Journal of Algebra 322 (2009) 21–24

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Finiteness of finitistic dimension is invariant under derived equivalences

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ARTICLE INFO

Article history:
Received 5 June 2008
Available online 27 March 2009
Communicated by Michel Van den Bergh

Keywords:
Finitistic dimension
Tilting complex
Derived equivalence
Left coherent ring

ABSTRACT

In this note, we show that the finiteness of the finitistic dimension of a left coherent ring is invariant under derived equivalences.

1. Introduction

Derived equivalence preserves many invariants of algebras, for example, the number of irreducible representations, the finiteness of global dimension, and the algebraic K-theory and G-theory (see [1,2,4,6–8]). In this short note, we point out that the finiteness of the finitistic dimension of an algebra is also preserved under derived equivalences. Recall that, given a ring A, the finitistic dimension of A, denoted by $\operatorname{fin.dim}(A)$, is the supremum of the projective dimensions of finitely generated left A-modules of finite projective dimension. The finitistic dimension conjecture, which is still open, states that every finite-dimensional algebra over a field should have finite finitistic dimension. It is well known that the conjecture is intimately related to many other homological conjectures in the representation theory of algebras. We refer the reader to [9] and the references therein for some new advances on the conjecture. Our result in this note may be used to understand the conjecture.

Theorem 1.1. *If two left coherent rings A and B are derived-equivalent, then the finitistic dimension of A is finite if and only if so is the finitistic dimension of B. More precisely, if $T^\bullet$ is a tilting complex over A with n non-zero terms such that $B \cong \operatorname{End}(T^\bullet)$, then* $\operatorname{fin.dim}(A) - n \leqslant \operatorname{fin.dim}(B) \leqslant \operatorname{fin.dim}(A) + n$.

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.03.017

Thus Theorem 1.1 extends a main result in [3], which states that the finiteness of finitistic dimension of is preserved by a tilting equivalence. The idea of our proof of Theorem 1.1 is similar to that in [3], but additional ingredients are needed.

2. Proof of the main result

In this section, we first recall briefly some basic definitions and results required in our proofs, and then give a proof of our main result.

Let $\mathcal{C}$ be an abelian category. A (cochain) complex $X^\bullet$ over $\mathcal{C}$ is a sequence of morphisms d_X^i between objects X^i in $\mathcal{C}: \cdots \to X^{i-1} \xrightarrow{d_X^{i-1}} X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} X^{i+2} \to \cdots$, such that $d_X^i d_X^{i+1} = 0$ for all $i \in \mathbb{Z}$. Note that the composition of two morphisms $f: X \to Y$ and $g: Y \to Z$ in the category $\mathcal{C}$ is denoted by fg in this paper. We write $X^\bullet = (X^i, d_X^i)$. We denote by $H^i(X^\bullet)$ the i-th homology of $X^\bullet$. The category of all complexes over $\mathcal{C}$ with the usual complex maps of degree zero is denoted by $\mathscr{C}(\mathcal{C})$. The homotopy and derived categories of complexes over $\mathcal{C}$ are denoted by $\mathscr{K}(\mathcal{C})$ and $\mathscr{D}(\mathcal{C})$, respectively. The full subcategory of $\mathscr{C}(\mathcal{C})$ consisting of bounded complexes over $\mathcal{C}$ is denoted by $\mathscr{C}^b(\mathcal{C})$. Similarly, $\mathscr{K}^b(\mathcal{C})$ and $\mathscr{D}^b(\mathcal{C})$ denote the full subcategories consisting of bounded complexes in $\mathscr{K}(\mathcal{C})$ and $\mathscr{D}(\mathcal{C})$, respectively.

Let A be a ring with identity. By an A-module we shall mean a left A-module. We denote by A-Mod the category of all A-modules, by A-mod the category of all finitely presented A-modules, and by A-proj the category of finitely generated projective A-modules. It is well known that $\mathscr{K}(A\text{-Mod})$, $\mathscr{K}^b(A\text{-Mod})$, $\mathscr{D}(A\text{-Mod})$ and $\mathscr{D}^b(A\text{-Mod})$ all are triangulated categories. Moreover, if $X \in \mathscr{K}^b(A\text{-proj})$, then $\mathrm{Hom}_{\mathscr{K}^b(A\text{-Mod})}(X, Z) \simeq \mathrm{Hom}_{\mathscr{D}^b(A\text{-Mod})}(X, Z)$ for every complex Z in $\mathscr{D}^b(A\text{-Mod})$.

Now we recall some definitions from Rickard [7]. Two ring A and B are called derived-equivalent if $\mathscr{D}(A\text{-Mod})$ and $\mathscr{D}(B\text{-Mod})$ are equivalent as triangulated categories. If A and B are left coherent rings, that is, rings for which the kernels of any homomorphisms between finitely generated projective modules are finitely generated, then A and B are derived-equivalent if $\mathscr{D}^b(A\text{-mod})$ and $\mathscr{D}^b(B\text{-mod})$ are equivalent as triangulated categories. A complex $T^\bullet$ in $\mathscr{K}^b(A\text{-proj})$ is called a tilting complex if $\mathrm{Hom}_{\mathscr{K}^b(A\text{-proj})}(T, T[n]) = 0$ for all integers $n \neq 0$, and $\mathrm{add}(T^\bullet)$, the full subcategory of direct summands of finite direct sums of copies of $T^\bullet$, generates $\mathscr{K}^b(A\text{-proj})$ as a triangulated category. For further information on triangulated categories, we refer to [4].

Let A be a left coherent ring, $T^\bullet \in \mathscr{K}^b(A\text{-proj})$ a tilting complex over A with $B = \mathrm{End}_{\mathscr{D}^b(A)\text{-mod}}(T^\bullet)$, and let

$$F: \mathscr{D}^b(B\text{-mod}) \to \mathscr{D}^b(A\text{-mod})$$

be an equivalence of triangulated categories such that $F(B) = T^\bullet$. Then F induces an equivalence of the homotopy categories from $\mathscr{K}^b(B\text{-proj})$ to $\mathscr{K}^b(A\text{-proj})$. Without less of generality, we may assume that $T^\bullet$ is of the form:

$$T^\bullet: \quad \cdots \to 0 \to T^{-n} \to \cdots \to T^{-1} \to T^0 \to 0 \to \cdots.$$

The following result is known in literature, for a proof one may see [6]. Note that in [6] the category A-Mod is considered, but, if we assume rings to be left coherent, then all arguments there work. From now on, we always assume that our rings are left coherent.

Lemma 2.1. *(See [6].) Assume that $n \geqslant 0$ and $T^i = 0$ for $i > 0$ and $i < -n$. Then*

(1) *If $G: \mathscr{D}^b(A\text{-mod}) \to \mathscr{D}^b(B\text{-mod})$ is the inverse of F, then there is a tilting complex $Q^\bullet$ over B such that $G(A) \simeq Q^\bullet$ and $Q^i = 0$ for all $i > n$ and $i < 0$.*

(2) *Let $m, n, d \in \mathbb{Z}$ and $d \geqslant 0$, and let $X^\bullet, Y^\bullet \in \mathscr{K}^b(A\text{-mod})$ such that $X^p = 0$ for all $p < m$, $Y^q = 0$ for all $q > n$, and $\mathrm{Ext}^i_A(X^r, Y^s) = 0$ for all $r, s \in \mathbb{Z}$ and $i \geqslant d$. Then $\mathrm{Hom}_{\mathscr{D}(A\text{-mod})}(X^\bullet, Y^\bullet[i]) = 0$ for all $i \geqslant d + n - m$.*

Proof of Theorem 1.1. We denote by $\mathrm{fin.dim}(A)$ the finitistic dimension of A, and by $\mathrm{proj.dim}(X)$ the projective dimension of a module X in A-mod. Suppose $\mathrm{fin.dim}(A) = d < \infty$. Let ${}_BM$ be a B-module of finite projective dimension, and let ${}_BN$ be an arbitrary B-module. As in [6], we show that $H^i(FN) = 0$ for $i < -n$ and $i > 0$. In fact, there is a tilting complex $Q^\bullet \in \mathscr{K}^b(A\text{-proj})$ such that $G(A) \simeq Q^\bullet$ and $Q^i = 0$ for $i > n$ and $i < 0$ by Lemma 2.1(1). Then we have

$$H^i(FN) \simeq \mathrm{Hom}_{\mathscr{D}(A\text{-mod})}\big(A, FN[i]\big) \simeq \mathrm{Hom}_{\mathscr{D}(B\text{-mod})}\big(Q^\bullet, N[i]\big) = 0$$

for $i < -n$ and $i > 0$. Hence it follows that the complex FN is quasi-isomorphic to the following complex

$$Y^\bullet: \quad \cdots \to 0 \to \mathrm{Coker}\big(d^{-n-1}\big) \to (FN)^{-n+1} \to \cdots \to (FN)^{-2} \to (FN)^{-1} \to \mathrm{Ker}\big(d^0\big) \to 0 \to \cdots.$$

Let $P^\bullet(M)$ be a minimal projective resolution of the B-module ${}_BM$. Since F is an equivalence between $\mathscr{K}^b(B\text{-proj})$ and $\mathscr{K}^b(A\text{-proj})$, we have $FP^\bullet(M) \in \mathscr{K}^b(A\text{-proj})$. It follows from $FM \simeq FP^\bullet(M)$ in $\mathscr{D}^b(A\text{-mod})$ that $H^i(P^\bullet(M)) = H^i(FM) = 0$ for $i < -n$ and $i > 0$. Thus, if we write $Z^\bullet = (Z^i, d^i) := FP^\bullet(M)$ with all Z^i finitely generated projective, then the complex $Z^\bullet$ is quasi-isomorphic to the following complex in $\mathscr{D}^b(A\text{-mod})$:

$$X^\bullet: \quad \cdots \to 0 \to \mathrm{Coker}\big(d^{-n-1}\big) \to Z^{-n+1} \to \cdots \to Z^{-2} \to Z^{-1} \to \mathrm{Ker}\big(d^0\big) \to 0 \to \cdots.$$

Since $H^i(Z^\bullet) = 0$ for $i < -n$ and $Z^\bullet \in \mathscr{K}^b(A\text{-proj})$, we see that the projective dimension of the finitely presented module $\mathrm{Coker}(d^{-n-1})$ is finite. Thus $\mathrm{proj.dim}_A(\mathrm{Coker}(d^{-n-1})) \leqslant d$. Similarly, since $H^i(Z^\bullet) = 0$ for $i > 0$, the corresponding complex

$$\cdots \to \mathrm{Ker}\big(d^0\big) \to Z^0 \to Z^1 \to \cdots \tag{$*$}$$

is exact. Since $Z^\bullet$ is a bounded complex and each Z^i is projective, the complex $(*)$ splits. Consequently, $\mathrm{Ker}(d^0)$ is projective and finitely generated since it is a direct summand of Z^0. So, if we replace FN by $Y^\bullet$, we see that $X^p = 0$ for $p < -n$, $Y^q = 0$ for $q > 0$, and that $\mathrm{Ext}^i_A(X^p, Y^q) = 0$ for $i \geqslant d + 1$ and $p, q \in \mathbb{Z}$. Hence $\mathrm{Hom}_{\mathscr{D}(A)}(X^\bullet, Y^\bullet[i]) = 0$ for all $i \geqslant d + 1 + n$ by Lemma 2.1(2). Thus we have the following:

$$\begin{aligned}
\mathrm{Ext}^i_B(M, N) &= \mathrm{Hom}_{\mathscr{D}(B\text{-mod})}\big(M, N[i]\big) \\
&\simeq \mathrm{Hom}_{\mathscr{D}(B\text{-mod})}\big(P^\bullet(M), N[i]\big) \\
&\simeq \mathrm{Hom}_{\mathscr{D}(A\text{-mod})}\big(FP^\bullet(M), FN[i]\big) \\
&\simeq \mathrm{Hom}_{\mathscr{D}(A\text{-mod})}\big(X^\bullet, Y^\bullet[i]\big) = 0
\end{aligned}$$

for all $i \geqslant d + n + 1$. Thus $\mathrm{proj.dim}(M) \leqslant d + n$, and $\mathrm{fin.dim}(B) \leqslant \mathrm{fin.dim}(A) + n$. Similarly, using the complex $Q^\bullet$ in Lemma 2.1, we can show that $\mathrm{fin.dim}(A) \leqslant \mathrm{fin.dim}(B) + n$. This finishes the proof. □

The following is a consequence of Theorem 1.1.

Corollary 2.2. *Suppose A is a self-injective Artin algebra and X is an A-module. If one of the algebras $\mathrm{End}_A(A \oplus \Omega^n(X))$, $n \geqslant 0$, has finite finitistic dimension, then all of them have finite finitistic dimension, where $\Omega^n(X)$ is the n-th syzygy of X.*

Proof. It is shown in [5, Corollary 3.8] that if A is a self-injective Artin algebra then all algebras of the form $\mathrm{End}_A(A \oplus \Omega^n(X))$, $n \geqslant 0$, are derived-equivalent. Thus the corollary follows from Theorem 1.1. □

Corollary 2.3. *Let A and B be two finite-dimensional k-algebras over a field k, $M \in A$-mod and $N \in B$-mod. Let $A[M]$ and $B[N]$ be the respective one-point extensions. If A and B are derived-equivalent, then the finiteness of finitistic dimension of any one of the algebras A, B, $A[M]$ and $B[N]$ implies that all of them have finite finitistic dimension.*

Proof. It is known that $\mathrm{fin.dim}(A[M]) < \infty$ if and only if $\mathrm{fin.dim}(A) < \infty$. By Theorem 1.1, we have $\mathrm{fin.dim}(A) < \infty$ if and only if $\mathrm{fin.dim}(B) < \infty$. Thus Corollary 2.3 follows. □

Acknowledgment

The authors thank NSFC (No. 10731070) for partial support.

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