A necessary condition for the finiteness of Δ-good module categories

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QUASI-HEREDITARY algebras were introduced by Cline *et al.*[1] in order to study the highest weight categories in representation theory of semisimple Lie algebras and algebraic groups. Many important algebras such as hereditary algebras, Schur algebras and algebras to blocks of the category $\mathcal{O}$ in ref. [2], are typical examples of quasi-hereditary algebras.

For a given quasi-hereditary algebra A, there is a partial order $(\Lambda, \leqslant)$ on the set of simple modules, and one studies the standard modules $\Delta = \{\Delta(\lambda) \mid \lambda \in \Lambda\}$. Of particular interest is the study of Δ-good module category $\mathcal{F}(\Delta)$ of all modules which have a Δ-filtration. Ringel[3] proved that $\mathcal{F}(\Delta)$ has almost split sequences. One of the interesting questions on the study of $\mathcal{F}(\Delta)$ is when $\mathcal{F}(\Delta)$ is finite (i.e. there are only finitely many pairwise non-isomorphic indecomposable modules in $\mathcal{F}(\Delta)$). In such a case, Xi[4] proved that the endomorphism ring of the direct sum of non-isomorphic indecomposable objects in $\mathcal{F}(\Delta)$ is again quasi-hereditary.

In this note we consider certain class of quasi-hereditary algebras and present a necessary condition for the finiteness of Δ-good module categories over these algebras. Our method is the use of factor-space categories. Note that this class of quasi-hereditary algebras covers the dual extension algebras of hereditary algebras[5], the Auslander algebras of representation-finite local algebras[6], the quadratic duals of representation-finite q-Schur algebras[7] and Temperley-Lieb algebras[8]1), and certain class of directed algebras.

The terminology used throughout is taken from refs. [9, 10]. We now recall some definitions and fix notation.

Let A be a finite-dimensional algebra over an algebraically closed field k. All modules in this note are finitely generated left A-modules (unless obviously not), and the composition of maps $f: M_1 \to M_2$ and $g: M_2 \to M_3$ will be denoted by fg. The category of all finitely generated A-modules is denoted by A-mod.

Let $\mathcal{K}$ be a Krull-Schmidt k-category and $|\cdot|: \mathcal{K} \to k$-mod an additive functor. The pair $(\mathcal{K}, |\cdot|)$ is called a vector-space category. The factor-space category $\check{\mathcal{U}}(\mathcal{K}, |\cdot|)$ of $(\mathcal{K}, |\cdot|)$ is the category of all triples $W = (W_\omega, W_0, \gamma_W: |W_0| \to W_\omega)$, where $W_0 \in \mathcal{K}$, $W_\omega \in k$-mod and γ_W is a k-linear map. A morphism from $W \to W'$ by definition is a pair (f_ω, f_0), where f_0: $W_0 \to W'_0$ and f_ω: $W_\omega \to W'_\omega$ such that $\gamma_W f_\omega = |f_0| \gamma_{W'}$.

If A_0 is an algebra over k and R is an A_0-module, one may form one-point coextension $[R]A_0$ which is defined to be the following matrix algebra:

$$C = \begin{bmatrix} k & DR \\ 0 & A_0 \end{bmatrix},$$

where $D = \mathrm{Hom}_k(-, k)$. The C-modules can be written as triples $W = (W_\omega, W_0, \delta_W)$, where W_ω is a k-vector-space, W_0 an A_0-module and δ_W: $DR \underset{A_0}{\otimes} W_0 \to W_\omega$ is a k-linear map. A morphism from W to U by definition is given by a pair (f_ω, f_0) with f_0 an A_0-module homomorphism and f_ω a k-linear map such that $\delta_W f_\omega = (1 \otimes f_0)\delta_U$. By the natural isomorphisms $DR \underset{A_0}{\otimes} W_0 \cong D\mathrm{Hom}(W_0, R)$ and $\mathrm{Hom}_k(DR \underset{A}{\otimes} W_0, W_\omega) \cong \mathrm{Hom}_k(D\mathrm{Hom}_A(W_0, R), W_\omega)$ we know that $[R]A_0$-mod and $\check{\nu}(A_0\text{-mod}, D\mathrm{Hom}_{A_0}(-, R))$ are equivalent. In the following we will simply identify $[R]A_0$-modules with the objects in $\check{\nu}(A_0\text{-mod}, D\mathrm{Hom}_{A_0}(-, R))$.

1 A necessary condition for the finiteness of $\mathscr{F}(\Delta)$

Let A be an algebra and C a factor algebra of A such that $C = [R]A_0$. Hence A_0 is also a factor algebra of A. Thus A_0-modules and C-modules can be considered as A-modules in a natural way. We denote by ω the coextension vertex of C.

In the following we take a full subcategory $\mathscr{C}$ of A_0-mod such that for every A_0-module M in $\mathscr{C}$ there holds $\mathrm{Ext}_A^2(M, \mathrm{rad}P(\omega)) = 0$ (for example, this condition is satisfied if proj. dim. $M \leqslant 1$ for all $M \in \mathscr{C}$), where $P(\omega)$ is the indecomposable projective A-module corresponding to the vertex ω, and we suppose $\mathrm{End}_A(P(\omega)) \cong k$. Recall that $\mathscr{F}(\mathscr{C} \vee P(\omega))$ denotes the full subcategory of A-mod consisting of all A-module M which have a filtration with factors in $\mathscr{C} \vee P(\omega)$.

Lemma 1. *Let A be an algebra with the above assumptions. If $\check{\nu}(\mathscr{C}, D\mathrm{Hom}_{A_0}(-, R))$ is infinite, then $\mathscr{F}(\mathscr{C} \vee P(\omega))$ is infinite.*

Proof. Let S be a set of the chosen representatives of the isoclasses of all objects in $\check{\nu}(\mathscr{C}, D\mathrm{Hom}_{A_0}(-, R))$. We shall construct a correspondence from S to the set of the isoclasses of objects in $\mathscr{F}(\mathscr{C} \vee P(\omega))$. For an element $W = (W_\omega, W_0, \delta_W)$ in S, one has the following exact sequence:

$$0 \to (W_\omega, 0, 0) \to (W_\omega, W_0, \delta_W) \to (0, W_0, 0) \to 0.$$

We then obtain an exact sequence in A-mod:

$$\varepsilon_W \qquad 0 \to E(\omega)^m \to W \to W_0 \to 0,$$

where $E(\omega)$ denotes the simple A-module corresponding to the vertex ω and $m = \dim_k W_\omega$.

On the other hand, in A-mod one has the following exact sequence:

$$0 \to R_1^m \to P(\omega)^m \xrightarrow{\pi} E(\omega)^m \to 0,$$

where π denotes the projective cover of $E(\omega)^m$ and R_1 the radical of $P(\omega)$.

Apply $\mathrm{Hom}_A(W_0, -)$ to the above sequence, we have a long exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_A(W_0, R_1^m) \to \mathrm{Hom}_A(W_0, P(\omega)^m) \to \mathrm{Hom}_A(W_0, E(\omega)^m) \\ &\to \mathrm{Ext}_A^1(W_0, R_1^m) \to \mathrm{Ext}_A^1(W_0, P(\omega)^m) \xrightarrow{\Phi} \mathrm{Ext}_A^1(W_0, E(\omega)^m) \\ &\to \mathrm{Ext}_A^2(W_0, R_1^m). \end{aligned}$$

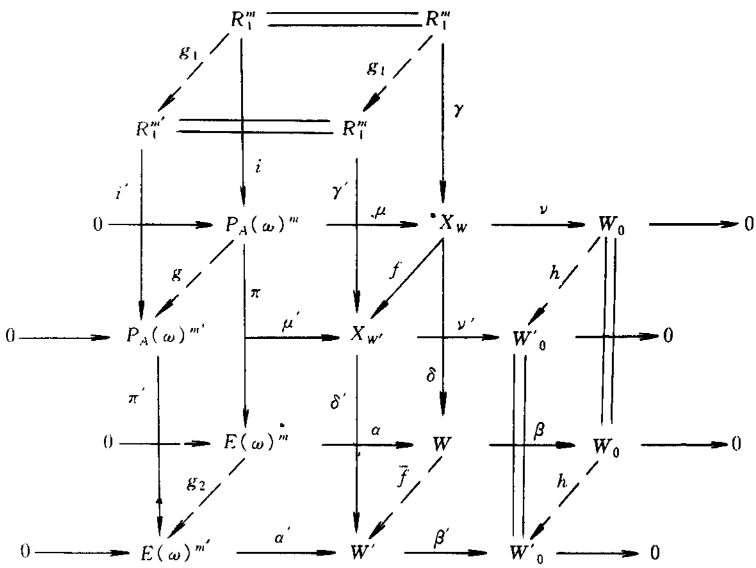
By assumption $\mathrm{Ext}_A^2(W_0, R_1^m) = 0$, we have that Φ is surjective. Now we take a fixed element $\eta_W \in \mathrm{Ext}_A^1(W_0, P(\omega)^m)$ such that $\Phi(\eta_W) = \varepsilon_W$ (note that the choice of η_W is not unique). Therefore, we get a diagram with commutative squares:

$$\begin{array}{lccccccccc}
 & & & R_1^m & = & R_1^m & & & & \\
 & & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \gamma} & & & & \\
\eta_W & 0 & \longrightarrow & P(\omega)^m & \longrightarrow & X_W & \longrightarrow & W_0 & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \delta} & & \| & & \\
\varepsilon_W & 0 & \longrightarrow & E(\omega)^m & \longrightarrow & W & \longrightarrow & W_0 & \longrightarrow & 0
\end{array}$$

Obviously, $X_W \in \mathscr{F}(\mathscr{C} \vee P(\omega))$. Thus one gets a correspondence $W \mapsto c(X_W)$ from S to the set of the isoclasses of objects in $\mathscr{F}(\mathscr{C} \vee P(\omega))$, where $c(X_W)$ denotes the isoclass of X_W.

Further, we give two properties of this correspondence.

(1) For $W, W' \in S$, the isomorphism $X_W \cong X_{W'}$ implies that $W = W'$. Indeed, let f be a morphism from X_W to X_W. The constructions of X_W and $X_{W'}$ yield the following diagram with exact rows and exact columns.



From $\mathrm{Hom}_A(P(\omega), W'_0) = 0$ it follows that there exists a morphism $g: P(\omega)^m \to P(\omega)^{m'}$ such that $\mu f = g\mu'$, then f induces a morphism $h: W_0 \to W'_0$ with $f\nu' = \nu h$. Since $R_1 = \mathrm{rad}(P(\omega))$, the morphism g induces morphisms $g_1: R_1^m \to R_1^{m'}$ and $g_2: E(\omega)^m \to E(\omega)^{m'}$ with $ig = g_1 i'$ and $\pi g_2 = g\pi'$. Thus $g_1\gamma' = g_1 i'\mu' = ig\mu' = i\mu f = \gamma f$. This implies that there is a morphism $\bar{f}: W \to W'$ with $f\delta' = \delta \bar{f}$. Finally, from $\pi\alpha \bar{f} = \mu\delta \bar{f} = \mu f\delta' = g\mu'\delta' = g\pi'\alpha' = \pi g_2\alpha'$ it follows that $\alpha \bar{f} = g_2\alpha'$ since π is surjective, and from $\delta\beta h = \nu h = f\nu' = f\delta'\beta' = \delta \bar{f}\beta'$ it follows that $\beta h = \bar{f}\beta'$ since δ is surjective. In fact, we have proved that all squares in the above diagram are commutative.

Moreover, if f is an isomorphism, then g is injective and $m = \dim_k \mathrm{Hom}_A(P(\omega), X_W) = \dim_k \mathrm{Hom}_A(P(\omega), X_{W'}) = m'$, thus g is an isomorphism. This implies that both g_2 and h are isomorphisms. Therefore, $\bar{f}$ is an isomorphism from W to W'. Since W, W' are in S, we have $W = W'$, as required.

(2) If $W \in S$ is indecomposable, so is X_W. Indeed, let $f \in \mathrm{End}_A(X_W)$. For intuitive thinking, we still use the diagram in (1) but identify all the symbols x' with x. By (1), f induces a morphism $\overline{f} \in \mathrm{End}_C(W)$. Since W is indecomposable, $\overline{f}$ is invertible or nilpotent. If $\overline{f}$ is invertible, then the morphisms g_2 and h are isomorphisms. Since $\mathrm{End}_A(P(\omega)) \cong k$, one gets that g is an isomorphism. Hence f is invertible. If $\overline{f}$ is nilpotent, then there is an $n \in \mathbb{N}$ such that $\to \overline{f}^n = 0$. This implies $g_2^n = 0$ and $h^n = 0$. Then $f^n \nu = \nu h^n = 0$; that is, there is an f_1: $X_W \to P(\omega)^m$ such that $f^n = f_1 \mu$. Again, by the assumption $\mathrm{End}_A(P(\omega)) \cong k$, it follows from $g_2^n = 0$ that $g^n = 0$. Hence $f^{2n} = f_1 \mu f^n = f_1 g^n \mu = 0$ and f is nilpotent. Therefore, X_W is indecomposable if W is indecomposable.

As a result of (1) and (2), the correspondnece $W \mapsto c(X_W)$ induces an injection from the set of the isoclasses of indecomposable objects in $\check{\nu}(\mathscr{C}, D\mathrm{Hom}_{A_0}(-, R))$ to that of the isoclasses of indecomposable objects in $\mathscr{F}(\mathscr{C} \vee (P(\omega))$. Therefore, the infiniteness of $\check{\nu}(\mathscr{C}, D\mathrm{Hom}_{A_0}(-, R))$ implie that of $\mathscr{F}(\mathscr{C} \vee (P(\omega))$. This finishes the proof.

As an application of the lemma to quasi-hereditary algebras, we obtain a necessary condition for the finiteness of Δ-good module categories.

Theorem 1. *Let A be a quasi-hereditary algebra with the weight poset* $\Lambda = \{1 < 2 < 3 < \cdots < n\}$ *and* $A_0 = A/Ae_nA$ *such that* $\mathrm{Ext}_A^2(M, \mathrm{rad}P_A(n)) = 0$ *for all* $M \in \mathscr{F}(\Delta_{A_0})$. *If* $\check{\nu}(\mathscr{F}(\Delta_{A_0}), D\mathrm{Hom}_{A_0}(-, R))$ *is infinite, then* $\mathscr{F}(\Delta_A)$ *is infinite, where R is the factor module of the indecomposable injective A-module $Q(n)$ corresponding to the weight n by its socle.*

Proof. By Lemma 1, it is enough to prove that the coextension $C := [R]A_0$ is a factor algebra of A. We consider the two-sided ideal $I = (\mathrm{rad}P(n))A = (\mathrm{rad}A)e_nA$ of A and set $A_1 = A/I$. Then $P_{A_1}(n) = Ae_n/Ie_n = Ae_n(\mathrm{rad}A)e_nAe_n = Ae_n/(\mathrm{rad}A)e_n$ is simple. Therefore

$$A_1 \cong [R_1]A_0 = \begin{bmatrix} k & DR_1 \\ 0 & A_0 \end{bmatrix},$$

where $D = \mathrm{Hom}_k(-, k)$, $R_1 = Q_1(n)/\mathrm{soc}Q_1(n)$ is an A_0-module, and $Q_1(n) = D(e_nA_1)$ is the injective A_1-module. Since $\mathrm{End}A(Ae_n) \cong k$, we have $e_n(\mathrm{rad}A)e_n = 0$. Then $e_nA_1 = e_nA/I = e_nA/e_n(\mathrm{rad}A)e_nA = e_nA$. Thus $Q_1(n) = D(e_nA_1)$ and $Q(n) = D(e_nA)$ coincide as A-modules.

Therefore, $[R]A_0 = [R_1]A_0 \cong A_1 = A/I$ is a factor algebra of A. This finishes the proof.

Remark 1. The converse of the theorem may not be true. For example, consider the algebra A given by the quiver

$$1 \underset{\alpha'}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta'}{\overset{\beta}{\rightleftarrows}} 3 \underset{\gamma'}{\overset{\gamma}{\rightleftarrows}} 4$$

with relations $\alpha\alpha' = \beta\beta' = \gamma\gamma' = 0$. According to Proposition 3.4 in ref. [11], the category $\mathscr{F}(\Delta_A)$ is infinite. But $\check{\nu}(\mathscr{F}(\Delta_{A_0}), D\mathrm{Hom}_A(-, R))$ is finite, where $A_0 = A/Ae_4A$, $R = D(e_4A)/\mathrm{soc}(D(e_4A))$.

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