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p-RADICAL IN *BCI*-ALGEBRAS

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Abstract. We introduce the notion of a radical of *BCI*-algebras, and obtain some properties of the radical.

Introduction. In 1966, Y. Imai and K. Isēki introduced the concept of a *BCK*-algebras in [1]. In the same year, K. Isēki introduced the concept of a *BCI*-algebra in [2] as follows.

Definition 1. Let $\langle X, *, 0 \rangle$ be an algebra of type $\langle 2, 0 \rangle$. If it satisfies the following properties:

(1) $((x * y) * (x * z)) * (z * y) = 0$,

(2) $(x * (x * y) * y = 0$,

(3) $x * x = 0$,

(4) $x * y = y * x = 0$ implies $x = y$,

then X is said to be a *BCI-algebra.*

In [3–6], a series of interesting notions concerning *BCI*-algebras were introduced and studied. Let us recall some basic definitions and results which we shall use in this paper.

On defines an *order* relation $\leqslant$ on a *BCI*-algebra by setting

$$x \leqslant y \rightleftarrows x * y = 0.$$

Then we have

(6) $x \leqslant y$ implies $z * y \leqslant z * x$,

(7) $x * 0 = x$,

(8) $(x * y) * z = (x * z) * y$.

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A subset S of a *BCI*-algebra X is a *subalgebra* of X, if it is closed by the operation $*$. A subset A of X is an *ideal*, if it satisfies (1) $0 \in A$, (2) $x * y$, $y \in A$ implies $x \in A$.

Let A be an ideal of X. For every $x, y \in X$, define as follows:

$$x \sim y \rightleftarrows x * y,\ y * x \in A$$

then $\sim$ is an equivalence relation on X. By C_x we denote the equivalence class containing x, and X/A the set of all equivalence classes. A binary $*$ on X/A is defined by

$$C_x * C_y = C_{x*y}$$

then $\langle X/A, *, C \rangle$ is also a *BCK*-algebra which is called the quotient algebra of X by A.

Definition 2. The set $B = \{x : 0 \leqslant x\}$ in a *BCI*-algebra X is called the *BCK-part* of X. Obviously B is a subalgebra of X, and it is also an ideal of X. In the quotient algebra X/B, $C_0 = B$.

Let X, Y be *BCI*-algebras. An operation $*$ on the cartesian product $X \times Y$ of X, Y is defined as follows:

$$(x_1, y_1) * (x_2, y_2) = (x_1 * x_2, y_1 * y_2),$$

$$0 = (0, 0).$$

Then $\langle X \times Y, *, 0 \rangle$ is a *BCI*-algebra, and it is called the *product* of X and Y.

I. Radicals of *BCI*-algebras.

Let $\langle X, *, 0 \rangle$ be a *BCI*-algebra.

Definition 3. If A is an ideal of X, and for every x in A, $0 \leqslant x$, then A is called a *positive ideal* of X, or beriefly a *p-ideal* of X. Clearly, *BCK*-part B of X contains all p-ideal of X, so it is a maximal p-ideal of X.

Definition 4. The *BCK*-part of X is called the *positive radical* of X, i. e., *p-radical* of X. If the p-radical of X is trivial, i. e., $B = \{0\}$, then X is called to be a *p-semisimple BCI-algebra*, or beriefly *p-semisimple algebra*.

Example 1. Let Z be the set of integers and $-$ the minus operation, then $\langle Z, -, 0 \rangle$ is a *BCI*-algebra. Since $0 - x = 0$ implies $x = 0$, its *BCK*-part must be trivial, i. e., the p-radical is trivial. Hence it is a p-semisimple algebra.

Remark 1. K. Isēki posed a problem in [6]. Let X be any proper *BCI*-algebra, B the *BCK*-part of X. Does $0 * a = a$ for every a of $X - B$ hold? This example gave a negative answer to the problem, for 1 is in $X - B$, but $0 - 1 \neq 1$.

Example 2. Let $X = \{0, a, b\}$. Define a binary operation $*$ by the following table:

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

Then X is a *BCI*-algebra, but not a p-semisimple algebra, because the *BCK*-part is $\{0, a\}$.

Theorem 1. *Let X be a BCI-algebra. The following properties of X are equivalent*:

1) *X is p-semisimple,*
2) $0 * x = 0$ *implies* $x = 0$,
3) $0 * (0 * x) = x$ *for every* x *in* X,
4) $x * (0 * y) = y * (0 * x)$ *for any* x, y *in* X.

Proof. 1) implies 2). Assume that X is p-semisimple, and $0 * x = 0$. Then we have $0 \leqslant x$. Therefore by the p-semisimplicity $x = 0$.

2) implies 3). Suppose that 2) holds in X. By (3) we have $(0 * x) * (0 * x) = 0$. By (8)

(10) $(0 * (0 * x)) * x = 0$,

which means $0 * (0 * x) \leqslant x$, and by (6) we have

$$0 = x * x \leqslant x * (0 * (0 * x)),$$

therefore $0 * (x * (0 * (0 * x))) = 0$. By 2) $x * (0 * (0 * x)) = 0$. From this equality together with (10) and (4), it follows that

$$0 * (0 * x) = x.$$

3) implies 4). Suppose that x has the property 3). By (8)

$$x * (0 * y) = (0 * (0 * x)) * (0 * y) = (0 * (0 * y)) * (0 * x) = y * (0 * x).$$

Hence 4) holds in X.

4) implies 1). Assume that 4) holds in X. Let x be an element of the *BCK*-part B of X, then $0 * x = 0$. Using (3) and (7), we obtain $x = x * (0 * 0) = 0 * (0 * x) = 0 * 0 = 0$. This implies $B = \{0\}$ which means X is p-semisimple.

Therefore we complete the proof of Theorem 1.

As a special case a *BCI*-algebra may coincides with its radical. Then the algebra is a *BCK*-algebra. Such an algebra is considered as a p-radical algebra.

Theorem 2. *If X is not p-semisimple, then*

1) *X is a BCK-algebra*

or

2) *X/B is p-semisimple, where B is the BCK-part of X.*

Proof. Suppose $X \neq B$. We shall show that the p-radical of X/B is trivial. If $C_0 * C_x = C_0$, i. e., $C_{0*x} = C_0$. Then $0 * x = (0 * x) * 0 \in B$, whence $0 = 0 * (0 * x)$. By (2), $(0 * (0 * x)) * x = 0$. Therefore $0 * x = 0$, which implies $x \in B$. Hence $C_x = C_0$. This implies that X/B is p-semisimple.

From Theorem 2 we can classify *BCI*-algebras as follows:

BCI-algebras
- p-semisimple
- non p-semisimple
  - *BCK*-algebras, i. e., p-radical algebras
  - non-*BCK*-algebras whose quotient algebras by B are p-semisimple

II. p-semisimple algebras and Abelian groups.

In this section, we will deal with the relations between p-semisimple algebras and Abelian groups.

Theorem 3. *Let X be a p-semisimple algebra. If we define*

$$x + y = x * (0 * y).$$

then $\langle X, +, 0\rangle$ is an Abelian group.

Proof. By using 4) of Theorem 1 and (8), we obtain

$$x+(y+z)=x*(0*(y*(0*z)))=(y*(0*z))*(0*x)$$
$$=(y*(0*x))*(0*z)=(x*(0*y))*(0*z)=(x+y)+z,$$

and

$$x+y=x*(0*y)=y*(0*x)=y+x.$$

Hence the operation $+$ is associative and commutative. Moreover,

$$x+0=0+x=0*(0*x)=x$$

and

$$x+(0*x)=(0*x)+x=(0*x)*(0*x)=0.$$

Therefore $0*x$ is the inverse of x. Thus X is an Abelian group with respect to $+$.

Conversely, we have the following

Theorem 4. *Any Abelian group is a p-semisimple algebra under the operation* $-$. *We omit the proof, as we can easily check the axioms of a BCI-algebra.*

The Abelian group induced by a p-semisimple algebra in Theorem 3 is called to be its *adjoint group*.

Theorem 5. *Let* $\langle X, +, 0\rangle$ *be the adjoint group of a p-semisimple algebra* $\langle X, *, 0\rangle$. *The p-semisimple algebra induced by* $\langle X, +, 0\rangle$ *coincides with* $\langle X, +, 0\rangle$.

By Theorems 3 and 4,

$$x-y=x+(-y)=x+(0*y)=x*(0*(0*y))=x*y,$$

which implies Theorem 5.

III. Some Properties of *p*-semisimple algebras.

In this section, we will give some results on a p-semisimple algebra. We may make use of Abelian group theory to study p-semisimple algebras.

Theorem 6. *Any subalgebra of a p-semisimple algebra is an ideal.*

Proof. Let S be a subalgebra of a p-semisimple algebra X. By Theorem 5 and the fact that S is a subalgebra, we obtain that $x, y \in S$ implies $x-y \in S$. This means that $\langle S, +, 0 \rangle$ is a subgroup of $\langle X, *, 0 \rangle$. Consequently, if $x * y, y \in S$, i. e., $x - y, y \in S$, then $x = (x - y) + y \in S$. Hence $\langle S, *, 0 \rangle$ is an ideal of X.

By Theorem 6, we know that any subalgebra of a p-semisimple algebra is also p-semisimple. The order of a subalgebra of a finite *BCI*-algebra X need not do a divisor of the order of X, but for a p-semisimple algebra, we have the following

Theorem 7. *If n is the order of a finite p-semisimple algebra X, then the order of its subalgebra is a divisor of n.*

This result is easily obtained from the order relation of adjoint groups.

The notion of a quasi-commutative *BCK*-algebra was introduced by H. Yutani [7]. This notion is also defined in a *BCI*-algebra by a similar way.

Theorem 8. *p-semisimple algebra is a quasi-commutative algebra of type* $(0, 1; 0, 0)$.

Proof. By Theorem 5, we have

$$Q_{0,1}(x,y) = (x * (x * y)) * (y * x) = (x - (x - y)) - (y - x) = x,$$

$$Q_{0,0}(y,x) = y * (y * x) = y - (y - x) = x.$$

Therefore, $Q_{0,1}(x, y) = Q_{0,0}(y, x)$, that is, the algebra is a quasi-commutative algebra of type $(0, 1; 0, 0)$.

Remark 2. It is easy to verify that any p-semisimple algebra is also of type $(0, 2; 1, 0)$, but it may not of type $(1, 0; 0, 0)$. For instance, in Example 1, consider 1 and 2, then

$$(1 - (1 - 2)) - (1 - 2) = 3, \quad 1 - (2 - 1) = 1.$$

Consequently, this implies $Q_{1,0}(1, 2) \neq Q_{0,0}(2, 1)$. This remark partly solve the question which K. Iséki posed in [5] that whether exist quasi-commutative algebras of higher type or not.

The fundamental theorem of Abelian groups implies the following

Theorem 9. *Let* $n = p_1^{k_1} \cdots p_r^{k_r}$ be the order of a finite p-semisimple algebra X, then X is isomorphic to a product of finite number of p-semisimple algebras of orders $p_i^{k_i}$.

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