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CONSTRUCTION OF STABLE EQUIVALENCES OF MORITA TYPE FOR FINITE-DIMENSIONAL ALGEBRAS I

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Abstract. In the representation theory of finite groups, the stable equivalence of Morita type plays an important role. For general finite-dimensional algebras, this notion is still of particular interest. However, except for the class of self-injective algebras, one does not know much on the existence of such equivalences between two finite-dimensional algebras; in fact, even a nontrivial example is not known. In this paper, we provide two methods to produce new stable equivalences of Morita type from given ones. The main results are Corollary 1.2 and Theorem 1.3. Here the algebras considered are not necessarily self-injective. As a consequence of our constructions, we give an example of a stable equivalence of Morita type between two algebras of global dimension 4, such that one of them is quasi-hereditary and the other is not. This shows that stable equivalences of Morita type do not preserve the quasi-heredity of algebras. As another by-product, we construct a Morita equivalence inside each given stable equivalence of Morita type between algebras A and B. This leads not only to a general formulation of a result by Linckelmann (1996), but also to a nice correspondence of some torsion pairs in A-mod with those in B-mod if both A and B are symmetric algebras. Moreover, under the assumption of symmetric algebras we can get a new stable equivalence of Morita type. Finally, we point out that stable equivalences of Morita type are preserved under separable extensions of ground fields.

1. Introduction

In the representation theory of finite groups, or more generally, finite-dimensional self-injective algebras, the stable equivalence of Morita type introduced by Broué is of considerable interest due to its connection with the celebrated conjecture of Broué (see [3, 7, 13]). It arises naturally for self-injective algebras, as was shown by a result of Rickard: if two self-injective algebras are derived equivalent, then they are stably equivalent of Morita type ([11, corollary 5.5]). Typical examples of stable equivalences of Morita type occur frequently in the block theory of finite groups. Other examples are the trivial extensions of two finite-dimensional algebras which are tilted from each other. Recently, it is shown that stable equivalences of Morita type are also of particular interest for general finite-dimensional algebras, for example, they preserve many interesting properties of algebras, such as the representation dimension [15], representation type [6] and Linckelmann's Theorem [8]. To understand stable equivalences of Morita type in general, it is necessary to have other new examples. However, as far as we know, all known examples of stable

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equivalences of Morita type are from the class of self-injective algebras. It seems to be an interesting project to have a method for constructing stable equivalences of Morita type for the general class of finite-dimensional algebras.

In the present paper, we shall provide several methods for constructing a new stable equivalence of Morita type from a given one. In this way, we can start with a stable equivalence of Morita type between self-injective algebras and obtain a stable equivalence of Morita type between non-self-injective algebras. Our first construction is the following result:

Theorem 1.1. *Let A and B be two finite-dimensional k-algebras with k a field. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B. If I is an ideal of A and if J is an ideal of B such that $JN = NI$ and $IM = MJ$, then the bimodules $(A/I)\otimes_A M\otimes_B (B/J)$ and $(B/J)\otimes_B N \otimes_A (A/I)$ define a stable equivalence of Morita type between A/I and B/J.*

As an application, we have the following construction, which shows that there do exist algebras satisfying all conditions in the above theorem.

Corollary 1.2. *Suppose that k is a field. Let A and B be two indecomposable non-simple self-injective k-algebras such that $A/rad(A)$ and $B/rad(B)$ are separable. Suppose that A has a decomposition $A = P_1\oplus P_2$, where P_1 is isomorphic to a direct sum of copies of an indecomposable projective A-module W, and where P_2 has no direct summands isomorphic to W. Similarly, suppose that B has a decomposition $B = P_1'\oplus P_2'$, where P_1' is isomorphic to a direct sum of copies of an indecomposable projective B-module W', and where P_2' has no direct summands isomorphic to W'. If two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B such that $N \otimes_A soc(W) \simeq soc(W')$, then there is a stable equivalence of Morita type between $A/soc(P_1)$ and $B/soc(P_1')$.*

As another consequence, we have a negative answer to the question of whether the quasi-heredity is preserved under stable equivalences of Morita type. In fact, we shall provide an example of a stable equivalence of Morita type between two representation-finite algebras of global dimension 4, such that one of them is quasi-hereditary, but the other is not. This question is one of the motivations of our study on stable equivalences of Morita type.

By Theorem 1.1, we can construct stable equivalences of Morita type between quotient algebras. Next, we shall show how to construct stable equivalences of Morita type between extension algebras. Our result in this direction is the following theorem, which admits a more general form (see Theorem 4.1).

Theorem 1.3. *Let A, B and C be three finite-dimensional k-algebras over a field k. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B. If R is an A-C-bimodule such that $M \otimes_B N \otimes_A R \simeq R$ as A-C-bimodules and that the automorphism group of B-C-bimodule $N \otimes_A R$ is $k\setminus 0$, then there is a stable equivalence of Morita type between the triangular matrix algebras $\begin{pmatrix} A & R \\ 0 & C \end{pmatrix}$ and $\begin{pmatrix} B & N\otimes_A R \\ 0 & C \end{pmatrix}$.*

The paper is organized in the following way: In Section 2 we recall some definitions and basic facts. Section 3 is devoted to the proofs of the Theorem 1.1 and its corollaries. Here we require the knowledge of an almost split sequence. In Section 4 we first prove a general result, Theorem 4.1, in terms of trivial extensions, and then give a proof of Theorem 1.3 as a specific case. In Section 5 we provide a method to get modules R satisfying conditions in Theorem 1.3; this leads to a more general formulation of Linckelmann's theorem in [7]. If, in addition, our algebras are symmetric, we also get two torsion pairs preserved under the stable equivalence of Morita type. The last section is designed to illustrate our results by some examples which are either blocks of finite groups or given by quivers with relations.

In the second paper [9], we shall prove that the stable equivalence of Morita type is preserved by forming Auslander algebras.

2. Preliminaries

In this section we shall fix notations and recall definitions and facts needed in our proofs.

Throughout this paper, we denote by k a fixed field and by k^* the multiplicative group of k. All algebras are assumed to be finite-dimensional k-algebras with 1. By a module we always mean a finitely-generated left module.

Given an algebra A, we denote by A-mod the category of all finitely-generated A-modules. The composition of two morphisms $f : L \to M$ and $g : M \to N$ is a morphism from L to N, which will be denoted by fg. The stable category A-$\underline{\text{mod}}$ of A is defined as follows: The objects of A-$\underline{\text{mod}}$ are the same as those of A-mod, and the morphisms between two objects X and Y are given by $\underline{\mathrm{Hom}}_A(X,Y) = \mathrm{Hom}_A(X,Y)/\mathcal{P}(X,Y)$, where $\mathcal{P}(X,Y)$ is the subspace of $\mathrm{Hom}_A(X,Y)$ consisting of those homomorphisms from X to Y which factor through a projective A-module. More generally, given a full subcategory $\mathcal{C}$ of A-mod, we denote by $\mathrm{Hom}_A(X,\mathcal{C},Y)$ the set of those homomorphisms from X to Y which factor through a module in $\mathcal{C}$.

Definition 2.1. Two algebras A and B are said to be stably equivalent if there is an equivalence $F : A$-$\underline{mod} \to$B-$\underline{mod}$ of the stable categories.

Two algebras A and B are stably equivalent of Morita type if there exist an A-B-bimodule ${}_AM_B$ and a B-A-bimodule ${}_BN_A$ such that

(1) M and N are projective as left and right modules, respectively, and

(2) $M \otimes_B N \simeq A \oplus P$ as A-A-bimodules for some projective A-A-bimodule P, and $N \otimes_A M \simeq B \oplus Q$ as B-B-bimodules for some projective B-B-bimodule Q.

Note that if two algebras A and B are stably equivalent of Morita type, then they are stably equivalent. In fact, the functor $N \otimes_A - : A$-mod $\to B$-mod induces an equivalence: A-$\underline{\text{mod}} \to B$-$\underline{\text{mod}}$, whose inverse is induced by $M \otimes_B - : B$-mod $\to A$-mod. If P and Q are zero, we arrive at a Morita equivalence. Thus the notion of a stable equivalence of Morita type is a combination of a Morita equivalence and a stable equivalence.

For a self-injective algebra A, the syzygy functor Ω_A is a stable equivalence: A-$\underline{\text{mod}} \to A$-$\underline{\text{mod}}$, whose inverse is given by the co-syzygy functor Ω_A^{-1} (see [2]). It is known that the syzygy functor Ω_A can be extended to a stable equivalence of Morita type induced by the bimodules $\Omega_{A\otimes_k A^{op}}(A)$ and $\Omega^{-1}_{A\otimes_k A^{op}}(A)$ when A is an indecomposable non-simple self-injective algebra, where A^{op} denotes the opposite algebra of A (see [7]).

The following lemma collects some properties on stable equivalences of Morita type.

Lemma 2.2 (see [7]). *Suppose k is a field. Let A and B be indecomposable non-simple k-algebras such that $A/rad(A)$ and $B/rad(B)$ are separable. If two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B, then we have the following:*

(1) *Up to isomorphism, N has a unique indecomposable non-projective direct summand N_1 as a B-A-bimodule, and M has a unique indecomposable non-projective direct summand M_1 as an A-B-bimodule. Moreover, M_1 and N_1 also induce a stable equivalence of Morita type between A and B.*

(2) *If A and B are self-injective and if N is indecomposable, then $N \otimes_A S$ is an indecomposable B-module for each simple A-module S. Similarly, if M is indecomposable, then $M \otimes_A S'$ is an indecomposable A-module for each simple B-module S'.* □

If two algebras A and B are stably equivalent, we have the following properties which are taken from [2, Chapter X].

Lemma 2.3. *Let $F : A\text{-}\underline{mod} \to B\text{-}\underline{mod}$ be a stable equivalence whose inverse is given by G. Then:*

(1) *F induces a one-to-one correspondence between the isomorphism classes of indecomposable non-projective modules in A-mod and that in B-mod.*

(2) *Suppose that A is an indecomposable non-simple symmetric Nakayama algebra and that B is a symmetric algebra with no semi-simple block. If we denote by F the induced correspondence on the indecomposable non-projective modules, then $F(S)$ is a uniserial B-module for each simple A-module S.*

For our purpose we also need the following fact. For a proof we refer to [5, section 9.2].

Lemma 2.4. *Let A be a finite-dimensional algebra with a decomposition $A = P_1 \oplus P_2$, where P_1 is isomorphic to a direct sum of copies of an indecomposable projective-injective A-module W and P_2 has no direct summands isomorphic to W. Then the socle of P_1, $soc(P_1)$, is an ideal in A. Moreover, an A-module X is an $A/soc(P_1)$-module if and only if X has no direct summands isomorphic to W.* □

Recall that a k-algebra A is *symmetric* if $A \simeq \mathrm{Hom}_k(A, k)$ as A-A-bimodules. It is known that symmetric algebras are self-injective algebras. An algebra A is said to be a *Nakayama algebra* if both the indecomposable projective and indecomposable injective modules are uniserial.

Finally, we remark that the definition of "stable equivalence of Morita type" in [6] is different from the one given in Definition 2.1. It seems that the conditions required in [6] are quite weaker than that in Definition 2.1.

3. Proofs of Theorem 1.1 and Corollary 1.2

In this section, we shall prove Theorem 1.1 and Corollary 1.2. In fact, we shall deduce a more general formulation of Corollary 1.2.

Let A be an algebra, and let I be an ideal of A. Recall that an A-module X has a natural A/I-module structure if X is annihilated by I, that is, $IX = 0$. The following lemma is well known.

Lemma 3.1. (1) *If X is a right A-module and if Y is a left A-module such that $XI = 0$ and $IY = 0$, then $X \otimes_A Y \simeq X \otimes_{A/I} Y$ by $x \otimes y \mapsto x \otimes y$.*

(2) *Let A and B be two algebras. If M is an A-B-bimodule such that ${}_AM$ is projective, then $M \otimes_B P$ is a projective A-module for any projective B-module P. Similarly, if M is an A-B-bimodule such that M_B is projective, then $Q \otimes_A M$ is a projective right B-module for any projective right A-module Q.*

First we shall prove Theorem 1.1, and then we turn to the proof of Corollary 1.2.

Proof of Theorem 1.1. Put $A' = A/I$ and $B' = B/J$. If we define $M' = A' \otimes_A (M \otimes_B B')$ and $N' = B' \otimes_B (N \otimes_A A')$, then M' is an A'-B'-bimodule and N' is a B'-A'-bimodule. Since $JN = NI$, it follows from Lemma 3.1(2) that the B-module $N \otimes_A A' \simeq N/NI = N/JN \simeq B' \otimes_B N$ is a projective B'-module. By Lemma 3.1(1), $N' = B' \otimes_B (N \otimes_A A') \simeq B' \otimes_{B'} (N \otimes_A A') \simeq N \otimes_A A'$ is a projective B'-module. Clearly, N' is also projective as a right A'-module by Lemma 3.1(2). Similarly, it follows from $IM = MJ$ that M' is projective as a left A'-module and as a right B'-module. Again by Lemma 3.1(1) together with the associativity of tensor products, we have the following isomorphisms of A'-A'-bimodules:

$$\begin{array}{rcl} M' \otimes_{B'} N' & \simeq & A' \otimes_A M \otimes_B (B' \otimes_{B'} B') \otimes_B N \otimes_A A' \\ & \simeq & A' \otimes_A M \otimes_B B' \otimes_B (N \otimes_A A') \\ & \simeq & A' \otimes_A M \otimes_B B' \otimes_{B'} (N \otimes_A A') \\ & \simeq & A' \otimes_A M \otimes_B (N \otimes_A A') \\ & \simeq & A' \otimes_A (M \otimes_B N) \otimes_A A' \\ & \simeq & A' \otimes_A (A \oplus P) \otimes_A A' \\ & \simeq & A' \otimes_A A' \oplus A' \otimes_A P \otimes_A A' \\ & \simeq & A' \otimes_{A'} A' \oplus A' \otimes_A P \otimes_A A' \\ & \simeq & A' \oplus A' \otimes_A P \otimes_A A'. \end{array}$$

Since P is a projective A-A-bimodule, it follows easily that $A' \otimes_A P \otimes_A A'$ is a projective A'-A'-bimodule. Similarly, we have a B'-B'-bimodule isomorphism

$$N' \otimes_{A'} M' \simeq B' \oplus B' \otimes_B Q \otimes_B B',$$

where $B' \otimes_B Q \otimes_B B'$ is a projective B'-B'-bimodule. Thus, by definition, the bimodules M' and N' define a stable equivalence of Morita type between A' and B'. □

Proof of Corollary 1.2. In the following we shall prove that under the assumption of Corollary 1.2, the ideals $I := \mathrm{soc}(P_1)$ in A and $J := \mathrm{soc}(P_1')$ in B satisfy the conditions in Theorem 1.1. The proof will be carried out in several steps.

(1) By Lemma 2.2, we can assume that M and N are indecomposable as bimodules. It follows from $N \otimes_A \mathrm{soc}(W) \simeq \mathrm{soc}(W')$ that $M \otimes_B \mathrm{soc}(W') \simeq \mathrm{soc}(W)$, since $M \otimes_B -$ and $N \otimes_A -$ induce mutually stable equivalences.

(2) For every indecomposable A-module X which is not isomorphic to W, the B-module $N \otimes_A X$ has no direct summands isomorphic to W'.

We use induction on the length $l(X)$ of X to prove (2). The claim holds for $l(X) = 1$ since $N \otimes_A X$ is indecomposable non-projective by Lemma 2.2. Assume now that $l(X) > 1$. There are three cases to be considered.

The first case: X is a projective module. By [2, proposition 5.5, p. 169], we have an almost split sequence:

$$0 \longrightarrow \mathrm{rad}(X) \longrightarrow X \oplus \mathrm{rad}(X)/\mathrm{soc}(X) \longrightarrow X/\mathrm{soc}(X) \longrightarrow 0,$$

where $\mathrm{rad}(X)$ and $X/\mathrm{soc}(X)$ are indecomposable non-projective modules, and $\mathrm{rad}(X)/\mathrm{soc}(X)$ has no non-zero projective summands. From this sequence we get an exact sequence

$$0 \longrightarrow N \otimes_A \mathrm{rad}(X) \longrightarrow N \otimes_A X \oplus N \otimes_A (\mathrm{rad}(X)/\mathrm{soc}(X)) \\ \longrightarrow N \otimes_A (X/\mathrm{soc}(X)) \longrightarrow 0,$$

where $N \otimes_A \mathrm{rad}(X)$, $N \otimes_A (X/\mathrm{soc}(X))$ and $N \otimes_A (\mathrm{rad}(X)/\mathrm{soc}(X))$ have no direct summands isomorphic to W' by induction. Now let us denote by P_U the maximal projective direct summand of a module U and by $U_\wp$ the non-projective complement of P_U in U. Thus, the above exact sequence canonically induces the following exact sequence:

$$(\pi) \quad 0 \longrightarrow (N \otimes_A \mathrm{rad}(X))_\wp \longrightarrow E \oplus (N \otimes_A (\mathrm{rad}(X)/\mathrm{soc}(X)))_\wp \\ \longrightarrow (N \otimes_A (X/\mathrm{soc}(X)))_\wp \longrightarrow 0,$$

where $(N \otimes_A \mathrm{rad}(X))_\wp$, $(N \otimes_A (X/\mathrm{soc}(X)))_\wp$ and $(N \otimes_A (\mathrm{rad}(X)/\mathrm{soc}(X)))_\wp$ are the non-projective parts of $N \otimes_A \mathrm{rad}(X)$, $N \otimes_A (X/\mathrm{soc}(X))$ and $N \otimes_A (\mathrm{rad}(X)/\mathrm{soc}(X))$, respectively, and where E is a projective module such that

$$N \otimes_A X \oplus P_{N \otimes_A (\mathrm{rad}(X)/\mathrm{soc}(X))} \simeq E \oplus P_{N \otimes_A \mathrm{rad}(X)} \oplus P_{N \otimes_A (X/\mathrm{soc}(X))}.$$

By induction, the module $P_{N \otimes_A \mathrm{rad}(X)} \oplus P_{N \otimes_A (X/\mathrm{soc}(X))}$ has no direct summands isomorphic to W'. To prove (2), it is sufficient to show that E has no direct summands isomorphic to W'. For this we first show that (π) is an almost split sequence. In the following, we may assume that $E \neq 0$.

If $\mathrm{rad}(X)/\mathrm{soc}(X) = 0$, then A is a Nakayama algebra of Loewy length 2 by [2, proposition 1.8, p. 341]. It follows from [10] that B is also a Nakayama algebra of Loewy length 2, and therefore

$$(\pi) \qquad 0 \longrightarrow (N \otimes_A \mathrm{rad}(X))_\wp \longrightarrow E \longrightarrow (N \otimes_A (X/\mathrm{soc}(X)))_\wp \longrightarrow 0$$

is an almost split sequence. If $\mathrm{rad}(X)/\mathrm{soc}(X) \neq 0$, then, by [2, proposition 1.6, p. 339], we have

$$\tau((N \otimes_A (X/\mathrm{soc}(X)))_\wp) \simeq (N \otimes_A \mathrm{rad}(X))_\wp,$$

where τ is the Auslander-Reiten translation. Since the co-syzygy functor Ω_A^{-1} is a stable equivalence and $\Omega_A^{-1}(\mathrm{soc}(X)) = X/\mathrm{soc}(X)$, we have that $\underline{\mathrm{End}}(X/\mathrm{soc}(X)) \simeq \underline{\mathrm{End}}(\mathrm{soc}(X))$ is a division ring. Since $N \otimes_A -$ induces a stable equivalence, we know that $\underline{\mathrm{End}}((N \otimes_A (X/\mathrm{soc}(X)))_\wp) \simeq \underline{\mathrm{End}}(X/\mathrm{soc}(X))$ is a division ring. It follows from [2, corollary 2.4, p. 149] that (π) is an almost split sequence.

Now by [2, proposition 5.5, p. 169], we know that E is indecomposable and that (π) is isomorphic to the sequence $0 \longrightarrow \mathrm{rad}(E) \longrightarrow E \oplus \mathrm{rad}(E)/\mathrm{soc}(E) \longrightarrow E/\mathrm{soc}(E) \longrightarrow 0$. Thus $E/\mathrm{soc}(E) \simeq (N \otimes_A (X/\mathrm{soc}(X)))_\wp$.

We consider the exact sequence $0 \longrightarrow \mathrm{soc}(X) \longrightarrow X \longrightarrow X/\mathrm{soc}(X) \longrightarrow 0$. From this we get the following exact sequence:

$$0 \longrightarrow N \otimes_A \mathrm{soc}(X) \longrightarrow N \otimes_A X \longrightarrow N \otimes_A (X/\mathrm{soc}(X)) \longrightarrow 0.$$

Note that $N \otimes_A \mathrm{soc}(X)$ is an indecomposable non-projective module and that

$$\begin{aligned} N \otimes_A (X/\mathrm{soc}(X)) &= (N \otimes_A (X/\mathrm{soc}(X)))_\wp \oplus P_{N \otimes_A (X/\mathrm{soc}(X))} \\ &\simeq E/\mathrm{soc}(E) \oplus P_{N \otimes_A (X/\mathrm{soc}(X))}. \end{aligned}$$

It follows easily that $N \otimes_A X \simeq E \oplus P_{N\otimes_A(X/\mathrm{soc}(X))}$ and $N \otimes_A \mathrm{soc}(X) \simeq \mathrm{soc}(E)$. Since $X \not\simeq W$ and $\mathrm{soc}(X) \not\simeq \mathrm{soc}(W)$, we have that $N \otimes_A \mathrm{soc}(X) \not\simeq N \otimes_A \mathrm{soc}(W)$ by Lemma 2.3(1), and thus $\mathrm{soc}(E) \not\simeq \mathrm{soc}(W')$. This implies that $E \not\simeq W'$. So we have proved (2) under the assumption that X is a projective module.

The second case: X is a non-projective module such that $\mathrm{soc}(X)$ has no direct summands isomorphic to $\mathrm{soc}(W)$. Consider the injective envelope of X: $X \longrightarrow I$, where I is a projective module and has no direct summands isomorphic to W. By applying $N \otimes_A -$ we get an injective map $N \otimes_A X \longrightarrow N \otimes_A I$. It follows that $N \otimes_A X$ has no direct summands isomorphic to W' since $N \otimes_A I$ has no direct summands isomorphic to W' by the proof of the first case.

The third case: X is a non-projective module such that $\mathrm{soc}(X)$ has a direct summand isomorphic to $\mathrm{soc}(W)$. We have an exact sequence: $0 \longrightarrow S \longrightarrow X \longrightarrow X/S \longrightarrow 0$, where the simple module S is isomorphic to $\mathrm{soc}(W)$, and X/S has non-zero projective summands. From this sequence we get a new exact sequence:

$$0 \longrightarrow N \otimes_A S \longrightarrow N \otimes_A X \longrightarrow N \otimes_A (X/S) \longrightarrow 0,$$

where $N \otimes_A S \simeq \mathrm{soc}(W')$ and $N \otimes_A (X/S)$ has no direct summands isomorphic to W' by induction. This new exact sequence induces canonically the following exact sequence:

$$(\varepsilon) \qquad 0 \longrightarrow N \otimes_A S \longrightarrow (N \otimes_A X)_0 \longrightarrow (N \otimes_A (X/S))_\wp \longrightarrow 0,$$

where $(N \otimes_A (X/S))_\wp$ is the non-projective part of $N \otimes_A (X/S)$, and $N \otimes_A X \simeq (N \otimes_A X)_0 \oplus P_{N\otimes_A(X/S)}$. To finish the proof of our claim, it suffices to show that $(N \otimes_A X)_0$ has no direct summands isomorphic to W'.

Suppose that $(N \otimes_A X)_0$ has a direct summand isomorphic to W'. Write $(N \otimes_A X)_0 = Y \oplus W'$. Since X is not projective, $N \otimes_A X$ is not projective by Lemma 2.3(1), and therefore Y is non-projective. Moreover, the homomorphism $N \otimes_A S \longrightarrow Y$ induced from (ε) is non-zero. For if it is not the case, then Y would be a direct summand of $(N \otimes_A (X/S))_\wp$, and thus the non-projective part of $N \otimes_A X$. Now if X_1 is the non-projective part of $M \otimes_B Y$, then X_1 would be isomorphic to X. On the other hand, X_1 being a direct summand of $M \otimes_B Y$ implies that X_1 is a direct summand of $M \otimes_B N \otimes_A (X/S)$, that is, X_1 is a direct summend of X/S; this contradicts that $X_1 \simeq X$. Thus we have proved that the homomorphism $N \otimes_A S \longrightarrow Y$ induced from (ε) is non-zero and injective.

Now we consider the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & W' & \xrightarrow{\sim} & Z & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N \otimes_A S & \longrightarrow & (N \otimes_A X)_0 & \longrightarrow & (N \otimes_A X/S)_\wp & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N \otimes_A S & \longrightarrow & Y & \longrightarrow & Y/(N \otimes_A S) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Then $W' \simeq Z$ by the Snake Lemma. This implies that $Z \longrightarrow (N \otimes_A (X/S))_{\wp}$ is a split monomorphism since W' is injective, and therefore the non-projective part of $(N \otimes_A (X/S))$ has a projective direct summand isomorphic to W'. This is a contradiction and also finishes the proof of (2).

(3) Similarly, we can show that for every indecomposable B-module X which is not isomorphic to W', the A-module $M \otimes_B X$ has no direct summands isomorphic to W.

(4) By Lemma 2.4, we have proved that for an A-module X, if $\mathrm{soc}(P_1)X = 0$, then $\mathrm{soc}(P_1')(N \otimes_A X) = 0$, and for a B-module X, if $\mathrm{soc}(P_1')X = 0$, then

$$\mathrm{soc}(P_1)(M \otimes_B X) = 0.$$

In particular, $\mathrm{soc}(P_1')(N \otimes_A (A/\mathrm{soc}(P_1))) = 0$ and $\mathrm{soc}(P_1)(M \otimes_B (B/\mathrm{soc}(P_1'))) = 0$, or equivalently, $\mathrm{soc}(P_1')N \subseteq N\mathrm{soc}(P_1)$ and $\mathrm{soc}(P_1)M \subseteq M\mathrm{soc}(P_1')$.

(5) Finally, we shall show that $\mathrm{soc}(P_1')N = N\mathrm{soc}(P_1)$ and $\mathrm{soc}(P_1)M = M\mathrm{soc}(P_1')$. Once this is done, the corollary follows by Theorem 1.1 immediately.

In fact, we assume that $P_1 \simeq W^n$. Since N is projective as a right A-module, $N\mathrm{soc}(P_1) \simeq N \otimes_A \mathrm{soc}(P_1) \simeq N \otimes_A \mathrm{soc}(W)^n \simeq \mathrm{soc}(W')^n$ as B-modules. By (4), $N/N\mathrm{soc}(P_1)$ has no summands isomorphic to W'. Therefore $N \simeq W'^n \oplus P'$ (where P' has no summands isomorphic to W') and $\mathrm{soc}(P_1')N \simeq \mathrm{soc}(P_1')W'^n$. On the other hand, W' is isomorphic to a summand of B, and $\mathrm{soc}(P_1')W'$ is isomorphic to a summand of $\mathrm{soc}(P_1')$. But $\mathrm{soc}(P_1')W'$ is also an indecomposable submodule of W' since W' has simple socle. This implies that $\mathrm{soc}(P_1')W' \simeq \mathrm{soc}(W')$ and $\mathrm{soc}(P_1')N \simeq \mathrm{soc}(P_1')W'^n \simeq \mathrm{soc}(W')^n$. It follows from (4) that $\mathrm{soc}(P_1')N = N\mathrm{soc}(P_1)$. Similarly, we have $\mathrm{soc}(P_1)M = M\mathrm{soc}(P_1')$. □

As a consequence of Corollary 1.2, we have the following more general result.

Corollary 3.2. *Let A and B be two indecomposable non-simple self-injective k-algebras such that $A/rad(A)$ and $B/rad(B)$ are separable. Suppose that A has a decomposition $A = P_1 \oplus P_2$, where P_1 is isomorphic to a direct sum of copies of indecomposable projective A-modules $W_1, ..., W_r$, and where P_2 has no direct summands isomorphic to W_j for all $1 \leq j \leq r$. Similarly, suppose that B has a decomposition $B = P_1' \oplus P_2'$, where P_1' is isomorphic to a direct sum of copies of indecomposable projective B-modules $W_1', ..., W_r'$, and where P_2' has no direct summands isomorphic to W_j' for all $1 \leq j \leq r$. If two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B such that $N \otimes_A soc(W_j) \simeq soc(W_j')$ for all j, then there is a stable equivalence of Morita type between $A/soc(P_1)$ and $B/soc(P_1')$.*

Proof. We assume that W_j are pairwise non-isomorphic and that W_j' are also pairwise non-isomorphic. For each s, we decompose P_1 as $P_1(s) \oplus Q_1(s)$ such that $P_1(s)$ is isomorphic to a direct sum of copies of the indecomposable projective A-module W_s and $Q_1(s)$ has no direct summands isomorphic to W_s. Similarly, we decompose P_1' as $P_1(s)' \oplus Q_1(s)'$ such that $P_1(s)'$ is isomorphic to a direct sum of copies of the indecomposable projective B-module W_s' and $Q_1(s)'$ has no direct summands isomorphic to W_s'.

Let I_s be the socle of $P_1(s)$ and let J_s be the socle of $P_1(s)'$. Then, by Lemma 2.4, I_s and J_s are ideals in A and B, respectively. Moreover, we have $J_sN = NI_s$ and $I_sM = MJ_s$ for all s by the proof of Corollary 1.2. Next we define $I = \sum_s I_s$ and $J = \sum_s J_s$. Then $JN = NI$ and $IM = MJ$. It is easy to see that $I = \mathrm{soc}(P_1)$ and $J = \mathrm{soc}(P_1')$. Hence Corollary 3.2 follows immediately from Theorem 1.1. □

As another direct consequence of Corollary 3.2, we have the following result on representation dimension. Recall that given an algebra A, the *representation dimension* of A is defined by Auslander in [1] as follows:

$$\begin{aligned}&\text{rep.dim}(A)\\&\quad = \inf \{\text{gl.dim}(\Lambda) \mid \Lambda \text{ is an algebra with dom.dim}(\Lambda) \geq 2 \text{ and End}({}_\Lambda T) \text{ is}\\&\qquad\qquad \text{Morita equivalent to } A, \text{where } T \text{ is the injective envelope of } \Lambda\}.\end{aligned}$$

Corollary 3.3. *Let A and B be as in Corollary 3.2. Then rep.dim$(A/soc(P_1))$ = rep.dim$(B/soc(P_1'))$.*

Proof. Since representation dimension is invariant under stable equivalences of Morita type, the corollary follows from Corollary 3.2. □

For further information on representation dimension we refer the reader to the original paper [1] or the recent papers [15, 16] and the references therein.

Finally, let us mention that stable equivalences of Morita type are preserved under separable field extensions.

Proposition 3.4. *Let A and B be two finite-dimensional k-algebras. Suppose that there is a stable equivalence of Morita type between A and B defined by two bimodules ${}_AM_B$ and ${}_BN_A$. If C is a separable k-algebra, then the modules $C \otimes_k M$ and $C \otimes_k N$ define a stable equivalence of Morita type between the two tensor products $C \otimes_k A$ and $C \otimes_k B$. In particular, if E is a separable field extension of k (for example, E is a finite extension of a perfect field k), then the two E-algebras $E \otimes_k A$ and $E \otimes_k B$ are stably equivalent of Morita type.*

Proof. It is clear that $C \otimes_k M$ is projective as a left $C \otimes_k A$-module and as a right $C \otimes_k B$-module. We need to verify condition (2) in Definition 2.1. For this we employ the following fact in [4, chap. XI, p.210]: if Λ and Γ are k-algebras, and if $X_\Lambda, X'_\Gamma, {}_\Lambda Y, {}_\Gamma Y'$ are modules, then

$$(X \otimes_\Lambda Y) \otimes_k (X' \otimes_\Gamma Y') \simeq (X \otimes_k X') \otimes_{\Lambda \otimes_k \Gamma} (Y \otimes_k Y').$$

Now it follows from this fact that

$$\begin{aligned}(C \otimes_k M) \otimes_{C \otimes_k B} (C \otimes_k N) &\simeq (C \otimes_C C) \otimes_k (M \otimes_B N)\\&\simeq C \otimes_k (A \oplus P)\\&\simeq C \otimes_k A \oplus C \otimes_k P.\end{aligned}$$

Since P is projective as an A-A-bimodule by definition and since C is projective as a C-C-bimodule by separability, we know that $C \otimes_k P$ is projective as a $(C \otimes_k A)$-$(C \otimes_k A)$-bimodule. Similarly, we have a desired assertion for $(C \otimes_k N) \otimes_{C \otimes_k A} (C \otimes_k M)$. Hence, by definition, the modules $C \otimes_k M$ and $C \otimes_k N$ define a stable equivalence of Morita type between the two tensor products $C \otimes_k A$ and $C \otimes_k B$. □

It is an open problem in [12] whether stable equivalences of Morita type are preserved by tensor products, that is, given two stable equivalences of Morita type between algebras A and B, and between C and D respectively, is there a stable equivalence of Morita type between $A \otimes_k C$ and $B \otimes_k D$? (Here we assume that all algebras are indecomposable.) This question is open even for block algebras of finite groups.

4. Stable equivalences of Morita type between triangular matrix algebras

In the previous section, we have seen how to get a new stable equivalence of Morita type between quotient algebras. This is a way, roughly speaking, of going from a pair of bigger algebras to a pair of smaller algebras. In this section we shall converse the way and construct a pair of bigger algebras from that of smaller ones. Our tool in this section is the so-called triangular matrix algebras; this is a special case of the trivial extensions.

Given an algebra A and an A-bimodule I, the *trivial extension* Λ of A by I is defined as follows: let $\Lambda := A \oplus I$. The multiplication on Λ is given by

$$(a,x)(b,y) = (ab, ay + xb)$$

for all $a, b \in A$ and $x, y \in I$. Then Λ is an associative algebra with the identity. Note that A is a subalgebra of Λ and I is an ideal in Λ with $I^2 = 0$.

Clearly, a left module over Λ is given by an A-module ${}_AX$ together with a morphism $\alpha : I \otimes_A X \longrightarrow {}_AX$ of A-modules, such that $(1_I \otimes \alpha) \circ \alpha = 0$. Similarly, a right Λ-module is given by a right A-module Y together with an A-module homomorphism $\beta : Y \otimes_A I \longrightarrow Y_A$, such that $(\beta \otimes 1_I) \circ \beta = 0$. If (X, α) and (X', α') are two Λ-modules, a morphism from (X, α) to (X', α') is an A-morphism $f : X \longrightarrow X'$ such that $\alpha \circ f = (1_I \otimes f) \circ \alpha'$.

Now we formulate our result in an abstract way.

Theorem 4.1. *Let A and B be two finite-dimensional k-algebras with k a field. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B. Suppose that an A-bimodule I and a B-bimodule J satisfy the following conditions:*

(1) *$M \otimes_B N \otimes_A I \simeq I$ and $N \otimes_A M \otimes_B J \simeq J$ as bimodules;*

(2) *there is a B-A-bimodule isomorphism $\psi : J \otimes_B N \xrightarrow{\sim} N \otimes_A I$ and an A-B-bimodule isomorphism $\varphi : I \otimes_A M \xrightarrow{\sim} M \otimes_B J$ such that the following diagrams are commutative:*

$$\begin{tikzcd}
I \otimes_A M \otimes_B N \arrow[rr, "1_I \otimes \rho_1"] \arrow[d, "\varphi \otimes 1_N"] & & I \otimes_A A \arrow[d, "\mu"] \\
M \otimes_B J \otimes_B N \arrow[r, "(1_M \otimes \psi)(\rho_1 \otimes 1_I)"] & A \otimes_A I \arrow[r, "\mu'"] & I\,,
\end{tikzcd}$$

$$\begin{tikzcd}
J \otimes_B N \otimes_A M \arrow[rr, "1_J \otimes \sigma_1"] \arrow[d, "\psi \otimes 1_M"] & & J \otimes_B B \arrow[d, "\tau"] \\
N \otimes_A I \otimes_A M \arrow[r, "(1_N \otimes \varphi)(\sigma_1 \otimes 1_J)"] & B \otimes_B J \arrow[r, "\tau'"] & J\,,
\end{tikzcd}$$

where μ, μ', τ, τ' are the multiplications, and where $(\rho_1, \rho_2) : M \otimes_B N \longrightarrow A \oplus P$ and $(\sigma_1, \sigma_2) : N \otimes_A M \longrightarrow B \oplus Q$ define the stable equivalence of Morita type between A and B.

Then there is a stable equivalence of Morita type between the trivial extension of A by I and that of B by J.

Proof. Let Λ be the trivial extension of A by I, and let Γ be the trivial extension of B by J. Note that our assumptions on I and J imply that $P\otimes_A I = I\otimes_A P = Q\otimes_B J = J\otimes_B Q = 0$. We define $\overline{M} = M\otimes_B \Gamma$ and $\overline{N} = N\otimes_A \Lambda$. By definition, $\overline{N} = N\otimes_A\Lambda \simeq N\oplus N\otimes_A I$. Since $N\otimes_A I\simeq J\otimes_B N$ as B-A-bimodules, the space $N\oplus N\otimes_A I$ has also a left Γ-module structure. Thus $\overline{N}$ is a Γ-Λ-bimodule. Since N is projective as each one-sided module, we see that $\overline{N}\simeq {}_\Gamma(\Gamma\otimes_B N)_\Lambda \simeq {}_\Gamma(N\otimes_A\Lambda)_\Lambda$ is projective as a left Γ-module and as a right Λ-module. A similar assertion holds for $\overline{M}$. Moreover, we have the following Λ-bimodule isomorphisms:

$$\begin{aligned}
\overline{M}\otimes_\Gamma \overline{N} &\simeq (M\otimes_B\Gamma)\otimes_\Gamma(\Gamma\otimes_B N)\\
&\simeq M\otimes_B\Gamma\otimes_B N\\
&\simeq M\otimes_B(B\oplus J)\otimes_B N\\
&\simeq M\otimes_B N\oplus M\otimes_B J\otimes_B N.
\end{aligned}$$

To see that $M\otimes_B N\oplus M\otimes_B J\otimes_B N$ is isomorphic to $A\oplus I\oplus P$ as Λ-bimodules, we need condition (2) above. Of course, as A-bimodules, they are isomorphic, but we have to show that they have the same Λ-bimodule structures. For this we shall use the description of morphisms of Λ-modules at the beginning of this section. Now we define $f := \begin{pmatrix} \rho_1 & 0 & \rho_2 \\ 0 & (\varphi^{-1}\otimes 1_N)(1_I\otimes\rho_1)\mu & 0\end{pmatrix} : M\otimes_B N\oplus M\otimes_B J\otimes_B N \longrightarrow A\oplus I\oplus P$ and show that this is a Λ-bimodule isomorphism. Clearly, this is an A-bimodule isomorphism. Moreover, the commutative diagram

$$\begin{tikzcd}
I\otimes_A M\otimes_B N\oplus I\otimes_A M\otimes_B J\otimes_B N \arrow[r, "\begin{pmatrix} 0 & \varphi\otimes 1_N \\ 0 & 0\end{pmatrix}"] \arrow[d, "1_I\otimes f"] & M\otimes_B N\oplus M\otimes_B J\otimes_B N \arrow[d, "f"] \\
I\otimes_A A\oplus I\otimes_A I\oplus I\otimes_A P \arrow[r, "\begin{pmatrix} 0 & \mu & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}"] & A\oplus I\oplus P
\end{tikzcd}$$

shows that f is a left Λ-homomorphism, and the commutative diagram

$$\begin{tikzcd}
M\otimes_B N\otimes_A I\oplus M\otimes_B J\otimes_B N\otimes_A I \arrow[r, "\begin{pmatrix} 0 & 1_M\otimes\psi^{-1} \\ 0 & 0\end{pmatrix}"] \arrow[d, "f\otimes 1_I"] & M\otimes_B N\oplus M\otimes_B J\otimes_B N \arrow[d, "f"] \\
A\otimes_A I\oplus I\otimes_A I\oplus P\otimes_A I \arrow[r, "\begin{pmatrix} 0 & \mu' & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}"'] & A\oplus I\oplus P
\end{tikzcd}$$

shows that f is a right Λ-module homomorphism. Thus we have shown that $\overline{M}\otimes_\Gamma \overline{N} \simeq A\oplus I\oplus P \simeq \Lambda\oplus\Lambda\otimes_A P\otimes_A\Lambda$ as Λ-Λ-bimodules. Similarly, we have

$\overline{N} \otimes_\Gamma \overline{M} \simeq \Gamma \oplus \Gamma \otimes_B Q \otimes_B \Gamma$ as Γ-Γ-bimodules. Note that $\Lambda \otimes_A P \otimes_A \Lambda$ and $\Gamma \otimes_B Q \otimes_B \Gamma$ are projective as bimodules. Thus the above isomorphisms show that $\overline{M}$ and $\overline{N}$ define a stable equivalence of Morita type between Λ and Γ. $\square$

As a special case, we consider the triangular matrix algebras. In this case, we have the following result.

Theorem 4.2. *Let A, B and C be three finite-dimensional k-algebras over a field k. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B. If R is an A-C-module such that $M \otimes_B N \otimes_A R \simeq R$ as A-C-bimodules and that the automorphism group of the module ${}_{B\otimes_k C^{op}}(N \otimes_A R)$ is k^*, then there is a stable equivalence of Morita type between the triangular matrix algebras $\begin{pmatrix} A & R \\ 0 & C \end{pmatrix}$ and $\begin{pmatrix} B & N \otimes_A R \\ 0 & C \end{pmatrix}$.*

Proof. We can prove this theorem by checking all conditions in Theorem 4.1. However, for the convenience of the reader, we give here a direct proof by using the description of modules over triangular matrix algebras.

Let $\Lambda = \begin{pmatrix} A & R \\ 0 & C \end{pmatrix}$ and $\Gamma = \begin{pmatrix} B & N \otimes_A R \\ 0 & C \end{pmatrix}$. To prove Theorem 4.2, we have to find two bimodules ${}_\Lambda\overline{M}_\Gamma$ and ${}_\Gamma\overline{N}_\Lambda$, which satisfy the conditions in Definition 2.1. For this purpose, we first introduce two exact functors and then use them to achieve our verification.

(1) Each Λ-module X can be described as a triple $X = (X_0, X_\omega, f)$, where X_0 is in A-mod, X_ω is in C-mod, and f is an A-homomorphism from $R \otimes_C X_\omega \longrightarrow X_0$. Each homomorphism from X to $Y = (Y_0, Y_\omega, g)$ is a pair $(\alpha_0, \alpha_\omega)$ in $\mathrm{Hom}_A(X_0, Y_0) \times \mathrm{Hom}_C(X_\omega, Y_\omega)$ such that $f\alpha_0 = (1_R \otimes \alpha_\omega)g$. From this description we see that every A-module X_0 can be considered as a Λ-module by identifying X_0 with $(X_0, 0, 0)$. Moreover, every projective A-module is also a projective Λ-module. The other indecomposable projective Λ-module, which is not projective A-module, is of the form $(R \otimes E, E, 1_{R\otimes_C E})$, where E is an indecomposable projective C-module. Finally, let us mention that a sequence $0 \longrightarrow (X_0, X_\omega, f) \xrightarrow{(\alpha_0,\alpha_\omega)} (Y_0, Y_\omega, g) \xrightarrow{(\beta_0,\beta_\omega)} (Z_0, Z_\omega, h) \longrightarrow 0$ of Λ-modules is exact if and only if both of the sequences $0 \longrightarrow X_0 \xrightarrow{\alpha_0} Y_0 \xrightarrow{\beta_0} Z_0 \longrightarrow 0$ in A-mod and $0 \longrightarrow X_\omega \xrightarrow{\alpha_\omega} Y_\omega \xrightarrow{\beta_\omega} Z_\omega \longrightarrow 0$ in C-mod are exact.

(2) We define two functors $F : \Lambda\text{-mod} \longrightarrow \Gamma\text{-mod}$ and $G : \Gamma\text{-mod} \longrightarrow \Lambda\text{-mod}$ as follows:

By assumption, we have an A-A-bimodule isomorphism $\rho = (\rho_1, \rho_2) : M \otimes_B N \simeq A \oplus P$ and a B-B-bimodule isomorphism $\sigma = (\sigma_1, \sigma_2) : N \otimes_A M \simeq B \oplus Q$, where P and Q are projective bimodules. From the A-C-bimodule isomorphism $M \otimes_B N \otimes_A R \simeq R$ we have two B-C-bimodule isomorphisms $1_N \otimes ((\rho_1 \otimes 1_R)\mu') : N \otimes_A M \otimes_B N \otimes_A R \simeq N \otimes_A R$ and $(\sigma_1 \otimes 1_{N\otimes_A R})\tau' : N \otimes_A M \otimes_B N \otimes_A R \simeq N \otimes_A R$, where $\mu' : A \otimes_A R \longrightarrow R$ and $\tau' : B \otimes_B (N \otimes_A R) \longrightarrow N \otimes_A R$ are the multiplication maps. Since the automorphism group of ${}_{B\otimes_k C^{op}}(N \otimes_A R)$ is k^*, there exists a nonzero element k_0 in k such that $1_N \otimes ((\rho_1 \otimes 1_R)\mu') = k_0((\sigma_1 \otimes 1_{N\otimes_A R})\tau')$. Without loss of generality, we may assume that $k_0 = 1$ (otherwise, we may replace σ by $k_0\sigma$).

Given a Λ-module $X = (X_0, X_\omega, f)$, we put $F(X) := (N \otimes_A X_0, X_\omega, 1_N \otimes f)$. For a morphism $(\alpha_0, \alpha_\omega) : X \longrightarrow Y$, we define $F(\alpha_0, \alpha_\omega) = (1_N \otimes \alpha_0, \alpha_\omega)$. It is easy to see that $F(\alpha_0, \alpha_\omega)$ is well-defined. Furthermore, F is a well-defined exact functor. By Watts' Theorem (see [14, theorem 3.33, p. 77]), $F \simeq {}_\Gamma F(\Lambda) \otimes_\Lambda -$, where the right Λ-module structure on $F(\Lambda)$ is induced by the right multiplication on Λ. Since F is exact, $F(\Lambda)$ is projective as a right Λ-module. It is easy to check that F takes projective modules to projective modules, and therefore $F(\Lambda) \simeq F(\Lambda) \otimes_\Lambda \Lambda$ is projective as a left Γ-module.

Now let us define the functor $G : \Gamma\text{-mod} \longrightarrow \Lambda\text{-mod}$. For (U, V, g) in Γ-mod with $g :\ {}_BN \otimes_A R \otimes_C V \longrightarrow\ {}_BU$, we define

$$G(U, V, g) := \big(M \otimes_B U, V, (((\rho_1 \otimes 1_R)\mu')^{-1} \otimes 1_V)(1_M \otimes g)\big),$$

and for $(\delta, \gamma) : (U, V, g) \longrightarrow (U', V', g')$, we define

$$\begin{aligned} G(\delta, \gamma) := (1_M \otimes \delta, \gamma) : (M \otimes_B U, V, (((\rho_1 \otimes 1_R)\mu')^{-1} \otimes 1_V)(1_M \otimes g)) \\ \longrightarrow (M \otimes_B U', V', (((\rho_1 \otimes 1_R)\mu')^{-1} \otimes 1_{V'})(1_M \otimes g')). \end{aligned}$$

Then G is well defined since $M \otimes_B N \otimes_A R \simeq R$, and exact since $M \otimes_B -$ is exact. By Watts' Theorem, $G \simeq\ {}_\Lambda G(\Gamma) \otimes_\Gamma -$, where the right Γ-module structure on $G(\Gamma)$ is induced by the right multiplication on Γ. Since G is exact, $G(\Gamma)$ is projective as a right Γ-module. It is easy to check that G takes projective modules to projective modules, and therefore $G(\Gamma) \simeq G(\Gamma) \otimes_\Gamma \Gamma$ is projective as a left Λ-module.

(3) We have that $G \circ F : \Lambda\text{-mod} \longrightarrow \Lambda\text{-mod}$ is an exact functor. Again, by Watts' Theorem, $G(F(\Lambda)) = M \otimes_B N \otimes_A A \oplus R \oplus C \simeq M \otimes_B N \oplus R \oplus C$ is a Λ-Λ-bimodule, where the left Λ-module structure is given by $\begin{pmatrix} a & r \\ 0 & c \end{pmatrix}(x, y, z) = (ax, ay + rz, cz)$, and the right Λ-module structure is given by $(x, y, z)\begin{pmatrix} a & r \\ 0 & c \end{pmatrix} = (xa, \rho_1(x)r + yc, zc)$ for all $a \in A$, $r, y \in R$, $c, z \in C$ and $x \in M \otimes_B N$. We want to show that this bimodule can be decomposed as a direct sum of the regular bimodule Λ and a projective Λ-bimodule.

We have the A-A-bimodule isomorphism $\rho = (\rho_1, \rho_2) : M \otimes_B N \simeq A \oplus P$. Note that the A-A-bimodule P has a natural Λ-Λ-bimodule structure, which is induced from the A-A-bimodule structure. Since P is a projective A-A-bimodule, P is a direct sum of the modules of the form $Ae_i \otimes_k e_jA$, where e_i and e_j are primitive idempotents in A. Assume now that $Ae_i \otimes_k e_jA$ is a direct summand of P. Since $P \otimes_A R = 0$, we have $e_jR \simeq e_jA \otimes_A R = 0$. It follows that $e_jA \simeq e_jA \oplus e_jR$ is a right projective Λ-module. Since Ae_i is a left projective Λ-module, we know that $Ae_i \otimes_k e_jA$ is a projective Λ-Λ-bimodule. Thus P is a projective Λ-Λ-bimodule.

We claim that there is a Λ-Λ-bimodule isomorphism $M \otimes_B N \oplus R \oplus C \simeq (A \oplus R \oplus C) \oplus P = \Lambda \oplus P$. In fact, we can explicitly give this map by $\bar\rho(x, y, z) = \big((\rho_1(x), y, z), \rho_2(x)\big)$. It is a straightforward exercise to show that $\bar\rho$ is a Λ-Λ-bimodule isomorphism. On the other hand, we have $G(F(\Lambda)) \simeq G(\Gamma) \otimes_\Gamma F(\Lambda)$ as Λ-Λ-bimodules. Thus we have proved that $G(\Gamma) \otimes_\Gamma F(\Lambda) \simeq \Lambda \oplus P$ as Λ-Λ-bimodules, where P is a projective Λ-Λ-bimodule.

(4) We have that $F \circ G : \Gamma\text{-mod} \longrightarrow \Gamma\text{-mod}$ is an exact functor. Again, by Watts' Theorem, $F(G(\Gamma)) = N \otimes_A M \otimes_B B \oplus N \otimes_A R \oplus C \simeq N \otimes_A M \oplus N \otimes_A R \oplus C$ is a Γ-Γ-bimodule , where the left Γ-module structure is given by $\begin{pmatrix} b & r' \\ 0 & c \end{pmatrix}(x, y, z) = (bx, by + r'z, cz)$, and the right Γ-module structure is given by $(x, y, z)\begin{pmatrix} b & r' \\ 0 & c \end{pmatrix} = (xb, (1_N \otimes ((\rho_1 \otimes 1_R)\mu'))(x \otimes r') + yc, zc) = (xb, \sigma_1(x)r' + yc, zc)$ for all $b \in B$, $c, z \in C$, $r', y \in N \otimes_A R$ and $x \in N \otimes_A M$. We want to show that this bimodule can be decomposed as a direct sum of the regular bimodule Γ and a projective bimodule.

We have a B-B-bimodule isomorphism $\sigma = (\sigma_1, \sigma_2) : N \otimes_A M \simeq B \oplus Q$. Note that the B-B-bimodule Q has a natural Γ-Γ-bimodule structure, which is induced from the B-B-bimodule structure. Since Q is a projective B-B-bimodule, Q is a direct sum of the modules of the form $Bf_i \otimes_k f_jB$, where f_i and f_j are primitive idempotents in B. Assume now that $Bf_i \otimes_k f_jB$ is a direct summand of Q. Since $Q \otimes_B N \otimes_A R = 0$, we have $f_j(N \otimes_A R) \simeq f_jB \otimes_B (N \otimes_A R) = 0$. It follows that $f_jB \simeq f_jB \oplus f_j(N \otimes_A R)$ is a right projective Γ-module. Since Bf_i is a projective Γ-module, we know that $Bf_i \otimes_k f_jB$ is a projective Γ-Γ-bimodule. Thus Q is a projective Γ-Γ-bimodule.

We claim that there is a Γ-Γ-bimodule isomorphism $N \otimes_A M \oplus N \otimes_A R \oplus C \simeq (B \oplus N \otimes_A R \oplus C) \oplus Q = \Gamma \oplus Q$. In fact, we can explicitly give this map by $\bar{\sigma}(x, y, z) = \big((\sigma_1(x), y, z), \sigma_2(x)\big)$. It is a straightforward exercise to show that $\bar{\sigma}$ is a Γ-Γ-bimodule isomorphism. Here we need the equality $1_N \otimes ((\rho_1 \otimes 1_R)\mu') = (\sigma_1 \otimes 1_{N \otimes_A R})\tau'$, but this is our assumption. On the other hand, we have $F(G(\Gamma)) \simeq F(\Lambda) \otimes_\Lambda G(\Gamma)$ as Γ-Γ-bimodules. Thus we have proved that $F(\Lambda) \otimes_\Lambda G(\Gamma) \simeq \Gamma \oplus Q$ as Γ-Γ-bimodules, where Q is a projective Γ-Γ-bimodule.

(5) If we define $\overline{M} := G(\Gamma)$ and $\overline{N} := F(\Lambda)$, then $\overline{M}$ and $\overline{N}$ satisfy all required conditions in the definition of stable equivalences of Morita type. □

The following is a corollary about the representation dimensions and the representation type of triangular matrix algebras.

Corollary 4.3. *Let A, B and C be three finite-dimensional k-algebras over a field k. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B. Suppose that R is an A-C-bimodule with $M \otimes_B N \otimes_A R \simeq R$ as A-C-bimodules and that the automorphism group of the B-C-bimodule $N \otimes_A R$ is isomorphic to k^*. Define $\Lambda = \begin{pmatrix} A & R \\ 0 & C \end{pmatrix}$ and $\Gamma = \begin{pmatrix} B & N \otimes_A R \\ 0 & C \end{pmatrix}$. Then:*

(1) *rep.dim(Λ) = rep.dim(Γ).*

(2) *If k is algebraically closed, then Λ and Γ have the same representation type.*

Proof. (1) follows from [15]. (2) follows from [6]. □

Remarks. (1) In the block theory of finite groups, the condition $M \otimes_B N \otimes_A R \simeq R$ can be satisfied for some modules R. For instance, a canonical candidate for R is that R is a simple A-module such that $N \otimes_A R$ is simple. In this case, $M \otimes_B N \otimes_A R$ is indecomposable by Lemma 2.2(2), and thus it is isomorphic to R. In the next section we shall discuss this condition in detail.

(2) The stable equivalence of Morita type between the triangular matrix algebras in Theorem 4.2 extends the original one between A and B.

(3) If $C = k$, the matrix algebra $\begin{pmatrix} A & R \\ 0 & C \end{pmatrix}$ is usually called the one-point extension of A by the A-module R. This special case of triangular matrix algebras will be used in the last section.

5. Morita equivalences inside stable equivalences of Morita type

In this section, we shall show that, given a stable equivalence of Morita type between two algebras A and B, there is a subcategory in A-mod and a subcategory in B-mod such that they are Morita equivalent. In many cases, this supplies us a lot of choices for modules R to satisfy the conditions in Theorem 4.2. We shall display several examples in the next section.

Throughout this section, we assume that A, B, M, N, P, Q are fixed as in Definition 2.1, and denote the functor $N \otimes_A -$ by F and the functor $M \otimes_B -$ by G.

Lemma 5.1. *If $R \in A$-mod satisfies $GF(R) \simeq R$, then so does any submodule and any quotient module of R. Conversely, if R has a submodule X such that $GF(X) \simeq X$ and $GF(R/X) \simeq R/X$, then $GF(R) \simeq R$.*

Proof. Since $GF(X) \simeq X \oplus P \otimes_A X$ for all $X \in A$-mod, the condition $GF(X) \simeq X$ is equivalent to $P \otimes_A X = 0$. Let X be a submodule of R. Then we have a natural exact sequence $0 \longrightarrow X \hookrightarrow R \longrightarrow Y \longrightarrow 0$. The module P is a projective right A-module; this yields the exact sequence $0 \longrightarrow P \otimes_A X \longrightarrow P \otimes_A R \longrightarrow P \otimes_A Y \longrightarrow 0$. Note that $P \otimes_A R = 0$ if and only if $P \otimes_A X = 0 = P \otimes_A Y$. This implies that $R \simeq GF(R)$ if and only if $GF(X) \simeq X$ and $GF(Y) \simeq Y$, which finishes the proof of Lemma 5.1. □

Let $\mathcal{C}$ be the full subcategory of A-mod consisting of all A-modules R satisfying $GF(R) \simeq R$. Let $\mathcal{C}_0$ be the full subcategory of A-mod consisting of all modules M such that each composition factor S of M satisfies $GF(S) \simeq S$. Then it follows from Lemma 5.1 that $\mathcal{C} = \mathcal{C}_0$.

Let $\mathcal{D}$ be the full subcategory of B-mod consisting of all modules Y such that $FG(Y) \simeq Y$. Similarly, we can see that $\mathcal{D}$ is the full subcategory of all modules Y such that all composition factors of Y are preserved under FG. Note that $\mathcal{C}$ and $\mathcal{D}$ are closed under submodules, factor modules and extensions (that is, if $0 \longrightarrow X \longrightarrow Z \longrightarrow Y \longrightarrow 0$ is an exact sequence in A-mod (respectively, in B-mod) such that X and Y are in $\mathcal{C}$ (respectively, in $\mathcal{D}$), then Z lies in $\mathcal{C}$ (respectively, in $\mathcal{D}$)). We also remark that if a simple A-module S lies in $\mathcal{C}$, then $F(S)$ must belong to $\mathcal{D}$.

Proposition 5.2. *F induces an equivalence between $\mathcal{C}$ and $\mathcal{D}$. In particular, for an A-module $R \in \mathcal{C}$, $End({}_BF(R)) \simeq k$ if and only if $End({}_AR) \simeq k$.*

Proof. If $X \in \mathcal{C}$, then $F(X)$ is in $\mathcal{D}$ since $FG(F(X)) \simeq F(GF(X)) \simeq F(X)$. So F is well defined. To see that F is an equivalence, we note that $P \otimes_A X = 0$ for all X in $\mathcal{C}$ and that $Q \otimes_B Y = 0$ for all Y in $\mathcal{D}$. This implies that if F' and G' are the restrictions of F and G to the subcategories $\mathcal{C}$ and $\mathcal{D}$, respectively, then $F' \circ G' \simeq 1_{\mathcal{D}}$ and $G' \circ F' \simeq 1_{\mathcal{C}}$. □

The above proposition may be regarded as a general formulation of Linckelmann's Theorem, which we now state as the following corollary.

Corollary 5.3 (see [7])**.** *Let A and B be indecomposable non-simple self-injective algebras such that $A/rad(A)$ and $B/rad(B)$ are separable. If F sends all simple A-modules to simple B-modules, then F is a Morita equivalence.*

Proof. We may assume that M and N are indecomposable as bimodules. Thus, by Lemma 2.2(2), $G(F(S))$ is indecomposable for each simple A-module S. This implies that $GF(S) \simeq S$ for all simple A-modules S. Hence $\mathcal{C} = A$-mod. We claim that $\mathcal{D} = B$-mod: if T is a simple B-module, then $G(T)$ is in $\mathcal{C}$ and $FG(T)$ belongs to $\mathcal{D}$ by Proposition 5.2. Since $\mathcal{D}$ is closed under direct summands, the module T is in $\mathcal{D}$. By Proposition 5.2, the corollary follows. □

Since each simple module is isomorphic to $Ae_i/\mathrm{rad}(Ae_i)$ for some primitive idempotent $e_i \in A$, we may define an ideal I in the following way: Let I be the ideal of A generated by those primitive idempotents e_i such that $GF(Ae_i/\mathrm{rad}(Ae_i)) \not\simeq Ae_i/\mathrm{rad}(Ae_i)$. Then A/I-mod can be regarded as a full subcategory of A-mod consisting of all those A-modules X such that $IX = 0$. Thus $A/I\text{-mod} = \mathcal{C}$. Similarly, there is an ideal J in B such that $B/J\text{-mod} = \mathcal{D}$. The proposition also implies that simple A/I-modules are mapped to simple B/J-modules under the functor F.

It is an open question whether A and B have the same number of simple modules (see [2], p. 409). Related to this question, we have the following corollary.

Corollary 5.4. *Suppose A and B are two algebras, and there is a stable equivalence of Morita type between A and B defined by F and G. If the number of isomorphism classes of simple A-modules not preserved by GF and that of simple B-modules not preserved by FG coincide, then A and B have the same number of isomorphism classes of simple modules.* $\square$

Associated with $\mathcal{C}$ are another two natural subcategories $\mathcal{C}'$ and $\mathcal{C}''$; they are related to torsion theory in A-mod. Recall that a pair $(\mathcal{T}, F)$ of module classes in A-mod is said to be a torsion pair provided $\mathcal{F}$ is the class of all modules X in A-mod with $\mathrm{Hom}_A(\mathcal{T}, X) = 0$, and $\mathcal{T}$ is the class of all modules Y in A-mod with $\mathrm{Hom}_A(Y, \mathcal{F}) = 0$. In this case, the modules in $\mathcal{T}$ are called torsion modules, and those in $\mathcal{F}$ are called torsion-free modules.

Now we define $\mathcal{C}'$ to be the full subcategory of A-mod consisting of all A-modules Y with $IY = Y$, and $\mathcal{D}'$ to be the full subcategory of B-mod consisting of all B-modules V with $JV = V$. Note that if $I = AeA$ and $J = BfB$ for an idempotent $e \in A$ and an idempotent $f \in B$, then $\mathcal{C}'$ and $\mathcal{D}'$ are generated by Ae and Bf, respectively. Furthermore, we define $\mathcal{C}''$ to be the full subcategory of A-mod consisting of all A-modules Y with $\mathrm{Hom}_A(\mathcal{C}, Y) = 0$, and $\mathcal{D}''$ to be the full subcategory of B-mod consisting of all B-modules V with $\mathrm{Hom}_B(\mathcal{D}, V) = 0$. Note that for each A-module X, there is a unique maximal submodule $t(X)$ of X such that $t(X) \in \mathcal{C}$ and $t(X/t(X)) = 0$.

Proposition 5.5. (1) *The pairs $(\mathcal{C}', C)$ and $(\mathcal{C}, C'')$ are torsion pairs in A-mod, and the pairs $(\mathcal{D}', D)$ and $(\mathcal{D}, D'')$ are torsion pairs in B-mod.*

(2) *Let $S_1, ..., S_n$ be a complete list of non-isomorphic simple A-modules which are not preserved by GF. Then a module X lies in $\mathcal{C}' \cap \mathcal{C}''$ if and only if $top(X) \oplus soc(X)$ lies in $add(\bigoplus_{j=1}^{n} S_j)$.*

(3) *If A and B are self-injective, then $F\mathcal{C}' \subseteq \mathcal{D}'$ and $G\mathcal{D}'' \subseteq \mathcal{C}''$.*

(4) *eAe-mod is equivalent to $\mathcal{C}' \cap \mathcal{C}''$, and fBf-mod is equivalent to $\mathcal{D}' \cap \mathcal{D}''$.*

Proof. (1) is clear. (2) follows from the definition of $\mathcal{C}'$ and the fact that $X \in \mathcal{C}''$ if and only if $t(X) = 0$. (3) follows from the fact that for self-injective algebras the functor G is a right adjoint to F, namely, $\mathrm{Hom}_B(F\mathcal{C}', \mathcal{D}) \simeq \mathrm{Hom}_A(\mathcal{C}', G\mathcal{D}) = \mathrm{Hom}_A(\mathcal{C}', \mathcal{C}) = 0$. Similarly, we have $G(\mathcal{D}'') \subseteq \mathcal{C}''$. (4) follows from a general result by Auslander. $\square$

If A and B are symmetric algebras (for example, the block algebras of finite groups), then we may say more on the torsion pairs above. The following result collects some of their properties.

Proposition 5.6. *Let A and B be symmetric algebras. Then:*
(1) $add(F\mathcal{C}') \subseteq \mathcal{D}', add(F\mathcal{C}'') \subseteq \mathcal{D}''$, *and* $add(G\mathcal{D}') \subseteq \mathcal{C}', add(G\mathcal{D}'') \subseteq \mathcal{C}''$.
(2) eAe *and* fBf *are stably equivalent of Morita type.*

Proof. (1) We know that (F, G) is an adjoint pair for A and B self-injective. If A and B are symmetric algebras, then (G, F) is also an adjoint pair. In fact, we know that G is isomorphic to $\mathrm{Hom}_B(N, -)$. This yields that $M \simeq \mathrm{Hom}_B(N, B) \simeq D(N)$ as bimodules since B is symmetric, where $D = \mathrm{Hom}_k(-, k)$. Thus

$$(N \otimes_A -, D(N) \otimes_B -) \simeq (F, G)$$

is an adjoint pair. This implies further that $(D(N) \otimes_B -, N \otimes_A -)$ is an adjoint pair, that is, (G, F) is an adjoint pair.

Now statement (1) follows from the adjoint pairs (F, G) and (G, F).

(2) We define a functor $H : eAe\text{-mod} \longrightarrow fBf\text{-mod}$ by

$$H(X) := \mathrm{Hom}_B(Bf, F(Ae \otimes_{eAe} X))$$

for each X in eAe-mod. Clearly, H is well defined. Moreover, we claim that H is an exact functor. In fact, since (G, F) is an adjoint pair, we have the k-space isomorphisms $\mathrm{Hom}_B(Bf, F(Ae \otimes_{eAe} X)) \simeq \mathrm{Hom}_A(G(Bf), Ae \otimes_{eAe} X) \simeq \mathrm{Hom}_A(G(Bf), Ae) \otimes_{eAe} X$ for any eAe-module X. Note that these isomorphisms are also functorial in X. Now let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be an exact sequence in eAe-mod. Applying the functor H and the above isomorphisms, we get the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H(X) \arrow[r] \arrow[d, "\simeq"] & H(Y) \arrow[r] \arrow[d, "\simeq"] & H(Z) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & (G(Bf), Ae) \otimes_{eAe} X \arrow[r] & (G(Bf), Ae) \otimes_{eAe} Y \arrow[r] & (G(Bf), Ae) \otimes_{eAe} Z \arrow[r] & 0
\end{tikzcd}$$

where $(G(Bf), Ae)$ denotes $\mathrm{Hom}_A(G(Bf), Ae)$. Since $G(Bf) \in \mathrm{add}(Ae)$ by Proposition 5.6(1), $\mathrm{Hom}_A(G(Bf), Ae)$ is a right projective eAe-module, and therefore the lower row is an exact sequence. It follows that the upper row is an exact sequence. Hence H is an exact functor.

Next, we show that the functor H induces a stable equivalence: eAe-$\underline{\mathrm{mod}} \longrightarrow fBf$-$\underline{\mathrm{mod}}$. Let $\mathrm{Pre}(Ae)$ be the full subcategory of A-mod whose objects are those X in A-mod which have projective presentations $P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ with the P_i in $\mathrm{add}(Ae)$ for $i = 0, 1$. Similarly, we have the full subcategory $\mathrm{Pre}(Bf)$ of B-mod. It is well known that the functor $Ae \otimes_{eAe} - : eAe\text{-mod} \longrightarrow A\text{-mod}$ induces an equivalence from eAe-mod to $\mathrm{Pre}(Ae)$ which is an inverse of $\mathrm{Hom}(Ae, -)$: $\mathrm{Pre}(Ae) \longrightarrow eAe$-mod. We have $\mathcal{P}(X, Y) = \mathrm{Hom}_A(X, \mathrm{add}(Ae), Y)$ for any two modules X and Y in $\mathrm{Pre}(Ae)$. Since the equivalent functor $\mathrm{Hom}(Ae, -)$: $\mathrm{Pre}(Ae) \longrightarrow eAe$-mod induces an isomorphism $\mathrm{Hom}_A(X, \mathrm{add}(Ae), Y) \longrightarrow \mathcal{P}(\mathrm{Hom}_A(Ae, X), \mathrm{Hom}_A(Ae, Y))$, we know that $\mathrm{Hom}(Ae, -)$ induces an equivalence between $\underline{\mathrm{Pre}}(Ae)$ and eAe-$\underline{\mathrm{mod}}$ where $\underline{\mathrm{Pre}}\,(Ae)$ denotes the full subcategory of A-$\underline{\mathrm{mod}}$. Thus we can identify the categories $\underline{\mathrm{Pre}}\,(Ae)$ and eAe-$\underline{\mathrm{mod}}$ by this equivalence. Similarly, the categories $\underline{\mathrm{Pre}}\,(Bf)$ and fBf-$\underline{\mathrm{mod}}$ are equivalent. Since $F(Ae) \in \mathrm{add}(Bf)$, we have $F(\mathrm{Pre}(Ae)) \subseteq \mathrm{Pre}(Bf)$, and therefore F induces an equivalence between $\underline{\mathrm{Pre}}\,(Ae)$ and $\underline{\mathrm{Pre}}\,(Bf)$. It follows that $H := \mathrm{Hom}_A(Bf, F(Ae \otimes_{eAe} -))$ induces a stable equivalence between eAe-$\underline{\mathrm{mod}}$ and fBf-$\underline{\mathrm{mod}}$.

Since the exact functor H induces a stable equivalence between two self-injective algebras eAe and fBf, we know that there is a stable equivalence of Morita type between eAe and fBf by [11, theorem 3.2, p. 167]. □

Summarizing the above discussions together, we get the main result of this section:

Theorem 5.7. *Let $e \in A$ and $f \in B$ be defined as before. Then:*

(1) *A/AeA and B/BfB are Morita equivalent;*

(2) *eAe and fBf are stably equivalent of Morita type if the algebras A and B are symmetric.*

Note that a further generalization of Theorem 5.7 will appear in [9].

6. Some examples

In this section, we first use Corollary 1.2 to construct a concrete example of a stable equivalence of Morita type between two algebras of global dimension 4, and then we display two examples from the Bock theory of groups. Finally, with the help of Theorem 4.2, we shall give an example of stable equivalence of Morita type between two one-point extension algebras.

Now let us recall some basic notions of Auslander-Reiten theory (see [2]).

Let A be an algebra. Then there is an Auslander-Reiten quiver of A, which we denote by Γ_A. The vertices of Γ_A are the isomorphism classes $[X]$ of indecomposable A-modules X. There is an arrow $[X] \longrightarrow [Y]$ if and only if there is an irreducible morphism $X \longrightarrow Y$. The arrow has valuation (a, b) if there is a minimal right almost split morphism $X^a \oplus Z \longrightarrow Y$ where X is not a summand of Z, and a minimal left almost split morphism $X \longrightarrow Y^b \oplus T$ where Y is not a summand of T. When A is of finite representation type, we omit the valuation since it is $(1, 1)$ for each arrow in Γ_A. We shall not distinguish between indecomposable A-modules and the corresponding vertices of Γ_A.

It is known that Γ_A is a translation quiver with Auslander-Reiten translation τ_A whose inverse will be denoted by τ_A^{-1}. We denote by Γ_A^s the stable part of Γ_A. It is obtained from Γ_A by removing the τ_A^{-1}-orbits of the projective vertices, the τ_A-orbits of the injective vertices and all arrows connected to vertices in the removed part.

Lemma 6.1 (see [2, corollary 1.9, p. 342])**.** *Let $F : A\text{-}\underline{mod} \to B\text{-}\underline{mod}$ be a stable equivalence between A and B. If A and B are self-injective algebras with no block of Loewy length 2, then F gives an isomorphism of translation quivers : $(\Gamma_A^s, \tau_A) \to (\Gamma_B^s, \tau_B)$. In particular, if we again denote by F the induced correspondence on the indecomposable non-projective modules in A-mod and B-mod, then F commutes with the Auslander-Reiten translation τ.* □

Recall that, given an algebra A, we may define a trivial extension $T(A)$ of A as follows: as a vector space, $T(A) := A \oplus D(A)$, where $D = \mathrm{Hom}_k(-, k)$; and the multiplication on $T(A)$ is given by

$$(a, f)(b, g) = (ab, ag + fb)$$

for $a, b \in A$ and $f, g \in D(A)$. It is known that $T(A)$ is always a symmetric algebra for any algebra A. The following result links tilting theory with trivial extension algebras.

Lemma 6.2 (see [11])**.** *Let A and B be two algebras. If A is tilted from B, then the trivial extensions $T(A)$ and $T(B)$ are stably equivalent of Morita type.* $\square$

Example 1. We now give our promised example. Note that all algebras appearing in this example are of finite representation type.

Suppose k is a field. Let A be the path algebra over k given by the quiver $1 \overset{\alpha}{\to} 2 \overset{\beta}{\to} 3$. It has a tilting module $T = P_1 \oplus P_3 \oplus S_1$, where P_i denotes the projective A-modules associated with vertex i, and S_i denotes the corresponding simple A-modules. The tilted algebra $B = \mathrm{End}({}_AT)$ is given by the same quiver $1 \overset{\alpha}{\to} 2 \overset{\beta}{\to} 3$ with relation $\alpha\beta = 0$. By Lemma 6.2, the trivial extensions $T(A)$ and $T(B)$ are stably equivalent of Morita type.

$T(A)$ is given by the quiver

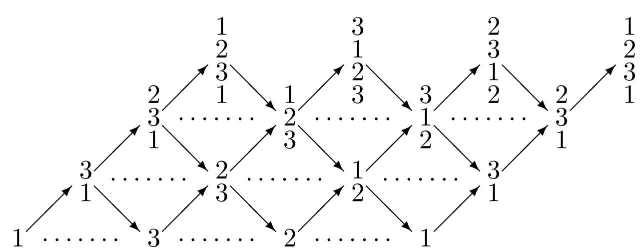
$$\nabla: \quad \begin{tikzcd} 1 \arrow[rr, "\alpha"] & & 2 \arrow[dl, "\beta"] \\ & 3 \arrow[ul, "\gamma"] & \end{tikzcd}$$

with relations $\alpha\beta\gamma\alpha = 0,\ \beta\gamma\alpha\beta = 0,\ \gamma\alpha\beta\gamma = 0.$

This algebra is a symmetric Nakayama algebra, and it has the following regular representation:

$$\begin{array}{ccccc} 1 & & 2 & & 3 \\ 2 & & 3 & & 1 \\ 3 & \oplus & 1 & \oplus & 2 \\ 1 & & 2 & & 3 \end{array}$$

We display the Auslander-Reiten quiver of $T(A)$ as follows:



where the dotted lines indicate the Auslander-Reiten translation, and the same vertices are identified.

$T(B)$ is given by the quiver

$$\Delta: \quad \begin{tikzcd} 1' \arrow[r, shift left, "\rho"] & 2' \arrow[l, shift left, "\rho'"] \arrow[r, shift left, "\delta"] & 3' \arrow[l, shift left, "\delta'"] \end{tikzcd}$$

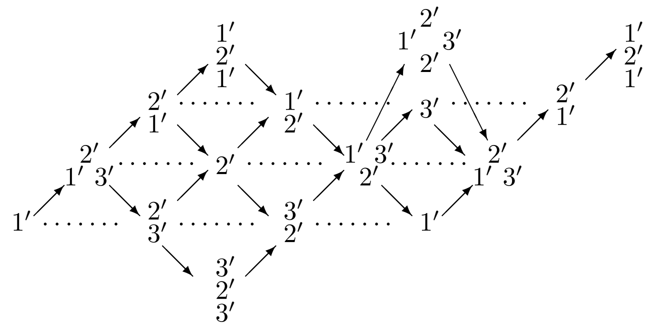
with relations

$$\rho\delta = \delta'\rho' = \rho'\rho - \delta\delta' = 0.$$

This algebra is symmetric and has the following regular representation:

$$\begin{array}{ccccc} 1' & & 2' & & 3' \\ 2' & \oplus & 1' \quad 3' & \oplus & 2' \\ 1' & & 2' & & 3' \end{array}$$

We display the Auslander-Reiten quiver of $T(B)$ as follows:



where the dotted lines indicate the Auslander-Reiten translation, and the same vertices are identified.

We denote by W the indecomposable projective $T(A)$-module corresponding to the vertex 1 in the quiver ∇. By Lemma 2.4, $\mathrm{soc}(W)$ is an ideal of $T(A)$, and the quotient algebra $T(A)' := T(A)/\mathrm{soc}(W)$ is given by the same quiver ∇ with relations $\alpha\beta\gamma = 0, \beta\gamma\alpha\beta = 0$. $T(A)'$ has global dimension 4 and the following regular representation:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \oplus \begin{array}{c} 2 \\ 3 \\ 1 \\ 2 \end{array} \oplus \begin{array}{c} 3 \\ 1 \\ 2 \\ 3 \end{array}$$

We denote by W' the indecomposable projective $T(B)$-module corresponding to the vertex $1'$ in the quiver of Δ. By Lemma 2.4, $\mathrm{soc}(W')$ is an ideal of $T(B)$ and the quotient algebra $T(B)' := T(B)/\mathrm{soc}(W')$ is given by the same quiver Δ with relations $\rho\delta = \delta'\rho' = \rho'\rho - \delta\delta' = \rho\rho' = 0$. $T(B)'$ has global dimension 4 and the following regular representation:

$$\begin{array}{c} 1' \\ 2' \end{array} \oplus \begin{array}{ccc} & 2' & \\ 1' & & 3' \\ & 2' & \end{array} \oplus \begin{array}{c} 3' \\ 2' \\ 3' \end{array}$$

Suppose that ${}_{T(A)}M_{T(B)}$ and ${}_{T(B)}N_{T(A)}$ define a stable equivalence of Morita type between $T(A)$ and $T(B)$. By Lemma 2.2, we can assume that N is indecomposable. Therefore $N \otimes_{T(A)} 1$ is an indecomposable non-projective $T(B)$-module. By Lemma 2.3(2), it can not be isomorphic to $\begin{array}{ccc} 1' & & 3' \\ & 2' & \end{array}$ since the latter is not uniserial. As $N \otimes_{T(A)} -$ commutes with the Auslander-Reiten translation τ and all simple $T(A)$-modules lie in the same τ-orbit, we know that $N \otimes_{T(A)} 1$ is isomorphic to neither $2'$ nor $\begin{array}{ccc} & 2' & \\ 1' & & 3' \end{array}$. Without loss of generality, we can assume that $N \otimes_{T(A)} 1 \simeq 1'$ or $N \otimes_{T(A)} 1 \simeq \begin{array}{c} 2' \\ 1' \end{array}$. In the first case, it follows from Corollary 1.2 that there is a stable equivalence of Morita type between $T(A)'$ and $T(B)'$. In the second case, we have that $N \otimes 2 \simeq 3'$ since $N \otimes_{T(A)} -$ commutes with τ. It also follows that there is a stable equivalence of Morita type between two algebras $T(A)''$ and $T(B)''$, where $T(A)''$ is the quotient algebra of $T(A)$ by the socle of the indecomposable projective module corresponding to the vertex 2 in the quiver of

$T(A)$, and $T(B)''$ is the quotient algebra of $T(B)$ by the socle of the indecomposable projective module corresponding to the vertex 3 in the quiver of $T(B)$. Note that $T(A)' \simeq T(A)''$ and $T(B)' \simeq T(B)''$ as algebras.

Remarks. (1) $N \otimes_{T(A)} 1 \simeq 1'$ and $N \otimes_{T(A)} 1 \simeq \begin{smallmatrix} 2' \\ 1' \end{smallmatrix}$ correspond to two different stable equivalences of Morita type between $T(A)$ and $T(B)$. It is easy to see that they are obtained from each other by composing the syzygy functor $\Omega_{T(B)}$.

(2) It is known that the class of quasi-hereditary algebras is not closed under tilting. In the above example, it is easy to verify that $T(B)'$ is a quasi-hereditary algebra, but $T(A)'$ is not. This example shows that the class of quasi-hereditary algebras is not closed under stable equivalences of Morita type.

Example 2. Now let us give two more examples from the block theory of finite groups. We follow the approach in [13].

Let k be an algebraically closed field of characteristic p.

(1) Suppose $p = 2$. The alternating group $G := A_5 \simeq SL(2,4) \simeq PSL(2,5)$ has Sylow 2-subgroup $P \simeq C_2 \times C_2$, with normalizer $H := N_G(P) \simeq A_4 \simeq P \rtimes C_3$. The principle block A of kG has three simple modules: the trivial module k and two 2-dimensional simple modules, which we shall denote by a and b; while the algebra $B := kH$ has three 1-dimensional simple modules, which we shall denote by k, 1, and 2. It is well known that there is an stable equivalence $F := N \otimes_A -$ of Morita type which coincides with Green correspondence on objects. Moreover, F sends the trivial module k of kG to the trivial kH-module k. Now if P_1 is the projective cover of the trivial kG-module and P_1' is the projective cover of kH-module of k, then, by Corollary 1.2, there is a stable equivalence of Morita type between the quotient algebras $A/\mathrm{soc}(P_1)$ and $B/\mathrm{soc}(P_1')$.

(2) Suppose $p = 3$. The alternating group $G := A_7$ has Sylow 3-subgroup $P \simeq C_3 \times C_3$, with normalizer $H := N_G(P) \simeq P \rtimes C_4$. The principle block A of kG has four simple modules: the trivial module, two 10-dimensional modules, and a 13-dimensional module. The algebra $B := kH$ has four 1-dimensional simple modules, which we shall denote by k, 1, 2 and 3. There is a stable equivalence $F := N \otimes_A -$ of Morita type which sends the trivial module to the trivial module, the 10-dimensional kG-modules to simple kH-modules 1 and 3, and the 13-dimensional module to a non-simple kH-module Y_2. Now if P_1 and P_3 are the projective covers of the trivial kG-module and one of the 10-dimensional modules, respectively, and if P_1' and P_3' are the projective covers of kH-modules of k and 3, respectively, then there is a stable equivalence of Morita type between $A/\mathrm{soc}(P_1 \oplus P_3)$ and $B/\mathrm{soc}(P_1' \oplus P_3')$ by Corollary 3.2.

Note that all quotient algebras in the above examples are representation-infinite and not self-injective. Let us also mention that the stable equivalences F of Morita type between the algebras A and B in the above examples are not Morita equivalences since the functors do not preserve all simple modules. On the other hand, if a stable equivalence of Morita type between two finite-dimensional algebras preserves all simple modules, then it must be a Morita equivalence. This was first proved in [7] for self-injective algebras, and then extended in [8] to arbitrary finite-dimensional algebras.

Example 3. Suppose k is an algebraically closed field. Then it is well known that the algebra A in Example 2(1) is Morita equivalent to the quotient algebra of the path algebra given by the quiver

$$\begin{tikzcd} a \arrow[r, shift left, "\rho"] & k \arrow[l, shift left, "\rho'"] \arrow[r, shift left, "\delta"] & b \arrow[l, shift left, "\delta'"] \end{tikzcd}$$

with relations

$$\rho\rho' = \delta'\delta = 0; \ \rho'\rho\delta\delta' = \delta\delta'\rho'\rho,$$

while the algebra B in Example 2(1) is Morita equivalent to the quotient algebra of the path algebra given by the quiver

$$\begin{tikzcd} 1 \arrow[rr, shift left, "\alpha"] \arrow[dr, shift left, "\eta"] & & 2 \arrow[ll, shift left, "\delta"] \arrow[dl, shift left, "\xi"] \\ & k \arrow[ul, shift left, "\gamma"] \arrow[ur, shift left, "\beta"] & \end{tikzcd}$$

with relations

$$\alpha\delta = \beta\gamma, \ \beta\xi = \delta\alpha, \ \gamma\eta = \xi\beta; \quad \alpha\beta = \beta\gamma = \gamma\alpha = \delta\eta = \eta\xi = \xi\delta = 0.$$

As we know from Example 2(1), there is a stable equivalence of Morita type between A and B, which is induced by $F := N \otimes_A - : A\text{-mod} \longrightarrow B\text{-mod}$, such that F sends the trivial module to the trivial module. Clearly, if we take R to be the trivial A-module, then it satisfies all conditions in Theorem 4.2. Note that $F(R) = k$ in this case. Thus there is a stable equivalence of Morita type between the two one-point extensions $\Lambda = \begin{pmatrix} A & k \\ 0 & k \end{pmatrix}$ and $\Gamma = \begin{pmatrix} B & k \\ 0 & k \end{pmatrix}$, where Λ can be described by the quiver

$$\begin{tikzcd} a \arrow[r, shift left, "\rho"] & k \arrow[l, shift left, "\rho'"] \arrow[r, shift left, "\delta"] & b \arrow[l, shift left, "\delta'"] \\ & c \arrow[u, "\kappa"'] & \end{tikzcd}$$

with relations

$$\rho\rho' = \delta'\delta = 0, \quad \kappa\rho' = \kappa\delta = 0, \quad \rho'\rho\delta\delta' = \delta\delta'\rho'\rho,$$

and the algebra Γ is given by the quiver

$$\begin{tikzcd} 1 \arrow[rr, shift left, "\alpha"] \arrow[dr, shift left, "\eta"] & & 2 \arrow[ll, shift left, "\delta"] \arrow[dl, shift left, "\xi"] \\ & k \arrow[ul, shift left, "\gamma"] \arrow[ur, shift left, "\beta"] & \\ & 3 \arrow[u, "\kappa"'] & \end{tikzcd}$$

with relations

$$\alpha\delta = \beta\gamma, \beta\xi = \delta\alpha, \gamma\eta = \xi\beta; \quad \alpha\beta = \beta\gamma = \gamma\alpha = \delta\eta = \eta\xi = \xi\delta = 0; \quad \kappa\gamma = \kappa\xi = 0.$$

Note that the algebras Λ and Γ are neither self-injective nor of finite global dimension. However, they have the same representation dimension.

If we consider Example 2(2), then, by Proposition 5.2, we can obtain many indecomposable A-modules R satisfying both $M \otimes_B N \otimes_A R \simeq R$ and $\mathrm{End}({}_A R) \simeq k$. Note that if A and B are self-injective algebras (this is the case for block algebras of finite groups), then these two conditions imply that $\mathrm{End}({}_B N \otimes_A R) \simeq k$ since $\mathrm{Hom}_B(F(R), F(R)) \simeq \mathrm{Hom}_A(R, GF(R)) \simeq \mathrm{Hom}_A(R, R)$. In this example, the category $\mathcal{D}$ consists of all B-modules with composition factors isomorphic either to k, 1 or 3, and the algebra eAe is isomorphic to the algebra fBf (see Proposition 5.5 for the notation); they are isomorphic to $k[x]/(x^3)$. We can get R by employing modules in $\mathcal{D}$. For instance, one can get several (uniserial) B-modules with trivial endomorphism algebra from the projective cover of the simple B-module 2, where the Loewy structure of this projective module is

$$\begin{array}{ccccccccc}
 & & & & 2 & & & & \\
 & & & \diagup & & \diagdown & & & \\
 & & 1 & & & & 3 & & \\
 & \diagup & & \diagdown & & \diagup & & \diagdown & \\
k & & & & 2 & & & & k \\
 & \diagdown & & \diagup & & \diagdown & & \diagup & \\
 & & 1 & & & & 3 & & \\
 & & & \diagdown & & \diagup & & & \\
 & & & & 2 & & & &
\end{array}$$

By Proposition 5.2, we obtain the corresponding modules R from the (uniserial) B-modules via the functor G. Hence there are stable equivalences of Morita type between some "complicated" one-point extensions of the algebras A and B.

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