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Constructions of stable equivalences of Morita type for finite dimensional algebras II

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Abstract. Suppose k is a field. Let A and B be two finite dimensional k-algebras such that there is a stable equivalence of Morita type between A and B. In this paper, we prove that (1) if A and B are representation-finite then their Auslander algebras are stably equivalent of Morita type; (2) The n-th Hochschild homology groups of A and B are isomorphic for all $n \geq 1$. A new proof is also provided for Hochschild cohomology groups of self-injective algebras under a stable equivalence of Morita type.

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1. Introduction

In the representation theory of artin algebras, there are three important equivalences: Morita equivalences, derived equivalences and stable equivalences. The classical Morita theory says that every Morita equivalence is induced by tensoring with a suitable bimodule, while a derived equivalence is induced by a suitable complex of bimodules by Rickard's Morita theory for derived categories. Stable equivalences are equivalences between stable categories. In general, they are not induced from bimodules. However, Rickard showed that a derived equivalence between self-injective algebras implies a stable equivalence induced by some bimodule. This led Broué into defining a special kind of stable equivalences, which are called stable equivalences of Morita type. They arise naturally for blocks of finite groups, or more generally, for self-injective algebras. For arbitrary finite dimensional algebras, this notion is still of particular interest, it preserves the representation dimension [22], representation type [9] and Linckelmann's Theorem [12]. However, how to produce such a stable equivalence for general finite dimensional algebras seems to be a difficult problem. Very recently, some advances in this direction are made in [13], where one starts from a stable equivalence of Morita type between two algebras and gets another one between certain quotient algebras or certain triangular matrix algebras. It turns out that one can produce a large class of stable equivalences of Morita type between non-selfinjective algebras.

The present paper has two aims. First, we shall provide another natural way to construct stable equivalences of Morita type. Here our starting point is a stable equivalence of Morita type between representation-finite algebras. We want to know whether there exists a stable equivalence of Morita type between their Auslander algebras. The second purpose of the paper is to compare the Hochschild homology or cohomology groups between two algebras which are stably equivalent of Morita type. The answer to the first question is the following positive result.

Theorem 1.1. *Suppose k is a field. Let A and B be two finite dimensional k-algebras of representation-finite type. Let Λ and Γ be the corresponding Auslander algebras of A and B, respectively. If A and B are stably equivalent of Morita type, then Λ and Γ are stably equivalent of Morita type.*

Note that a stable equivalence of Morita type between representation-finite self-injective algebras occurs frequently and is better understood (see [2], [19] and others). Thus, based on this fact, the above result provides us a plenty of new examples of stable equivalences of Morita type between finite dimensional algebras of global dimension at most two.

Since Auslander algebras are quasi-hereditary, their Hochschild homology groups H^n vanish except $n = 0$ by a result of Zacharia in [24]. Particularly, in Theorem 1.1 no matter Λ and Γ are stably equivalent of Morita type or not, the n-th Hochschild homology groups of Λ and Γ are automatically isomorphic for $n \geq 1$. An inverse question is how about the Hochschild Homology groups of A and B.

To this question, we shall prove that the Hochschild homology groups are in fact invariant under a stable equivalence of Morita type. More precisely, we have

Theorem 1.2. *Suppose A and B are two finite dimensional k-algebras. If A and B are stably equivalent of Morita type, then the n-th Hochschild homology group of A is isomorphic to that of B for all $n \geq 1$.*

To deal with Hochschild cohomology groups, we first prove some general properties for a stable equivalence of Morita type. From this we get a new proof of the fact that the Hochschild cohomology groups of self-injective algebras are invariant under a stable equivalence of Morita type. This was first proved in [14], but our proof here is more direct.

Theorem 1.3. *Let A and B be two self-injective k-algebras. If A and B are stably equivalent of Morita type, then $H^n(A) \simeq H^n(B)$ for all $n \geq 1$.*

We remark that neither the 0-th Hochschild homology group, nor the 0-th Hochschild cohomology group is invariant under a stable equivalence of Morita type.

The paper is broken down into sections as follows. After some elementary preparations in section two we prove Theorem 1.1 in section three and Theorem 1.2 and Theorem 1.3 in section four. Some examples to illustrate the necessity of some assumptions in the main results are displayed in section five. In the last section, we use a property developed in this paper to extend a result in [13].

2. Preliminaries

In this section we shall fix notations and recall definitions and facts needed in the proofs of our main results.

Throughout this paper, k will stand for a fixed field. All our categories will be k-categories and all functors are k-functors. All algebras will be assumed to be finite dimensional k-algebras with the identity. Unless stated otherwise, by a module we shall mean a finitely generated left module.

Given an algebra A, we denote by A-mod the category of finite dimensional A-modules and by A-$\underline{\text{mod}}$ the stable module category, which is the quotient of A-mod modulo the ideal of maps that factor through projective A-modules. The global dimension and the dominant dimension of A will be denoted by gl.dim(A) and dom.dim(A), respectively. Here the dominant dimension of A is defined as the maximal number n in a minimal injective resolution of the regular A-module ${}_AA$:

$$0 \longrightarrow {}_AA \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \ldots \longrightarrow I_{n-1} \longrightarrow \ldots,$$

such that $I_0, I_1, ..., I_{n-1}$ are projective.

In this paper, the composition of two morphisms $f : X \to Y$ and $g : Y \to Z$ will be denoted by fg. If there is no confusion, the k-duality $\mathrm{Hom}_k(-, k)$ will be denoted by D.

Now let us recall the definition of a stable equivalence of Morita type. This notion is due to Broué [5] and is a combination of the notion of a Morita equivalence and a stable equivalence. It was first noted to be useful for blocks in the representation theory of finite groups, or more generally, for finite dimensional self-injective algebras.

Definition 2.1. *Let A and B be two (arbitrary) k-algebras. We say that A and B are stably equivalent of Morita type if there exist an A-B-bimodule ${}_AM_B$ and a B-A-bimodule ${}_BN_A$ such that*

(1) *M and N are projective as one-sided modules, and*
(2) *$M \otimes_B N \simeq A \oplus P$ as A-A-bimodules for some projective A-A-bimodule P, and $N \otimes_A M \simeq B \oplus Q$ as B-B-bimodules for some projective B-B-bimodule Q.*

Note that if A and B are stably equivalent of Morita type, then their opposite algebras A^{op} and B^{op} are also stably equivalent of Morita type.

Suppose that two algebras A and B are stably equivalent of Morita type. We can define functors $T_N : A\text{-mod} \to B\text{-mod}$ by $X \mapsto N \otimes_A X$ and $T_M : B\text{-mod} \to A\text{-mod}$ by $Y \mapsto M \otimes_B Y$. Similarly, we have functors T_P and T_Q.

Lemma 2.2. (see [22])

(1) *T_M, T_N, T_P and T_Q are exact functors.*
(2) *$T_M \circ T_N \to id_{A-mod} \oplus T_P$ and $T_N \circ T_M \to id_{B-mod} \oplus T_Q$ are natural isomorphisms.*
(3) *The images of T_P and T_Q consist of projective modules.* ■

Clearly, if two algebras A and B are stably equivalent of Morita type, then they are stably equivalent. In fact, the functor $T_N : A\text{-mod} \to B\text{-mod}$ induces an equivalence: $A\text{-}\underline{\text{mod}} \to B\text{-}\underline{\text{mod}}$, whose inverse is induced by $T_M : B\text{-mod} \to A\text{-mod}$.

We say that an algebra is representation-finite if there are only finitely many non-isomorphic indecomposable A-modules.

Definition 2.3. *Let A be a representation-finite algebra.*

(1) *An A-module X is said to be an additive generator for A-mod if $add(X) = A$-mod, that is, every indecomposable A-module is isomorphic to a direct summand of X.*
(2) *Let X be an additive generator for A-mod. The endomorphism algebra $\Lambda = End_A(X)$ of X is called the Auslander algebra of A. (This is unique up to Morita equivalence.)*

Note that an algebra A is representation-finite if and only if A-mod has an additive generator. As we know, Auslander algebras might be of any representation type. However, they were characterized in [3] by the following homological properties: their global dimensions are at most 2 and their dominant dimensions are at least 2. More precisely, Auslander proved the following theorem.

Theorem 2.4. (see [3]) *Let $\mathcal{C}$ be the class of Morita-equivalence classes $[A]$ of representation-finite algebras. Let $\mathcal{D}$ be the class of Morita-equivalence classes $[\Lambda]$ of algebras satisfying gl.dim$(\Lambda) \leq 2$ and dom.dim$(\Lambda) \geq 2$. Then there is a one to one correspondence between $\mathcal{C}$ and $\mathcal{D}$ given as follows:*

(1) *If A is a representation-finite algebra and R is an additive generator for A-mod, then we send A to $\Lambda = End_A(R)$.*
(2) *If Λ is an algebra with gl.dim$(\Lambda) \leq 2$ and dom.dim$(\Lambda) \geq 2$ and I is a projective-injective Λ-modules such that add(I) is precisely the category of projective-injective Λ-modules, then we send Λ to $A = End_\Lambda(I)$.* ■

Auslander algebras are closely related to certain triangular matrix algebras. So, for our purpose, we also need some basic facts on triangular matrix algebras and morphism category (see [4] for more details).

Let A be an algebra. The triangular matrix algebra of A is defined as follows:

$$T_2(A) = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b.c \in A \right\}$$

with the usual matrix addition and multiplication. It is well known that each $T_2(A)$-module U can be described as a triple $U = (U_1, U_2, f)$, where U_1 and U_2 are in A-mod, and $f : U_2 \to U_1$ is an A-homomorphism; and each homomorphism from U to $V = (V_1, V_2, g)$ can be interpreted as a pair (α_1, α_2) in $\text{Hom}_A(U_1, V_1) \times \text{Hom}_A(U_2, V_2)$ such that $f\alpha_1 = \alpha_2 g$.

Let A be an algebra. Suppose $\mathcal{C}$ is a full subcategory of A-mod. The morphism category of $\mathcal{C}$ is the k-category Morph($\mathcal{C}$) defined by the following data. The objects of Morph($\mathcal{C}$) are the morphisms $f : C_2 \to C_1$ in $\mathcal{C}$; and the morphisms from an object $f : C_2 \to C_1$ to another object $f' : C_2' \to C_1'$ are pairs (g_1, g_2) where

$g_i : C_i \to C_i'$ is a homomorphism in $\mathcal{C}$ for $i = 1, 2$ such that $fg_1 = g_2 f'$. The composition of two morphisms are defined in a trivial way.

The relationship of $T_2(A)$-mod and A-mod was described in the following way(see [3]).

Lemma 2.5. *The category Morph(A-mod) is equivalent to $T_2(A)$-mod as exact categories.* ■

Note that the equivalence functor in Lemma 2.5 sends each object $f : M_2 \longrightarrow M_1$ in Morph(A-mod) to the $T_2(A)$-module (M_1, M_2, f).

Let $\wp(A)$ be the full subcategory of A-mod consisting of all projective modules. For objects $f : P_2 \longrightarrow P_1$ and $f' : P_2' \longrightarrow P_1'$ in Morph($\wp(A)$), we define $\Re_A(f, f') = \{(g_1, g_2) : f \to f' \mid$ there is an $h : P_1 \to P_2'$ such that $hf' = g_1\}$. Then $\Re_A$ gives a relation on Morph($\wp(A)$). We can define the factor category Morph($\wp(A)$)/$\Re_A$. The objects of Morph($\wp(A)$)/$\Re_A$ are the same as those of Morph($\wp(A)$). The morphisms from f to f' in Morph($\wp(A)$)/$\Re_A$ are the elements of the k-space $\mathrm{Hom}(f, f')/\Re_A(f, f')$. By [4, proposition 1.2, p.102], the natural functor Coker_A: Morph($\wp(A)$) $\to$ A-mod defined by $\mathrm{Coker}_A(f : P_2 \to P_1) = \mathrm{Coker}(f)$ induces an equivalence of categories: Morph($\wp(A)$)/$\Re_A \to A$-mod.

Now let A be a representation-finite algebra and let R be an additive generator for A-mod. By Λ we denote the Auslander algebra of A. By [4, proposition 2.1, p.33], $\mathrm{Hom}_A(R, -) : A\text{-mod} \to \Lambda\text{-mod}$ induces an equivalence: A-mod $\to \wp(\Lambda)$. It follows that $\mathrm{Hom}_A(R, -)$ induces an equivalence of categories: $T_2(A)$-mod $\to$ Morph($\wp(\Lambda)$), which is defined by $(U_1, U_2, f) \mapsto (R, f)$: $(R, U_2) \to (R, U_1)$. Here and in the sequel we denote $\mathrm{Hom}_A(R, *)$ by $(R, *)$.

Lemma 2.6. *Let A be a representation-finite algebra, R an additive generator and Λ the Auslander algebra of A. Then the composition functor $\mathrm{Coker}_\Lambda \circ \mathrm{Hom}_A(R, -)$: $T_2(A)$-mod $\to$ Λ-mod induces an equivalence $H_A : T_2(A)\text{-mod}/\Re_A' \to \Lambda$-mod, where $\Re_A'$ is the relation on $T_2(A)$-mod defined by $\Re_A'(U, V) = \{(\alpha_1, \alpha_2) : U \to V \mid$ there is a homomorphism $\gamma : U_1 \to V_2$ such that $\gamma g = \alpha_1\}$ for modules $U = (U_1, U_2, f)$ and $V = (V_1, V_2, g)$.*

Proof. It is straightforward to see that $\mathrm{Coker}_\Lambda \circ \mathrm{Hom}_A(R, -)$ is full and dense. On the other hand, one can verify that $\mathrm{Coker}_\Lambda \circ \mathrm{Hom}_A(R, -)(\alpha_1, \alpha_2) = \mathrm{Coker}_\Lambda((R, \alpha_1), (R, \alpha_2)) = 0$ if and only if there is some $h : (R, U_1) \to (R, V_2)$ such that $h(R, g) = (R, \alpha_1)$. But this is equivalent to saying that there is a homomorphism $\gamma : U_1 \to V_2$ of A-modules such that $(R, \gamma) = h$ and $\gamma g = \alpha_1$, since $\mathrm{Hom}_A(R, -) : A\text{-mod} \to \wp(\Lambda)$ is an equivalence. ■

In the next section we shall use the above functor as a bridge to prove our main result Theorem 1.1.

3. Proof of Theorem 1.1

In this section, we shall show that forming Auslander algebras will provide a convenient way to get stable equivalences of Morita type.

We need the following homological facts, which are easy to check. Let us remind the reader that the modules in our paper are always assumed to be finitely generated.

Lemma 3.1. *Let C, D and E be three algebras and ${}_CX_D$ and ${}_DY_E$ bimodules, where X_D is projective. Then the natural morphism ϕ: ${}_CX \otimes_D Y_E \to Hom_D({}_DX^*{}_C, {}_DY_E)$, where $X^* = Hom_D(X, D)$ and $\phi(x \otimes y)(f) = f(x)y$ for $x \in X, y \in Y$ and $f \in X^*$ is an isomorphism of C-E-bimodules.* ■

Lemma 3.2. *Let k be a field. Let C, D and E be three algebras. For every triple of $({}_CX_D, {}_CY, Z_E)$, there is an D-E-bimodule isomorphism ψ: $Hom_C({}_CX_D, {}_CY) \otimes_k Z_E \to Hom_C({}_CX_D, {}_CY \otimes_k Z_E)$ defined by $\psi(f \otimes z)(x) = f(x) \otimes z$ for $x \in X, z \in Z$ and $f \in Hom_C(X, Y)$.* ■

Let us remark that the two lemmas would be false if the bimodules are not finitely generated. We thank the referee for pointing out this fact.

From now on, we assume that A, B, M, N, P, Q are fixed as in Definition 2.1. Furthermore, we assume that A and B are representation-finite. We choose an additive generator R for A-mod and an additive generator S for B-mod, and denote by $\Lambda = \mathrm{End}_A(R)$ and by $\Gamma = \mathrm{End}_B(S)$ the corresponding Auslander algebras of A and B, respectively.

Recall from Section 2 that we have the following equivalences of categories: $H_A : T_2(A)\text{-mod}/\Re'_A \to \Lambda\text{-mod}$ and $H_B : T_2(B)\text{-mod}/\Re'_B \to \Gamma\text{-mod}$. In order to link the two categories Λ-mod and Γ-mod together, we define two functors $\widetilde{T_N} : T_2(A)\text{-mod}/\Re'_A \to T_2(B)\text{-mod}/\Re'_B$ and $\widetilde{T_M} : T_2(B)\text{-mod}/\Re'_B \to T_2(A)\text{-mod}/\Re'_A$. For $U = (U_1, U_2, f)$ in $T_2(A)\text{-mod}/\Re'_A$, we define $\widetilde{T_N}(U) = (T_N(U_1), T_N(U_2), T_N(f))$. For a morphism $(\alpha_1, \alpha_2) + \Re'_A(U, V) : U \to V = (V_1, V_2, g)$, we set $\widetilde{T_N}((\alpha_1, \alpha_2) + \Re'_A(U, V)) = (T_N(\alpha_1), T_N(\alpha_2)) + \Re'_B(T_N(U), T_N(V)) : \widetilde{T_N}(U) \to \widetilde{T_N}(V)$. Since $\widetilde{T_N}(\Re'_A(U, V)) \subseteq \Re'_B(T_N(U), T_N(V))$, it is easy to see that $\widetilde{T_N}$ is well-defined. The functor $\widetilde{T_M}$ can be defined similarly.

Now we can define two new functors F and G between Λ-mod and Γ-mod. Let $F : \Lambda\text{-mod} \to \Gamma\text{-mod}$ be the compositions: $\Lambda\text{-mod} \xrightarrow{H_A^{-1}} T_2(A)\text{-mod}/\Re'_A \xrightarrow{\widetilde{T_N}} T_2(B)\text{-mod}/\Re'_B \xrightarrow{H_B} \Gamma\text{-mod}$, where H_A^{-1} is the inverse of H_A. Similarly, we define $G : \Gamma\text{-mod} \to \Lambda\text{-mod}$ as the compositions: $\Gamma\text{-mod} \xrightarrow{H_B^{-1}} T_2(B)\text{-mod}/\Re'_B \xrightarrow{\widetilde{T_M}} T_2(A)\text{-mod}/\Re'_A \xrightarrow{H_A} \Lambda\text{-mod}$, where H_B^{-1} is the inverse of H_B. So we are in the following situation:

$$\begin{array}{ccc} \Lambda\text{-mod} & \xrightarrow{F} & \Gamma\text{-mod} \\ \downarrow H_A^{-1} & & \uparrow H_B \\ T_2(A)\text{-mod}/\Re'_A & \xrightarrow{\widetilde{T_N}} & T_2(B)\text{-mod}/\Re'_B \end{array} \qquad \begin{array}{ccc} \Gamma\text{-mod} & \xrightarrow{G} & \Lambda\text{-mod} \\ \downarrow H_B^{-1} & & \uparrow H_A \\ T_2(B)\text{-mod}/\Re'_B & \xrightarrow{\widetilde{T_M}} & T_2(A)\text{-mod}/\Re'_A \end{array}$$

We claim that F and G take projective modules to projective modules. Indeed, let $X \simeq \mathrm{Hom}_A(R, U_0)$ be a projective Λ-module with U_0 an A-module.

Then $H_A^{-1}(X) \simeq (U_0, 0, 0)$ in $T_2(A)$-mod/$\mathfrak{R}'_A$ and $\widetilde{T_N} H_A^{-1}(X) \simeq (T_N(U_0), 0, 0)$ in $T_2(B)$-mod/$\mathfrak{R}'_B$ with $T_N(U_0)$ a B-module. Therefore $F(X) \simeq \mathrm{Hom}_B(S, T_N(U_0))$ is a projective Γ-module. This implies that the functor F takes projective modules to projective modules. Similarly, the functor G takes projective modules to projective modules. In the following we shall prove that F and G are exact functors.

Lemma 3.3. *The above defined functors F and G are exact.*

Proof. We only prove that F is an exact functor since the argument for G will be similar to that for F.

Let $\delta : 0 \to X \to Y \to Z \to 0$ be a short exact sequence in Λ-mod. Since gl.dim$(\Lambda) \leq 2$, by the Horseshoe Lemma (see [20, lemma 6.20, p.187]), we have an exact commutative diagram in Λ-mod:

$$\begin{array}{lccccccccc}
 & & & 0 & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\theta: & 0 & \to & P_2 & \to & P_2 \oplus Q_2 & \to & Q_2 & \to & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\varepsilon: & 0 & \to & P_1 & \to & P_1 \oplus Q_1 & \to & Q_1 & \to & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\eta: & 0 & \to & P_0 & \to & P_0 \oplus Q_0 & \to & Q_0 & \to & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\delta: & 0 & \to & X & \to & Y & \to & Z & \to & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & 0 & & 0 & & 0 & &
\end{array}$$

where P_i, Q_i $(i = 0, 1, 2)$ are projective Λ-modules, and the short exact sequences θ, ε and η are canonical split exact sequences. Since $\mathrm{Hom}_A(R, -) : A\text{-mod} \to \wp(\Lambda)$ is an equivalence, the exact commutative diagram in Λ-mod:

$$\begin{array}{lccccccccc}
 & & & 0 & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\theta: & 0 & \to & P_2 & \to & P_2 \oplus Q_2 & \to & Q_2 & \to & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\varepsilon: & 0 & \to & P_1 & \to & P_1 \oplus Q_1 & \to & Q_1 & \to & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\eta: & 0 & \to & P_0 & \to & P_0 \oplus Q_0 & \to & Q_0 & \to & 0
\end{array}$$

corresponds to a commutative diagram in A-mod:

$$\begin{array}{lccccccccc}
 & & & 0 & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\theta': & 0 & \to & U_2 & \to & U_2 \oplus V_2 & \to & V_2 & \to & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\varepsilon': & 0 & \to & U_1 & \to & U_1 \oplus V_1 & \to & V_1 & \to & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\eta': & 0 & \to & U_0 & \to & U_0 \oplus V_0 & \to & V_0 & \to & 0,
\end{array}$$

where U_i, V_i $(i = 0, 1, 2)$ are A-modules, and the short exact sequences θ', ε' and η' are canonical split exact sequences. Since $\mathrm{Hom}_A(R, -) : A\text{-mod} \to \Lambda\text{-mod}$ is a faithful functor, and since a faithful functor between abelian categories reflects exact sequences (see [8, proposition 3, p.94]), the columns in the above diagram are also exact. Using the functor T_N, we get an exact commutative diagram in B-mod:

$$\begin{array}{lccccccc}
 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\theta'': & 0\to & T_N(U_2) & \to & T_N(U_2)\oplus T_N(V_2) & \to & T_N(V_2) & \to 0\\
 & & \downarrow & & \downarrow & & \downarrow & \\
\varepsilon'': & 0\to & T_N(U_1) & \to & T_N(U_1)\oplus T_N(V_1) & \to & T_N(V_1) & \to 0\\
 & & \downarrow & & \downarrow & & \downarrow & \\
\eta'': & 0\to & T_N(U_0) & \to & T_N(U_0)\oplus T_N(V_0) & \to & T_N(V_0) & \to 0,
\end{array}$$

where θ'', ε'' and η'' are canonical split exact sequences. Applying the left exact functor $\mathrm{Hom}_B(S,-) : B\text{-mod} \to \Gamma\text{-mod}$ to the above diagram and taking the cokernels of the columns, we get an exact commutative diagram in Γ-mod:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \longrightarrow (S,T_N(U_2)) & \longrightarrow & (S,T_N(U_2))\oplus(S,T_N(V_2)) & \longrightarrow & (S,T_N(V_2)) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \longrightarrow (S,T_N(U_1)) & \longrightarrow & (S,T_N(U_1))\oplus(S,T_N(V_1)) & \longrightarrow & (S,T_N(V_1)) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
 & 0 & \longrightarrow (S,T_N(U_0)) & \longrightarrow & (S,T_N(U_0))\oplus(S,T_N(V_0)) & \longrightarrow & (S,T_N(V_0)) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\widetilde{\delta}: & & \mathrm{cok}(f) & \stackrel{\chi}{\longrightarrow} & \mathrm{cok}(g) & \longrightarrow & \mathrm{cok}(h) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $(S,*)$ denotes $\mathrm{Hom}_B(S,*)$. By the Snake Lemma, χ is an injective homomorphism. Thus the row $\widetilde{\delta}$ is a short exact sequence in Γ-mod. On the other hand, we know by definition that $\widetilde{\delta}$ is just the image of δ under the functor F. It follows that F is an exact functor. ■

Proof of Theorem 1.1. We shall prove that F and G define a stable equivalence of Morita type between Λ and Γ. By Lemma 3.3, the functors $F : \Lambda\text{-mod} \to \Gamma\text{-mod}$ and $G : \Gamma\text{-mod} \to \Lambda\text{-mod}$ are exact. By Watts Theorem (see [20, theorem 3.33, p.77]), we know that $F \simeq {}_\Gamma F(\Lambda)\otimes_\Lambda -$, where the right Λ-module structure on $F(\Lambda)$ is induced by the right multiplication on Λ. Since F is exact, $F(\Lambda)$ is projective as a right Λ-module. Since F takes projective modules to projective modules, $F(\Lambda) \simeq F(\Lambda)\otimes_\Lambda \Lambda$ is projective as a left Γ-module. Similarly, $G \simeq {}_\Lambda G(\Gamma)\otimes_\Gamma -$, and $G(\Gamma)$ is projective as left and right modules.

Since the composition of exact functors is again exact, the functor $G\circ F : \Lambda$-mod $\to \Lambda$-mod is an exact functor. Thus ${}_\Lambda G(F(\Lambda))_\Gamma = \mathrm{Hom}_A(R_\Lambda, M\otimes_B N\otimes_A R_\Lambda)$ is a Λ-Λ-bimodule. It is straightforward to verify that the Λ-Λ-bimodule structure on $G(F(\Lambda))$ are naturally from the Hom structure. Since we have an

A-A-bimodule isomorphism $\rho = (\rho_1, \rho_2) : M \otimes_B N \simeq A \oplus P$, it follows that the natural isomorphism $\overline{\rho} : \mathrm{Hom}_A(R_\Lambda, M \otimes_B N \otimes_A R_\Lambda) \simeq \mathrm{Hom}_A(R_\Lambda, R_\Lambda) \oplus \mathrm{Hom}_A(R_\Lambda, P \otimes_A R_\Lambda)$ is an isomorphism of Λ-Λ-bimodules, where $\overline{\rho}(f) = (f(\rho_1 \otimes 1_R)\mu, f(\rho_2 \otimes 1_R))$ and $\mu : A \otimes_A R \to R$ is the multiplication map. Note that $\mathrm{Hom}_A({}_A R_\Lambda, {}_A R_\Lambda) = \Lambda$ is the regular Λ-Λ-bimodule. We claim that $\mathrm{Hom}_A(R_\Lambda, P \otimes_A R_\Lambda)$ is a projective Λ-Λ-bimodule.

In fact, we know that P is a projective A-A-bimodule by assumption. Thus P is isomorphic to a direct summand of a module $(A \otimes_k A^{op})^m$ for some positive number m. By Lemma 3.1 and Lemma 3.2, we have the following Λ-Λ-bimodule isomorphisms: $\mathrm{Hom}_A(R_\Lambda, A \otimes_k A^{op} \otimes_A R_\Lambda) \simeq \mathrm{Hom}_A(R_\Lambda, A) \otimes_k A^{op} \otimes_A R_\Lambda \simeq \mathrm{Hom}_A(R_\Lambda, A) \otimes_k \mathrm{Hom}_A(\mathrm{Hom}_A(A^{op}, A), R_\Lambda)$. Clearly, $\mathrm{Hom}_A(R_\Lambda, A) \otimes_k \mathrm{Hom}_A(\mathrm{Hom}_A(A^{op}, A), R_\Lambda)$ is a projective Λ-Λ-bimodule since $\mathrm{Hom}_A(R_\Lambda, A)$ is a projective left Λ-module and $\mathrm{Hom}_A(\mathrm{Hom}_A(A^{op}, A), R_\Lambda)$ is a projective right Λ-module. This shows that for any free A-A-bimodule W the Λ-Λ-bimodule $\mathrm{Hom}_A(R_\Lambda, W \otimes_A R_\Lambda)$ is projective, and therefore $\mathrm{Hom}_A(R_\Lambda, P \otimes_A R_\Lambda)$ is projective for any projective A-A-bimodule P.

Since $G(F(\Lambda)) \simeq G(\Gamma) \otimes_\Gamma F(\Lambda)$ as Λ-Λ-bimodules, we have proved that $G(\Gamma) \otimes_\Gamma F(\Lambda) \simeq \Lambda \oplus \mathrm{Hom}_A(R, P \otimes_A R)$ as Λ-Λ-bimodules, where $\mathrm{Hom}_A(R, P \otimes_A R)$ is a projective Λ-Λ-bimodule. Similarly, we have Γ-Γ-bimodule isomorphism: $F(\Lambda) \otimes_\Lambda G(\Gamma) \simeq \Gamma \oplus \mathrm{Hom}_B(S, Q \otimes_B S)$, where $\mathrm{Hom}_B(S, Q \otimes_B S)$ is a projective Γ-Γ-bimodule since Q is a projective B-B-bimodule. Thus $F(\Lambda)$ and $G(\Gamma)$ define a stable equivalence of Morita type between Λ and Γ. ■

Remark. For self-injective algebras, Rickard proved that any stable equivalence induced from an exact functor is of Morita type. But it is not known if this statement holds true for non-selfinjective algebras. If so, the proof of Theorem 1.1 could be simplified.

As an immediate application of Theorem 1.1 together with the results in [9] and [22], we have the following corollary.

Corollary 3.4. *Suppose that two finite dimensional k-algebras A and B are representation-finite. If they are stably equivalent of Morita type, then their Auslander algebras have the same representation dimension and the same representation type.* ■

For the definition of representation dimension we refer the reader to the original paper of Auslander [3], or to [22,23]. For the detailed definition of representation type we refer the reader to [9], for example.

As another application of Theorem 1.1, we determine, up to stable equivalence of Morita type, the Auslander algebras of Brauer tree algebras.

The Brauer tree algebras are of particular interest in modular representation theory of finite groups due to the fact that they describe the structure of blocks with cyclic defect groups of finite groups (see, for example, [1]). By [17] and [15], we know that a stable equivalence between Brauer tree algebras is in fact a derived equivalence and that a derived equivalence implies a stable equivalence of Morita type. Hence, for Brauer tree algebras, stable equivalences of Morita type and derived equivalences coincide. Each Brauer tree algebra is derived equivalent to a unique symmetric Nakayama algebra by a result of Rickard, and the structure of Auslander algebras of symmetric Nakayama algebras can be described by quivers and relations. So the following proposition follows from Theorem 1.1 immediately.

Proposition 3.5. *If Λ is the Auslander algebra of a Brauer tree algebra, then there is a unique symmetric Nakayama algebra C such that Λ and the Auslander algebra of C are stably equivalent of Morita type.* ■

Thus, by this proposition, we can determine the quivers and relations of the Auslander algebras of Brauer tree algebras up to a stable equivalence of Morita type.

We remark that Theorem 1.1 is not true for stable equivalences or derived equivalences in general. One may find counterexamples in the section five.

4. Proofs of Theorem 1.2 and Theorem 1.3

In the previous section we have seen that if two algebras A and B are stably equivalent of Morita type, then their Auslander algebras are also stably equivalent of Morita type. Furthermore, in this section we shall show that the Hochschild homology groups of A and B are isomorphic.

Definition 4.1. *Let Λ be a finite dimensional algebra over a field k. If X is a Λ-Λ-bimodule, then the Hochschild homology of Λ with coefficients in X is defined as*

$$H_n(\Lambda, X) = Tor_n^{\Lambda^e}(X, \Lambda)$$

for all $n \geq 0$, where $\Lambda^e = \Lambda \otimes_k \Lambda^{op}$ is the enveloping algebra of Λ. If $X = \Lambda$ we obtain the Hochschild homology of $\Lambda : H_*(\Lambda) = Tor_*^{\Lambda^e}(\Lambda, \Lambda)$.

Dually, the Hochschild cohomology of Λ with coefficients in X is defined as

$$H^n(\Lambda, X) = Ext_{\Lambda^e}^n(\Lambda, X)$$

for all $n \geq 0$. If $X = \Lambda$ we obtain the Hochschild cohomology of $\Lambda : H^*(\Lambda) = Ext_{\Lambda^e}^*(\Lambda, \Lambda)$.

The following lemma provides a way to get projective bimodules.

Lemma 4.2. *Let A, B and C be three k-algebras. Suppose P is a projective A-B-bimodule. Then:*

(1) *If M is a C-A-bimodule such that ${}_CM$ and M_A are projective modules, then $M \otimes_A P$ is a projective C-B-bimodule. Similarly, if M is a B-C-bimodule such that ${}_BM$ and M_C are projective modules, then $P \otimes_B M$ is a projective A-C-bimodule.*
(2) *For A-modules X_A and ${}_AY$, we have $Tor_n^A(X_A, {}_AY) \simeq Tor_n^{A^{op}}(Y, X)$ for all n.*
(3) *If A and B are self-injective, then $A \otimes_k B$ is self-injective.*

Proof. (2) and (3) are well-known in homological algebra. Here we sketch a proof of (1). First, if $P = A \otimes_k B^{op}$, then we have $M \otimes_A P \simeq M \otimes_k B^{op}$. This implies that $M \otimes_A P$ is a projective C-B-bimodule for any free $A \otimes_k B^{op}$-module P. Hence (1) follows if P is a direct summand of a free $A \otimes_k B^{op}$-module. ■

To calculate the Hochschild homology groups, the following result, taken from [6, theorem 2.8, 2.8a, p.167], may be useful sometimes.

Lemma 4.3. (see [6, p.167])

(1) *Let Λ, Γ and Σ be three k-algebras. In the situation $(X_{\Lambda-\Gamma}, {}_\Lambda Y_\Sigma, {}_{\Gamma-\Sigma}Z)$ assume that $Tor_n^\Lambda(X, Y) = 0 = Tor_n^\Sigma(Y, Z)$ for $n \geq 1$. Then there is an isomorphism*

$$Tor^{\Gamma\otimes_k\Sigma}(X \otimes_\Lambda Y, Z) \simeq Tor^{\Lambda\otimes_k\Gamma}(X, Y \otimes_\Sigma Z)$$

(2) *Let Λ, Γ and Σ be three k-algebras. In the situation $(X_{\Lambda-\Gamma}, {}_\Lambda Y_\Sigma, Z_{\Gamma-\Sigma})$ assume that $Tor_n^\Lambda(X, Y) = 0 = Ext_\Sigma^n(Y, Z)$ for $n \geq 1$. Then there is an isomorphism*

$$Ext_{\Gamma\otimes_k\Sigma}(X \otimes_\Lambda Y, Z) \simeq Ext_{\Lambda\otimes_k\Gamma}(X, Hom_\Sigma(Y, Z)). \quad ■$$

The following result says that the Hochschild homology groups are invariants of a stable equivalence of Morita type.

Theorem 4.4. *Let A and B be two k-algebras. If A and B are stably equivalent of Morita type, then $H_n(A) \simeq H_n(B)$ for all $n \geq 1$.*

Proof. Note that any bimodule ${}_AX_B$ can be identified with the left $A \otimes_k B^{op}$-module X or with the right $B \otimes_k A^{op}$-module X. With this convention we shall calculate the homology groups of A and B.

In Lemma 4.3 (1) we let $\Lambda = B$, $\Sigma = A$ and $\Gamma = B^{op}$, and define $X = {}_BB_B = (B)_{\Lambda-\Gamma}$, $Y = N = {}_\Lambda N_\Sigma$ and $Z = {}_AM_B = {}_{\Gamma-\Sigma}M$. By the definition of stable equivalence of Morita type, ${}_BN$ and N_A are projective, that is, ${}_\Lambda Y$ and Y_Σ are projective. Thus $\mathrm{Tor}_n^\Lambda(X, Y) = 0 = \mathrm{Tor}_n^\Sigma(Y, Z)$ for $n \geq 1$. This implies that the condition in Lemma 4.3 (1) is fulfilled. Hence there is an isomorphism

$$\begin{aligned}\mathrm{Tor}_i^{B\otimes_k B^{op}}(B, N \otimes_A M) &\simeq \mathrm{Tor}_i^{B^{op}\otimes_k A}(B \otimes_B N, M)\\ &\simeq \mathrm{Tor}_i^{B^{op}\otimes_k A}(N, M).\end{aligned}$$

Similarly, we have the following isomorphism

$$\mathrm{Tor}_i^{A\otimes_k A^{op}}(A, M \otimes_B N) \simeq \mathrm{Tor}_i^{A^{op}\otimes_k B}(M, N).$$

By definition, we have $M \otimes_B N \simeq A \oplus P$ as A-A-bimodules for some projective A-A-bimodule P, and $N \otimes_A M \simeq B \oplus Q$ as B-B-bimodules for some projective B-B-bimodule Q. By Lemma 4.2(2), we have $\mathrm{Tor}_i^{A^{op}\otimes_k B}(M, N) \simeq \mathrm{Tor}_i^{B^{op}\otimes_k A}(N, M)$. Thus $\mathrm{Tor}_n^{A^e}(A, A \oplus P) \simeq \mathrm{Tor}_n^{B^e}(B, B \oplus Q)$. This yields that

$$\begin{aligned} H_n(A) &= \mathrm{Tor}_n^{A^e}(A, A) = \mathrm{Tor}_n^{A^e}(A, A \oplus P) \simeq \mathrm{Tor}_n^{B^e}(B, B \oplus Q) \\ &= \mathrm{Tor}_n^{B^e}(B, B) = H_n(B) \end{aligned}$$

for all $n \geq 1$. ∎

As a consequence of Theorem 4.4, we consider the homology groups of products of two algebras.

Given two stable equivalences of Morita type between algebras A and B, and between C and D, it is an open question whether there is a stable equivalence of Morita type between the tensor products $A \otimes_k C$ and $B \otimes_k D$. However, we may have the following result on the homology groups of $A \otimes_k C$ and $B \otimes_k D$.

Corollary 4.5. *Suppose that A and B are stably equivalent of Morita type, and that C and D are stably equivalent of Morita type. If $H_0(A) \simeq H_0(B)$ and $H_0(C) \simeq H_0(D)$, then for all $n \geq 0$, we have $H_n(A \otimes_k C) \simeq H_n(B \otimes_k D)$.*

Proof. By Theorem 4.4, $H_p(A) \simeq H_p(B)$ and $H_q(C) \simeq H_q(D)$ for all $p, q \geq 1$. Since $H_n(A \otimes_k C) \simeq \bigoplus_{p+q=n} H_p(A) \otimes_k H_q(C)$ by [21, proposition 9.4.1, p.319], it follows from Theorem 4.4 that

$$\begin{aligned} H_n(A \otimes_k C) &\simeq \bigoplus_{p+q=n} H_p(A) \otimes_k H_q(C) \simeq \bigoplus_{p+q=n} H_p(B) \otimes_k H_q(D) \\ &\simeq H_n(B \otimes_k D). \end{aligned}$$

This implies the corollary. ∎

Next, recall that an algebra is called quasi-hereditary if there is a finite chain $0 = J_0 \subseteq J_1 \subseteq \ldots \subseteq J_n = A$ of ideals in A such that J_i/J_{i-1} is a projective idempotent ideal in A/J_{i-1} and $\mathrm{End}_{A/J_{i-1}}(J_i/J_{i-1})$ is semi-simple. For a quasi-hereditary algebra A, it was proved in [24] that $H_n(A) = 0$ for all $n \geq 1$ if $A/\mathrm{rad}(A)$ is separable. Thus we have the following corollary.

Corollary 4.6. *Suppose A and B are stably equivalent of Morita type. If B is quasi-hereditary such that $B/rad(B)$ is separable, then $H_n(A) = 0$ for all $n \geq 1$.* ∎

Remark. (1) The Hochschild homology group $H_0(A)$ is isomorphic to the quotient of A modulo the ideal $[A, A]$ generated by all elements of the form $xy - yx$ with $x, y \in A$. In particular, if A is an algebra given by a quiver with relations, such that there is no loop in the quiver, then $[A, A]$ is just the radical of A and $H_0(A)$ is isomorphic to the Grothendieck group of A. Thus the Hochschild homology groups are of interest when one considers the open problem whether A and B have the same number of non-projective simple modules if they are stably equivalent of Morita type (see [4], p.409).

(2) If A and B are stably equivalent of Morita type, then

$$H_0(A) \oplus H_0(A, P) \oplus H_0(A, P) \oplus P \otimes_{A^e} P$$
$$\simeq H_0(B) \oplus H_0(B, Q) \oplus H_0(B, Q) \quad \oplus Q \otimes_{B^e} Q,$$

where P and Q are as in 2.1. This follows from the proof of Theorem 4.4. In general, we could not have $H_0(A) \simeq H_0(B)$. A counterexample is displayed in the next section.

Now let us apply the idea in the proof of Theorem 4.4 to consider the Hochschild cohomology groups. We shall demonstrate the following result proved first in [14]. Here we shall supply a different proof which seems to be simpler and more elementary.

Theorem 4.7. *Let A and B be two self-injective k-algebras. If A and B are stably equivalent of Morita type, then $H^n(A) \simeq H^n(B)$ for all $n \geq 1$.* ∎

Before we start the proof of Theorem 4.7, we first introduce some notations and then reveal some properties of a stable equivalence of Morita type between arbitrary algebras.

For an A-module X, we have a unique (up to isomorphism) decomposition $X = X_1 \oplus X_0$, where X_1 has no nonzero projective summands and X_0 is projective. We call X_1 the stable part of X. An algebra A is said to be separable if A is a projective A-A-bimodule.

Lemma 4.8. *Let A and B be two algebras with no separable summands. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B. As bimodules, $M = M_1 \oplus M_0$ and $N = N_1 \oplus N_0$, where M_1 and N_1 are the stable parts of M and N, respectively. Then we have the following.*

(1) *M_1 and N_1 define a stable equivalence of Morita type between A and B.*
(2) *If ${}_BY_A$ is a bimodule such that M and Y define a stable equivalence of Morita type between A and B, then $Y_1 \simeq N_1$. Similarly, if ${}_AX_B$ is a bimodule such that X and N define a stable equivalence of Morita type between A and B, then $X_1 \simeq M_1$.*
(3) *If A and B are self-injective algebras, then $(N_1 \otimes_A -, M_1 \otimes_B -)$ and $(M_1 \otimes_B -, N_1 \otimes_A -)$ are adjoint pairs of functors.*

Proof. (1) By definition, we have bimodule isomorphisms $M \otimes_B N \simeq A \oplus P$ and $N \otimes_A M \simeq B \oplus Q$, where P and Q are projective as bimodules. Therefore $A \oplus P \simeq (M_1 \oplus M_0) \otimes_B (N_1 \oplus N_0) \simeq (M_1 \otimes_B N_1) \oplus (M_1 \otimes_B N_0) \oplus (M_0 \otimes_B N_1) \oplus (M_0 \otimes_B N_0)$. Since $(M_1 \otimes_B N_0) \oplus (M_0 \otimes_B N_1) \oplus (M_0 \otimes_B N_0)$ is a projective A-A-bimodule, and since A has no projective A-A-summands by the assumption, A must be the stable part of $M_1 \otimes_B N_1$, that is, $M_1 \otimes_B N_1 \simeq A \oplus P'$ for some projective A-A-bimodule P'. Similarly, we have that $N_1 \otimes_A M_1 \simeq B \oplus Q'$ for some projective B-B-bimodule Q'. This proves that M_1 and N_1 define a stable equivalence of Morita type between A and B.

(2) By (1) we have bimodule isomorphisms: $M_1 \otimes_B N_1 \simeq A \oplus P'$, $N_1 \otimes_A M_1 \simeq B \oplus Q'$, $M_1 \otimes_B Y_1 \simeq A \oplus P''$, $Y_1 \otimes_A M_1 \simeq B \oplus Q''$, where P', Q', P'', Q'' are projective as bimodules. On the one hand, $Y_1 \otimes_A M_1 \otimes_B N_1 \simeq Y_1 \otimes_A (M_1 \otimes_B N_1) \simeq Y_1 \otimes_A (A \oplus P') \simeq Y_1 \oplus Y_1 \otimes_A P'$. On the other hand, $Y_1 \otimes_A M_1 \otimes_B N_1 \simeq (Y_1 \otimes_A M_1) \otimes_B N_1 \simeq (B \oplus Q'') \otimes_B N_1 \simeq N_1 \oplus Q'' \otimes_B N_1$. Since $Y_1 \otimes_A P'$ and $Q'' \otimes_B N_1$ are projective as bimodules, $Y_1 \simeq N_1$ must be the stable part of $Y_1 \otimes_A M_1 \otimes_B N_1$.
Similarly, we can prove the second statement of (2).

(3) Since ${}_B N_1$ is projective, we have a B-module isomorphism α: $N_1 \simeq \mathrm{Hom}_{B^{op}}(\mathrm{Hom}_B(N_1, B), B)$ defined by $\alpha(x)(f) = f(x)$ for $x \in N_1$ and $f \in \mathrm{Hom}_B(N_1, B)$. It is easy to verify that α is also a right A-module homomorphism. Therefore $N_1 \simeq \mathrm{Hom}_{B^{op}}(\mathrm{Hom}_B(N_1, B), B)$ as B-A-bimodules.

We claim that $\mathrm{Hom}_B(N_1, B)$ has no projective summands as A-B-bimodules. Otherwise, suppose $\mathrm{Hom}_B(N_1, B)$ has a projective direct summand U. Then there is an A-B-bimodule V such that $U \oplus V \simeq (A \otimes_k B^{op})^m$ for some positive number m. Note that with A and B also $A \otimes_k B^{op}$ is self-injective by Lemma 4.2(3). Since $\mathrm{Hom}_B(U \oplus V, B_B) \simeq \mathrm{Hom}_B((A \otimes_k B^{op})^m, B_B)$ and since $\mathrm{Hom}_B(A \otimes_k B^{op}, B_B) \simeq \mathrm{Hom}_k(A, \mathrm{Hom}_B(B^{op}, B_B)) \simeq D(A \otimes_k D\mathrm{Hom}_B(B^{op}, B_B))$ and the module $\mathrm{Hom}_B(B^{op}, B_B)$ is an injective B-module, we know that $\mathrm{Hom}_B(U, B_B)$ is a projective B-A-bimodule. (Here we use D to denote the k-duality.) This contradicts to the fact that $N_1 \simeq \mathrm{Hom}_B(\mathrm{Hom}_B(N_1, B), B)$ has no projective direct summands as bimodules.

Since A and B are self-injective, the bimodules $\mathrm{Hom}_B(N_1, B)$ and N_1 define a stable equivalence of Morita type between A and B by [18, theorem 3.2]. By (2), $\mathrm{Hom}_B(N_1, B) \simeq M_1$. This shows that $M_1 \otimes_B - \simeq \mathrm{Hom}_B(N_1, B) \otimes_B - \simeq \mathrm{Hom}_B(N_1, -)$ since ${}_B N_1$ is projective. Thus the right adjoint of the functor $N_1 \otimes_A -$ is naturally isomorphic to $M_1 \otimes_B -$. Hence we have an adjoint pair $(N_1 \otimes_A -, M_1 \otimes_B -)$. Similarly, since the bimodules $\mathrm{Hom}_A(M_1, A)$ and M_1 define a stable equivalence of Morita type between B and A. By (2), $\mathrm{Hom}_A(M_1, A) \simeq N_1$. This shows that $(M_1 \otimes_B -, N_1 \otimes_A -) \simeq (M_1 \otimes_B -, \mathrm{Hom}_A(M_1, A) \otimes_A -) \simeq (M_1 \otimes_B -, \mathrm{Hom}_A(M_1, -))$ is an adjoint pair, too. ■

Lemma 4.9. *Let A and B be two algebras such that B is a self-injective algebra. Then the functor* δ: $(B \otimes_k A^{op})$*-mod* $\to$ $(A \otimes_k B^{op})$*-mod defined by* $\delta(X) = \mathrm{Hom}_B(X, B)$ *is a duality.*

Proof. Clearly δ: $(B \otimes_k A^{op})$-mod $\to$ $(A \otimes_k B^{op})$-mod is a contravariant functor. Since B is a self-injective algebra, we have a B-module isomorphism α_X: $X \simeq \mathrm{Hom}_{B^{op}}(\mathrm{Hom}_B(X, B), B)$ defined by $\alpha_X(x)(f) = f(x)$ for $x \in X$ and $f \in \mathrm{Hom}_B(X, B)$. It is easy to verify that α_X is also a right A-module homomorphism. Therefore $X \simeq \mathrm{Hom}_{B^{op}}(\mathrm{Hom}_B(X, B), B)$ as B-A-bimodules. Hence we have a natural isomorphism α: $1_{B\otimes_k A^{op}-mod} \to \mathrm{Hom}_{B^{op}}(-, B)\mathrm{Hom}_B(-, B)$. Similarly, since B^{op} is a self-injective algebra, we have a natural isomorphism β: $1_{A\otimes_k B^{op}-mod} \to \mathrm{Hom}_B(-, B)\,\mathrm{Hom}_{B^{op}}(-, B)$. It follows that δ: $B \otimes_k A^{op}$-mod $\to A \otimes_k B^{op}$-mod is a duality. ■

The following lemma is trivial (by dimension shifting).

Lemma 4.10. *Let A be an algebra with δ: A-mod $\to A^{op}$-mod a duality. Then we have $Ext^i_A(X, Y) \simeq Ext^i_{A^{op}}(\delta(Y), \delta(X))$ for all i and all A-modules X and Y.* ■

Proof of Theorem 4.7. Let $A = A_1 \times A_0$ such that A_1 has no separable summands and A_0 is a separable algebra. Similarly, we have a decomposition $B = B_1 \times B_0$ such that B_1 has no separable summands and B_0 is a separable algebra. Using the same technique as in the proof of Lemma 4.8, one can easily show that the algebras A_1 and B_1 are stably equivalent of Morita type. Since A_0 is a projective-injective A-A-bimodule, we have $H^n(A) \simeq H^n(A_1)$ for all $n \geq 1$. Similarly, we have $H^n(B) \simeq H^n(B_1)$ for all $n \geq 1$. Thus, without loss of generality, we may assume that A and B are self-injective algebras with no separable summands, and that $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ are adjoint pairs by 4.8.

In the left module version of Lemma 4.3(2), we let $\Lambda = A$, $\Sigma = B$ and $\Gamma = B^{op}$, and define $X = {}_AM_B = {}_{\Lambda-\Gamma} M$, $Y = {}_BN_A = {}_\Sigma N_\Lambda$ and $Z = {}_BB_B = {}_{\Gamma-\Sigma}B$. By the definition of a stable equivalence of Morita type, the modules ${}_BN$ and N_A are projective, that is, ${}_\Sigma Y$ and Y_Λ are projective. Thus $\mathrm{Tor}_i^\Lambda(Y, X) = 0 = \mathrm{Ext}^i_\Sigma(Y, Z)$ for $i \geq 1$. Hence there is an isomorphism $\mathrm{Ext}^i_{B\otimes_k B^{op}}(N \otimes_A M, B) \simeq \mathrm{Ext}^i_{A\otimes_k B^{op}}(M, \mathrm{Hom}_B(N, B))$. By Lemma 4.8(3), we have $\mathrm{Hom}_B(N, B) \simeq M$, and therefore $\mathrm{Ext}^i_{B\otimes_k B^{op}}(N \otimes_A M, B) \simeq \mathrm{Ext}^i_{A\otimes_k B^{op}}(M, M)$. Similarly, we have $\mathrm{Ext}^i_{A\otimes_k A^{op}}(M \otimes_B N, A) \simeq \mathrm{Ext}^i_{B\otimes_k A^{op}}(N, N)$. Note that δ is a duality between $(B \otimes_k A^{op})$-mod and $(A \otimes_k B^{op})$-mod with $\delta(N) = M$ by Lemma 4.9 . Thus, by Lemma 4.10, $\mathrm{Ext}^i_{B\otimes_k A^{op}}(N, N) \simeq \mathrm{Ext}^i_{A\otimes_k B^{op}}(M, M)$.

By definition, we have $M \otimes_B N \simeq A \oplus P$ as A-A-bimodules for some projective A-A-bimodule P, and $N \otimes_A M \simeq B \oplus Q$ as B-B-bimodules for some projective B-B-bimodule Q. Thus, for all $n \geq 1$, we have that

$$\begin{aligned} H^n(A) = \mathrm{Ext}^n_{A^e}(A, A) &\simeq \mathrm{Ext}^n_{A^e}(A \oplus P, A) \simeq \mathrm{Ext}^i_{A^e}(M \otimes_B N, A) \\ &\simeq \mathrm{Ext}^i_{B\otimes_k A^{op}}(N, N) \simeq \mathrm{Ext}^i_{A\otimes_k B^{op}}(M, M) \\ &\simeq \mathrm{Ext}^i_{B^e}(N \otimes_A M, B) \simeq \mathrm{Ext}^n_{B^e}(B \oplus Q, B) \\ &\simeq \mathrm{Ext}^n_{B^e}(B, B) = H^n(B). \end{aligned}$$

This finishes the proof of Theorem 4.7. ■

Remark. (1) It is known that the 0-th Hochschild cohomology of an algebra Λ is characterized by the center of Λ. Note that even though A and B are symmetric and stably equivalent of Morita type, the centers of A and B may not be isomorphic. A trivial counterexample is that algebras have semisimple summands. For non-trivial counterexample over a discrete valuation ring one may look at [7]. (We thank the referee for pointing out this reference). However, we don't know any non-trivial counterexample over a field. Note that the Hochschild cohomology groups are invariances of a derived equivalence.

(2) In Theorem 1.1 of [14], we find the statement " ... then their Hochschild cohomology algebras are isomorphic." This statement seems to be too strong, and we don't know how to deduce the isomorphism for 0-th Hochschild cohomology.

(3) A direct consequence of Lemma 4.8(3) is that the second statement of [13, Theorem 5.7] holds true instead for self-injective algebras (see the Appendix at the end of this paper).

5. Examples

In this section we exhibit some examples to illustrate the necessity of some assumptions in the previous results.

Example 1. The term “stable equivalence of Morita type" cannot be weakened to “stable equivalence" in Theorem 1.1.

Let us consider the algebras $A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ and $B = k[X]/(X^2)$. They are stably equivalent; and their Auslander algebras Λ and Γ are given by the following quivers and relations, respectively:

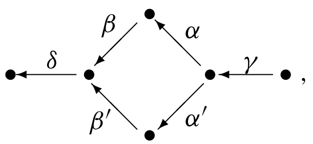
$$\Lambda: \ \bullet \xleftarrow{\beta} \bullet \xleftarrow{\alpha} \bullet\,, \quad \alpha\beta = 0; \qquad\qquad \Gamma: \ \bullet \underset{\gamma}{\overset{\delta}{\leftrightarrows}} \bullet\,, \quad \gamma\delta = 0.$$

Clearly, Λ and Γ are not stably equivalent since the numbers of isomorphism classes of non-projective indecomposable modules of Λ and Γ are not the same. This shows that two stably equivalent algebras may have non-stably equivalent Auslander algebras.

Note also that for A and B themselves neither the Hochschild homology groups, nor the Hochschild cohomology groups are isomorphic. Thus Theorem 4.4 is not true for stable equivalences.

Example 2. Theorem 1.1 does not holds in general for derived equivalences. Let A be the path algebra given by the quiver $\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet$ and let B be the algebra given by the same quiver with relation $\alpha\beta = 0$. Then A and B are derived equivalent since B is a tilted algebra of A. The Auslander algebra Λ of A and the Auslander algebra Γ of B are given by the following quivers and relations, respectively :

$\Lambda:$



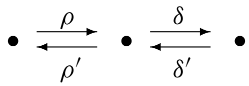
$$\alpha\beta - \alpha'\beta' = \gamma\alpha' = \beta'\delta = 0.$$

$$\Gamma: \quad \bullet \xleftarrow{\eta} \bullet \xleftarrow{\xi} \bullet \xleftarrow{\varphi} \bullet \xleftarrow{\rho} \bullet\,, \qquad \xi\eta = \rho\varphi = 0.$$

Clearly, Λ and Γ are not derived equivalent, since the numbers of isomorphic classes of simple modules on Λ and Γ are not the same.

Example 3. In general, the 0-th Hochschild homology groups are not preserved under a stable equivalence of Morita type. A trivial example is that algebras have semisimple summands. However, the following non-trivial example shows that $H_0(A) \not\simeq H_0(B)$ even though A and B are indecomposable and symmetric.

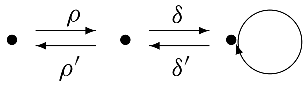
Let A be an algebra given by the following quiver



with relations:

$$\rho\rho' = \delta'\delta = 0;\ (\rho'\rho\delta\delta')^2 = (\delta\delta'\rho'\rho)^2.$$

Let B be an algebra given by the quiver



with relations:

$$\rho\rho' = \delta'\delta = \delta c = c\delta' = 0;\ c^2 = \delta'\rho'\rho\delta,\ \delta\delta'\rho'\rho = \rho'\rho\delta\delta'.$$

These two algebras are the A_2 and C_2 in the notation of [11] and proved to be derived equivalent by Linckelmann. They appear as blocks of group algebras. Hence they are stably equivalent of Morita type. Clearly, we have $\mathrm{H}_0(A) \simeq k^3$, and $\mathrm{H}_0(B) \simeq k^4$ because the loop c does not lie in $[B, B]$. Hence the 0-th Hochschild homology groups of A and B are not isomorphic. However, since the two algebras are derived equivalent, the centers of A and B are isomorphic.

6. Appendix (March 16, 2004)

In this appendix we shall use the property developed in this paper to show that Theorem 5.7 in [13] can be extended to self-injective algebras. In the following we keep all notations as in the section five of [13].

Theorem. *Let A and B be self-injective k-algebras with no separable summands. If $A/rad(A)$ and $B/rad(B)$ are spilt semisimple (i.e., every simple module has k as its endomorphism algebra), then*

(1) $F\mathcal{C}' \subseteq \mathcal{D}'$, $F\mathcal{C}'' \subseteq \mathcal{D}''$, *and* $G\mathcal{D}' \subseteq \mathcal{C}'$, $G\mathcal{D}'' \subseteq \mathcal{C}''$.
(2) *eAe and fBf are stably equivalent of Morita type.*

Proof. (1) By Lemma 4.8(3), we can assume that (F, G) and (G, F) are adjoint pairs. Since the functor G is a right adjoint to F, namely, $\mathrm{Hom}_B(F\mathcal{C}', \mathcal{D}) \simeq \mathrm{Hom}_A(\mathcal{C}', G\mathcal{D}) = \mathrm{Hom}_A(\mathcal{C}', \mathcal{C}) = 0$. Therefore $F(\mathcal{C}') \subseteq \mathcal{D}'$. Similarly, we have $G(\mathcal{D}'') \subseteq \mathcal{C}''$. Since (G, F) is also an adjoint pair, we have similarly that $F(\mathcal{C}'') \subseteq \mathcal{D}''$ and $G(\mathcal{D}') \subseteq \mathcal{C}'$.

(2) We define $\overline{M} = eMf$ and $\overline{N} = fNe$. Clearly, $\overline{M}$ is an eAe-fBf-bimodule and $\overline{N}$ is a fBf-eAe-bimodule. Since $F(Ae) \in \mathrm{add}(Bf)$ by (1), $\overline{N} = fNe \simeq \mathrm{Hom}_B(Bf, F(Ae)$ is projective as a left fBf-module. Since (G, F) is an adjoint pair and $G(Bf) \in \mathrm{add}(Ae)$, $\overline{N} = fNe \simeq \mathrm{Hom}_B(Bf, F(Ae) \simeq \mathrm{Hom}_A(G(Bf), Ae)$ is projective as a right eAe-module. Similarly, it follows from the adjoint pair (F, G) that $\overline{M}$ is projective as

a left eAe-module and as a right fBf-module. By the associativity of tensor products, we have the following isomorphisms of eAe-eAe-bimodules: $\overline{M} \otimes_{fBf} \overline{N} = eMf \otimes_{fBf} fNe \simeq eA \otimes_A M \otimes_B Bf \otimes_{fBf} fB \otimes_B N \otimes_A Ae \simeq eA \otimes_A M \otimes_B (Bf \otimes_{fBf} fB \otimes_B N \otimes_A Ae) \simeq eA \otimes_A M \otimes_B (Bf \otimes_{fBf} \mathrm{Hom}_B(Bf, F(Ae)))$.

We claim that the natural morphism ϕ: $Bf \otimes_{fBf} \mathrm{Hom}_B(Bf, F(Ae)) \to F(Ae)$ define by $\phi(x \otimes g) = g(x)$ for $x \in Bf, g \in \mathrm{Hom}_B(Bf, F(Ae))$ is an isomorphism of B-eAe-bimodules. In fact, we have a natural B-isomorphism ϕ: $Bf \otimes_{fBf} \mathrm{Hom}_B(Bf, Bf) \to Bf$ (by $\phi(x \otimes g) = g(x)$). Therefore for any B-module $Y \in \mathrm{add}(Bf)$, there is a B-isomorphism ϕ: $Bf \otimes_{fBf} \mathrm{Hom}_B(Bf, Bf) \to Bf$ ($\phi(x \otimes g) = g(x)$) by additivity. It follows that ϕ: $Bf \otimes_{fBf} \mathrm{Hom}_B(Bf, F(Ae)) \to F(Ae)$ is a B-isomorphism. It is straightforward to verify that ϕ is also a right eAe-module morphism. Thus ϕ: $Bf \otimes_{fBf} \mathrm{Hom}_B(Bf, F(Ae)) \to F(Ae)$ is an isomorphism of B-eAe-bimodules.

The above B-eAe-bimodule isomorphisms lead to the following eAe-eAe-bimodule isomorphisms: $\overline{M} \otimes_{fBf} \overline{N} \simeq eA \otimes_A M \otimes_B N \otimes_A Ae \simeq eA \otimes_A (A \oplus P) \otimes_A Ae \simeq eAe \oplus ePe$. Since P is a projective A-A-bimodule and therefore is isomorphic to a direct sum of the modules of the form $P_1 \otimes_k P_2$ where P_1 is a left projective A-module and P_2 is a right projective A-module. Since $\overline{M} \otimes_{fBf} \overline{N}$ and eAe are projective as left eAe-modules, $eP_1 \otimes_k P_2e$ is a projective eAe-module. It follows that eP_1 is a projective eAe-module. Similarly P_2e is a right projective eAe-module. Thus ePe is isomorphic to a direct sum of the modules of the form $eP_1 \otimes_k P_2e$ where eP_1 is a left projective eAe-module and P_2e is a right projective eAe-module. This implies that ePe is a projective eAe-eAe-bimodule. Similarly, we have a fBf-fBf-bimodule isomorphism: $\overline{N} \otimes_{eAe} \overline{M} \simeq fBf \oplus fQf$, where fQf is a projective fBf-fBf-bimodule. Thus, by definition, the bimodules $\overline{M}$ and $\overline{N}$ define a stable equivalence of Morita type between eAe and fBf. ■

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Finally, we would like to point out that the finitistic dimension is invariant under a stable equivalence of Morita type and that the finitistic dimension conjecture is still open.

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