{Ext}_A^1(Q_0, Q_0) = 0$.

Proof.
(1) If $Q = Q' \oplus Q''$, then $\Sigma Q = \Sigma Q' \oplus \Sigma Q''$.
(2) See [**1**, 5.8].
(3) If we apply $\operatorname{Hom}_A(Q_0, -)$ to the exact sequence

$$0 \longrightarrow B \longrightarrow Q_0 \longrightarrow Q(x) \longrightarrow 0,$$

then we get (3), because $\operatorname{Ext}_A^1(Q_0, Q(x)) = 0$ and $\operatorname{Ext}_A^1(Q_0, B) = 0$ by (2).

2.2 LEMMA. *Let A be an algebra. The following are equivalent:*
(1) $Q(x) \in S_x^A$.
(2) *Q_0 is indecomposable and $Q_0 \cong Q(x)$.*
(3) *If $\bigoplus_{i=1}^n R_i \to Q(x)$ is a sink map ending at $Q(x)$ with indecomposables R_i, then $n = 1$ and* $\dim \operatorname{Hom}(R_1, Q(x)) = \dim \operatorname{End}(Q(x))$.

The proof is obvious.

2.3 DEFINITION [**1**]. (1) A morphism $f: M \to N$ in A-mod is said to be *right minimal* if an endomorphism $g: M \to M$ is an automorphism whenever $gf = f$.

(2) Let $\mathscr{X}$ be a full subcategory of A-mod which is closed under isomorphisms and summands. A morphism $f: X \to C$ in A-mod is said to be a *right $\mathscr{X}$-approximation of C* if $X \in \mathscr{X}$ and the sequence

$$\operatorname{Hom}_A(M, X) \longrightarrow \operatorname{Hom}_A(M, C) \longrightarrow 0$$

induced by f is exact for every $M \in \mathscr{X}$. If f is, in addition, right minimal, then f is said to be a *minimal* right $\mathscr{X}$-approximation.

2.4 LEMMA. *Let* $\mathscr{X} = \operatorname{add} S_x^A$. *Then π_Q is a minimal $\mathscr{X}$-approximation for $Q(x)$.*

Proof. Let $g: Q_0 \to Q_0$ with $g\pi_Q = \pi_Q$, then we can get the following diagram.

$$\begin{tikzcd}
P(x) \otimes Q_\omega \arrow[r] \arrow[d] & Q_0 \arrow[r, "\pi_Q"] \arrow[d, "g"] & Q(x) \arrow[r] \arrow[d, equal] & 0 \\
P(x) \otimes Q_\omega \arrow[r] & Q_0 \arrow[r, "\pi_Q"] & Q(x) \arrow[r] & 0
\end{tikzcd}$$

Since the endomorphism ring of Q is a local ring, we deduce that g is an automorphism. Since $\operatorname{Ext}_A^1(M, B) = 0$ for all M in S_x^A by Lemma 2.1(2), it is clear by definition that π_Q is a right $\mathscr{X}$-approximation.

2.5 DEFINITION [**6**]. An indecomposable A-module M is called *directing* if there is no finite sequence $(M_0 = M, M_1, \ldots, M_t = M)$ of indecomposable modules M_i, $0 \leqslant i \leqslant t$, with $\operatorname{rad}(M_{i-1}, M_i) \neq 0$ for all $1 \leqslant i \leqslant t$. An algebra is called representation-directed provided every indecomposable module is directing.

LEMMA. *Suppose that $Q(x)$ is directing. Then the functor* $\operatorname{Hom}(P(x), -)$ *is faithful on the additive full subcategory* $\operatorname{add} Q_0$.

Proof. By 2.2 we may suppose that $Q(x) \notin S_x^A$. Let $f: X_1 \to X_2$ be a homomorphism with $\operatorname{Hom}(P(x), f) = 0$. Then we consider the following diagram.

$$\begin{tikzcd}
P(x) \otimes_k Q_\omega \arrow[r, "\gamma_Q"] \arrow[dr, "0"'] & X_1 \oplus \bigoplus_{i=2}^{m} X_1 \arrow[r, "\pi_Q"] \arrow[d, "f' = \begin{bmatrix} f \\ 0 \end{bmatrix}"] & Q(x) \arrow[r] \arrow[dl, dashed, "f''"] & 0 \\
& X_2 & &
\end{tikzcd}$$

Since $\gamma_Q f' = 0$, there is a morphism $f'': Q(x) \to X_2$ with $f' = \pi_Q f''$. But

$$\operatorname{Hom}(Q(x), X_2) = 0$$

since $Q(x)$ is directing. Therefore $f = 0$.

2.6 Lemma. *Let A be an algebra and S_x^A scalar. Then there is no homomorphism $g_{ij}: X_i \to X_j$ with $i \neq j$ such that* $\operatorname{Hom}(P(x), g_{ij}) \neq 0$.

Proof. We suppose that for $i = 1$ and $j = 2$ there are $g_{12}: X_1 \to X_2$ and $g_2: X_2 \to Q(x)$ such that $g_{12} g_2 \neq 0$. Note that we have $\dim \operatorname{Hom}_A(X_1, Q(x)) = 1$. Now we consider the diagram

$$\begin{tikzcd}[column sep=huge]
X_1 \oplus X_2 \oplus \bigoplus_{i=3}^{m} X_i \arrow[r, "{[g_{12} g_2, g_2, f]^t = \pi_Q}"] \arrow[d, "\begin{bmatrix} g_2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}"'] & Q(x) \arrow[d, equal] \\
X_2 \oplus \bigoplus_{i=3}^{m} X_i \arrow[r, "\alpha := \begin{bmatrix} g_2 \\ f \end{bmatrix}"'] & Q(x)
\end{tikzcd}$$

which is commutative. It is easy to see that α is a right $\mathscr{X}$-approximation. But this contradicts that π_Q is a minimal $\mathscr{X}$-approximation of $Q(x)$.

2.7 Theorem. *Let A be an algebra and S_x^A scalar. Put $\mathscr{X} = \operatorname{add} S_x^A$. Let $Q_0 \to Q(x)$ be a minimal right $\mathscr{X}$-approximation. If $Q(x)$ is directing, then the endomorphism ring $\operatorname{End}_A(Q_0)$ of Q_0 is a semisimple finite-dimensional algebra.*

Proof. This theorem follows immediately from 2.5 and 2.6.

2.8 Corollary. *If the algebra A is representation-directed, then*
(1) *$\{X_i \mid i = 1, \ldots, m\}$ is the set of all minimal elements in of $(S_x^A, \leqslant)$,*
(2) *$m \leqslant 3$.*

Proof. (1) This is obvious.
(2) By [7, 10, Theorem], the category of S_x^A-spaces of the poset S_x^A is representation-finite, thus it follows from (1) and Kleiner's theorem [**6**, 2.6, Theorem 1] that $m \leqslant 3$.

To complete this section we point out the following fact.

2.9 THEOREM. *Let $P(x)$ be directing and $P(x) \to N$ a source map starting from $P(x)$. Set*

$$\mathcal{M} = \{M \in A-\mathrm{ind} \mid M \text{ is a direct summand of } N\}$$

and assume that every module in $\mathcal{M}$ is directing. If $(S_x^A, \leqslant)$ is scalar, then $\mathcal{M}$ is just the set of all maximal elements of $(S_x^A, \leqslant)$.

The proof of this theorem is dual to that of Theorem 2.7. Since every module in $\mathcal{M}$ is directing, one can easily see that the functor $\mathrm{Hom}(P(x), -)$ is faithful on the full subcategory $\mathrm{add}\, N$. Note that the source map $P(x) \to N$ is a minimal left $\mathcal{X}$-approximation for $P(x)$ since N belongs to $\mathcal{X} := \mathrm{add}\, S_x^A$. Since S_x^A is scalar, one can show as in 2.6 that if M_1 and M_2 are indecomposable summands of N with $M_1 \ncong M_2$, then $\mathrm{Hom}(M_1, M_2) = 0$. Thus the theorem follows.

3. *The minimal elements in S_x^A*

Let

$$\bigoplus_{i=1}^{n} R_i^{d_i} \xrightarrow{(g_i)} Q(x)$$

be a sink map for $Q(x)$ with indecomposable modules R_i and let $R_i \ncong R_j$ if $i \neq j$.

In this section we assume that $Q(x)$ and all the R_i are directing and that

$$d_i = \dim \mathrm{Hom}_A(R_i, Q(x)).$$

Let

$$g_i = (g_j^{(i)})_{1 \leqslant j \leqslant d_i}^t,$$

where the upper index t stands for transpose, and

$$f_i = (g_j^{(i)}) : R_i \longrightarrow Q(x)^{d_i}.$$

Put $K_i = \mathrm{Ker}(f_i)$ and $C_i = \mathrm{Coker}(f_i)$.

3.1 LEMMA. *If f_i is a surjective map, then K_i is indecomposable and $\tau^- K_i \in S_x^A$.*

Proof. Note that f_i is an irreducible map. Thus K_i is indecomposable. The rest of the statement 3.1 follows from the following diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i & \longrightarrow & E_i & \longrightarrow & \tau^- K_i & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_i & \longrightarrow & R_i & \longrightarrow & Q(x)^{d_i} & \longrightarrow & 0 \end{array}$$

where the upper row is an Auslander–Reiten sequence. It is clear that $\mathrm{Hom}_A(K_i, Q(x)) = 0$.

3.2 LEMMA. *If f_i is injective, then C_i is simple. In this case we denote by $P(y_i)$ the projective cover of C_i and have $P(y_i) \in S_x^A$.*

Proof. Suppose that C_i is not simple. Then we can take a simple module S which is a factor module of C_i and consider the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_i & \xrightarrow{f_i} & Q(x)^{d_i} & \xrightarrow{\epsilon_i} & C_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g_1'} & & \| & & \downarrow{\scriptstyle p_i} & & \\ 0 & \longrightarrow & Y & \xrightarrow{g_2'} & Q(x)^{d_i} & \xrightarrow{\epsilon_i p_i} & S & \longrightarrow & 0 \end{array}$$

It is clear that g_2' is not split epimorphism. From the equalities

$$\dim \operatorname{Hom}_A(Q(x)^{d_i}, Q(x)) = \dim \operatorname{Hom}_A(R_i, Q(x)) = \dim \operatorname{Hom}_A(Y, Q(x)) = d_i$$

it follows that g_1' is not split monomorphism. Thus the irreducible map f_i has a non-trivial decomposition. This is a contradiction.

It is obvious that $P(y_i) \in S_x^A$.

3.3 LEMMA. *Let A be representation-directed; then*
(1) *if f_i is surjective,* $\operatorname{Hom}_A(\tau^- K_i, R_i) = 0$,
(2) *if f_i is injective,* $\operatorname{Hom}_A(P(y_i), R_i) = 0$.

Proof. (1) We apply $\operatorname{Hom}_A(\tau^- K_i, -)$ to the exact sequence

$$0 \longrightarrow K_i \longrightarrow R_i \longrightarrow Q(x) \longrightarrow 0$$

and obtain the following long exact sequence

$$0 \longrightarrow (\tau^- K_i, R_i) \longrightarrow (\tau^- K_i, Q(x)) \xrightarrow{\delta} {}^1(\tau^- K_i, K_i) \longrightarrow \cdots$$

(we abbreviate (X, Y) for $\operatorname{Hom}_A(X, Y)$ and ${}^i(X, Y)$ for $\operatorname{Ext}_A^i(X, Y)$ for $i \geq 1$). Since $\dim \operatorname{Hom}_A(\tau^- K_i, Q(x)) = 1$ and the map δ is non-zero, we get (1).

(2) Applying $\operatorname{Hom}_A(P(y_i), -)$ to the exact sequence

$$0 \longrightarrow R_i \longrightarrow Q(x) \longrightarrow C_i \longrightarrow 0,$$

we have the following exact sequence:

$$0 \longrightarrow (P(y_i), R_i) \longrightarrow (P(y_i), Q(x)) \longrightarrow (P(y_i), C_i) \longrightarrow 0.$$

Since $\dim \operatorname{Hom}_A(P(y_i), Q(x)) = 1 = \dim \operatorname{Hom}_A(P(y_i), C_i)$, one has (2).

3.4 LEMMA. *Suppose that $K_i \neq 0$ for $i = 1, 2$. Then $K_1 \cong K_2$ if and only if $R_1 \cong R_2$.*

Proof. We have the following exact sequences:

$$0 \longrightarrow K_1 \longrightarrow R_1 \xrightarrow{f_1} Q(x)^{d_1} \longrightarrow 0,$$
$$0 \longrightarrow K_2 \longrightarrow R_2 \xrightarrow{f_2} Q(x)^{d_2} \longrightarrow 0.$$

Applying $\operatorname{Hom}_A(R_2, -)$ to the first of these sequences, we obtain the exact sequence

$$0 \longrightarrow (R_2, K_1) \longrightarrow (R_2, R_1) \longrightarrow (R_2, Q(x)^{d_1}) \longrightarrow {}^1(R_2, K_1) \longrightarrow \cdots.$$

Suppose that $K_1 \cong K_2$. Then $\operatorname{Hom}_A(R_2, K_1) = 0$, since R_2 is directing. From the Auslander–Reiten formula $\operatorname{Ext}_A^1(R_2, K_1) \cong D\underline{\operatorname{Hom}}_A(\tau^- K_1, R_2)$, together with 3.3(2), we obtain $\operatorname{Ext}_A^1(R_2, K_1) = 0$. Hence $\operatorname{Hom}_A(R_2, R_1) \neq 0$. Similarly, $\operatorname{Hom}_A(R_1, R_2) \neq 0$. Since R_1 is directing, it follows that $R_1 \cong R_2$.

Now suppose that $R_1 \cong R_2$. Then we have

$$0 \longrightarrow (Q(x)^{d_1}, Q(x)^{d_2}) \longrightarrow (R_1, Q(x)^{d_2}) \longrightarrow (K_1, Q(x)^{d_2}) \longrightarrow \cdots.$$

Since $\dim \operatorname{Hom}_A(R_1, Q(x)) = d_1$, the first non-zero map of the above sequence is surjective. So we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & K_1 \arrow[r] \arrow[d] & R_1 \arrow[r, "f_1"] \arrow[d, "\wr"] & Q(x)^{d_1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K_2 \arrow[r] & R_2 \arrow[r, "f_2"] & Q(x)^{d_2} \arrow[r] & 0, \end{tikzcd}$$

which shows that the right vertical map $Q(x)^{d_1} \to Q(x)^{d_2} \cong Q(x)^{d_1}$ has to be split epimorphism since f_2 is not a split monomorphism and f_1 is irreducible. Hence $K_1 \cong K_2$.

3.5 LEMMA. *Suppose that $C_i \neq 0$ for $i = 1, 2$. Then $C_1 \cong C_2$ if and only if $R_1 \cong R_2$.*

The proof is similar to the proof of 3.4.

3.6 LEMMA. *Suppose that $C_1 \neq 0$ and $K_2 \neq 0$. Then $P(y_1)$ is not isomorphic to $\tau^- K_2$.*

From 3.4 to 3.6 we know that all non-zero modules $\tau^- K_i$ and $P(y_j)$ are distinct elements of S_x^A.

3.7 LEMMA. *Let K be the kernel of an irreducible surjective map $f: R \to Q(x)$ between directing indecomposable modules. Suppose that K is directing and $\dim \operatorname{Hom}_A(\tau^- K, Q(x)) = 1$. If $M \in S_x^A$ satisfies $M \leqslant \tau^- K$, then $M \cong \tau^- K$.*

Proof. Suppose, to the contrary, that $M \not\cong \tau^- K$. Then we consider the following pull-back diagram with $f'g \neq 0$:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & \tau^- K \arrow[r] \arrow[d, "f'"] & 0 \\ 0 \arrow[r] & K \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & K \arrow[r] & R \arrow[r, "f"] & Q(x) \arrow[r] & 0. \end{tikzcd} \tag{*}$$

Since $f'g \neq 0$ and $\dim \operatorname{Hom}_A(\tau^- K, Q(x)) = 1$, the first-row exact sequence is an Auslander–Reiten sequence, thus the second row is not split and so M is not projective. Let $0 \to \tau M \to E'' \to M \to 0$ be an Auslander–Reiten sequence. Then we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "g'"] \arrow[d] & E' \arrow[r, "p"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \tau M \arrow[r] & E'' \arrow[r] & M \arrow[r] & 0 \end{tikzcd} \tag{**}$$

which shows that $\operatorname{Hom}_A(K, \tau M) \neq 0$. Since $M \not\cong \tau^- K$, there is a non-invertible map $0 \neq h: K \to \tau M$.

Now we consider the following push-out diagram:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "h \neq 0"] & R \arrow[r, "f"] \arrow[d, "g_1"] & Q(x) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \tau M \arrow[r] & R' \arrow[r, "g_2"] & Q(x) \arrow[r] & 0. \end{tikzcd}$$

We claim that g_1 is not a split monomorphism. Otherwise, we could have $\mathrm{Hom}_A(\tau M, R) \neq 0$ but, on the other hand, we know from

$$0 \longrightarrow (\tau M, K) \longrightarrow (\tau M, R) \longrightarrow (\tau M, Q(x))$$

and $\mathrm{Hom}_A(\tau M, K) = 0$ that $\mathrm{Hom}_A(\tau M, R) = 0$, because $M \in S_x^A$. This is a contradiction. Suppose now that g_2 is a split epimorphism. Then $\mathrm{Hom}_A(R, \tau M) \neq 0$ and so, since R is directing, $\mathrm{Ext}_A^1(R, \tau M) = 0$. Since $\mathrm{Hom}_A(\tau M, R) = 0$ and h is not a split monomorphism, it follows from [2, 2.11] that the sequence

$$0 \longrightarrow \mathrm{Ext}_A^1(Q(x), \tau M) \longrightarrow \mathrm{Ext}_A^1(R, \tau M)$$

is exact. Thus we get $\mathrm{Ext}_A^1(Q(x), \tau M) = 0$. Applying $\mathrm{Hom}_A(-, \tau M)$ to the diagram $(*)$ we obtain the following commutative diagram:

$$\begin{tikzcd}
\ldots \arrow[r] \arrow[d] & (K, \tau M) \arrow[r] \arrow[d] & {}^1(Q(x), \tau M) = 0 \arrow[r] \arrow[d, equal] & \ldots \arrow[d] \\
\ldots \arrow[r] & (K, \tau M) \arrow[r, "\delta"] & {}^1(M, \tau M) \arrow[r] & \ldots
\end{tikzcd}$$

which implies that $\delta = 0$. But the diagram $(**)$ tells us that $\delta \neq 0$. This contradiction shows that g_2 is not a split epimorphism, which is a contradiction to f being irreducible, and so $M \cong \tau^- K$.

In order to carry out a further discussion we now introduce some notation.

Let M be an indecomposable module. We denote by $\mathrm{Supp}(M)$ the set of all vertices y of Δ_0 such that $\mathrm{Hom}_A(P(y), M) \neq 0$. If $y \in \Delta_0$, we denote by e_y the primitive idempotent element corresponding to y. Put

$$e = \sum_{y \in \mathrm{Supp}(P(x))} e_y,$$

$\bar{e} = 1 - e$ and $\bar{A} = \bar{e} A \bar{e}$. We assume that there is an irreducible injective map $f: R \to Q(x)$, where R is indecomposable. Let $E(y)$ be the cokernel of f (see the proof of 3.2).

3.8 LEMMA. *Suppose that $Q(x)$ is a directing module. Then $\tau^- R$ is an indecomposable projective $\bar{A}$-module.*

Proof. We can regard $\bar{A}$-mod as a full subcategory of A-mod which is closed under factor modules and extensions. That $Q(x)$ is directing implies that $\tau^- R$ is an $\bar{A}$-module. Since with $Q(x)$ also R has no submodule which is isomorphic to a module in $\bar{A}$-mod, it follows from [3, 3.7] that $\tau^- R$ is $\bar{A}$-projective, and is indecomposable since it is indecomposable as an A-module.

3.9 LEMMA. *Suppose that $Q(x)$ is directing. Then $P(y) \cong A\bar{e} \otimes_{\bar{A}} \tau^- R$ and* $\mathrm{Hom}(P(x), P(y)) \neq 0$.

Proof. Since $\tau^- R$ is an indecomposable projective $\bar{A}$-module, it is of the form $\bar{A} e_z$, where $z \in \mathrm{Supp}(P(x))$ and so $\bar{e} e_z = e_z$. Hence $A\bar{e} \otimes_{\bar{A}} \tau^- R \cong A e_z = P(z)$. Now the exact sequence

$$0 \longrightarrow R \longrightarrow Q(x) \longrightarrow E(y) \longrightarrow 0$$

induces an epimorphism

$$\mathrm{Hom}_A(\tau^- R, E(y)) \longrightarrow \mathrm{Ext}_A^1(\tau^- R, R).$$

Hence $\mathrm{Hom}_A(\tau^- R, E(y)) \neq 0$ and, since $\mathrm{Top}(\tau^- R) = E(y)$, it follows that $z = y$.

Since $Q(x)$ is injective, any irreducible map $Q(x) \to \tau^- R$ is an epimorphism. Hence $E(y)$ is a composition factor of $Q(x)$. Therefore $\mathrm{Hom}_A(P(x), P(y)) \cong \mathrm{Hom}_A(P(y), Q(x)) \neq 0$.

3.10 LEMMA. *Suppose that S_x^A is scalar and that $Q(x)$ and $P(y)$ are directing. If $X \in S_x^A$ with $X \leqslant P(y)$, then $X \cong P(y)$.*

Proof. We know that $\mathrm{Hom}(P(y), R) = 0$. Thus $\mathrm{Hom}(X, R) = 0$, because S_x^A is scalar. Let K be the kernel of an irreducible map $g: Q(x) \to \tau^- R$, and $\alpha: \tau^- R \to E(y) = \mathrm{Top}(\tau^- R)$ be the canonical projection. Then, since $Q(x)$ is directing, αg factors through the cokernel of the irreducible map $R \to Q(x)$ and we get an exact commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "\gamma"] & Q(x) \arrow[r, "g"] \arrow[d, "\wr"] & \tau^- R \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & R \arrow[r] & Q(x) \arrow[r] & E(y) \arrow[r] & 0 \end{tikzcd}$$

Hence γ is a monomorphism and so we must have $\mathrm{Hom}_A(X, K) = 0$.

Now we apply $\mathrm{Hom}(X, -)$ to the first exact sequence above and get the following exact sequence:

$$0 \longrightarrow (X, Q(x)) \longrightarrow (X, \tau^- R) \longrightarrow {}^1(X, K) \longrightarrow 0$$

which shows that $\mathrm{Hom}(X, \tau^- R) \neq 0$.

Since $\mathrm{Top}(\tau^- R) = E(y)$ and $\tau^- R = \bar{e}(\tau^- R)$, there is an epimorphism $h: P(y) \to \tau^- R = \bar{e}P(y)$ taking m to $\bar{e}m$. Now there is a short exact sequence of abelian groups

$$0 \longrightarrow eAe_y \longrightarrow Ae_y = P(y) \xrightarrow{h} \bar{e}Ae_y = \tau^- R \longrightarrow 0 \qquad (*)$$

and since h is in A-mod, this is in fact a short exact sequence in A-mod, with $B' := eAe_y$ an A-module with support contained in $\mathrm{Supp}(P(x))$. It follows from (*) that $\mathrm{Top}(B') = \mathrm{Soc}(R) = E(x)$. Hence, by Lemma 2.1(2), $\mathrm{Hom}_A(X, B') = 0$ and so, applying $\mathrm{Hom}_A(X, -)$ to (*) we obtain

$$\mathrm{Hom}_A(X, P(y)) \cong \mathrm{Hom}_A(X, \tau^- R) \neq 0.$$

Since $P(y)$ is directing, it follows that $P(y) \cong X$.

Now we apply the above considerations to the particular case when A is representation-directed and get the main result mentioned in the introduction.

3.11 LEMMA. *Suppose that A is a representation-directed algebra. Then $m = n$.*

Proof. Now we consider the opposite algebra A^{op} of A and the set $S_x^{A^{\mathrm{op}}}$. By 2.9 the set $S_x^{A^{\mathrm{op}}}$ has n maximal elements, and therefore the set

$$\bar{S}_x^A := \{X \in A\text{-ind} \mid X \not\cong Q(x), \mathrm{Hom}_A(X, Q(x)) \neq 0 \text{ and } \mathrm{Hom}_A(\tau^- X, Q(x)) = 0\}$$

has n minimal elements. It was shown in [7] that $S_x^A \cong \bar{S}_x^A$ as poset. This implies that $m = n$.

3.12 THEOREM. *Let A be a basic connected representation-directed algebra and x a vertex. Let*

$$\bigoplus_{i=1}^{m} R_i \xrightarrow{(f_i)_i^t} Q(x)$$

be a sink map ending at $Q(x)$, where R_i are indecomposables. Put

$$M_i = \begin{cases} \tau^- \operatorname{Ker}(f_i), & \text{if } f_i \text{ is surjective,} \\ P(y), & \text{if } \operatorname{Cok}(f_i) \cong E(y). \end{cases}$$

Then $\{M_i \mid i = 1, \ldots, m\}$ is the set of all minimal elements in S_x^A.

3.13 COROLLARY (Baustista and Brenner). *Let A be a representation-directed algebra and $\bigoplus_{i=1}^{d} R_i \to Q(x)$ a sink map. Then $d \leqslant 3$.*

Proof. This follows from 3.12 [7, 10, theorem] and Kleiner's theorem.

4. *Conclusion*

We keep the notation introduced before. We point out the following fact which suggests that the construction in 3.12 might be true for some classes of non-representation-directed algebras.

4.1 PROPOSITION. *Let A be a tame concealed algebra (see* [6] *for the definition). Suppose that S_x^A is scalar and $d_i = 1$. Then*

(1) *$n \leqslant m$;*

(2) *the set of all M_i constructed in* 3.12 *is a subset of the set of all minimal elements of $(S_x^A, \leqslant)$.*

The proof of 4.1 follows from 3.10 and the following.

4.2 LEMMA. *Let A be as in* 4.1. *If S_x^A is scalar, then* 3.7 *holds without the assumption that K is directing.*

Proof. Note that for two non-isomorphic regular modules M and N in S_x^A we never have $\operatorname{Hom}(M,N) \neq 0$ and $\operatorname{Hom}(N,M) \neq 0$ by [9, 4.2]. Thus we can copy the proof of 3.7 to prove 4.2.

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