Good tilting modules and recollements of derived module categories

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Abstract

Let T be an infinitely generated tilting module of projective dimension at most one over an arbitrary associative ring A, and let B be the endomorphism ring of T. We prove that if T is good, then there exists a ring C, a homological ring epimorphism $B \to C$ and a recollement among the (unbounded) derived module categories $\mathscr{D}(C)$ of C, $\mathscr{D}(B)$ of B and $\mathscr{D}(A)$ of A. In particular, the kernel of the total left-derived functor $T \otimes_B^{\mathbb{L}}$- is triangle equivalent to the derived module category $\mathscr{D}(C)$. Conversely, if $T \otimes_B^{\mathbb{L}}$- admits a fully faithful left adjoint functor, then T is good. Moreover, if T arises from an injective ring epimorphism, then C is isomorphic to the coproduct of two relevant rings. In the case of commutative rings, the ring C can be strengthened as the tensor product of two commutative rings. Consequently, we produce a large variety of examples (from Dedekind domains and p-adic number theory, or Kronecker algebra) to show that two different stratifications of the derived module category of a ring by derived module categories of rings may have completely different derived composition factors (even up to ordering and up to derived equivalence), or different lengths. This shows that the Jordan–Hölder theorem fails even for stratifications by derived module categories.

Contents

1. *Introduction*

The theory of finitely generated tilting modules has been successfully applied, in the representation theory of algebras and groups, to understand different aspects of algebraic structure and homological features of (algebraic) groups, algebras and modules (for instance, see [**14**, **16**, **17**, **24**, **27**–**31**]). Recently, infinitely generated tilting modules over arbitrary associated rings have become of interest in and attracted increasing attentions towards understanding derived categories and equivalences of general rings [**2**–**5**, **7**, **9**–**11**, **21**, **22**, **37**–**39**]. In this general situation, many classical results in the tilting theory appear in a very different new fashion. For example, Happel's Theorem (see also [**17**]) on derived equivalences induced by infinitely

generated tilting modules comes up with a new formulation in which quotient categories are involved (see [**9**]). This more general context of tilting theory not only renews our view on features of finitely generated tilting modules, but also provides us completely different information about the whole tilting theory. Let us recall the definition of tilting modules over an arbitrary ring from [**22**].

Let A be a ring with identity, and let T be a left A-module which may be infinitely generated. The module T is called a tilting module (of projective dimension at most 1) provided that:

(T1) T has projective dimension at most 1,
(T2) $\mathrm{Ext}_A^i(T, T^{(\alpha)}) = 0$, for each $i \geqslant 1$ and each cardinal α, and
(T3) there exists an exact sequence $0 \to A \to T_0 \to T_1 \to 0$ of left A-modules, where T_0 and T_1 are isomorphic to direct summands of arbitrary direct sums of copies of T.

If, in addition, T is finitely presented, then we say that T is a classical tilting module. If the modules T_0 and T_1 in $(T3)$ are isomorphic to direct summands of finite direct sums of copies of T, then we say that T is a good tilting module, following [**11**]. Actually, each classical tilting module is good, furthermore, it is proved in [**11**] that, for an arbitrary tilting A-module T, there exists a good tilting A-module T' which is equivalent to T, that is, T and T' generate the same full subcategories in the category of all left A-modules.

One of the realizations of tilting modules is universal localizations. It is shown in [**1**] that every tilting module over a ring is associated in a canonical manner with a ring epimorphism which can be interpreted as a universal localization at a set of finitely presented modules of projective dimension at most 1.

As in the theory of classical tilting modules, a natural context for studying infinitely generated tilting modules is the relationship of derived categories and equivalences induced by infinitely generated tilting modules. In fact, if T is a good tilting module over a ring A, and if B is the endomorphism ring of T, then Bazzoni proves in [**9**] that the total right-derived functor $\mathbb{R}\mathrm{Hom}_A(T, -)$ induces an equivalence between the (unbounded) derived category $\mathscr{D}(A)$ of A and the quotient category of the derived category $\mathscr{D}(B)$ of B modulo the full triangulated subcategory $\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -)$ which is the kernel of the total left derived functor $T \otimes_B^{\mathbb{L}} -$. Thus, in general, the total right-derived functor $\mathbb{R}\mathrm{Hom}_A(T, -)$ does not define a derived equivalence between A and B. This is a contrary phenomenon to the classical situation (see [**17**]). The condition for A and B to be derived-equivalent depends on the vanishing of $\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -)$. It is shown in [**9**] that $\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -)$ vanishes if and only if T is a classical tilting module. From this point of view, the triangulated category $\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -)$ measures how far a good tilting module is from being classical, in other words, the difference between the two derived categories, $\mathscr{D}(A)$ and $\mathscr{D}(B)$. It is certainly of interest to have a little bit knowledge about the categories $\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -)$ for infinitely generated tilting modules T. This might help us to understand some new aspects of the tilting theory of infinitely generated tilting modules.

The main purpose of this paper is to give a characterization of the triangulated categories $\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -)$ for infinitely generated tilting modules T, namely we show that if the tilting module T is good, then the triangulated category $\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -)$ is equivalent to the derived category of a ring C, and therefore, there is a recollement among the derived categories of rings A, B and C. Conversely, the existence of such a recollement implies that the given tilting module T is good. More precisely, our result can be stated as follows:

THEOREM 1.1. *Let A be a ring, T a tilting A-module of projective dimension at most 1 and B the endomorphism ring of T.*

(1) *If T is good, then there is a ring C, a homological ring epimorphism $\lambda : B \to C$ and a recollement among the unbounded derived categories of the rings A, B and C :*

$$\mathscr{D}(C) \longrightarrow \mathscr{D}(B) \xrightarrow{\ j^!\ } \mathscr{D}(A)$$

such that the triangle functor $j^!$ is isomorphic to the total left-derived functor ${}_AT \otimes^{\mathbb{L}}_B -$. In this case, the kernel of the functor $T \otimes^{\mathbb{L}}_B -$ is equivalent to the unbounded derived category $\mathscr{D}(C)$ of C as triangulated categories.

(2) *If the triangle functor $T \otimes^{\mathbb{L}}_B - : \mathscr{D}(B) \to \mathscr{D}(A)$ admits a fully faithful left adjoint $j_! : \mathscr{D}(A) \to \mathscr{D}(B)$, then the given tilting module T is good.*

Let us remark that a noteworthy difference of Theorem 1.1(1) from the result [**4**, Proposition 1.7] is that our recollement is over derived module categories of precisely determined rings, while the recollement in [**4**, Proposition 1.7] involves a triangulated category. Theorem 1.1(1) realizes this abstract triangulated category by a derived module category via describing the kernel of the functor $T \otimes^{\mathbb{L}}_B -$. Our result also distinguishes itself from the one in [**42**], where C is a differential graded ring instead of a usual ring, and where the consideration is restricted to ground ring being a field.

If we apply Theorem 1.1 to tilting modules arising from ring epimorphisms, then we can see that, in most cases, the recollements given in Theorem 1.1 are different from the usual ones induced from the structure of triangular matrix rings. The following corollary is a consequence of the proof of Theorem 1.1.

COROLLARY 1.2. (1) *Let $R \to S$ be an injective ring epimorphism such that $\mathrm{Tor}_1^R(S, S) = 0$ and that ${}_RS$ has projective dimension at most one. Then there is a recollement of derived module categories:*

$$\mathscr{D}(S \sqcup_R S') \longrightarrow \mathscr{D}(\mathrm{End}_R(S \oplus S/R)) \longrightarrow \mathscr{D}(R)$$

where S' is the endomorphism ring of the R-module S/R, and $S \sqcup_R S'$ is the coproduct of S and S' over R. If, in addition, R is commutative, then $S \sqcup_R S'$ is isomorphic to the tensor product $S \otimes_R S'$ of S and S' over R.

(2) *For every prime number $p \geqslant 2$, the derived category of the ring $\left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q}_p \\ 0 & \mathbb{Z}_p \end{smallmatrix} \right)$ admits two stratifications, one of which clearly has composition factors $\mathbb{Q}$ and $\mathbb{Z}_p$, and the other has composition factors $\mathbb{Q}_{(p)}$ and $\mathbb{Q}_p$, where $\mathbb{Q}_{(p)}$, $\mathbb{Q}$, $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the rings of p-integers, rational numbers, p-adic integers and p-adic numbers, respectively.*

As pointed out in [**5**], the Jordan–Hölder theorem fails for stratifications of derived module categories by triangulated categories. Our Corollary 1.2(2) (see also the example in Section 8) shows that the Jordan–Hölder theorem fails even for stratifications of derived module categories by derived module categories, and therefore the problem posed in [**5**] gets a negative answer.

The paper is organized as follows: In Section 2, we recall some definitions, notations and useful results which are needed for our proofs. In Section 3, we shall first establish a connection between universal localizations and recollements of triangulated categories, and then prove Proposition 3.6 which is crucial for the proof of the main result. In Section 4, we discuss some homological properties of good tilting modules, and establish another crucial result, Proposition 4.6, for the proof of the main result Theorem 1.1. After these preparations, we apply the results obtained in Section 3 to prove Theorem 1.1(1). In Section 5, we prove the

second part of Theorem 1.1. This may be regarded as a converse statement of the first part. In Section 6, we apply Theorem 1.1 to good tilting modules arising from ring epimorphisms, and prove Corollary 1.2(1). In these cases, the universal localization rings in Theorem 1.1 can be given by coproducts of rings. In Section 7, we consider the existence of the recollements in Theorem 1.1 for Dedekind domains, and prove Corollary 1.2(2). It turns out that many derived module categories of rings possess stratifications by derived module categories of rings, such that, even up to ordering and up to derived equivalence, not all of their composition factors are the same; for instance, the derived category of the endomorphism ring of the abelian group $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ (or its variation $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Q}_{(p)}$). Note that, in the examples presented in this section, the two stratifications all have the same lengths. In Section 8, we give an example of a non-commutative algebra over which the derived category of the endomorphism ring of a tilting module has two stratifications of different finite lengths. This, together with the examples in Section 7, gives a complete answer to an open problem in [**5**] negatively.

2. *Preliminaries*

In this section, we shall recall some definitions, notation and basic results which are related to our proofs. In particular, we recall the notions of recollements and torsion torsion-free (TTF) triples as well as their relationship.

2.1. *Some conventions*

All rings considered in this paper are assumed to be associative and with identity, and all ring homomorphisms preserve identity.

Let A be a ring. We denote by A-Mod the category of all unitary left A-modules. For an A-module M, we denote by add(M) (respectively, Add(M)) the full subcategory of A-Mod consisting of all direct summands of finite (respectively, arbitrary) direct sums of copies of M. In many circumstances, we shall write A-proj and A-Proj for add$(_A A)$ and Add$(_A A)$, respectively. If I is an index set, we denote by $M^{(I)}$ the direct sum of I copies of M. If there is a surjective homomorphism from $M^{(I)}$ to an A-module X, we say that X is generated by M, or M generates X. By Gen(M), we denote the full subcategory of A-Mod generated by M.

If $f : M \to N$ is a homomorphism of A-modules, then the image of $x \in M$ under f is denoted by $(x)f$ instead of $f(x)$. Also, for any A-module X, the induced morphisms $\mathrm{Hom}_A(X, f) : \mathrm{Hom}_A(X, M) \to \mathrm{Hom}_A(X, N)$ and $\mathrm{Hom}_A(f, X) : \mathrm{Hom}_A(N, X) \to \mathrm{Hom}_A(M, X)$ are denoted by f^* and f_*, respectively.

Let $\mathcal{C}$ be an additive category.

Given two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, we denote the composition of f and g by fg which is a morphism from X to Z, while we denote the composition of a functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ with a functor $G : \mathcal{D} \to \mathcal{E}$ between categories $\mathcal{D}$ and $\mathcal{E}$ by GF which is a functor from $\mathcal{C}$ to $\mathcal{E}$. The image of the functor F is denoted by $\mathrm{Im}(F)$ which is a full subcategory of $\mathcal{D}$.

Throughout the paper, a full subcategory $\mathcal{D}$ of $\mathcal{C}$ is always assumed to be closed under isomorphisms, that is, if X and Y are objects in $\mathcal{C}$, then $Y \in \mathcal{D}$ whenever $Y \simeq X$ with $X \in \mathcal{D}$.

Let $\mathcal{Y}$ be a full subcategory of $\mathcal{C}$. By $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{Y}))$, we denote the left orthogonal subcategory with respect to $\mathcal{Y}$, that is, the full subcategory of $\mathcal{C}$ consisting of the objects X such that $\mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$ for all objects Y in $\mathcal{Y}$. Similarly, $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(\mathcal{Y}, -))$ stands for the right orthogonal subcategory of $\mathcal{C}$ with respect to $\mathcal{Y}$.

By a complex $X^{\bullet}$ over $\mathcal{C}$, we mean a sequence of morphisms d_X^i between objects X^i in $\mathcal{C} : \cdots \to X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} X^{i+2} \to \cdots$, such that $d_X^i d_X^{i+1} = 0$, for all $i \in \mathbb{Z}$. In this case, we write $X^{\bullet} = (X^i, d_X^i)_{i \in \mathbb{Z}}$, and call d_X^i a differential of $X^{\bullet}$. Sometimes, for simplicity, we write $(X^i)_{i \in \mathbb{Z}}$ for $X^{\bullet}$ without mentioning the morphisms d_X^i. For a fixed integer n, we denote by

$X^\bullet[n]$ the complex obtained from $X^\bullet$ by shifting n degrees, that is, $(X^\bullet[n])^0 = X^n$, and by $H^n(X^\bullet)$ the cohomology of $X^\bullet$ in degree n.

Let $\mathscr{C}(\mathcal{C})$ be the category of all complexes over $\mathcal{C}$ with chain maps, and $\mathscr{K}(\mathcal{C})$ the homotopy category of $\mathscr{C}(\mathcal{C})$. We denote by $\mathscr{C}^{\mathrm{b}}(\mathcal{C})$ and $\mathscr{K}^{\mathrm{b}}(\mathcal{C})$ the full subcategories of $\mathscr{C}(\mathcal{C})$ and $\mathscr{K}(\mathcal{C})$ consisting of bounded complexes over $\mathcal{C}$, respectively. When $\mathcal{C}$ is abelian, the derived category of $\mathcal{C}$ is denoted by $\mathscr{D}(\mathcal{C})$, which is the localization of $\mathscr{K}(\mathcal{C})$ at all quasi-isomorphisms. The full subcategory of $\mathscr{D}(\mathcal{C})$ consisting of bounded complexes over $\mathcal{C}$ is denoted by $\mathscr{D}^{\mathrm{b}}(\mathcal{C})$. As usual, for a ring A, we simply write $\mathscr{C}(A)$ for $\mathscr{C}(A\text{-Mod})$, $\mathscr{K}(A)$ for $\mathscr{K}(A\text{-Mod})$, $\mathscr{C}^{\mathrm{b}}(A)$ for $\mathscr{C}^{\mathrm{b}}(A\text{-Mod})$ and $\mathscr{K}^{\mathrm{b}}(A)$ for $\mathscr{K}^{\mathrm{b}}(A\text{-Mod})$. Similarly, we write $\mathscr{D}(A)$ and $\mathscr{D}^{\mathrm{b}}(A)$ for $\mathscr{D}(A\text{-Mod})$ and $\mathscr{D}^{\mathrm{b}}(A\text{-Mod})$, respectively. Furthermore, we always identify A-Mod with the full subcategory of $\mathscr{D}(A)$ consisting of all stalk complexes concentrated on degree 0.

Now we recall some basic facts about derived functors defined on derived module categories. We refer to [**15**] for details and proofs.

Let R and S be rings, and let H be an additive functor from R-Mod to S-Mod.

(1) For each complex $X^\bullet$ in $\mathscr{D}(R)$, there is a complex $I^\bullet \in \mathscr{C}(R\text{-Inj})$ such that $X^\bullet$ is quasi-isomorphic to $I^\bullet$, where R-Inj is the full subcategory of R-Mod consisting of all injective R-modules. Dually, for each complex $Y^\bullet$ in $\mathscr{D}(R)$, there is a complex $P^\bullet \in \mathscr{C}(R\text{-Proj})$ such that $P^\bullet$ is quasi-isomorphic to $Y^\bullet$.

(2) There is a total right-derived functor $\mathbb{R}H$ and a total left-derived functor $\mathbb{L}H$ defined on $\mathscr{D}(R)$. If $X^\bullet, Y^\bullet \in \mathscr{D}(R)$, then $\mathbb{R}H(X^\bullet) = H(I^\bullet)$ and $\mathbb{L}H(Y^\bullet) = H(P^\bullet)$, where $I^\bullet$ and $P^\bullet$ are chosen as in (1). Here, we think of H as an induced functor between homotopy categories, and if $X^\bullet = (X^i, d_X^i)_{i \in \mathbb{Z}}$, then $H(X^\bullet) := (H(X^i), H(d_X^i))_{i \in \mathbb{Z}}$.

In case T is an R-S-bimodule, the total right-derived functor of $\mathrm{Hom}_R(T, -)$ is denoted by $\mathbb{R}\mathrm{Hom}_R(T, -)$, and the total left-derived functor of $T \otimes_S -$ is denoted by $T \otimes_S^{\mathbb{L}} -$.

(3) Any adjoint pair (G, H) of additive functors G and H between R-Mod and S-Mod induces an adjoint pair $(\mathbb{L}G, \mathbb{R}H)$ between the unbounded derived categories of R and S.

2.2. *Homological ring epimorphisms*

Let R and S be rings. Recall that a homomorphism $\lambda : R \to S$ of rings is called a ring epimorphism if, for any two homomorphisms $f_1, f_2 : S \to T$ of rings, the equality $\lambda f_1 = \lambda f_2$ implies that $f_1 = f_2$. It is known that λ is a ring epimorphism if and only if the multiplication map $S \otimes_R S \to S$ is an isomorphism as S–S-bimodules if and only if $x \otimes 1 = 1 \otimes x$ in $S \otimes_R S$, for any $x \in S$. It follows that, for a ring epimorphism, we have $X \otimes_S Y \simeq X \otimes_R Y$, for any S-modules X_S and $_SY$. An example of ring epimorphisms is the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Note that $\mathbb{Q}$ is an injective and a flat $\mathbb{Z}$-module.

Given a ring epimorphism $\lambda : R \to S$ between two rings R and S, we can regard S-Mod as a full subcategory of R-Mod via λ. This means that $\mathrm{Hom}_S(X, Y) \simeq \mathrm{Hom}_R(X, Y)$, for all S-modules X and Y.

Two ring epimorphisms $\lambda : R \to S$ and $\lambda' : R \to S'$ are said to be equivalent, if there is a ring isomorphism $\psi : S \to S'$ such that $\lambda' = \lambda \psi$. This defines an equivalence relation on the class of ring epimorphisms $R \to S$ with R fixed. The equivalence classes with respect to this equivalence relation are called the epiclasses of R. This notion is associated with bireflective subcategories of module categories.

Recall that a full subcategory $\mathcal{D}$ of R-Mod is said to be coreflective, if every R-module X admits a $\mathcal{D}$-reflection, that is, there exists an R-module $D' \in \mathcal{D}$ and a homomorphism $f : D' \to X$ of R-modules such that $\mathrm{Hom}_R(D, f) : \mathrm{Hom}_R(D, D') \to \mathrm{Hom}_R(D, X)$ is an isomorphism as abelian groups for any module $D \in \mathcal{D}$. Dually, one defines the notion of reflective subcategories of R-Mod. The full subcategory $\mathcal{D}$ of R-Mod is called bireflective, if it is both reflective and coreflective.

Ring epimorphisms are related to bireflective subcategories in the following way.

LEMMA 2.1 [**1**, Theorem 1.4]. *For a full subcategory $\mathcal{D}$ of R-Mod, the following statements are equivalent.*

(1) *There is a ring epimorphism $\lambda : R \to S$ such that the category $\mathcal{D}$ is the image of the restriction functor $\lambda_* : S$-Mod $\to R$-Mod.*

(2) *The category $\mathcal{D}$ is a bireflective subcategory of R-Mod.*

(3) *The category $\mathcal{D}$ is closed under direct sums, products, kernels and cokernels.*

Thus, there is a bijection between the epiclasses of R and the bireflective subcategories of R-Mod. Furthermore, the map $\lambda : R \to S$ in (1), viewed as a homomorphism of R-modules, is a $\mathcal{D}$-reflection of R.

Following Geigle and Lenzing [**26**], we say that a ring epimorphism $\lambda : R \to S$ is homological, if $\operatorname{Tor}_i^R(S, S) = 0$ for all $i > 0$. This is equivalent to saying that the restriction functor $\lambda_* : \mathscr{D}(S) \to \mathscr{D}(R)$ induced by λ is fully faithful. In [**26**, Theorem 4.4], the following lemma is proved.

LEMMA 2.2. *For a homomorphism $\lambda : R \to S$ of rings, the following assertions are equivalent.*

(1) *The homomorphism λ is homological, .*

(2) *For all right S-modules X and all left S-modules Y, the natural map $\operatorname{Tor}_i^R(X, Y) \to \operatorname{Tor}_i^S(X, Y)$ is an isomorphism for all $i \geqslant 0$.*

(3) *For all S-modules X and Y, the natural map $\operatorname{Ext}_S^i(X, Y) \to \operatorname{Ext}_R^i(X, Y)$ is an isomorphism for all $i \geqslant 0$.*

Note that the condition (3) in Lemma 2.2 can be replaced by the corresponding version of right modules. For more details, one may look at [**26**] and [**36**, Section 5.3].

2.3. *Recollements and TTF triples*

In this section, we first recall the definitions of recollements and TTF triples, and then state a correspondence between them.

From now on, $\mathcal{D}$ denotes a triangulated category with small coproducts (that is, coproducts indexed over sets exist in $\mathcal{D}$), and with [1] the shift functor of $\mathcal{D}$.

The notion of recollements was first defined by Beilinson *et al.* [**12**] to study 'exact sequences' of derived categories of coherent sheaves over geometric objects.

DEFINITION 2.3. Let $\mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories. We say that $\mathcal{D}$ is a recollement of $\mathcal{D}'$ and $\mathcal{D}''$, if there are six triangle functors as in the following diagram:

$$\mathcal{D}'' \xrightarrow{i_* = i_!} \mathcal{D} \xrightarrow{j^! = j^*} \mathcal{D}'$$
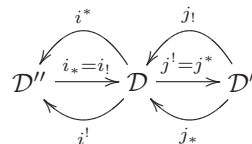
such that

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs;

(2) i_*, j_* and $j_!$ are fully faithful functors;

(3) $i^! j_* = 0$ (and thus also $j^! i_! = 0$ and $i^* j_! = 0$); and
(4) for each object $C \in \mathcal{D}$, there are two triangles in $\mathcal{D}$:

$$i_! i^!(C) \longrightarrow C \longrightarrow j_* j^*(C) \longrightarrow i_! i^!(C)[1],$$
$$j_! j^!(C) \longrightarrow C \longrightarrow i_* i^*(C) \longrightarrow j_! j^!(C)[1].$$

Recollements are closely related to TTF triples which are defined in terms of torsion pairs. So, let us first recall the notion of torsion pairs in triangulated categories.

DEFINITION 2.4 [**14**]. A torsion pair in $\mathcal{D}$ is a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{D}$ satisfying the following conditions:

(1) $\mathrm{Hom}_{\mathcal{D}}(\mathcal{X}, \mathcal{Y}) = 0$;
(2) $\mathcal{X}[1] \subseteq \mathcal{X}$ and $\mathcal{Y}[-1] \subseteq \mathcal{Y}$; and
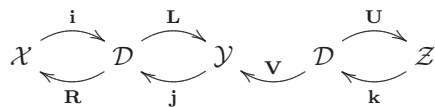(3) for each object $C \in \mathcal{D}$, there is a triangle

$$X_C \longrightarrow C \longrightarrow Y^C \longrightarrow X_C[1]$$

in $\mathcal{D}$ such that $X_C \in \mathcal{X}$ and $Y^C \in \mathcal{Y}$. In this case, $\mathcal{X}$ is called a torsion class and $\mathcal{Y}$ is called a torsion-free class. If, in addition, $\mathcal{X}$ is a triangulated subcategory of $\mathcal{D}$ (or equivalently, $\mathcal{Y}$ is a triangulated subcategory of $\mathcal{D}$), then the torsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be hereditary (see [**14**, Chapter I, Proposition 2.6]).

Note that, if $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in $\mathcal{D}$, then $\mathcal{X} = \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{Y}))$ which is closed under small coproducts, and $\mathcal{Y} = \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(\mathcal{X}, -))$ which is closed under small products.

DEFINITION 2.5 [**14**]. A TTF triple in $\mathcal{D}$ is a triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of full subcategories $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ of $\mathcal{D}$ such that both $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are torsion pairs. In this case, $\mathcal{X}$ is said to be a smashing subcategory of $\mathcal{D}$.

It follows from [**14**, Chapter I.2.] that, associated with a TTF triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ in $\mathcal{D}$, there are seven triangle functors demonstrated in the following diagram:

$$\mathcal{X} \underset{\mathbf{R}}{\overset{\mathbf{i}}{\rightleftarrows}} \mathcal{D} \underset{\mathbf{j}}{\overset{\mathbf{L}}{\rightleftarrows}} \mathcal{Y} \underset{\mathbf{V}}{\overset{}{\rightleftarrows}} \mathcal{D} \underset{\mathbf{k}}{\overset{\mathbf{U}}{\rightleftarrows}} \mathcal{Z}$$

such that

(1) $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are canonical inclusions;
(2) $(\mathbf{i}, \mathbf{R}), (\mathbf{L}, \mathbf{j}), (\mathbf{j}, \mathbf{V})$ and $(\mathbf{U}, \mathbf{k})$ are adjoint pairs; and
(3) the composition functor $\mathbf{Ui} : \mathcal{X} \to \mathcal{Z}$ of the functors $\mathbf{i}$ and $\mathbf{U}$ is a triangle equivalence with the quasi-inverse functor $\mathbf{Rk}$ which is the composition of the functors $\mathbf{k}$ and $\mathbf{R}$.

Note that if $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF triple in $\mathcal{D}$, then it is easy to check that $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are automatically triangulated subcategories of $\mathcal{D}$.

Observe also that the existence of the functors $\mathbf{R}$ and $\mathbf{L}$ in the above diagram follows from the fact that $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in $\mathcal{D}$ (see [**14**, Chapter I, Proposition 2.3] for details). Furthermore, $\mathcal{Y}$ is closed under small coproducts and products.

Now, we state a correspondence between recollements and TTF triples given in [**33**, Section 9.2,**36**, Section 4.2]. For more details, we refer the reader to these papers.

LEMMA 2.6. (1) *If $\mathcal{D}$ is a recollement of $\mathcal{D}'$ and $\mathcal{D}''$ in Definition 2.3, then $(j_!(\mathcal{D}'), i_*(\mathcal{D}''), j_*(\mathcal{D}'))$ is a TTF triple in $\mathcal{D}$.*

(2) *If $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF triple in $\mathcal{D}$, then $\mathcal{D}$ is a recollement of $\mathcal{X}$ and $\mathcal{Y}$ as follows:*

$$\mathcal{Y} \underset{V}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\underset{j}{\longrightarrow}}}}} \mathcal{D} \underset{kUi}{\overset{i}{\underset{\longrightarrow}{\overset{\longleftarrow}{\underset{R}{\longrightarrow}}}}} \mathcal{X}$$

2.4. *Generators and compact objects*

In this section, we shall recall some definitions and facts on generators in triangulated categories.

Given a class of objects $\mathcal{U}$ in $\mathcal{D}$, we denote by $\mathrm{Tria}(\mathcal{U})$ the smallest full triangulated subcategory of $\mathcal{D}$ which contains $\mathcal{U}$ and is closed under small coproducts. If $\mathcal{U}$ consists of only one single object U, then we simply write $\mathrm{Tria}(\mathrm{U})$ for $\mathrm{Tria}(\{\mathrm{U}\})$.

DEFINITION 2.7. A class $\mathcal{U}$ of objects in $\mathcal{D}$ is called a class of generators of $\mathcal{D}$ if an object D in $\mathcal{D}$ is zero whenever $\mathrm{Hom}_{\mathcal{D}}(U[n], D) = 0$ for every object U of $\mathcal{U}$ and every n in $\mathbb{Z}$.

An object P in $\mathcal{D}$ is called compact if the functor $\mathrm{Hom}_{\mathcal{D}}(P, -)$ preserves small coproducts, that is, $\mathrm{Hom}_{\mathcal{D}}(P, \bigoplus_{i \in I} X_i) \simeq \bigoplus_{i \in I} \mathrm{Hom}_{\mathcal{D}}(P, X_i)$, where I is a set; and exceptional if $\mathrm{Hom}_{\mathcal{D}}(P, P[i]) = 0$, for all $i \neq 0$. The object P is called a tilting object if P is compact, exceptional and a generator of $\mathcal{D}$. Note that, for a compact generator P, we have $\mathrm{Tria}(P) = \mathcal{D}$ (see [**36**], for instance).

The category $\mathcal{D}$ is said to be compactly generated if $\mathcal{D}$ admits a set $\mathcal{V}$ of compact generators. In this case, $\mathcal{D} = \mathrm{Tria}(\mathcal{V})$, and we say that $\mathcal{D}$ is compactly generated by $\mathcal{V}$.

It is well known that, for a ring A, the unbounded derived category $\mathscr{D}(A)$ is a compactly generated triangulated category, and one of its compact generators is $_AA$. Moreover, a complex $P^{\bullet} \in \mathscr{D}(A)$ is compact if and only if it is quasi-isomorphic to a bounded complex of finitely generated projective A-modules.

The relationship between compact objects and TTF triples is explained in the next result, which states that any set of compact objects in a triangulated category with small coproducts gives rise to a TTF triple. For more details, we refer the reader to [**14**, Chapter III, Theorem 2.3; Chapter IV, Proposition 1.1].

LEMMA 2.8. *Let $\mathcal{C}$ be a compactly generated triangulated category which admits all small coproducts. Suppose that $\mathcal{P}$ is a set of compact objects in $\mathcal{C}$. Set $\mathcal{X} := \mathrm{Tria}(\mathcal{P}), \mathcal{Y} := \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(\mathcal{X}, -))$ and $\mathcal{Z} := \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(\mathcal{Y}, -))$. Then $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF triple in $\mathcal{C}$. Moreover, $\mathcal{Y}$ coincides with the full subcategory of $\mathcal{C}$ consisting of the objects Y such that $\mathrm{Hom}_{\mathcal{C}}(P[n], Y) = 0$, for every $P \in \mathcal{P}$ and $n \in \mathbb{Z}$.*

3. *Universal localizations and recollements*

In this section, we shall further generalize and develop some known results and connections between universal localizations and recollements of triangulated categories in the literature. In this consideration, homological ring epimorphisms and perpendicular categories will play a role.

Now, we fix a ring R, and suppose that Σ is a set of homomorphisms between finitely generated projective R-modules. For each $f : P^{-1} \to P^0$ in Σ, we denote by $P_f^{\bullet}$ the following

complex of R-modules:

$$\cdots \longrightarrow 0 \longrightarrow P^{-1} \stackrel{f}{\longrightarrow} P^0 \longrightarrow 0 \longrightarrow \cdots,$$

where P^{-1} and P^0 are of degrees -1 and 0, respectively.

Set

$$\Sigma^\bullet := \{P_f^\bullet \mid f \in \Sigma\},$$
$$\Sigma^\perp := \{X \in R\text{-Mod} \mid \mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, X[i]) = 0 \text{ for all } P^\bullet \in \Sigma^\bullet \text{ and all } i \in \mathbb{Z}\},$$
$$\mathscr{D}(R)_{\Sigma^\perp} := \{Y^\bullet \in \mathscr{D}(R) \mid H^n(Y^\bullet) \in \Sigma^\perp \text{ for all } n \in \mathbb{Z}\},$$

where $H^n(Y^\bullet)$ is the nth cohomology of the complex $Y^\bullet$. Note that some special cases of $\Sigma^\perp$ have been discussed in the literature (see, for example, [1, 4, 21, 26]). For example, the set Σ consists of injective homomorphisms or only one single homomorphism. In some papers, such a category $\Sigma^\perp$ is called the perpendicular category of Σ.

Universal localizations were pioneered by Ore and Cohn, in order to study embedding of noncommutative rings in skew fields.

Before recalling the definition of universal localizations, we mention the following result, due initially to Cohn (see also [38]), which explains how universal localizations arise.

THEOREM 3.1 [20]. *Let R and Σ be as above. Then there is a ring R_Σ and a homomorphism $\lambda : R \to R_\Sigma$ of rings with the following properties:*

(1) *λ is Σ-inverting, that is, if $\alpha : P \to Q$ belongs to Σ, then $R_\Sigma \otimes_R \alpha : R_\Sigma \otimes_R P \to R_\Sigma \otimes_R Q$ is an isomorphism of R_Σ-modules, and*

(2) *λ is universal Σ-inverting, that is, if S is a ring such that there exists a Σ-inverting homomorphism $\varphi : R \to S$, then there exists a unique homomorphism $\psi : R_\Sigma \to S$ of rings such that $\varphi = \lambda\psi$.*

The homomorphism $\lambda : R \to R_\Sigma$ in Theorem 3.1 is a ring epimorphism with $\mathrm{Tor}_1^R(R_\Sigma, R_\Sigma) = 0$. It is called the universal localization of R at Σ.

The left–right symmetry holds for universal localizations (see [38, Chapter 4, pp. 51–52]).

LEMMA 3.2. *Let R_Σ be the universal localization of R at Σ in Theorem 3.1, and let $\Gamma := \{\mathrm{Hom}_R(f, R) \mid f \in \Sigma\}$ which is a set of homomorphisms between finitely generated projective right R-modules. Then R_Σ is isomorphic to the universal localization of R at Γ.*

The proof of this lemma is actually a consequence of the following two observations: (a) For any finitely generated projective R-module P, we have a natural isomorphism: $\mathrm{Hom}_R(P, R) \otimes_R - \simeq \mathrm{Hom}_R(P, -)$, and (b) the functor $\mathrm{Hom}_R(-, {}_R R) : \mathrm{add}({}_R R) \to \mathrm{add}(R_R)$ defines an equivalence of categories.

It is easy to see that if R has weak dimension at most 1, then the localization $\lambda : R \to R_\Sigma$ of R at any set Σ is homological, and moreover, the weak dimension of R_Σ is also at most 1 by Lemma 2.2.

If Σ is a finite set, then we may assume that Σ contains only one homomorphism since the universal localization at Σ is the same as the universal localization at the direct sum of the homomorphisms in Σ.

The following result is a more general formulation of the case discussed in [1, 4]. Nevertheless, many arguments of the proof there work in this general situation. We outline here a modified proof.

PROPOSITION 3.3. (1) $\Sigma^\perp$ *is closed under isomorphic images, extensions, kernels, cokernels, direct sums and products.*

(2) $\Sigma^\perp$ *coincides with the image of the restriction functor* $\lambda_* : R_\Sigma\text{-Mod} \to R\text{-Mod}$ *induced by the ring homomorphism* λ *defined in Theorem 3.1. In this sense, we can identify* $\Sigma^\perp$ *with* $R_\Sigma\text{-Mod}$ *via* λ.

(3) $\mathscr{D}(R)_{\Sigma^\perp} = \mathrm{Ker}(\mathrm{Hom}_{\mathscr{D}(R)}(\mathrm{Tria}(\Sigma^\bullet), -))$.

In order to prove Proposition 3.3, we need the following known homological result.

LEMMA 3.4. *Suppose that* $W^\bullet = (W^i)_{i\in\mathbb{Z}}$ *is a complex in* $\mathscr{C}(R\text{-Proj})$ *such that* $W^i = 0$, *for all* $i \in \mathbb{Z}\backslash\{-1, 0\}$. *Then, for each* $X^\bullet \in \mathscr{D}(R)$ *and* $n \in \mathbb{Z}$, *there is an exact sequence of abelian groups:*

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(W^\bullet, H^{n-1}(X^\bullet)[1]) \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(W^\bullet, X^\bullet[n])$$
$$\longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(W^\bullet, H^n(X^\bullet)) \longrightarrow 0.$$

Proof. It is sufficient to show the statement for $n = 0$. In this case, it follows from the triangle $W^{-1} \to W^0 \to W^\bullet \to W^{-1}[1]$ that the following diagram is commutative and exact:

$$\mathrm{Hom}_{\mathscr{K}(R)}(W^0[1], X^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{K}(R)}(W^{-1}[1], X^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{K}(R)}(W^\bullet, X^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{K}(R)}(W^0, X^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{K}(R)}(W^{-1}, X^\bullet)$$

$$\downarrow\simeq \qquad\qquad \downarrow\simeq \qquad\qquad\qquad\qquad\qquad \simeq\downarrow \qquad\qquad \simeq\downarrow$$

$$\mathrm{Hom}_{\mathscr{K}(R)}(W^0, H^{-1}(X^\bullet)) \longrightarrow \mathrm{Hom}_{\mathscr{K}(R)}(W^{-1}, H^{-1}(X^\bullet)) \qquad\qquad \mathrm{Hom}_{\mathscr{K}(R)}(W^0, H^0(X^\bullet)) \longrightarrow \mathrm{Hom}_{\mathscr{K}(R)}(W^{-1}, H^0(X^\bullet)).$$

Here, we use the fact that $\mathrm{Hom}_{\mathscr{D}(R)}(P, X^\bullet[n]) = \mathrm{Hom}_{\mathscr{K}(R)}(P, X^\bullet[n]) \simeq \mathrm{Hom}_R(P, H^n(X^\bullet))$ for every projective module P and $n \in \mathbb{Z}$. Thus, Lemma 3.4 follows. $\square$

Proof of Proposition 3.3. (1) Clearly, $\Sigma^\perp$ is closed under isomorphic images and extensions. In the following, we shall prove that $\Sigma^\perp$ is closed under kernels and cokernels. Recall that $\Sigma^\perp$ is defined to be the full subcategory of $R\text{-Mod}$ consisting of those R-modules X that $\mathrm{Hom}_{\mathscr{D}(R)}(U^\bullet, X) = \mathrm{Hom}_{\mathscr{D}(R)}(U^\bullet, X[1]) = 0$ for all $U^\bullet \in \Sigma^\bullet$. Suppose that $f : Y \to Z$ is a homomorphism between two modules Y and Z in $\Sigma^\perp$. Set $K := \mathrm{Ker}(f), I := \mathrm{Im}(f)$ and $C := \mathrm{Coker}(f)$. Then we have two exact sequences of R-modules:

$$0 \longrightarrow K \longrightarrow Y \longrightarrow I \longrightarrow 0 \quad\text{and}\quad 0 \longrightarrow I \longrightarrow Z \longrightarrow C \longrightarrow 0.$$

Since every short exact sequence in $R\text{-Mod}$ can canonically be extended to a triangle in $\mathscr{D}(R)$, we get two triangles in $\mathscr{D}(R)$:

$$K \longrightarrow Y \longrightarrow I \longrightarrow K[1] \quad\text{and}\quad I \longrightarrow Z \longrightarrow C \longrightarrow I[1].$$

For convenience, we will write $_{\mathscr{D}(R)}(X^\bullet, Y^\bullet)$ for the Hom-set $\mathrm{Hom}_{\mathscr{D}(R)}(X^\bullet, Y^\bullet)$, with $X^\bullet, Y^\bullet \in \mathscr{D}(R)$. Let $P^\bullet \in \Sigma^\bullet$. Then, by applying $_{\mathscr{D}(R)}(P^\bullet, -)$ to these triangles, we obtain two long exact sequences of abelian groups

$$0 \longrightarrow {}_{\mathscr{D}(R)}(P^\bullet, K) \longrightarrow {}_{\mathscr{D}(R)}(P^\bullet, Y) \longrightarrow {}_{\mathscr{D}(R)}(P^\bullet, I) \longrightarrow {}_{\mathscr{D}(R)}(P^\bullet, K[1])$$
$$\longrightarrow {}_{\mathscr{D}(R)}(P^\bullet, Y[1]) \longrightarrow {}_{\mathscr{D}(R)}(P^\bullet, I[1]) \longrightarrow 0;$$
$$0 \longrightarrow {}_{\mathscr{D}(R)}(P^\bullet, I) \longrightarrow {}_{\mathscr{D}(R)}(P^\bullet, Z) \longrightarrow {}_{\mathscr{D}(R)}(P^\bullet, C) \longrightarrow {}_{\mathscr{D}(R)}(P^\bullet, I[1])$$
$$\longrightarrow {}_{\mathscr{D}(R)}(P^\bullet, Z[1]) \longrightarrow {}_{\mathscr{D}(R)}(P^\bullet, C[1]) \longrightarrow 0.$$

Since Y and Z lie in $\Sigma^\perp$, we know $_{\mathscr{D}(R)}(P^\bullet, Y) = {}_{\mathscr{D}(R)}(P^\bullet, Z) = {}_{\mathscr{D}(R)}(P^\bullet, Y[1]) = {}_{\mathscr{D}(R)}(P^\bullet, Z[1]) = 0$. It follows that $_{\mathscr{D}(R)}(P^\bullet, K) = {}_{\mathscr{D}(R)}(P^\bullet, I) = 0$, and so $_{\mathscr{D}(R)}(P^\bullet, K[1]) = 0$. This implies $K \in \Sigma^\perp$. Similarly, we can conclude that I and C belong to $\Sigma^\perp$. Hence $\Sigma^\perp$ is

closed under kernels, images and cokernels. By the definition of $\Sigma^\perp$ and the fact that Hom-functors commute with products, we infer that $\Sigma^\perp$ is closed under products. Since $\Sigma^\bullet$ is a set of bounded complexes over finitely generated projective R-modules, these complexes are compact, and therefore $\Sigma^\perp$ is closed under direct sums.

(2) Observe that, for each element $f : P^{-1} \to P^0$ in Σ, there is a canonical triangle in $\mathscr{D}(R)$:

$$(*) \quad P^{-1} \xrightarrow{f} P^0 \longrightarrow P_f^\bullet \longrightarrow P^{-1}[1].$$

If, in addition, f is injective, then we have a short exact sequence of R-modules:

$$(**) \quad 0 \longrightarrow P^{-1} \xrightarrow{f} P^0 \longrightarrow \mathrm{Coker}(f) \longrightarrow 0.$$

In this case, we get $P_f^\bullet \simeq \mathrm{Coker}(f)$ in $\mathscr{D}(R)$. Note that the same statement as (2) is obtained in [**1**, Lemma 1.6, Proposition 1.7] under the extra assumption that each element in Σ is injective, where the sequence $(**)$ is used. In fact, this assumption is not necessary since we can replace $(**)$ by $(*)$ and modify the proof there to show the general case. For more details, we refer the reader to [**1**].

(3) This follows directly from Lemmas 2.8 and 3.4. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Combining Lemma 2.1 with Proposition 3.3, we have the following result, which says that, in some sense, Morita equivalences preserve universal localizations.

COROLLARY 3.5.   *Let* $\lambda : R \to R_\Sigma$ *be the universal localization of the ring R at the set Σ. Suppose that P is a finitely generated projective generator for R-Mod. Set $\Delta := \{\mathrm{Hom}_R(P, f) \mid f \in \Sigma\}$. Then the ring homomorphism $\mu : \mathrm{End}_R(P) \to \mathrm{End}_{R_\Sigma}(R_\Sigma \otimes_R P)$, defined by $g \mapsto R_\Sigma \otimes_R g$ for any $g \in \mathrm{End}_R(P)$, is the universal localization of the ring $\mathrm{End}_R(P)$ at the set Δ.*

Proof.   Let $S := \mathrm{End}_R(P)$. Since $_RP$ is a finitely generated projective generator for R-Mod, the Hom-functor $\mathrm{Hom}_R(P, -) : R\text{-Mod} \to S\text{-Mod}$ is an equivalence, which extends to a triangle equivalence between $\mathscr{D}(R)$ and $\mathscr{D}(S)$. By the definitions of $\Sigma^\perp$ and $\Delta^\perp$, the restriction of $\mathrm{Hom}_R(P, -)$ induces an equivalence from $\Sigma^\perp$ to $\Delta^\perp$. Note that $R_\Sigma \otimes_R P$ is a finitely generated projective generator for R_Σ-Mod. Since the functor $\lambda_* : R_\Sigma\text{-Mod} \to R\text{-Mod}$ is fully faithful and since the image of λ_* coincides with $\Sigma^\perp$ by Proposition 3.3(2), it follows from the following commutative diagram of functors:

$$
\begin{array}{ccc}
R_\Sigma\text{-Mod} & \xrightarrow[\simeq]{\mathrm{Hom}_{R_\Sigma}(R_\Sigma \otimes_R P, -)} & \mathrm{End}_{R_\Sigma}(R_\Sigma \otimes_R P)\text{-Mod} \\
\lambda_* \downarrow & & \downarrow \mu_* \\
R\text{-Mod} & \xrightarrow[\simeq]{\mathrm{Hom}_R(P, -)} & S\text{-Mod}
\end{array}
$$

that μ_* is fully faithful, and that the image of μ_* coincides with $\Delta^\perp$. This implies also that μ is a ring epimorphism. Note that, under our conventions, full subcategories are always closed under isomorphic images.

On the other hand, if $\varphi : S \to S_\Delta$ is the universal localization of S at Δ, then, by Proposition 3.3(2), the image of φ_* coincides with $\Delta^\perp$. Thus, the two ring epimorphisms μ and φ are equivalent by Lemma 2.1. This means that the two rings S_Δ and $\mathrm{End}_{R_\Sigma}(R_\Sigma \otimes_R P)$ are isomorphic. Thus, μ is the universal localization of S at Δ. $\qquad\qquad\qquad\square$

Motivated by Neeman and Ranicki [**34**, Theorem 0.7 and Proposition 5.6], see also [**4**, Theorem 4.8 (3)], we shall establish the following connection between universal localizations and recollements of triangulated categories. The last condition (5) of Proposition 3.6 below seems to appear for the first time in the work, and will be used in Section 4 to prove Theorem 1.1(1).

PROPOSITION 3.6. *Let $\lambda : R \to R_\Sigma$ be the universal localization of R at Σ.*

(a) *Let $\mathbf{j}$ be the canonical embedding of $\mathscr{D}(R)_{\Sigma^\perp}$ into $\mathscr{D}(R)$. Then there is a recollement*

$$\mathscr{D}(R)_{\Sigma^\perp} \xrightarrow{\ \ \mathbf{j}\ \ } \mathscr{D}(R) \longrightarrow \mathrm{Tria}(\Sigma^\bullet)$$

such that $\mathbf{L}$ is the left adjoint of $\mathbf{j}$ and $T^\bullet := \mathbf{L}(R)$ is a compact generator of $\mathscr{D}(R)_{\Sigma^\perp}$.

(b) *The following statements are equivalent:*

(1) *$\lambda : R \to R_\Sigma$ is a homological epimorphism of rings;*
(2) *$\lambda_* : \mathscr{D}(R_\Sigma) \xrightarrow{\sim} \mathscr{D}(R)_{\Sigma^\perp}$;*
(3) *the complex $T^\bullet$ in (a) is a tilting object in $\mathscr{D}(R)_{\Sigma^\perp}$;*
(4) *the complex $T^\bullet$ in (a) is isomorphic to R_Σ in $\mathscr{D}(R)$;*
(5) *the complex $T^\bullet$ in (a) is isomorphic in $\mathscr{D}(R)$ to a complex $X^\bullet := (X^i)_{i \in \mathbb{Z}}$ such that $X^i \in \Sigma^\perp$ for all $i \in \mathbb{Z}$.*

Proof. The existence of the above recollement is an immediate consequence of Lemmas 2.6(2), 2.8 and Proposition 3.3. The property in (a) follows from the proof in [**14**, Chapter IV, Proposition 1.1]. As to the property (b), we note that the equivalences among the first four statements in (b) can be deduced from [**4**, Proposition 1.7, Lemma 4.6]. Clearly, the statement (4) implies the statement (5). We shall show that (5) implies (4).

Let $\lambda : R \to R_\Sigma$ be the universal localization of R at Σ. In what follows, we always identify $\Sigma^\perp$ with R_Σ-Mod via λ. This is due to Proposition 3.3(2).

Suppose that $T^\bullet \simeq X^\bullet := (X^i)_{i \in \mathbb{Z}}$ in $\mathscr{D}(R)$ such that $X^i \in R_\Sigma$-Mod for all $i \in \mathbb{Z}$. Since λ is a ring epimorphism, we get $\mathrm{Hom}_{R_\Sigma}(X, Y) \simeq \mathrm{Hom}_R(X, Y)$ for all $X, Y \in R_\Sigma$-Mod. Thus, $X^\bullet$ can be considered as a complex over R_Σ-Mod, that is, $X^\bullet \in \mathscr{C}(R_\Sigma)$. Let λ_1 be the map $\mathrm{Hom}_{\mathscr{D}(R)}(\lambda, X^\bullet) : \mathrm{Hom}_{\mathscr{D}(R)}(R_\Sigma, X^\bullet) \to \mathrm{Hom}_{\mathscr{D}(R)}(R, X^\bullet)$. We claim that λ_1 is surjective. In fact, there is a commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathscr{K}(R)}(R_\Sigma, X^\bullet) & \xrightarrow{\ q_1\ } & \mathrm{Hom}_{\mathscr{D}(R)}(R_\Sigma, X^\bullet) \\
\downarrow{\lambda_2} & & \downarrow{\lambda_1} \\
\mathrm{Hom}_{\mathscr{K}(R)}(R, X^\bullet) & \xrightarrow{\ q_2\ } & \mathrm{Hom}_{\mathscr{D}(R)}(R, X^\bullet)
\end{array}$$

where $\lambda_2 = \mathrm{Hom}_{\mathscr{K}(R)}(\lambda, X^\bullet)$, and where q_1 and q_2 are induced by the localization functor $q : \mathscr{K}(R) \to \mathscr{D}(R)$. Clearly, q_2 is a bijection. To prove that λ_1 is surjective, it suffices to show that λ_2 is bijective. Indeed, λ_2 is a composition of the following series of isomorphisms:

$$\mathrm{Hom}_{\mathscr{K}(R)}(R_\Sigma, X^\bullet) \simeq H^0(\mathrm{Hom}_R(R_\Sigma, X^\bullet)) = H^0(\mathrm{Hom}_{R_\Sigma}(R_\Sigma, X^\bullet))$$
$$\simeq \mathrm{Hom}_{\mathscr{K}(R)}(R, X^\bullet),$$

where the equality follows from the fact that λ is a ring epimorphism. More precisely, for $\bar{f}^\bullet := \overline{(f^i)} \in \mathrm{Hom}_{\mathscr{K}(R)}(R_\Sigma, X^\bullet)$ with $(f^i)_{i \in \mathbb{Z}}$ a chain map, the series of the above maps are defined by

$$\overline{(f^i)} \longmapsto \overline{f^0} = \overline{f^0} \longmapsto \overline{\lambda_*(f^\bullet)},$$

where $\lambda_*(f^\bullet)$ is a chain map from R to $X^\bullet$ with λf^0 in degree 0 and zero in all other degrees. Thus λ_2 is bijective. This implies that λ_1 is surjective. Now, let λ' be the map $\mathrm{Hom}_{\mathscr{D}(R)}(\lambda, T^\bullet) : \mathrm{Hom}_{\mathscr{D}(R)}(R_\Sigma, T^\bullet) \to \mathrm{Hom}_{\mathscr{D}(R)}(R, T^\bullet)$. Since $T^\bullet \simeq X^\bullet$ in $\mathscr{D}(R)$, we know that λ' is also surjective. Suppose that $\varphi : R \to T^\bullet := \mathbf{L}(R)$ is the unit adjunction morphism with respect to the adjoint pair $(\mathbf{L}, \mathbf{j})$. Then there exists $g : R_\Sigma \to T^\bullet$ in $\mathscr{D}(R)$ such that $\varphi = \lambda g$. Since R_Σ belongs to $\mathscr{D}(R)_{\Sigma^\perp}$, there exists $f : T^\bullet \to R_\Sigma$ in $\mathscr{D}(R)$ such that $\lambda = \varphi f$. This gives

rise to the following commutative diagram in $\mathscr{D}(R)$:

$$
\begin{array}{ccccc}
R & = & R & = & R \\
\varphi \downarrow & & \lambda \downarrow & & \downarrow \varphi \\
T^\bullet & \overset{f}{\dashrightarrow} R_\Sigma & & \overset{g}{\dashrightarrow} & T^\bullet.
\end{array}
$$

Consequently, $\varphi = \varphi fg$ and $\lambda = \lambda gf$. On the one hand, since φ is the unit adjunction morphism, we have $fg = 1_{T^\bullet}$. On the other hand, it follows from [**1**, Theorem 1.4] that λ is an R_Σ-Mod-reflection of R, that is, the morphism of abelian groups $\operatorname{Hom}_R(\lambda, Z) : \operatorname{Hom}_R(R_\Sigma, Z) \to \operatorname{Hom}_R(R, Z)$ is bijective, for any $Z \in R_\Sigma$-Mod. This yields $gf = 1_{R_\Sigma}$. Thus, f is an isomorphism. In other words, $T^\bullet \simeq R_\Sigma$ in $\mathscr{D}(R)$. Therefore, (5) implies (4). $\qquad\square$

REMARK. Note that every tilting module is associated to a class of finitely presented modules of projective dimension at most 1 (see [**1**, **10**]) and thus to a universal localization since each finitely presented module of projective dimension at most 1 is determined by an injective homomorphism between finitely generated projective modules. In Proposition 3.6, we do not require that each homomorphism in Σ is injective. From this point of view, the formulation of Proposition 3.6(b) seems to be more general than that in [**4**, Theorem 4.8(3)].

COROLLARY 3.7. *Let $R \subseteq S$ be an extension of rings, that is, R is a subring of the ring S with the same identity, and let B be the endomorphism ring of the R-module $S \oplus S/R$. Then there is a recollement of triangulated categories:*

$$
\mathscr{D}(B)_{\Sigma^\perp} \longrightarrow \mathscr{D}(B) \longrightarrow \mathscr{D}(R)
$$

where $\Sigma := \{\pi^*\}$, and the homomorphism $\pi^* : \operatorname{Hom}_R(S \oplus S/R, S) \to \operatorname{Hom}_R(S \oplus S/R, S/R)$ of B-modules is defined by $f \mapsto f\pi$, for any $f \in \operatorname{Hom}_R(S \oplus S/R, S)$, which is induced by the canonical map $\pi : S \to S/R$.

Proof. It follows from Proposition 3.6(a) that we have the following recollement:

$$
\mathscr{D}(B)_{\Sigma^\perp} \longrightarrow \mathscr{D}(B) \longrightarrow \operatorname{Tria}(\Sigma^\bullet)
$$

To show that $\operatorname{Tria}(\Sigma^\bullet)$ is equivalent to $\mathscr{D}(R)$ as triangulated categories, it suffices to prove that the complex $\Sigma^\bullet \in \mathscr{K}^{\mathrm{b}}(B\text{-proj})$ is exceptional with $\operatorname{End}_{\mathscr{D}(B)}(\Sigma^\bullet) \simeq R$.

In fact, let $_R T := S \oplus S/R$ and $B := \operatorname{End}_R(T)$. Then $\operatorname{add}(_R T)$ and B-proj are equivalent, and therefore $\mathscr{K}^{\mathrm{b}}(\operatorname{add}(_R T))$ and $\mathscr{K}^{\mathrm{b}}(B\text{-proj})$ are equivalent as triangulated categories via the functor $\operatorname{Hom}_R(T, -)$. Thus, to show that the complex $\Sigma^\bullet \in \mathscr{K}^{\mathrm{b}}(B\text{-proj})$ is exceptional with $\operatorname{End}_{\mathscr{D}(B)}(\Sigma^\bullet) \simeq R$, it is sufficient to show that the complex

$$
\Pi^\bullet : \quad 0 \longrightarrow S \overset{\pi}{\longrightarrow} S/R \longrightarrow 0
$$

in $\mathscr{K}^{\mathrm{b}}(\operatorname{add}(T))$ is exceptional with $\operatorname{End}_{\mathscr{K}^{\mathrm{b}}(\operatorname{add}(T))}(\Pi^\bullet) \simeq R$ since $\operatorname{Hom}_R(T, \Pi^\bullet) = \Sigma^\bullet$.

It is easy to see $\operatorname{Hom}_{\mathscr{K}^{\mathrm{b}}(\operatorname{add}(T))}(\Pi^\bullet, \Pi^\bullet[-1]) = 0$. To show $\operatorname{Hom}_{\mathscr{K}^{\mathrm{b}}(\operatorname{add}(T))}(\Pi^\bullet, \Pi^\bullet[1]) = 0$, we pick up a homomorphism $f : S \to S/R$ of R-modules, suppose $(1)f = s + R \in S/R$ and define $g : _R S \to _R S$ by $x \mapsto xs$ for $x \in S$. Clearly, g is a homomorphism of R-modules and $(f - g)|_R = 0$. Thus, there exists a homomorphism $h : S/R \to S/R$ such that $f - g = \pi h$. This implies that f is zero in $\mathscr{K}^{\mathrm{b}}(\operatorname{add}(T))$, that is, $\operatorname{Hom}_{\mathscr{K}^{\mathrm{b}}(\operatorname{add}(T))}(\Pi^\bullet, \Pi^\bullet[1]) = 0$. Hence we have shown that $\Pi^\bullet$ is exceptional.

Now, we define a ring homomorphism α from $\operatorname{End}_{\mathscr{K}^{\mathrm{b}}(\operatorname{add}(T))}(\Pi^\bullet)$ to R as follows: Given $f = (f^0, f^1) \in \operatorname{End}_{\mathscr{K}^{\mathrm{b}}(\operatorname{add}(T))}(\Pi^\bullet)$, let $(f)\alpha$ be the unique map determined by the following exact commutative diagram of R-modules:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{\ \lambda\ } & S & \xrightarrow{\ \pi\ } & S/R & \longrightarrow & 0 \\
 & & {\scriptstyle (f)\alpha}\big\downarrow & & {\scriptstyle f^0}\big\downarrow & & {\scriptstyle f^1}\big\downarrow & & \\
0 & \longrightarrow & R & \xrightarrow{\ \lambda\ } & S & \xrightarrow{\ \pi\ } & S/R & \longrightarrow & 0
\end{array}
$$

Note that if f is null-homotopic, then $(f)\alpha$ is zero. This means that α is well defined. Clearly, α is a ring homomorphism. We claim that α is an isomorphism of rings. It is easy to check that α is injective. We shall show that α is surjective. Let $r \in R$. We define $f^0 : S \to S$ to be the right multiplication of r. Then there is a homomorphism $f^1 : S/R \to S/R$ of R-modules such that $f^0 \pi = \pi f^1$. This means that α is surjective. Hence α is an isomorphism of rings. So, $\Sigma^\bullet$ is compact and exceptional with $\operatorname{End}_{\mathscr{D}(B)}(\Sigma^\bullet) \simeq R$. Now, it follows from [**31**, Corollary 8.4, Theorem 8.5] that $\operatorname{Tria}(\Sigma^\bullet)$ is equivalent to $\mathscr{D}(R)$ as triangulated categories. This proves Corollary 3.7. $\qquad\square$

As another corollary of Proposition 3.6, we have the following result.

COROLLARY 3.8. *If the weak dimension of R is at most 1, then there is a recollement*

$$
\mathscr{D}(R_\Sigma) \rightleftarrows \mathscr{D}(R) \rightleftarrows \operatorname{Tria}(\Sigma^\bullet)
$$

where Σ is a set of homomorphisms between finitely generated projective R-modules.

Proof. Under the assumption, the universal localization map λ_Σ is trivially a homological ring epimorphism. So, this corollary follows from Proposition 3.6(b). $\qquad\square$

4. *Recollements of derived categories and infinitely generated tilting modules*

In this section, we shall use our results in Section 3 to show the first statement of the main result, Theorem 1.1. More precisely, we first recall the definition of infinitely generated tilting modules, and then discuss some of their homological properties. Especially, we shall establish a crucial result, Proposition 4.6, which will play a role in our proof of the main result.

Let A be a ring with identity.

DEFINITION 4.1 [**22**]. An A-module T is called a tilting module (of projective dimension at most one) if the following conditions are satisfied.

(T1) The projective dimension of T is at most 1, that is, there exists a projective resolution of T: $0 \to P_1 \to P_0 \to T \to 0$, where P_i is projective for $i = 0, 1$.

(T2) The module T has no self-extensions, that is, $\operatorname{Ext}^i_A(T, T^{(\alpha)}) = 0$ for each $i \geqslant 1$ and every cardinal α; where $T^{(\alpha)}$ stands for the direct sum of α copies of T.

(T3) There exists an exact sequence

$$
0 \longrightarrow {}_A A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0
$$

of A-modules such that $T_i \in \operatorname{Add}(T)$ for $i = 0, 1$.

If P_1 and P_0 in the condition $(T1)$ are finitely generated, then the tilting module T is called a classical tilting module (see [**16**, **28**]).

Two tilting A-modules T and T' are said to be equivalent if $\mathrm{Add}(T) = \mathrm{Add}(T')$, or equivalently, $\mathrm{Gen}(T) = \mathrm{Gen}(T')$, where $\mathrm{Gen}(T)$ denotes the full subcategory of A-Mod generated by T. Recall that an A-module M is generated by T, if there is an index set I and a surjective homomorphism $f : T^{(I)} \to M$.

An A-module T is said to be good if it satisfies $(T1)$, $(T2)$ and

$(T3)'$ there is an exact sequence

$$0 \longrightarrow {}_A A \longrightarrow T_0 \overset{\varphi}{\longrightarrow} T_1 \longrightarrow 0$$

in A-Mod such that $T_i \in \mathrm{add}(T)$ for $i = 0, 1$.

Note that each classical tilting module is good. Moreover, for any given tilting module ${}_A T$ as in Definition 4.1, the module $T' := T_0 \oplus T_1$ is a good tilting module which is equivalent to the given one.

From now on, we assume in this section that T is a *good* tilting A-module. Let $B := \mathrm{End}_A(T)$. We define

$$T^\perp := \{X \in A\text{-Mod} \mid \mathrm{Ext}^i_A(T, X) = 0 \text{ for all } i \geqslant 1\},$$
$$\mathscr{E} := \{Y \in B\text{-Mod} \mid \mathrm{Tor}^B_i(T, Y) = 0 \text{ for all } i \geqslant 0\};$$
$$G := {}_A T \otimes^{\mathbb{L}}_B - : \mathscr{D}(B) \longrightarrow \mathscr{D}(A), \quad H := \mathbb{R}\mathrm{Hom}_A(T, -) : \mathscr{D}(A) \longrightarrow \mathscr{D}(B);$$
$$\mathcal{Y} := \mathrm{Ker}(G), \qquad \mathcal{Z} := \mathrm{Im}(H),$$
$$Q^\bullet := \cdots \longrightarrow 0 \longrightarrow \mathrm{Hom}_A(T, T_0) \overset{\varphi^*}{\longrightarrow} \mathrm{Hom}_A(T, T_1) \longrightarrow 0 \longrightarrow \cdots \in \mathscr{C}^{\mathrm{b}}(B\text{-proj}),$$

where $\varphi^* := \mathrm{Hom}_A(T, \varphi)$, and where the finitely generated projective B-modules $\mathrm{Hom}_A(T, T_0)$ and $\mathrm{Hom}_A(T, T_1)$, as terms of the complex $Q^\bullet$, are of degrees 0 and 1, respectively. Clearly, $H(A) = Q^\bullet$ in $\mathscr{D}(B)$.

In the next lemma, we mention a few basic properties of tilting modules. For proofs, we refer to [**11**, Proposition 1.4, Lemma 1.5, **9**].

LEMMA 4.2. *Let T be a tilting A-module. Then, we have the following.*

(1) *The right B-module T_B has a projective resolution $0 \longrightarrow Q_1 \overset{\psi}{\longrightarrow} Q_0 \longrightarrow T_B \longrightarrow 0$ such that $Q_i \in \mathrm{add}(B_B)$ for $0 \leqslant i \leqslant 1$.*
(2) *For the right B-module T, we have $\mathrm{End}_{B^{\mathrm{op}}}(T) \simeq A^{\mathrm{op}}$ and $\mathrm{Ext}^i_{B^{\mathrm{op}}}(T, T) = 0$ for all $i \geqslant 1$.*
(3) *For each $Y \in \mathrm{Add}({}_B B)$, we have $\mathrm{Ext}^i_A(T, {}_A T \otimes_B Y) = 0$ for all $i \geqslant 1$.*
(4) *For each $X \in T^\perp$, we have $\mathrm{Tor}^B_i({}_A T_B, \mathrm{Hom}_A(T, X)) \simeq \begin{cases} X, & i=0, \\ 0, & i>0. \end{cases}$*
(5) *The full subcategory $T^\perp$ is closed under direct sums.*

The following result is shown in [**9**, Theorem 5.1], which says that the unbounded derived category of B-Mod is bigger than that of A-Mod in general.

THEOREM 4.3. *The functor H is fully faithful, and the functor G induces a triangle equivalence between $\mathscr{D}(B)/\mathrm{Ker}(G)$ and $\mathscr{D}(A)$. Here, we denote by $\mathscr{D}(B)/\mathrm{Ker}(G)$ the Verdier quotient of $\mathscr{D}(B)$ by the subcategory $\mathrm{Ker}(G)$.*

The following lemma supplies a method to obtain modules in $\mathscr{E}$, and is also useful for our later calculations.

LEMMA 4.4. *Suppose that I is a cardinal and $X_i \in T^{\perp}$ for each $i \in I$. Consider the canonical exact sequence*

$$0 \longrightarrow \bigoplus_{i \in I} \operatorname{Hom}_A(T, X_i) \xrightarrow{\delta_I} \operatorname{Hom}_A\left(T, \bigoplus_{j \in I} X_j\right) \longrightarrow \operatorname{Coker}(\delta_I) \longrightarrow 0$$

in B-Mod, where δ_I is defined by $(f_i)_{i \in I} \mapsto \sum_{i \in I} f_i \lambda_i$ with $f_i \in \operatorname{Hom}_A(T, X_i)$ and $\lambda_i : X_i \to \bigoplus_{j \in I} X_j$ the canonical inclusion for each $i \in I$. Then $\operatorname{Coker}(\delta_I) \in \mathscr{E}$. Particularly, for each projective B-module P, the unit adjunction morphism $\eta'_P : P \to \operatorname{Hom}_A(T, T \otimes_B P)$ is injective with $\operatorname{Coker}(\eta'_P) \in \mathscr{E}$.

Proof. Note that δ_I is well defined. By the definition of δ_I, we can see easily that δ_I is injective. So, there is a canonical exact sequence

$$(*) \quad 0 \longrightarrow \bigoplus_{i \in I} \operatorname{Hom}_A(T, X_i) \xrightarrow{\delta_I} \operatorname{Hom}_A\left(T, \bigoplus_{j \in I} X_j\right) \longrightarrow \operatorname{Coker}(\delta_I) \longrightarrow 0.$$

Since $T^{\perp}$ is closed under direct sums by Lemma 4.2(5), we have $\bigoplus_{j \in I} X_j \in T^{\perp}$. It then follows from Lemma 4.2(4) that

$$\operatorname{Tor}^B_m\left(T, \operatorname{Hom}_A\left(T, \bigoplus_{j \in I} X_j\right)\right) \simeq \begin{cases} \bigoplus_{j \in I} X_j, & m = 0, \\ 0, & m > 0. \end{cases}$$

Similarly, for any $i \in I$, we have

$$\operatorname{Tor}^B_n(T, \operatorname{Hom}_A(T, X_i)) \simeq \begin{cases} X_i, & n = 0, \\ 0, & n > 0. \end{cases}$$

Since the right module T_B has a projective dimension at most 1, we see that $\operatorname{Tor}^B_t(T, \operatorname{Coker}(\delta_I)) = 0$, for any $t > 1$. By applying the functor $_A T \otimes_B - : B\text{-Mod} \to A\text{-Mod}$ to the sequence $(*)$, we can easily form the following exact commutative diagram:

$$0 \twoheadrightarrow \operatorname{Tor}^B_1(T, \operatorname{Coker}(\delta_I)) \twoheadrightarrow T \otimes_B (\bigoplus_{i \in I} \operatorname{Hom}_A(T, X_i)) \twoheadrightarrow T \otimes_B \operatorname{Hom}_A(T, \bigoplus_{j \in I} X_j) \twoheadrightarrow T \otimes_B \operatorname{Coker}(\delta_I) \twoheadrightarrow 0$$

$$\downarrow \simeq \qquad\qquad\qquad\qquad \downarrow \simeq$$

$$\bigoplus_{i \in I} X_i \,=\!=\!=\!=\!=\!=\!=\!=\!=\, \bigoplus_{j \in I} X_j$$

This implies $T \otimes_B \operatorname{Coker}(\delta_I) = 0 = \operatorname{Tor}^B_1(T, \operatorname{Coker}(\delta_I))$. Hence $\operatorname{Coker}(\delta_I) \in \mathscr{E}$.

To prove the last statement of Lemma 4.4, we note that the unit adjunction

$$\eta' : 1_{B\text{-Mod}} \longrightarrow \operatorname{Hom}_A(T, T \otimes_B -)$$

is a natural transformation of functors from B-Mod to itself, and that $\mathscr{E}$ is closed under direct summands. Thus, it is sufficient to show that the statement holds for free B-modules. Let α be any cardinal. Then we may form the following exact commutative diagram:

$$\begin{array}{ccc} B^{(\alpha)} & \xrightarrow{\eta'_{B^{(\alpha)}}} & \operatorname{Hom}_A(T, T \otimes_B B^{(\alpha)}) \\ \| & & \downarrow \simeq \\ 0 \longrightarrow \operatorname{Hom}_A(T, T)^{(\alpha)} & \xrightarrow{\delta_\alpha} & \operatorname{Hom}_A(T, T^{(\alpha)}) \longrightarrow \operatorname{Coker}(\delta_\alpha) \longrightarrow 0 \end{array}$$

Since δ_α is injective, we conclude that $\eta'_{B^{(\alpha)}}$ also is injective, and therefore $\operatorname{Coker}(\eta'_{B^{(\alpha)}}) \simeq \operatorname{Coker}(\delta_\alpha) \in \mathscr{E}$. This finishes the whole proof. $\square$

In the next lemma, we give a description of the category $\mathscr{E}$.

LEMMA 4.5. *The following statements hold.*

(1) *The category* $\mathscr{E} = \{X \in B\text{-Mod} \mid \operatorname{Hom}_{\mathscr{D}(B)}(Q^{\bullet}, X[i]) = 0 \text{ for all } i \in \mathbb{Z}\}$. *In particular,* $\mathscr{E}$ *is closed under direct sums and products.*

(2) *The category* $\mathscr{E}$ *is closed under isomorphic images, extensions, kernels and cokernels. In particular,* $\mathscr{E}$ *is an abelian subcategory of* B*-Mod.*

Proof. (1) Let X be a B-module and i an integer. Then

$$\operatorname{Hom}_{\mathscr{D}(B)}(Q^{\bullet}, X[i]) \simeq \operatorname{Hom}_{\mathscr{K}(B)}(Q^{\bullet}, X[i]) \simeq H^{i}(\operatorname{Hom}_{B}(Q^{\bullet}, X))$$
$$\simeq H^{i}(\operatorname{Hom}_{B}(Q^{\bullet}, B) \otimes_{B} X),$$

where the last isomorphism follows from the fact that the restriction of the natural transformation $\operatorname{Hom}_{B}(-, B) \otimes_{B} X \to \operatorname{Hom}_{B}(-, X)$ to $\mathscr{C}(B\text{-proj})$ is a natural isomorphism. By the definition of $Q^{\bullet}$, we know that $\operatorname{Hom}_{B}(Q^{\bullet}, B)$ is the complex:

$$\cdots \longrightarrow 0 \longrightarrow \operatorname{Hom}_{A}(T_{1}, T) \xrightarrow{\varphi_{*}} \operatorname{Hom}_{A}(T_{0}, T) \longrightarrow 0 \longrightarrow \cdots$$

in $\mathscr{C}^{\mathrm{b}}(B^{\mathrm{op}}\text{-proj})$, where $\varphi_{*} := \operatorname{Hom}_{A}(\varphi, T)$, and where the finitely generated projective B^{op}-modules $\operatorname{Hom}_{A}(T_{1}, T)$ and $\operatorname{Hom}_{A}(T_{0}, T)$ are of degrees -1 and 0, respectively. Note that the conditions (T_{2}) and (T_{3}) in Definition 4.1 imply that the sequence

$$0 \longrightarrow \operatorname{Hom}_{A}(T_{1}, T) \xrightarrow{\varphi_{*}} \operatorname{Hom}_{A}(T_{0}, T) \longrightarrow T \longrightarrow 0$$

is exact. In other words, the complex $\operatorname{Hom}_{B}(Q^{\bullet}, {}_{B}B)$ is quasi-isomorphic to T_{B}. Here, we use the fact that the functor $\operatorname{Hom}_{A}(-, T) : \operatorname{add}({}_{A}T) \to \operatorname{add}(B_{B})$ is an equivalence of categories. It follows from the definition of $\operatorname{Tor}_{i}^{B}$ that

$$H^{i}(\operatorname{Hom}_{B}(Q^{\bullet}, B) \otimes_{B} X) \simeq \begin{cases} 0 & \text{if } i > 0, \\ \operatorname{Tor}_{-i}^{B}(T, X) & \text{if } i \leqslant 0. \end{cases}$$

This means that $\operatorname{Hom}_{\mathscr{D}(B)}(Q^{\bullet}, X[i]) = 0$ if and only if $\operatorname{Tor}_{-i}^{B}(T, X) = 0$. Hence

$$\mathscr{E} = \{X \in B\text{-Mod} \mid \operatorname{Hom}_{\mathscr{D}(B)}(Q^{\bullet}, X[i]) = 0 \text{ for all } i \in \mathbb{Z}\}.$$

Consequently, $\mathscr{E}$ is closed under direct products. Further, since $Q^{\bullet}$ is a bounded complex of finitely generated projective B-modules, we know that $\mathscr{E}$ is closed under direct sums, too.

(2) This statement follows directly from Proposition 3.3(1). $\square$

The following proposition is crucial to the proof of Theorem 1.1(1).

PROPOSITION 4.6. *The triple* $(\operatorname{Tria}(Q^{\bullet}), \operatorname{Ker}(G), \operatorname{Im}(H))$ *is a TTF triple in* $\mathscr{D}(B)$. *Moreover,*

$$\operatorname{Ker}(G) = \{\overline{Y}^{\bullet} \in \mathscr{D}(B) \mid \overline{Y}^{\bullet} \simeq Y^{\bullet} \text{ in } \mathscr{D}(B) \text{ with } Y^{i} \in \mathscr{E} \text{ for all } i \in \mathbb{Z}\};$$
$$\operatorname{Im}(H) = \{\overline{Z}^{\bullet} \in \mathscr{D}(B) \mid \overline{Z}^{\bullet} \simeq Z^{\bullet} \text{ in } \mathscr{D}(B) \text{ with } Z^{i} \in \operatorname{Hom}_{A}(T, \operatorname{Add}(T)) \text{ for all } i \in \mathbb{Z}\},$$

where $\operatorname{Hom}_{A}(T, \operatorname{Add}(T))$ *stands for the full subcategory of* B*-Mod consisting of all the modules* $\operatorname{Hom}_{A}(T, T')$ *with* T' *in* $\operatorname{Add}(T)$.

Proof. Recall that we have denoted $\operatorname{Ker}(G)$ by $\mathcal{Y}$, and $\operatorname{Im}(H)$ by $\mathcal{Z}$. The whole proof of this proposition will be divided into three steps.

Step (1). We prove that the pair $(\mathcal{Y}, \mathcal{Z})$ is a torsion pair in $\mathscr{D}(B)$. In fact, for any $Y^{\bullet} \in \mathcal{Y}$ and $W^{\bullet} \in \mathscr{D}(A)$, we have $\operatorname{Hom}_{\mathscr{D}(B)}(Y^{\bullet}, H(W^{\bullet})) \simeq \operatorname{Hom}_{\mathscr{D}(A)}(G(Y^{\bullet}), W^{\bullet}) = \operatorname{Hom}_{\mathscr{D}(A)}(0, W^{\bullet}) = 0$ because the pair (G, H) is an adjoint pair of triangle functors. This shows $\operatorname{Hom}_{\mathscr{D}(B)}(\mathcal{Y}, \mathcal{Z}) = 0$.

Let $\eta : Id_{\mathscr{D}(B)} \to HG$ be the unit adjunction, and let $\varepsilon : GH \to Id_{\mathscr{D}(A)}$ be the counit adjunction. By Theorem 4.3, we know that ε is invertible. For any $M^\bullet$ in $\mathscr{D}(B)$, the canonical morphism $\eta_{M^\bullet} : M^\bullet \to HG(M^\bullet)$ can be extended to a triangle in $\mathscr{D}(B)$:

$$M^\bullet \xrightarrow{\eta_{M^\bullet}} HG(M^\bullet) \longrightarrow N^\bullet \longrightarrow M^\bullet[1].$$

By applying the functor G to the above triangle, we obtain a triangle in $\mathscr{D}(A)$:

$$G(M^\bullet) \xrightarrow{G(\eta_{M^\bullet})} GHG(M^\bullet) \longrightarrow G(N^\bullet) \longrightarrow G(M^\bullet)[1].$$

Since ε is invertible, we see that $G(\eta_{M^\bullet})$ is an isomorphism. This shows $G(N^\bullet) = 0$, that is, $N^\bullet \in \mathcal{Y}$. Since $\mathcal{Y}$ is a triangulated subcategory of $\mathscr{D}(B)$, we have $N^\bullet[-1] \in \mathcal{Y}$. Thus, the following triangle:

$$(*) \quad N^\bullet[-1] \longrightarrow M^\bullet \xrightarrow{\eta_{M^\bullet}} HG(M^\bullet) \longrightarrow N^\bullet$$

in $\mathscr{D}(B)$ with $HG(M^\bullet) \in \mathcal{Z}$ shows that the third condition of Definition 2.4 is satisfied. Hence the pair $(\mathcal{Y}, \mathcal{Z})$ is a torsion pair in $\mathscr{D}(B)$ by Definition 2.4. Since $\mathcal{Y}$ is a triangulated category, the torsion pair $(\mathcal{Y}, \mathcal{Z})$ is hereditary.

Step (2). We calculate the categories $\mathcal{Y}$ and $\mathcal{Z}$. Before starting our calculations, we mention the following result in [**41**, Theorem 10.5.9, Corollary 10.5.11]:

For every complex $X^\bullet$ in $\mathscr{D}(B)$, there exists a quasi-isomorphism $\overline{X}^\bullet \to X^\bullet$ with $\overline{X}^\bullet$ a complex of $({}_A T \otimes_B -)$-acyclic B-modules such that $G(X^\bullet) \simeq T \otimes_B \overline{X}^\bullet$. Here, a B-module N is said to be $({}_A T \otimes_B -)$-acyclic if $\mathrm{Tor}_i^B(T, N) = 0$ for any $i > 0$. Thus, the action of the left-derived functor G on any complex $U^\bullet$ of $({}_A T \otimes_B -)$-acyclic B-modules is the same as that of the functor ${}_A T \otimes_B -$ which acts by T tensoring each term of $U^\bullet$.

A similar statement holds for the right–derived functor H.

Now let us first interpret the triangle $(*)$ in terms of objects in $\mathscr{C}(B\text{-Proj})$. For the complex $M^\bullet$, we choose $P^\bullet \in \mathscr{C}(B\text{-Proj})$ such that $P^\bullet$ is quasi-isomorphic to $M^\bullet$. Then $G(M^\bullet) \simeq T \otimes_B P^\bullet$. By Lemma 4.2(3), we have $HG(M^\bullet) = \mathrm{Hom}_A(T, T \otimes_B P^\bullet)$ because the A-module $T \otimes_B P$ is $\mathrm{Hom}_A(T, -)$-acyclic, for any projective B-module P. Note that the homomorphism $\eta_{P^\bullet}$ coincides with $(\eta'_{P^n})_{n \in \mathbb{Z}}$, where P^n is the nth term of the complex $P^\bullet$ and $\eta'_{P^n} : P^n \to \mathrm{Hom}_A(T, T \otimes_B P^n)$ is the unit adjunction morphism for each $n \in \mathbb{Z}$. By Lemma 4.4, there is a short exact sequence of complexes

$$0 \longrightarrow P^\bullet \xrightarrow{\eta_{P^\bullet}} \mathrm{Hom}_A(T, T \otimes_B P^\bullet) \longrightarrow \mathrm{Coker}(\eta_{P^\bullet}) \longrightarrow 0$$

such that $(\mathrm{Coker}(\eta_{P^\bullet}))^i = \mathrm{Coker}(\eta'_{P^i}) \in \mathscr{E}$ for each $i \in \mathbb{Z}$. Thus, we can form the following commutative diagram of triangles in $\mathscr{D}(B)$:

$$
\begin{array}{ccccccc}
\mathrm{Coker}(\eta_{P^\bullet})[-1] & \longrightarrow & P^\bullet & \xrightarrow{\eta_{P^\bullet}} & \mathrm{Hom}_A(T, T \otimes_B P^\bullet) & \longrightarrow & \mathrm{Coker}(\eta_{P^\bullet}) \\
\downarrow \simeq & & \downarrow \simeq & & \| & & \downarrow \simeq \\
N^\bullet[-1] & \longrightarrow & M^\bullet & \xrightarrow{\eta_{M^\bullet}} & \mathbb{R}\mathrm{Hom}_A(T, T \otimes_B^{\mathbb{L}} M^\bullet) & \longrightarrow & N^\bullet
\end{array}
$$

On the one hand, if $M^\bullet \in \mathcal{Y}$, then $T \otimes_B^{\mathbb{L}} M^\bullet = 0$ by definition, and so $M^\bullet \simeq \mathrm{Coker}(\eta_{P^\bullet})[-1]$ in $\mathscr{D}(B)$. On the other hand, if $M^\bullet \simeq Y^\bullet$ in $\mathscr{D}(B)$ for some complex $Y^\bullet$ with $Y^i \in \mathscr{E}$, for each $i \in \mathbb{Z}$, then $T \otimes_B^{\mathbb{L}} M^\bullet \simeq T \otimes_B^{\mathbb{L}} Y^\bullet = T \otimes_B Y^\bullet = 0$ by the above-mentioned fact. This means $M^\bullet \in \mathcal{Y}$. Hence the first equality in Proposition 4.6 holds.

To prove the second equality, we observe that, by Lemma 4.2(4), $\mathrm{Hom}_A(T, T \otimes_B \mathrm{Hom}_A(T, T')) \simeq \mathrm{Hom}_A(T, T')$ for any $T' \in \mathrm{Add}(T)$. Let $Z^\bullet$ be a complex in $\mathscr{D}(B)$ such that $Z^i \in \mathrm{Hom}_A(T, \mathrm{Add}(T))$. Then $HG(Z^\bullet) \simeq \mathrm{Hom}_A(T, T \otimes_B Z^\bullet) \simeq Z^\bullet$ in $\mathscr{D}(B)$ because every B-module in $\mathrm{Hom}_A(T, \mathrm{Add}(T))$ is $(T \otimes_B -)$-acyclic by Lemmata 4.2(3) and 4.2(4). This implies $Z^\bullet \in \mathcal{Z}$. Conversely, for any $W^\bullet \in \mathscr{D}(A)$, we can choose a complex $L^\bullet \in \mathscr{C}(B\text{-Proj})$ such that $L^\bullet$ is quasi-isomorphic to $H(W^\bullet)$. By Theorem 4.3, we conclude that

$H(W^\bullet) \simeq HG(H(W^\bullet)) \simeq HG(L^\bullet)$ in $\mathscr{D}(B)$. Since $HG(L^\bullet) = H(T \otimes_B^{\mathbb{L}} L^\bullet) = H(T \otimes_B L^\bullet) \simeq$ $\mathrm{Hom}_A(T, T \otimes_B L^\bullet)$, where the last isomorphism follows from Lemma 4.2(3) and the above-mentioned fact about the functor H. Clearly, the complex $\mathrm{Hom}_A(T, T \otimes_B L^\bullet)$ has each term in $\mathrm{Hom}_A(T, \mathrm{Add}(T))$. Thus, the second equality in Proposition 4.6 holds.

Step (3). We claim that there is a full subcategory $\mathcal{X}$ of $\mathscr{D}(B)$ such that $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF triple in $\mathscr{D}(B)$. Furthermore, we have $\mathcal{X} = \mathrm{Tria}(Q^\bullet)$.

Indeed, since $\mathscr{E}$ is closed under direct sums and products by Lemma 4.5, we conclude that the triangulated full subcategory $\mathcal{Y}$ is closed under all small coproducts and products. Then the existence of the TTF triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ in $\mathscr{D}(B)$ follows straightforward from [**13**, Proposition 5.14]. Moreover, $\mathcal{X} = \mathrm{Ker}(\mathrm{Hom}_{\mathscr{D}(B)}(-, \mathcal{Y}))$ and $\mathcal{Y} = \mathrm{Ker}(\mathrm{Hom}_{\mathscr{D}(B)}(\mathcal{X}, -))$. Now we shall prove $\mathcal{X} = \mathrm{Tria}(Q^\bullet)$. First, we show $Q^\bullet \in \mathcal{X}$. This is equivalent to verifying $\mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, \mathcal{Y}) = 0$. Let $\mathcal{Y}' := \mathrm{Ker}(\mathrm{Hom}_{\mathscr{D}(B)}(\mathrm{Tria}(Q^\bullet), -))$. By Lemma 2.8, we see that $(\mathrm{Tria}(Q^\bullet), \mathcal{Y}')$ is a torsion pair in $\mathscr{D}(B)$ with

$$\mathcal{Y}' = \{Y^\bullet \in \mathscr{D}(B) \mid \mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, Y^\bullet[i]) = 0 \text{ for all } i \in \mathbb{Z}\}.$$

Recall that $\varphi^* := \mathrm{Hom}_A(T, \varphi) : \mathrm{Hom}_A(T, T_0) \to \mathrm{Hom}_A(T, T_1)$ is a homomorphism between finitely generated projective B-modules. We define $\Sigma := \{\varphi^*\}$. Then $\Sigma^\bullet = \{Q^\bullet[1]\}$ (see notation in Section 3). By Lemma 4.5(1), we have $\Sigma^\perp = \mathscr{E}$. Thus, it follows from Proposition 3.3 that

$$\mathcal{Y}' = \mathscr{D}(B)_{\mathscr{E}} := \{Y^\bullet \in \mathscr{D}(B) \mid H^i(Y^\bullet) \in \mathscr{E} \text{ for all } i \in \mathbb{Z}\}.$$

According to Lemma 4.5(2), $\mathscr{E}$ is an abelian subcategory of B-Mod. This forces $\mathcal{Y} \subseteq \mathcal{Y}'$. In particular, we have $\mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, \mathcal{Y}) = 0$, which yields $Q^\bullet \in \mathcal{X}$. Therefore, $\mathrm{Tria}(Q^\bullet) \subseteq \mathcal{X}$ since $\mathcal{X}$ is a full triangulated subcategory of $\mathscr{D}(B)$.

Let $\mathbf{i} : \mathcal{X} \to \mathscr{D}(B)$ and $\mathbf{k} : \mathcal{Z} \to \mathscr{D}(B)$ be the canonical inclusions. Then the functor $\mathbf{i}$ has a right adjoint functor $\mathbf{R} : \mathscr{D}(B) \to \mathcal{X}$. Since $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF triple in $\mathscr{D}(B)$, the functor $\mathbf{Rk} : \mathcal{Z} \to \mathcal{X}$ is an equivalence (see the statements after Definition 2.5 in Section 2.3). So the composition functor $\mathbf{Rk}H : \mathscr{D}(A) \to \mathcal{X}$ is an equivalence because $H : \mathscr{D}(A) \to \mathcal{Z}$ is an equivalence. Since a functor possessing a right adjoint functor preserves coproducts, we know that the functor $\mathbf{Rk}H$ commutes with coproducts. Note that coproducts depend on the category where coproducts are taken. We know that coproducts in $\mathcal{Z}$ exist since $\mathscr{D}(A)$ admits all small coproducts, but we do not know if these coproducts in $\mathcal{Z}$ coincide with those in $\mathscr{D}(B)$. In general, $\mathcal{Z}$ is not closed under coproducts in $\mathscr{D}(B)$.

Since a torsion class in $\mathscr{D}(B)$ is always closed under coproducts, this means that coproducts in $\mathcal{X}$ exist and coincide with that in $\mathscr{D}(B)$.

Since $H(A) \simeq Q^\bullet \in \mathcal{X}$, we have $\mathbf{Rk}H(A) \simeq \mathbf{R}(Q^\bullet) = Q^\bullet$. Note that $\mathscr{D}(A) = \mathrm{Tria}(A)$ and that the triangle functor $\mathbf{Rk}H : \mathscr{D}(A) \to \mathcal{X}$ is an equivalence under which $\mathrm{Tria}(A)$ has the image $\mathrm{Tria}(Q^\bullet)$ since the functor $\mathbf{Rk}H$ commutes with coproducts. It follows that $\mathcal{X} = \mathrm{Tria}(Q^\bullet)$ and $\mathcal{Y} = \mathcal{Y}'$. Hence $(\mathrm{Tria}(Q^\bullet), \mathrm{Ker}(G), \mathrm{Im}(H))$ is a TTF triple in $\mathscr{D}(B)$. $\square$

As a consequence of Proposition 4.6, we give an alternative proof of the fact that finitely generated tilting modules are classical. A known proof of this fact is a combination of Angeleri Hügel and Herbera [**2**, Corollary 9.13(5)] together with a result of Bazzoni and Herbera [**10**]. We thank Lidia Angeleri-Hügel for pointing out these references.

COROLLARY 4.7. *Suppose that T is a tilting A-module. If T is finitely generated, then T is classical.*

Proof. In general, the following facts are true for a finitely generated A-module M, with $B := \mathrm{End}_A(M)$.

(1) For any index set δ and $X_i \in A$-Mod with $i \in \delta$, the canonical homomorphism $\bigoplus_{i \in \delta} \operatorname{Hom}_A(M, X_i) \to \operatorname{Hom}_A(M, \bigoplus_{i \in \delta} X_i)$, given by $(f_i)_{i \in \delta} \mapsto [m \mapsto (mf_i)_{i \in \delta}]$, is an isomorphism.

(2) The functor $\operatorname{Hom}_A(M, -): \operatorname{Add}(M) \longrightarrow \operatorname{Add}({}_B B)$ is an equivalence of additive categories.

(3) If $M \in \operatorname{Add}(N)$ for some A-module N, then $M \in \operatorname{add}(N)$.

The proofs of (1) and (3) are standard. It is easy to see that (2) follows from (1) together with the natural isomorphism $\operatorname{Hom}_A(M, U) \otimes_B - \longrightarrow \operatorname{Hom}_A(M, U \otimes_B -)$ of the functors from $\operatorname{Add}({}_B B)$ to B-Mod, where U is an A-B-bimodule.

We shall first prove that the tilting A-module T is good. Indeed, it follows from $(T3)$ in Definition 4.1 that there exists an exact sequence

$$0 \longrightarrow {}_A A \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \longrightarrow 0$$

of A-modules such that $T_i \in \operatorname{Add}(T)$ for $i = 0, 1$. Without loss of generality, we can assume that $T_0 = T^{(\alpha)}$ for some index set α. Clearly, there exists a finite subset β of α such that $(1)f_0 \in T^{(\beta)}$ which is a direct summand of $T^{(\alpha)}$. This implies $\operatorname{Im}(f_0) \subseteq T^{(\beta)}$. Consequently, we have $f_0 = (f', 0): A \to T^{(\alpha)} = T^{(\beta)} \oplus T^{(\alpha \setminus \beta)}$, where $f': A \to T^{(\beta)}$ is the right multiplication map defined by $(1)f_0$. Thus $T_1 \simeq \operatorname{Coker}(f') \oplus T^{(\alpha \setminus \beta)}$ as A-modules, and therefore $\operatorname{Coker}(f') \in \operatorname{Add}(T)$. Further, $\operatorname{Coker}(f')$ is finitely generated since T is finitely generated. By (3), we have $\operatorname{Coker}(f') \in \operatorname{add}(T)$. As a result, there exists an exact sequence $0 \to A \to T^{(\beta)} \to \operatorname{Coker}(f') \to 0$ of A-modules such that both $T^{(\beta)}$ and $\operatorname{Coker}(f')$ belong to $\operatorname{add}(T)$. Thus, T is a good tilting A-module.

By Theorem 4.3, the total right-derived functor $H: \mathscr{D}(A) \to \mathscr{D}(B)$ with $B := \operatorname{End}_A(T)$ is fully faithful. Thus, to show that T is a classical tilting A-module, it suffices to show $\operatorname{Im}(H) = \mathscr{D}(B)$. In fact, since T is finitely generated, we know that $\operatorname{Hom}_A(T, -): \operatorname{Add}(T) \to \operatorname{Add}({}_B B)$ is an equivalence by (2). Let $\operatorname{Hom}_A(T, \operatorname{Add}(T))$ stand for the full subcategory of B-Mod consisting of all the modules $\operatorname{Hom}_A(T, T')$ with T' in $\operatorname{Add}(T)$. Then $\operatorname{Hom}_A(T, \operatorname{Add}(T)) = \operatorname{Add}({}_B B)$ by (1). Note that, for each complex $Y^\bullet$ in $\mathscr{D}(B)$, there is a complex $P^\bullet$ in $\mathscr{C}(\operatorname{Add}({}_B B))$ such that $P^\bullet$ is quasi-isomorphic to $Y^\bullet$. Thus, we conclude from Proposition 4.6 that $\operatorname{Im}(H) = \mathscr{D}(B)$. This shows that T is classical, finishing the proof. $\square$

With the above preparations, now we prove Theorem 1.1(1).

Proof of Theorem 1.1(1).   By Proposition 4.6, we know that the triple $(\operatorname{Tria}(Q^\bullet), \operatorname{Ker}(G), \operatorname{Im}(H))$ is a TTF triple in $\mathscr{D}(B)$. Moreover, $\mathscr{D}(A)$ and $\operatorname{Tria}(Q^\bullet)$ are equivalent as triangulated categories, due to Keller in [**31**, Corollary 8.4, Theorem 8.5]. According to the correspondence between recollements and TTF triples in Lemma 2.6(2), we can form the following recollement:

$$\operatorname{Ker}(G) \xrightarrow{\ \ \mathbf{j}\ \ } \mathscr{D}(B) \longrightarrow \mathscr{D}(A)$$

where $\mathbf{j}$ is the canonical embedding and $\mathbf{L}$ is the left adjoint of $\mathbf{j}$. Recall that $\varphi^* := \operatorname{Hom}_A(T, \varphi)$ is the homomorphism between the finitely generated projective B-modules $\operatorname{Hom}_A(T, T_0)$ and $\operatorname{Hom}_A(T, T_1)$. As in Section 3, we define $\Sigma := \{\varphi^*\}$. By Lemma 4.5(1), we have $\Sigma^\perp = \mathscr{E}$. By Step (3) in the proof of Proposition 4.6, we have $\operatorname{Ker}(G) = \mathscr{D}(B)_{\Sigma^\perp}$. Let $\lambda: B \to B_\Sigma$ be the universal localization of B at Σ. Since $\mathbf{L}$ is a functor from $\mathscr{D}(B)$ to $\operatorname{Ker}(G)$, we have $\mathbf{L}(B) \in \operatorname{Ker}(G)$, and therefore it satisfies the condition (5) of Proposition 3.6, according to Proposition 4.6. Thus, by Proposition 3.6, we know that $\lambda_*: \mathscr{D}(B_\Sigma) \xrightarrow{\sim} \mathscr{D}(B)_{\Sigma^\perp}$ is an equivalence of triangulated categories, and that the homomorphism λ is a homological ring epimorphism. Set $C := B_\Sigma$. Then $\operatorname{Ker}(G)$ and $\mathscr{D}(C)$ are equivalent as triangulated categories.

Consequently, we can get the following recollement from the above one:

$$\mathscr{D}(C) \longrightarrow \mathscr{D}(B) \longrightarrow \mathscr{D}(A)$$

In the following, we shall explicitly describe the six triangle functors arising in the above recollement.

Here, we follow the notation used in Definition 2.3, and take $\mathcal{D} = \mathscr{D}(B), \mathcal{D}' = \mathscr{D}(A)$ and $\mathcal{D}'' = \mathscr{D}(C)$. Then we can define $i^* = C \otimes_B^{\mathbb{L}} -$, $i_* = \lambda_*$ and $i^! = \mathbb{R}\mathrm{Hom}_B(C, -)$. As for the other three functors, we put $j_! = \mathbf{iRk}H$, $j^! = G$ and $j_* = H$ up to natural isomorphism. Let $\mathbf{U} : \mathscr{D}(B) \to \mathcal{Z}$ be a left adjoint of the inclusion $\mathbf{k} : \mathcal{Z} \to \mathscr{D}(B)$. By Lemma 2.6 and the proof of Proposition 4.6, we get the following diagram of functors:

$$\mathscr{D}(B) \xrightarrow[\mathbf{kUi}]{\overset{\mathbf{i}}{\underset{\mathbf{R}}{\rightleftarrows}}} \mathcal{X} \xleftarrow{\mathbf{Rk}H} \mathscr{D}(A)$$

with the properties:

  (i) $(\mathbf{i}, \mathbf{R})$ and $(\mathbf{R}, \mathbf{kUi})$ are adjoint pairs,
  (ii) $\mathbf{Rk}H$ is an equivalence of triangulated categories.

This implies that $j_! = \mathbf{iRk}H$ and $j_* = (\mathbf{kUi})(\mathbf{Rk}H)$. Note that the composition functor $\mathbf{UiRk} : \mathcal{Z} \to \mathcal{Z}$ of the functors $\mathbf{Ui}$ and $\mathbf{Rk}$ is natural isomorphic to the identity functor $1_{\mathcal{Z}}$ by the property (3) of a TTF triple (see Section 2.3). Consequently, we can choose $j_* = H$. Since (G, H) is an adjoint pair of functors, we can choose $j^* = G$. Thus, the proof of the first part of Theorem 1.1 is completed. $\qquad\square$

REMARKS. (1) The ring C in Theorem 1.1 equals zero if and only if T is a classical tilting module. In fact, $C = 0$ if and only if $\mathrm{Ker}(G) = 0$ if and only if G is an equivalence if and only if T is classical.

(2) From the proof of Theorem 1.1(1), we know that a good tilting module T has the property: The functor G admits a fully faithful left adjoint $j_!$. In the next section, we shall show that this property guarantees that the tilting module T is good.

(3) The ring C in Theorem 1.1 is the universal localization of B at $\{\varphi^*\}$. However, by Lemma 3.2, C is also isomorphic to the universal localization of B at the ψ in Lemma 4.2.

5. *Existence of recollements implies goodness of tilting modules*

In this section, we shall prove the second part of Theorem 1.1, which is a converse of the first part in some sense. Our proof depends on the property that the total left-derived functor G admits a fully faithful left adjoint $j_!$.

Proof of Theorem 1.1(2). Let T be a tilting A-module and B the endomorphism ring of T. Recall that G and H stand for the triangle functors $T \otimes_B^{\mathbb{L}} - : \mathscr{D}(B) \to \mathscr{D}(A)$ and $\mathbb{R}\mathrm{Hom}_A(T, -) : \mathscr{D}(A) \to \mathscr{D}(B)$, respectively. Suppose that G admits a fully faithful left adjoint $j_! : \mathscr{D}(A) \to \mathscr{D}(B)$. We want to show that T is a good tilting module.

To prove that T is good, it suffices to find a short exact sequence of A-modules,

$$0 \longrightarrow A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0,$$

such that $T_i \in \mathrm{add}(T)$ for $i = 0, 1$.

First, we observe some consequences of the assumption that $j_!$ is fully faithful. Set $W^\bullet :=$ $j_!(A)$. Since the total left-derived functor G commutes with coproducts, we can easily show that the functor $j_!$ preserves compact objects. In particular, the complex $W^\bullet$ is compact in $\mathscr{D}(B)$, which implies $W^\bullet \simeq Q^\bullet$ in $\mathscr{D}(B)$, for some $Q^\bullet \in \mathscr{C}^{\mathrm{b}}(B\text{-proj})$. Since the Hom-functor $\mathrm{Hom}_A(T, -)$ induces an equivalence between $\mathrm{add}(T)$ and B-proj, we can assume that $Q^\bullet = \mathrm{Hom}_A(T, X^\bullet)$, where $X^\bullet \in \mathscr{C}^{\mathrm{b}}(\mathrm{add}(T))$ is of the following form:

$$0 \longrightarrow X^s \longrightarrow \cdots \longrightarrow X^i \xrightarrow{d^i} X^{i+1} \longrightarrow \cdots \longrightarrow X^t \longrightarrow 0$$

for $s \leqslant 0 \leqslant t$. Since the functor $j_!$ is fully faithful, we conclude from [**32**, Chapter IV, Section 3, Theorem 1, p.90] that the unit adjunction morphism $\widetilde{\eta} : Id_{\mathscr{D}(A)} \to Gj_!$ is invertible. Thus, $A \simeq G(W^\bullet) \simeq G(Q^\bullet)$ in $\mathscr{D}(A)$. Note that $T \otimes_B \mathrm{Hom}_A(T, X^\bullet) \simeq X^\bullet$ in $\mathscr{C}^{\mathrm{b}}(A\text{-mod})$ since $X^i \in$ $\mathrm{add}(T)$, for each $s \leqslant i \leqslant t$. Consequently, we have $A \simeq X^\bullet$ in $\mathscr{D}(A)$. It follows that $H^0(X^\bullet) \simeq A$ and $H^i(X^\bullet) = 0$, for any $i \neq 0$.

Second, if $t = 0$, then the homomorphism $X^0 \to H^0(X^\bullet)$ splits, this implies $A \in \mathrm{add}(T)$. Hence T is a good tilting module. Now we assume $t \neq 0$. Then we can decompose $X^\bullet$ into two long exact sequences of A-modules:

$$0 \longrightarrow X^s \xrightarrow{d^s} \cdots \longrightarrow X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{\pi} M \longrightarrow 0,$$

$$0 \longrightarrow A \longrightarrow M \xrightarrow{\mu} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^t} X^t \longrightarrow 0;$$

where $d^0 = \pi\mu$ and M is the cokernel of d^{-1}. We claim $\mathrm{Im}(\mu) \in \mathrm{add}(T)$. In fact, we have a long exact sequence

$$0 \longrightarrow \mathrm{Im}(\mu) \xrightarrow{\nu} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^t} X^t \longrightarrow 0,$$

where ν is the canonical inclusion. For each $1 \leqslant i \leqslant t$, since $X^i \in \mathrm{add}(T)$, we have $\mathrm{Im}(d^i) \in \mathrm{Gen}(T)$. As we know, $T^\perp = \mathrm{Gen}(T)$ for a tilting module T. Consequently, we see that $\mathrm{Ext}_A^1(T, \mathrm{Im}(d^i)) = 0$ for any $1 \leqslant i \leqslant t$. Note that $\mathrm{Im}(d^t) = X^t \in \mathrm{add}(T)$. Thus, we can easily show $\mathrm{Im}(\mu) \in \mathrm{add}(T)$ by induction on t.

Finally, we shall prove $M \in \mathrm{add}(T)$. If $s = 0$, then $M = X^0 \in \mathrm{add}(T)$. Suppose $s < 0$. Since $\mathrm{Im}(\mu) \in \mathrm{add}(T)$ and the sequence $0 \to A \to M \to \mathrm{Im}(\mu) \to 0$ is exact, we know that $\mathrm{Ext}_A^1(M, T) = 0$ and M has projective dimension at most 1. In addition, $\mathrm{Im}(d^{-1})$ is a quotient module of X^{-1}. It follows that $\mathrm{Ext}_A^1(M, \mathrm{Im}(d^{-1})) = 0$, which implies that the homomorphism π splits. Thus, $M \in \mathrm{add}(X^0) \subseteq \mathrm{add}(T)$.

Now we define $T_0 = M$ and $T_1 = \mathrm{Im}(\mu)$. Then the sequence $0 \to A \to T_0 \xrightarrow{\mu} T_1 \to 0$ satisfies $T_i \in \mathrm{add}(T)$ for $i = 0, 1$. Thus, T is a good tilting module, and the proof is completed. $\square$

REMARK. Suppose that G admits a fully faithful left adjoint $j_! : \mathscr{D}(A) \to \mathscr{D}(B)$. Then there exists a TTF triple $(j_!(\mathscr{D}(A)), \mathrm{Ker}(G), H(\mathscr{D}(A)))$ in $\mathscr{D}(B)$ (see [**14**, Chapter I, Proposition 2.11] for details), where $j_!(\mathscr{D}(A))$ and $H(\mathscr{D}(A))$ denote the images of $j_!$ and H, respectively. By Lemma 2.6, we know that the derived category $\mathscr{D}(B)$ is a recollement of the derived category $\mathscr{D}(A)$ and the triangulated category $\mathrm{Ker}(G)$. Since T is good by Theorem 1.1(2), it follows from Theorem 1.1(1) that $\mathrm{Ker}(G)$ is triangle equivalent to the derived category $\mathscr{D}(C)$ of a ring C. Thus, we get a recollement of derived module categories as in Theorem 1.1(1).

6. *Applications to tilting modules arising from ring epimorphisms*

In this section we apply our main result Theorem 1.1 to tilting modules arising from ring epimorphisms. In this case, we shall describe the universal localization rings appearing in the main result by coproducts defined by Cohn [**19**].

We start with recalling of some definitions.

Let R_0 be a ring with identity. An R_0-ring is a ring R together with a ring homomorphism $\lambda_R : R_0 \to R$. An R_0-homomorphism from an R_0-ring R to another R_0-ring S is a ring homomorphism $f : R \to S$ such that $\lambda_S = \lambda_R f$. If R_0 is commutative and the image of $\lambda_R : R_0 \to R$ is contained in the centre $Z(R)$ of R, then we say that R is an R_0-algebra.

Recall that the coproduct of a family $\{R_i \mid i \in I\}$ of R_0-rings with I an index set is an R_0-ring R together with a family $\{\rho_i : R_i \to R \mid i \in I\}$ of R_0-homomorphisms such that, for any R_0-ring S with a family of R_0-homomorphisms $\{\tau_i : R_i \to S \mid i \in I\}$, there is a unique R_0-homomorphism $\delta : R \to S$ such that $\tau_i = \rho_i \delta$, for all $i \in I$.

It is well known that the coproduct of a family $\{R_i \mid i \in I\}$ of R_0-rings exists. In this case, we denote their coproduct by $\sqcup_{R_0} R_i$. For example, the coproduct of the polynomial rings $k[x]$ and $k[y]$ over a field k is the free ring $k\langle x, y\rangle$ in two variables x and y over k. Note that $R_0 \sqcup_{R_0} S = S = S \sqcup_{R_0} R_0$, for every R_0-ring S.

Let R_i be an R_0-ring for $i = 1, 2$. We denote by B the matrix ring $\begin{pmatrix} R_1 & R_1 \otimes_{R_0} R_2 \\ 0 & R_2 \end{pmatrix}$. Let $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in B$, and let $\varphi : Be_1 \to Be_2$ be the map sending $\begin{pmatrix} r_1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} r_1 \otimes 1 \\ 0 \end{pmatrix}$ for $r_1 \in R_1$. Let $\rho_i : R_i \to R_1 \sqcup_{R_0} R_2$ be the canonical R_0-homomorphism for $i = 1, 2$.

The following lemma reveals a relationship between coproducts and universal localizations.

LEMMA 6.1 [**38**, Theorem 4.10, p. 59]. *The universal localization B_φ of B at φ is equal to $M_2(R_1 \sqcup_{R_0} R_2)$, the 2×2 matrix ring over the coproduct $R_1 \sqcup_{R_0} R_2$ of R_1 and R_2 over R_0. Furthermore, the corresponding ring homomorphism $\lambda_\varphi : B \to B_\varphi$ is given by $\begin{pmatrix} r_1 & x_1 \otimes x_2 \\ 0 & r_2 \end{pmatrix} \mapsto \begin{pmatrix} (r_1)\rho_1 & (x_1)\rho_1(x_2)\rho_2 \\ 0 & (r_2)\rho_2 \end{pmatrix}$ for $r_i, x_i \in R_i$ with $i = 1, 2$.*

The next result says, in some sense, that taking coproducts of rings preserves universal localizations.

LEMMA 6.2. *Let R_0 be a ring, Σ a set of homomorphisms between finitely generated projective R_0-modules, and $\lambda_\Sigma : R_0 \to R_1 := (R_0)_\Sigma$ the universal localization of R_0 at Σ. Then, for any R_0-ring R_2, the coproduct $R_1 \sqcup_{R_0} R_2$ is isomorphic to the universal localization $(R_2)_\Delta$ of R_2 at the set $\Delta := \{R_2 \otimes_{R_0} f \mid f \in \Sigma\}$.*

Proof. Let $R := (R_2)_\Delta$ and $\lambda_\Delta : R_2 \to R$ the universal localization of R_2 at Δ. Suppose that $\lambda_{R_2} : R_0 \to R_2$ is the ring homomorphism defining the R_0-ring R_2. Then R is an R_0-ring via the composition $\lambda_{R_2}\lambda_\Delta$ of λ_{R_2} with λ_Δ. Moreover, we shall prove that there is a unique R_0-ring homomorphism $\nu : R_1 \to R$, that is, a ring homomorphism ν with $\lambda_{R_2}\lambda_\Delta = \lambda_\Sigma \nu$. In fact, for any $f : P_1 \to P_0$ in Σ, the map $R \otimes_{R_0} f : R \otimes_{R_0} P_1 \to R \otimes_{R_0} P_0$ of R-modules is an isomorphism because $R \otimes_{R_0} f \simeq R \otimes_{R_2} (R_2 \otimes_{R_0} f)$ and the latter is an isomorphism. Thus, by the universal property of λ_Σ, there is a unique ring homomorphism $\nu : R_1 \to R$ such that $\lambda_{R_2}\lambda_\Delta = \lambda_\Sigma \nu$, as desired.

Now, we show that R together with the two ring homomorphisms λ_Δ and ν satisfies the definition of coproducts, and therefore $R_1 \sqcup_{R_0} R_2$ is isomorphic to R.

Indeed, suppose that S is an arbitrary R_0-ring with two R_0-homomorphisms $\tau_i : R_i \to S$ for $i = 1, 2$. Then $\lambda_\Sigma \tau_1 = \lambda_{R_2}\tau_2$. Further, since we have

$$S \otimes_{R_2} (R_2 \otimes_{R_0} h) \simeq S \otimes_{R_0} h \simeq S \otimes_{R_1} (R_1 \otimes_{R_0} h),$$

and since $R_1 \otimes_{R_0} h$ is an isomorphism for any $h \in \Sigma$, we infer that $S \otimes_{R_2} (R_2 \otimes_{R_0} h)$ is an isomorphism, for any $h \in \Sigma$. It follows from the property of universal localizations that there is a unique ring homomorphism $\delta : R \to S$ such that $\tau_2 = \lambda_\Delta \delta$. Clearly, $\lambda_\Sigma \tau_1 = \lambda_\Sigma \nu \delta$, and $\tau_1 = \nu\delta$ since λ_Σ is a ring epimorphism. Note that δ is also an R_0-ring homomorphism. Thus,

$\delta : R \to S$ is actually a unique R_0-homomorphism such that $\tau_1 = \nu\delta$ and $\tau_2 = \lambda_\Delta\delta$. This shows that R is isomorphic to the coproduct $R_1 \sqcup_{R_0} R_2$ of R_1 and R_2 over R_0. $\qquad\square$

Sometimes, coproducts can be interpreted as tensor products of rings.

LEMMA 6.3.   *Let R_0 be a commutative ring, and let R_i be an R_0-algebra for $i = 1, 2$. If one of the homomorphisms $\lambda_{R_1} : R_0 \to R_1$ and $\lambda_{R_2} : R_0 \to R_2$ is a ring epimorphism, then the coproduct $R_1 \sqcup_{R_0} R_2$ is isomorphic to the tensor product $R_1 \otimes_{R_0} R_2$.*

Proof.   It is known that the tensor product $R_1 \otimes_{R_0} R_2$ of two rings R_1 and R_2 over R_0 has the following universal property: If $f_i : R_i \to R$ is a homomorphism of R_0-rings for $i = 1, 2$, such that $(r_2)f_2(r_1)f_1 = (r_1)f_1(r_2)f_2$ for all $r_i \in R_i$ with $i = 1, 2$, then there is a unique ring homomorphism $f : R_1 \otimes_{R_0} R_2 \to R$ of R_0-rings that satisfies $(x_1 \otimes x_2)f = (x_1)f_1(x_2)f_2$ for $x_i \in R_i$ with $i = 1, 2$. In particular, if $\lambda_1 : R_1 \to R_1 \otimes_{R_0} R_2$ is the map given by $r_1 \mapsto r_1 \otimes 1$ for $r_1 \in R_1$, and if $\lambda_2 : R_2 \to R_1 \otimes_{R_0} R_2$ is the one given by $r_2 \mapsto 1 \otimes r_2$ for $r_2 \in R_2$, then $f_i = \lambda_i f$ for $i = 1, 2$.

To prove Lemma 6.3, it suffices to show that, for any R_0-homomorphisms $f_i : R_i \to R$ for $i = 1, 2$, the condition $(r_2)f_2(r_1)f_1 = (r_1)f_1(r_2)f_2$ holds true for all $r_i \in R_i$ with $i = 1, 2$.

Assume that $\lambda_{R_1} : R_0 \to R_1$ is a ring epimorphism. For any element $y \in R_2$, we define two ring homomorphisms $\theta_1 : R_1 \to M_2(R)$ and $\theta_2 : R_1 \to M_2(R)$ as follows:

$$(x)\theta_1 = \begin{pmatrix} (x)f_1 & 0 \\ 0 & (x)f_1 \end{pmatrix}$$

and

$$(x)\theta_2 = \begin{pmatrix} 1 & 0 \\ (y)f_2 & 1 \end{pmatrix} \begin{pmatrix} (x)f_1 & 0 \\ 0 & (x)f_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -(y)f_2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} (x)f_1 & 0 \\ (y)f_2(x)f_1 - (x)f_1(y)f_2 & (x)f_1 \end{pmatrix},$$

for $x \in R_1$. Now, we verify $\lambda_{R_1}\theta_1 = \lambda_{R_1}\theta_2$. This is equivalent to showing that, if $x = (r)\lambda_{R_1}$ with $r \in R_0$, then $(y)f_2\,(x)f_1 = (x)f_1(y)f_2$. In fact, we always have

$$(y)f_2\,(x)f_1 = (y)f_2((r)\lambda_{R_1})f_1 = (y)f_2((r)\lambda_{R_2})f_2 = (y(r)\lambda_{R_2})f_2,$$
$$(x)f_1(y)f_2 = ((r)\lambda_{R_1})f_1(y)f_2 = ((r)\lambda_{R_2})f_2(y)f_2 = ((r)\lambda_{R_2}y)f_2.$$

Since R_2 is an R_0-algebra, it follows from $\mathrm{Im}(\lambda_{R_2}) \subseteq Z(R_2)$ that $y(r)\lambda_{R_2} = (r)\lambda_{R_2}y$, and so $(y)f_2(x)f_1 = (x)f_1(y)f_2$ whenever $x = (r)\lambda_{R_1}$ with $r \in R_0$. This shows $\lambda_{R_1}\theta_1 = \lambda_{R_1}\theta_2$ and $\theta_1 = \theta_2$ since $\lambda_{R_1} : R_0 \to R_1$ is a ring epimorphism. Thus, $(y)f_2(x)f_1 = (x)f_1(y)f_2$ for any $x \in R_1$. Note that y is an arbitrary element of R_2. Hence $(y)f_2(x)f_1 = (x)f_1(y)f_2$ for any $x \in R_1$ and $y \in R_2$. $\qquad\square$

From now on, $\lambda : R \to S$ denotes an injective ring homomorphism from R to S. We define B to be the endomorphism ring of the R-module $S \oplus S/R$, and S' the endomorphism ring of the R-module S/R. Let π stands for the canonical surjective map $S \to S/R$ of R-modules. Then we have an exact sequence of R-modules:

$$(*) \quad 0 \longrightarrow R \longrightarrow S \overset{\pi}{\longrightarrow} S/R \longrightarrow 0.$$

In the next two lemmas, we collect some facts on ring epimorphisms.

LEMMA 6.4.   *Let $\lambda : R \to S$ be an injective ring epimorphism with $\mathrm{Tor}_1^R(S, S) = 0$. Then we have the following.*

(1) *An R-module X belongs to S-Mod if and only if $\operatorname{Ext}_R^i(S/R, X) = 0$, for $i = 0, 1$.*

(2) *Let $T := S \oplus S/R$. Then*

$$\operatorname{End}_R(T) \simeq \left(\begin{array}{cc} S & \operatorname{Hom}_R(S, S/R) \\ 0 & \operatorname{End}_R(S/R) \end{array} \right).$$

Moreover, if e_1 and e_2 are the idempotent elements in $\operatorname{End}_R(T)$ corresponding to the summands S and S/R, respectively, then the homomorphism $\pi^* : \operatorname{End}_R(T)e_1 \to \operatorname{End}_R(T)e_2$ induced from the canonical surjection $\pi : S \to S/R$ is given by $\left(\begin{smallmatrix} s \\ 0 \end{smallmatrix} \right) \mapsto \left(\begin{smallmatrix} x \mapsto (xs)\pi \\ 0 \end{smallmatrix} \right)$ for $s, x \in S$.

Proof. (1) follows from [**26**]. For (2), it follows from (1) that $\operatorname{Hom}_R(S/R, S) = 0$. By applying $\operatorname{Hom}_R(-, S)$ to the exact sequence $(*)$, we get $\operatorname{Hom}_R(S, S) \simeq \operatorname{Hom}_R(R, S) \simeq S$. $\square$

LEMMA 6.5. *Suppose that $\lambda : R \to S$ is an injective ring epimorphism with $\operatorname{Tor}_1^R(S, S) = 0$.*

(1) *The right multiplication map $\mu : R \to S'$ defined by $r \mapsto (x \mapsto xr)$ for $r \in R$ and $x \in S/R$, is a ring homomorphism. Consequently, S' can be regarded as an R-ring via the map μ. Further, μ is an isomorphism if and only if $\operatorname{Ext}_R^i(S, R) = 0$ for $i = 0, 1$.*

(2) *There is an isomorphism $\theta : S \otimes_R S' \simeq \operatorname{Hom}_R(S, S/R)$ of S-S'-bimodules such that $1 \otimes 1$ is mapped to the canonical surjection $\pi : S \to S/R$.*

(3) *There is an exact sequence of R-S'-modules:*

$$0 \to S' \xrightarrow{\lambda'} S \otimes_R S' \xrightarrow{\pi \otimes S'} (S/R) \otimes_R S' \to 0,$$

where the map λ' is defined by $f \mapsto 1 \otimes f$ for any $f \in S'$. Moreover, the evaluation map $\psi : (S/R) \otimes_R S' \to S/R$ defined by $y \otimes g \mapsto (y)g$ for $y \in S/R$ and $g \in S'$, is an isomorphism of R-S'-bimodules.

(4) *If $\lambda : R \to S$ is homological, then $\operatorname{Tor}_i^R(S, S') = 0$ for any $i > 0$.*

(5) *If R is commutative, then so is S'.*

Proof. (1) It is easy to check that the right multiplication map μ is a ring homomorphism since S/R is an R-R-bimodule. Clearly, μ is injective if and only if $\operatorname{Hom}_R(S, R) = 0$. For μ to be surjective, we use the following exact sequence:

$$0 \longrightarrow \operatorname{Hom}_R(S, R) \longrightarrow \operatorname{Hom}_R(S, S) \xrightarrow{\pi^*} \operatorname{Hom}_R(S, S/R)$$
$$\longrightarrow \operatorname{Ext}_R^1(S, R) \longrightarrow \operatorname{Ext}_R^1(S, S),$$

where $\operatorname{Ext}_R^1(S, S) = 0$ by Lemma 6.4(1). This sequence shows that $\operatorname{Ext}_R^1(S, R) = 0$ if and only if π^* is surjective. Now we claim that π^* is surjective if and only if μ is surjective. In fact, suppose first that μ is surjective. Then, for any $f \in \operatorname{Hom}_R(S, S/R)$, we may use the argument in the proof of Corollary 3.7 to get two homomorphisms $f_1 : S \to S$ and $f_2 : S/R \to S/R$ such that $f = f_1\pi + \pi f_2$. Since μ is surjective, there is an element $r \in R$ such that f_2 is the right multiplication of r. This implies that πf_2 factorizes through π. Hence π^* is surjective. Conversely, Suppose that π^* is surjective. For any $h \in S'$, there is an element $s \in S$ such that $h : x + R \mapsto xs + R$ for $x \in S$. In particular, $0 = 1 + R \in S/R$ is mapped to $s + R = 0 \in S/R$. Hence $s \in R$, and therefore μ is surjective.

(2) Recall that a ring homomorphism is an epimorphism if and only if the multiplication map $S \otimes_R S \to S$ is an isomorphism as S–S-bimodules. Since λ is injective, it follows from the exact sequence $(*)$ that we have a long exact sequence of S–R-bimodules:

$$0 \longrightarrow \operatorname{Tor}_1^R(S, S) \longrightarrow \operatorname{Tor}_1^R(S, S/R) \longrightarrow S \otimes_R R \xrightarrow{1 \otimes \lambda} S \otimes_R S \longrightarrow S \otimes_R (S/R) \longrightarrow 0.$$

Since $\mathrm{Tor}_1^R(S,S) = 0$ and $1 \otimes_R \lambda$ is an isomorphism of S-R-modules, we have $S \otimes_R (S/R) = 0 = \mathrm{Tor}_1^R(S, S/R)$.

Now, by applying $\mathrm{Hom}_R(-, S/R)$ to $(*)$, we get another exact sequence of R-$\mathrm{End}_R(S/R)$-bimodules:

$$(**) \quad 0 \longrightarrow \mathrm{Hom}_R(S/R, S/R) \longrightarrow \mathrm{Hom}_R(S, S/R) \longrightarrow \mathrm{Hom}_R(R, S/R).$$

One can check that the last homomorphism in the above sequence $(**)$ is surjective because each element $s + R$ in S/R gives rise to at least one homomorphism from the R-module S to the R-module S/R by $x \mapsto xs + R$ for $x \in S$. This yields the following exact sequence of S-$\mathrm{End}_R(S/R)$-bimodules:

$$0 \longrightarrow S \otimes_R \mathrm{Hom}_R(S/R, S/R) \longrightarrow S \otimes_R \mathrm{Hom}_R(S, S/R) \longrightarrow S \otimes_R (S/R) \longrightarrow 0,$$

which shows that $S \otimes_R \mathrm{Hom}_R(S/R, S/R) \xrightarrow{\sim} S \otimes_R \mathrm{Hom}_R(S, S/R)$. Clearly, under this isomorphism the element $1 \otimes_R 1$ in $S \otimes \mathrm{Hom}_R(S/R, S/R)$ is sent to $1 \otimes \pi$. Since the multiplication map: $S \otimes_R S \longrightarrow S$ is an isomorphism of S–S-bimodules, we see that the multiplication map: $S \otimes_R X \to X$ is an isomorphism for every S-module X. Clearly, $\mathrm{Hom}_R({}_R S_S, S/R)$ is an S-module. So, it follows that $S \otimes_R \mathrm{Hom}_R(S, S/R) \to \mathrm{Hom}_R(S, S/R)$ is an isomorphism under which $1 \otimes \pi$ is sent to π. As a result, the map $\theta: S \otimes_R S' \to \mathrm{Hom}_R(S, S/R)$ defined by $s \otimes f \mapsto (t \mapsto (ts)(\pi f))$ for $s, t \in S$ and $f \in S'$, is an isomorphism of S-S'-bimodules. Clearly, under this isomorphism, the element $1 \otimes 1$ in $S \otimes_R S'$ is sent to π.

(3) Applying $- \otimes_R S'$ to the sequence $(*)$ and identifying $R \otimes_R S'$ with S', we then obtain the following right exact sequence of R-S'-bimodules:

$$(\spadesuit) \quad S' \xrightarrow{\lambda'} S \otimes_R S' \xrightarrow{\pi \otimes S'} (S/R) \otimes_R S' \to 0,$$

where the map λ' is defined by $f \mapsto 1 \otimes f$ for any $f \in S'$. Combining this sequence with $(**)$, one can check that the following diagram of R-S'-bimodules is exact and commutative:

$$
\begin{array}{ccccccccc}
S' & \xrightarrow{\lambda'} & S \otimes_R S' & \xrightarrow{\pi \otimes S'} & (S/R) \otimes_R S' & \longrightarrow & 0 \\
\| & & \theta \downarrow & & \psi \downarrow & & \\
0 \longrightarrow & \mathrm{Hom}_R(S/R, S/R) & \xrightarrow{\pi_*} & \mathrm{Hom}_R(S, S/R) & \longrightarrow & S/R & \longrightarrow & 0
\end{array}
$$

where ψ is the evaluation map, and where $\mathrm{Hom}_R(R, S/R)$ is identified with S/R as R-S'-bimodules. Since θ is an isomorphism, we infer that λ' is injective, and that ψ is an isomorphism of R–S'-bimodules.

(4) Suppose that λ is an injective homological ring epimorphism. Then $\mathrm{Tor}_i^R(S, S) = 0$ for $i > 0$. Recall that we have proved that $S \otimes_R (S/R) = 0 = \mathrm{Tor}_1^R(S, S/R)$ in (2). Thus, by applying the tensor functor $S \otimes_R -$ to the canonical sequence $(*)$, we conclude that $\mathrm{Tor}_i^R(S, S/R) = 0$. By (3), we know that $(S/R) \otimes_R S' \simeq S/R$ as left R-modules. Thus, $\mathrm{Tor}_i^R(S, (S/R) \otimes_R S') = 0$. Since $S \otimes_R S'$ is a left S-module, it follows from Lemma 2.2(2) that $\mathrm{Tor}_i^R(S, S \otimes_R S') = 0$. Now, applying the tensor functor $S \otimes_R -$ to the exact sequence $(\spadesuit)$, we obtain $\mathrm{Tor}_i^R(S, S') = 0$.

(5) Since R is commutative, the tensor product $S \otimes_R S'$ of S and S' over R is a ring, which is well defined. By Lemma 6.5(3), there exists an exact sequence of R–S'-modules:

$$0 \to S' \xrightarrow{\lambda'} S \otimes_R S' \xrightarrow{\pi \otimes S'} (S/R) \otimes_R S' \to 0.$$

Since λ' is a ring homomorphism, the ring $S \otimes_R S'$ can be considered an S'–S'-bimodule via λ', and therefore, $(S/R) \otimes_R S'$ can also be regarded as an S'–S'-bimodule. In addition, by Lemma 6.5(3), the evaluation map $\psi: (S/R) \otimes_R S' \to S/R$, defined by $y \otimes g \mapsto (y)g$, for any $y \in S/R$ and $g \in S'$, is an isomorphism of R-S'-bimodules. Since the image of $(y)g \otimes 1$ under ψ is also equal to $(y)g$, we have $(y)g \otimes 1 = y \otimes g$ in $(S/R) \otimes_R S'$. Consequently, for any $f, g \in S'$ and $y \in S/R$, we get $y \otimes fg = f(y \otimes g) = f((y)g \otimes 1) = (y)g \otimes f$ in $(S/R) \otimes_R S'$,

where the first and third equalities follow from the left S'-module structure of $(S/R) \otimes_R S'$. This yields that $(y)fg = (y \otimes fg)\psi = ((y)g \otimes f)\psi = (y)gf$ in S/R. Thus, $fg = gf$. Since f and g are arbitrary elements in S', we see that S' is a commutative ring. $\square$

As a consequence of Theorem 1.1, we have the following corollary.

COROLLARY 6.6. (1) *Let* $\lambda : R \to S$ *be an injective ring epimorphism such that* $\mathrm{Tor}_1^R(S, S) = 0$. *If* $_RS$ *has projective dimension at most one, then there is a recollement of derived module categories*:

$$\mathscr{D}(S \sqcup_R S') \longrightarrow \mathscr{D}(B) \longrightarrow \mathscr{D}(R)$$

where $S \sqcup_R S'$ *is the coproduct of* S *and* S' *over* R. *If, in addition,* R *is commutative, then the coproduct* $S \sqcup_R S'$ *can be replaced by the tensor product* $S \otimes_R S'$ *of* S *and* S' *over* R.

(2) *Let* R *be a ring,* Σ *a left Ore set of regular elements of* R, *and* $S := \Sigma^{-1}R$ *the localization of* R *at* Σ. *If* $_RS$ *has projective dimension at most 1, then the recollement in* (1) *exists.*

Proof. (1) Let $R_0 = R$, $R_1 = S$, $R_2 = \mathrm{End}_R(S/R)$, $T := S \oplus S/R$, and $B = \mathrm{End}_R(T)$. Under the isomorphism θ of Lemma 6.5(2), B is identified with the matrix ring $\left(\begin{smallmatrix} S & S \otimes_R S' \\ 0 & S' \end{smallmatrix} \right)$, and the map π^* from Lemma 6.4(2) becomes a map $\varphi : \left(\begin{smallmatrix} s \\ 0 \end{smallmatrix} \right) \mapsto \left(\begin{smallmatrix} cs \otimes 1 \\ 0 \end{smallmatrix} \right)$ as in Lemma 6.1. Thus, the first statement in Corollary 6.6(1) follows from Theorem 1.1(1) and Lemma 6.1.

Now, assume that R is commutative. It is well known that if $R \to \Lambda$ is a ring epimorphism, then Λ is commutative, too. This means that S is commutative, and therefore S' is commutative by Lemma 6.5(5). Thus, the second statement in Corollary 6.6(1) follows immediately from Lemma 6.3.

(2) follows from (1). $\square$

7. Dedekind domains and recollements of derived module categories

In this section, we shall first discuss recollements of derived module categories arising from injective homological ring epimorphisms $\lambda : R \to S$ between arbitrary Dedekind domains. As a consequence, we can produce examples to show that two different stratifications of a derived module category by derived module categories of rings may have different derived composition factors, which answers negatively a question in [**5**] and shows that the Jordan–Hölder theorem fails for derived module categories with simple derived module categories as composition factors.

Note that if R is a commutative ring and $\lambda : R \to S$ is a ring epimorphism, then S must be commutative. So, in the following, we can assume that both rings R and S are commutative rings.

7.1. Recollements constructed from Dedekind domains

Throughout this subsection, R always stands for a Dedekind domain. Recall that R is called a Dedekind domain if it is a hereditary integral domain. A typical example of Dedekind domains is the ring $\mathbb{Z}$ of all rational integers. Note that, for an integer domain R which is not a field, it is a Dedekind domain if and only if R is noetherian, and the localization of R at each maximal ideal is a discrete valuation ring. Furthermore, if R is a Dedekind domain, then it must be a 1-Gorenstein ring, which is by definition noetherian and the injective dimension of R is at most 1.

We denote by $\mathrm{Spec}(R)$ (respectively, $\mathrm{mSpec}(R)$) the set of all prime (respectively, maximal) ideals of R. It is known that the Krull dimension of R is at most one, that is, each non-zero prime ideal of R is maximal. Moreover, for an R-module M, we denote by $E(M)$ the injective envelope of it.

For a multiplicative subset Σ of R, we denote by $\Sigma^{-1}R$ the localization of R at Σ, and by $f_\Sigma : R \to \Sigma^{-1}R$ the canonical homological ring epimorphism. Clearly, the map f_Σ is injective since R is an integral domain. If Φ is the multiplicative set of all non-zero elements of R, then $\Phi^{-1}R$ is a field, which is called the quotient field of R, and simply denoted by Q. Note that $\Sigma^{-1}R$ can be regarded as an intermediate ring between R and Q, that is, $R \subseteq \Sigma^{-1}R \subseteq Q$.

For a prime ideal $\mathfrak{p}$ of R, we always write $R_\mathfrak{p}$ for $(R \setminus \mathfrak{p})^{-1}R$, and $f_\mathfrak{p}$ for $f_{R \setminus \mathfrak{p}}$, and say that $R_\mathfrak{p}$ is the localization of R at $\mathfrak{p}$. For example, the localization $\mathbb{Z}_\mathfrak{p}$ of $\mathbb{Z}$ at the maximal ideal $\mathfrak{p} = p\mathbb{Z}$ coincides with $\mathbb{Q}_{(p)}$ for every prime $p \in \mathbb{N}$, where $\mathbb{Q}_{(p)}$ is the set of all p-integers. Recall that $q = n/m \in \mathbb{Q}$ with m, n a pair of coprime integers is called a p-integer if p does not divide m.

Further, we define $J_\mathfrak{p} := \varprojlim_i R/\mathfrak{p}^i$, the $\mathfrak{p}$-adic completion of R. It is clear that, for each $\mathfrak{p} \in \mathrm{mSpec}(R)$, we have $J_\mathfrak{p} \simeq \varprojlim_i R_\mathfrak{p}/\mathfrak{p}^i R_\mathfrak{p} \simeq \mathrm{End}_R(E(R/\mathfrak{p}))$ as rings (see [**25**, Theorem 3.4.1, **35**, Corollary 11.2]).

According to Bass [**8**, Theorems 1, 6.2], we see that the regular module R has a minimal injective resolution of the form:

$$0 \longrightarrow R \xrightarrow{f_\Phi} Q \xrightarrow{\pi} \bigoplus_{\mathfrak{p} \in \mathrm{mSpec}(R)} E(R/\mathfrak{p}) \longrightarrow 0,$$

where π is the canonical surjective map which is regarded as a homomorphism of R-modules.

In the following, we shall consider the so-called Bass tilting modules, as mentioned in [**3**]. Now, let us recall the construction.

Let Δ be a subset of $\mathrm{mSpec}(R)$, and define

$$\Delta' := \mathrm{mSpec}(R) \setminus \Delta, \quad R_{(\Delta)} := \pi^{-1}\left(\bigoplus_{\mathfrak{p} \in \Delta} E(R/\mathfrak{p})\right)$$

$$\text{and} \quad T_{(\Delta)} := R_{(\Delta)} \oplus \bigoplus_{\mathfrak{p} \in \Delta} E(R/\mathfrak{p}).$$

Then, we get two associated exact sequences of R-modules

$$(a) \quad 0 \longrightarrow R \longrightarrow R_{(\Delta)} \xrightarrow{\pi} \bigoplus_{\mathfrak{p} \in \Delta} E(R/\mathfrak{p}) \longrightarrow 0;$$

$$(b) \quad 0 \longrightarrow R_{(\Delta)} \longrightarrow Q \longrightarrow \bigoplus_{\mathfrak{p} \in \Delta'} E(R/\mathfrak{p}) \longrightarrow 0.$$

Note that $R_{(\Delta)}$ is just an R-submodule of Q. It is shown in [**3**, Section 4] that the R-module $T_{(\Delta)}$ is a tilting module, which is called a Bass tilting module over R. Further, Trlifaj and Pospíšil [**39**] prove that every tilting module over R is equivalent to a Bass tilting module (see also [**7**, Corollary 6.12]). Clearly, the sequence (a) implies that the R-tilting module $T_{(\Delta)}$ is good.

The next lemma describes some properties relevant to Bass tilting modules, which will be useful for our later discussions.

LEMMA 7.1. (1) $R_{(\Delta)} = \bigcap_{\mathfrak{p} \in \Delta'} R_\mathfrak{p}$, which is a flat R-module. Hence $R_{(\Delta)}$ can be regarded as a subring of Q containing R. In particular, the quotient field of $R_{(\Delta)}$ also equals Q. (Note that we set $\bigcap_{\mathfrak{p} \in \emptyset} R_\mathfrak{p} = Q$.)

(2) *The canonical inclusions $R \to R_{(\Delta)}$ and $R_{(\Delta)} \to Q$ are homological ring epimorphisms. In particular, $R_{(\Delta)}$ is a Dedekind domain.*

(3) *The $R_{(\Delta)}$-module*

$$T'_{(\Delta)} := Q \oplus \bigoplus_{\mathfrak{p} \in \Delta'} E(R/\mathfrak{p})$$

is a good tilting $R_{(\Delta)}$-module.

(4)

$$B_\Delta := \operatorname{End}_R(T_{(\Delta)}) \simeq \left(\begin{array}{cc} R_{(\Delta)} & R_{(\Delta)} \otimes_R J_\Delta \\ 0 & J_\Delta \end{array} \right),$$

where $J_\Delta := \operatorname{End}_R(R_{(\Delta)}/R) \simeq \prod_{\mathfrak{p} \in \Delta} J_\mathfrak{p}$.

(5)

$$B'_\Delta := \operatorname{End}_{R_{(\Delta)}}(T'_{(\Delta)}) \simeq \left(\begin{array}{cc} Q & Q \otimes_R J'_\Delta \\ 0 & J'_\Delta \end{array} \right),$$

where $J'_\Delta := \operatorname{End}_{R_{(\Delta)}}(Q/R_{(\Delta)}) \simeq \prod_{\mathfrak{p} \in \Delta'} J_\mathfrak{p}$.

(6) *The canonical inclusion $R_{(\Delta)} \to Q$ induces a ring isomorphism*

$$R_{(\Delta)} \otimes_R J_\Delta \simeq Q \otimes_R J_\Delta.$$

(7) *For any $P \subseteq \operatorname{mSpec}(R)$, the canonical map*

$$\Theta_P : Q \otimes_R \prod_{\mathfrak{p} \in P} J_\mathfrak{p} \longrightarrow \prod_{\mathfrak{p} \in P} Q \otimes_R J_\mathfrak{p},$$

defined by $q \otimes (x_\mathfrak{p})_{\mathfrak{p} \in P} \mapsto (q \otimes x_\mathfrak{p})_{\mathfrak{p} \in P}$ for $q \in Q$ and $x_\mathfrak{p} \in J_\mathfrak{p}$, is an injective ring homomorphism.

Proof. (1) is contained in [**40**].

(2) Recall that a Dedekind domain is a hereditary noetherian prime ring. Then, by Crawley-Boevey [**23**, Remark 3.3], we know that if $R \subseteq \Lambda \subseteq Q$ are extensions of rings, then $R \to \Lambda$ is a universal localization, and therefore it is a homological ring epimorphism. This also implies that Λ is a Dedekind domain. Thus, (2) follows directly.

(3) follows from [**3**, Section 4] since $R_{(\Delta)}$ is a Dedekind domain and the quotient field of $R_{(\Delta)}$ is equal to Q.

(4) Let $\mathfrak{p}$ and $\mathfrak{q}$ be two arbitrary maximal ideals of R. By Enochs and Jenda [**25**, Theorem 3.3.8], it follows that $\operatorname{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{q})) \neq 0$ if and only if $\mathfrak{p} = \mathfrak{q}$. Furthermore, we have $\operatorname{End}_R(E(R/\mathfrak{p})) \simeq J_\mathfrak{p}$ as rings. Consequently, we infer that $J_\Delta \simeq \operatorname{End}_R(\bigoplus_{\mathfrak{p} \in \Delta} E(R/\mathfrak{p})) \simeq \prod_{\mathfrak{p} \in \Delta} J_\mathfrak{p}$ as rings. According to Lemma 6.5(2), we have $\operatorname{Hom}_R(R_{(\Delta)}, \bigoplus_{\mathfrak{p} \in \Delta} E(R/\mathfrak{p})) \simeq R_{(\Delta)} \otimes_R J_\Delta$ as $R_{(\Delta)}$-J_Δ-bimodules. Now, (4) follows from Lemma 6.4(2) immediately.

(5) We first observe that $\operatorname{Hom}_R(X, Y) \simeq \operatorname{Hom}_{R_{(\Delta)}}(X, Y)$ and $X \otimes_R Y \simeq X \otimes_{R_{(\Delta)}} Y$, for any $R_{(\Delta)}$-modules X and Y since the canonical inclusion $R \to R_{(\Delta)}$ is a ring epimorphism, and then we can apply the argument in the proof of (4) to showing (5).

(6) Note that if C is a commutative noetherian ring and if I is an ideal of C, then (i) the I-adic completion of C is a flat C-module, and (ii) the product of flat C-modules is flat (see [**25**, Corollary 2.5.15, Theorem 3.2.24]). This implies that J_Δ is a flat R-module. In order to prove that $\mu_\Delta \otimes_R J_\Delta : R_{(\Delta)} \otimes_R J_\Delta \to Q \otimes_R J_\Delta$ is an isomorphism, where $\mu_\Delta : R_{(\Delta)} \to Q$ is the canonical inclusion, it suffices to show $(\bigoplus_{\mathfrak{p} \in \Delta'} E(R/\mathfrak{p})) \otimes_R J_\Delta = 0$. This is equivalent to $E(R/\mathfrak{p}) \otimes_R J_\Delta = 0$, for any $\mathfrak{p} \in \Delta'$. However, the latter is a direct consequence of [**25**, Lemma 6.7.7].

(7) Clearly, the map Θ_P is a ring homomorphism. It remains to show that Θ_P is injective. Definitely, it suffices to prove that Θ_P is injective for $P = \operatorname{mSpec}(R)$. In this case, by applying

[**2**, Corollary 9.8] to the tilting R-module $T_{(P)}$, we can verify the injectivity of Θ_P. This finishes the proof of (7). □

We remark that $R_{(\Delta)}$ is always an intersection of localizations by Lemma 7.1(2). But, in general, it may not be a localization of R at any multiplicative set. For a counterexample, we refer the reader to [**40**].

Combining Corollary 6.6(1) with Lemma 7.1, we have the following result on recollements of derived module categories of endomorphism rings.

PROPOSITION 7.2.   *Let R be a Dedekind domain and let Δ be a subset of $\mathrm{mSpec}(R)$. Then we get the following recollements of derived module categories:*

$$\mathscr{D}(Q \otimes_R J_\Delta) \longrightarrow \mathscr{D}(B_\Delta) \longrightarrow \mathscr{D}(R)$$

$$\mathscr{D}(Q \otimes_R J'_\Delta) \longrightarrow \mathscr{D}(B'_\Delta) \longrightarrow \mathscr{D}(R_{(\Delta)})$$

In the rest of this subsection, we consider the ring $\mathbb{Z}$, which is a Dedekind domain. In this case, we can have a much better formulation than Proposition 7.2. Our discussion below uses some basic results on p-adic numbers in algebraic number theory.

Fix a prime number $p \geqslant 2$. A p-adic integer is a formal infinite series $\sum_{i=0}^\infty a_i p^i$, where $0 \leqslant a_i < p$ for all $i \geqslant 0$. A p-adic number is a formal infinite series of the form $\sum_{j=-m}^\infty a_j p^j$, where $m \in \mathbb{Z}$ and $0 \leqslant a_j < p$ for all $j \geqslant -m$. The sets of all p-adic integers and p-adic numbers are denoted by $\mathbb{Z}_p$ and $\mathbb{Q}_p$, respectively. Note that $\mathbb{Z}_p$ is a discrete valuation ring of global dimension 1 with the unique maximal ideal $p\mathbb{Z}_p$, and that $\mathbb{Q}_p$ is a field.

If $f \in \mathbb{Q}$ is a rational number, then we can write $f = (g/h)p^{-m}$, where $g, h \in \mathbb{Z}$, $(gh, p) = 1$. Taking the p-adic expression $g/h = \sum_{i=0}^\infty a_i p^i$ of the rational number g/h, we have

$$f = \sum_{i=0}^\infty a_i p^{-m+i} \in \mathbb{Q}_p.$$

In this way, we can regard $\mathbb{Q}$ as a subfield of $\mathbb{Q}_p$. This implies that, for $f \in \mathbb{Q}$, there are at most finitely many prime numbers q such that $f \in \mathbb{Q}_q \setminus \mathbb{Z}_q$, or equivalently, $f \in \mathbb{Z}_q$ for almost all prime number q. It is well-known that $\mathbb{Q} \otimes_\mathbb{Z} \mathbb{Z}_p \simeq \mathbb{Q}_p$ by the multiplication map since $\mathbb{Q}_p = \{p^m y \mid m \in \mathbb{Z}, y \in \mathbb{Z}_p\}$. Clearly, $\mathbb{Q} \subset \mathbb{Q}_p$ and $\mathbb{Z} \subset \mathbb{Z}_{(p)} \subset \mathbb{Z}_p \subset \mathbb{Q}_p$, for every prime $p \in \mathbb{N} := \{0, 1, 2, \ldots\}$.

An alternative definition of $\mathbb{Z}_p$ is that $\mathbb{Z}_p$ is the p-adic completion $\varprojlim_i \mathbb{Z}/p^i\mathbb{Z}$ of $\mathbb{Z}$. Another algebraic definition of $\mathbb{Z}_p$ is that $\mathbb{Z}_p$ is isomorphic to the quotient of the formal power series ring $\mathbb{Z}[[X]]$ by the ideal generated by $X - p$. Note that $\mathbb{Q}_p$ is the field of fractions of $\mathbb{Z}_p$. For more details about p-adic numbers, one may refer to [**35**, Chapter II, Section 2]. We denote by $\widehat{\mathbb{Z}}$ the product $\prod_p \mathbb{Z}_p$ of all $\mathbb{Z}_p$ with p positive prime numbers. This is a commutative ring, which is isomorphic to $\mathrm{End}_\mathbb{Z}(\mathbb{Q}/\mathbb{Z})$.

Now, let Λ be the set of all prime numbers in $\mathbb{N}$, and let I be a subset of Λ. Set $I' := \Lambda \setminus I$, $\Delta := \{\mathfrak{p} = p\mathbb{Z} \mid p \in I\}$ and $\mathbb{Z}_{(I)} := \mathbb{Z}_{(\Delta)}$.

LEMMA 7.3.   *The following statements hold true for the ring $\mathbb{Z}$ of integers.*

(1) *Let $\Sigma := \mathbb{Z} \setminus \bigcup_{q \in I'} \mathfrak{q}$. Then $\mathbb{Z}_{(I)} = \Sigma^{-1}\mathbb{Z}$, which is the smallest subring of $\mathbb{Q}$ containing $1/p$ for all $p \in I$.*

(2) *The injective ring homomorphism*

$$\Theta_I : \mathbb{Q} \otimes_{\mathbb{Z}} \prod_{p \in I} \mathbb{Z}_p \longrightarrow \prod_{p \in I} \mathbb{Q}_p$$

defined by $q \otimes (x_p)_{p \in I} \mapsto (qx_p)_{p \in I}$ *for* $q \in \mathbb{Q}$ *and* $x_p \in \mathbb{Z}_p$ *satisfies that*

$$\mathrm{Im}(\Theta_I) = \mathbb{A}_I := \left\{ (y_p)_{p \in I} \in \prod_{p \in I} \mathbb{Q}_p \mid y_p \in \mathbb{Z}_p \text{ for almost all } p \in I \right\}.$$

In particular, if I *is a finite set, then* $\mathrm{Im}(\Theta_I) = \mathbb{A}_I = \prod_{p \in I} \mathbb{Q}_p$. *Note that* $\mathbb{A}_I$ *is a kind of adéle in global class field theory (see [**35**, Chapter VI]).*

(3) *There are ring isomorphisms:*

$$\mathbb{Q} \otimes_{\mathbb{Z}} \prod_{p \in I} \mathbb{Z}_p \simeq \mathbb{A}_I, \quad \mathbb{Q} \otimes_{\mathbb{Z}} \prod_{p \in I'} \mathbb{Z}_p \simeq \mathbb{A}_{I'}.$$

Proof. (1) Let $q \in I'$. By Lemma 7.1(1), we have $\mathbb{Z}_{(I)} = \bigcap_{q \in I'} \mathbb{Z}_{\mathfrak{q}}$, where $\mathbb{Z}_{\mathfrak{q}}$ is the localization of $\mathbb{Z}$ at $\mathfrak{q}$ with $\mathfrak{q} = q\mathbb{Z}$. It follows from $\mathbb{Z}_{\mathfrak{q}} = \mathbb{Q}_{(q)}$ that

$$\mathbb{Z}_{(I)} = \bigcap_{q \in I'} \mathbb{Q}_{(q)} = \mathbb{Z}[p^{-1} \mid p \in I] = \Sigma^{-1}\mathbb{Z}.$$

(2) For each prime number p, the canonical ring homomorphism $\mu : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathbb{Q}_p$, defined by $f \otimes x_p \mapsto f x_p$ for any $f \in \mathbb{Q}$, and $x_p \in \mathbb{Z}_p$, is an isomorphism. Moreover, for each $f \in \mathbb{Q}$, there are at most finitely many prime numbers q such that $f \in \mathbb{Q}_q \setminus \mathbb{Z}_q$. In other words, $f \in \mathbb{Z}_q$ for almost all prime number q. This implies $\mathrm{Im}(\Theta_I) = \mathbb{A}_I$.

(3) This follows from (2). □

With the help of Lemma 7.3, we can state Proposition 7.2 for $R = \mathbb{Z}$ more explicitly.

COROLLARY 7.4. *We have the following recollements of derived module categories:*

where $B_I := \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_{(I)} \oplus \mathbb{Z}_{(I)}/\mathbb{Z})$ *and* $B_I' := \mathrm{End}_{\mathbb{Z}_{(I)}}(\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}_{(I)})$.

7.2. *Examples*

In the following, we shall exploit Corollary 7.4 to give a couple of concrete examples of derived module categories that have two different stratifications by derived module categories of rings with different composition factors. This is related to the following problem proposed in [**5**], and initially came up around 1988 in the work of Cline, Parshall and Scott [**18**] on highest weight categories and quasi-hereditary algebras.

PROBLEM. Given a ring R, do all stratifications of $\mathscr{D}(R)$ by derived module categories of rings have the same finite number of factors, and are these factors the same for all stratifications, up to ordering and up to derived equivalence?

A negative partial solution to this problem can be seen from Examples 7.5 and 7.6 below.
Let us first recall the definition of a stratification of $\mathscr{D}(R)$ for R a ring in [**5**].
Let R be a ring. If there are rings R_1 and R_2 such that a recollement

$$(*) \quad \mathscr{D}(R_1) \longrightarrow \mathscr{D}(R) \longrightarrow \mathscr{D}(R_2)$$

exists, then R_i or $\mathscr{D}(R_i)$, with $1 \leqslant i \leqslant 2$, are called factors of R or $\mathscr{D}(R)$. In this case, we also
say that $(*)$ is a recollement of R. The ring R is called derived simple if $\mathscr{D}(R)$ does not admit
any non-trivial recollement whose factors are derived categories of rings. It is pointed out in
[**4**] that every Dedekind domain (thus every discrete valuation ring) is derived simple.

A stratification of $\mathscr{D}(R)$ is defined to be a sequence of iterated recollements of the following
form: a recollement of R, if it is not derived simple,

$$\mathscr{D}(R_0) \longrightarrow \mathscr{D}(R) \longrightarrow \mathscr{D}(R_1)$$

a recollement of R_0, if it is not derived simple,

$$\mathscr{D}(R_{00}) \longrightarrow \mathscr{D}(R_0) \longrightarrow \mathscr{D}(R_{01})$$

and a reollement of R_2, if it is not derived simple,

$$\mathscr{D}(R_{10}) \longrightarrow \mathscr{D}(R_1) \longrightarrow \mathscr{D}(R_{11})$$

and recollements of R_{ij} with $0 \leqslant i, j \leqslant 1$, if they are not derived simple, and so on, until
one arrives at derived simple rings at all positions, or continue to infinitum. All the derived
simple rings appearing in this procedure are called composition factors of the stratification.
The cardinality of the set of all composition factors (counting the multiplicity) is called the
length of the stratification. If this procedure stops after finitely many steps, we say that this
stratification is finite or of finite length.

The first example below shows two stratifications of a derived module category with infinitely
many different derived simple module categories as composition factors

EXAMPLE 7.5. Let $\mathbb{Z} \hookrightarrow \mathbb{Q}$ be the inclusion. Then $T = \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ is a tilting $\mathbb{Z}$-module, and

$$B := \operatorname{End}_R(T) = \begin{pmatrix} \mathbb{Q} & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \\ 0 & \widehat{\mathbb{Z}} \end{pmatrix}.$$

Note that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \simeq \mathbb{R}$ as abelian groups (see [**29**, Exercise 6.2, p. 109]), where $\mathbb{R}$ is
the field of real numbers.
We take $\Delta := \operatorname{mSpec}(\mathbb{Z})$. By Corollary 6.6(1), we have a recollement:

$$\mathscr{D}(\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}) \longrightarrow \mathscr{D}(B) \longrightarrow \mathscr{D}(\mathbb{Z})$$

Let $e_2 = (1, 0, \ldots) \in \widehat{\mathbb{Z}}$. Then $\widehat{\mathbb{Z}} = \widehat{\mathbb{Z}}e_2 \oplus \widehat{\mathbb{Z}}(1 - e_2)$. This is a decomposition of ideals of $\widehat{\mathbb{Z}}$. Thus, we have a decomposition of ideals of the ring $\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$:

$$\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_2 \oplus \mathbb{Q} \otimes_{\mathbb{Z}} \prod_{p \geqslant 3} \mathbb{Z}_p = \mathbb{Q}_2 \oplus \mathbb{Q} \otimes_{\mathbb{Z}} \prod_{p \geqslant 3} \mathbb{Z}_p.$$

This procedure can be repeated infinitely many times. Then it follows that $\mathscr{D}(\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$ has a derived composition series with infinitely many simple factors $\mathscr{D}(\mathbb{Q}_p)$. This shows that $\mathscr{D}(B)$ has a stratification with derived composition factors equivalent to either $\mathscr{D}(\mathbb{Z})$ or $\mathscr{D}(\mathbb{Q}_p)$, both are derived simple, that is, each of them cannot be a middle term in any proper recollement of derived module categories of rings.

Transparently, it follows from the triangular form of B that $\mathscr{D}(B)$ has a stratification with infinitely many composition factors equivalent to either $\mathscr{D}(\mathbb{Q})$ or $\mathscr{D}(\mathbb{Z}_p)$. Clearly, $\mathscr{D}(\mathbb{Z})$ and $\mathscr{D}(\mathbb{Q})$ are not equivalent as triangulated categories since derived equivalences preserve the semisimplicity of rings. Thus $\mathscr{D}(B)$ has two stratifications which have different composition factors. This gives a negative answer to the second question of the above-mentioned problem.

In Example 7.5, the two stratifications of the category $\mathscr{D}(B)$ by derived module categories have infinite many composition factors. In the next example, we shall see that even if one requires finiteness of stratifications of a derived module category, their composition factors still may be different. This is contrary to the well known Jordan–Hölder theorem which says that any two (finite) composition series of a group have the same list of composition factors (up to the ordering and up to isomorphism).

EXAMPLE 7.6. (1) Let I be a non-empty finite subset of $\mathrm{mSpec}(\mathbb{Z})$. We consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}_{(I)} \to \bigoplus_{p \in I} E(\mathbb{Z}/p) \to 0$$

of abelian groups. Then $T := \mathbb{Z}_{(I)} \oplus \bigoplus_{p \in I} E(\mathbb{Z}/p\mathbb{Z})$ is a tilting module. On the one hand, by Lemmas 6.4(2) and 7.1(4), we have

$$\mathrm{End}_{\mathbb{Z}}(T) \simeq \begin{pmatrix} \mathbb{Z}_{(I)} & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_I, \mathbb{Z}_{(I)}/\mathbb{Z}) \\ 0 & \bigoplus_{p \in I} \mathbb{Z}_p \end{pmatrix}.$$

On the other hand, since I is a finite set, by Corollary 7.4, $\mathrm{End}_{\mathbb{Z}}(T)$ admits a recollement

$$\mathscr{D}(\bigoplus_{p \in I} \mathbb{Q}_p) \longrightarrow \mathscr{D}(\mathrm{End}_{\mathbb{Z}}(T)) \longrightarrow \mathscr{D}(\mathbb{Z})$$

Thus, $\mathscr{D}(\mathrm{End}_{\mathbb{Z}}(T))$ admits two stratifications, one of which has the composition factors $\mathbb{Z}_{(I)}$ and $\mathbb{Z}_p$ with $p \in I$, and the other has the composition factors $\mathbb{Z}$ and $\mathbb{Q}_p$ with $p \in I$. Since $\mathbb{Z}_{(I)}$ is a localization of $\mathbb{Z}$ by Lemma 7.3, it is of global dimension 1. Note that derived equivalences preserve the centers of rings. This shows that all the rings $\mathbb{Z}, \mathbb{Z}_{(I)}, \mathbb{Z}_p$ and $\mathbb{Q}_p$ are pairwise not derived-equivalent. Hence the two stratifications have completely different composition factors.

(2) Let $\mathfrak{p} = p\mathbb{Z} \subset \mathbb{Z}$ with p a prime number in $\mathbb{N}$. We consider the exact sequence of $\mathbb{Z}_{\mathfrak{p}}$-modules:

$$0 \to \mathbb{Z}_{\mathfrak{p}} \to \mathbb{Q} \to E(\mathbb{Z}_{\mathfrak{p}}/p\mathbb{Z}_{\mathfrak{p}}) \to 0.$$

Define $T := \mathbb{Q} \oplus E(\mathbb{Z}_{\mathfrak{p}}/p\mathbb{Z}_{\mathfrak{p}})$. Thus, by Lemma 7.1, Lemma 7.3 and Corollary 7.4, we have

$$\operatorname{End}_{\mathbb{Z}_{\mathfrak{p}}}(T) \simeq \operatorname{End}_{\mathbb{Z}}(T) \simeq \left(\begin{array}{cc} \mathbb{Q} & \mathbb{Q}_p \\ 0 & \mathbb{Z}_p \end{array} \right),$$

and a recollement:

$$\mathscr{D}(\mathbb{Q}_p) \longrightarrow \mathscr{D}(\operatorname{End}_{\mathbb{Z}_{\mathfrak{p}}}(T)) \longrightarrow \mathscr{D}(\mathbb{Z}_{\mathfrak{p}})$$

Note that the ring $\operatorname{End}_{\mathbb{Z}_{\mathfrak{p}}}(T)$ is left hereditary, but not left noetherian.

On the one hand, $\mathscr{D}(\operatorname{End}_{\mathbb{Z}_{\mathfrak{p}}}(T))$ has clearly a stratification of length 2 with the composition factors $\mathbb{Q}$ and $\mathbb{Z}_p$. On the other hand, it admits another stratification of length 2 with the composition factors $\mathbb{Q}_p$ and $\mathbb{Z}_{\mathfrak{p}}$. Note that $\mathbb{Z}_{\mathfrak{p}} = \mathbb{Q}_{(p)}$. Since $\mathbb{Z}_p$ and $\mathbb{Q}_p$ are uncountable sets and since derived equivalences preserve the centers of rings, we deduce that neither $\mathbb{Q}$ and $\mathbb{Q}_{(p)}$, nor $\mathbb{Z}_p$ and $\mathbb{Q}_{(p)}$ are derived equivalent. Clearly, the global dimensions of $\mathbb{Z}_p$ and $\mathbb{Q}_{(p)}$ are one. Thus, we have proved that the derived category of the ring $\operatorname{End}_{\mathbb{Z}_{\mathfrak{p}}}(T)$ has two stratifications of length 2 without any common composition factors.

Thus, this example shows also that the main result in [**5**, Theorem 6.1] for hereditary artin algebras cannot be extended to left hereditary rings.

Note that in each example given in this section the sets of composition factors of the two stratifications of the derived module category have the same cardinalities. In the next section, we shall see that this phenomenon is not always true.

8. *Further examples and open questions*

The main purpose of this section is to present examples of derived module categories of rings such that they possess two stratifications (by derived module categories of rings) with different finite lengths. Namely, we consider the following:

QUESTION.   Is there a ring R such that $\mathscr{D}(R)$ has two stratifications of different finite lengths by derived module categories of rings ?

Thus, we solve the whole problem in [**5**] negatively.

Let k be a field. We denote by $k[x]$ and $k[[x]]$ the polynomial and formal power-series algebras over k in one variable x, respectively, and by $k((x))$ the Laurent power series algebra in one variable x, that is, $k((x)) := \{x^{-n}a \mid n \in \mathbb{N}, a \in k[[x]]\}$.

Now, let k be an algebraically closed field, and let R be the Kronecker algebra $\left(\begin{smallmatrix} k & k^2 \\ 0 & k \end{smallmatrix} \right)$. It is known that R can be given by the following quiver:

$$Q: \quad 2 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 1 \ ,$$

and that R-Mod is equivalent to the category of representations of Q over k.

Let V be a simple regular R-module. For each $m > 0$, we denote by $V[m]$ the module of regular length m on the ray

$$V = V[1] \subset V[2] \subset \ldots \subset V[m] \subset V[m+1] \subset \ldots,$$

and let $V[\infty] = \varinjlim V[m]$ be the corresponding Prüfer modules. Note that the only regular submodule of $V[\infty]$ of regular length m is $V[m]$ with its canonical inclusion in $V[\infty]$, and that each endomorphism of $V[\infty]$ in R-Mod restricts to an endomorphism of $V[m]$ for any $m > 0$. Thus, $V[\infty]$ admits a unique chain of regular submodules. For more details, we refer to [**37**, Section 4.5].

From now on, we denote by V the simple regular R-module: $k \underset{1}{\overset{0}{\rightrightarrows}} k$. Let $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Since $\mathrm{Hom}_R(Re_1, Re_2) \simeq k^2$, we can identify a homomorphism from Re_1 to Re_2 in R-Mod with an element in k^2. Fix a minimal projective resolution of V:

$$ 0 \longrightarrow Re_1 \xrightarrow{\partial := (1,0)} Re_2 \longrightarrow V \longrightarrow 0, $$

and denote by $\lambda : R \to R_V$ the universal localization of R at the set $\Sigma := \{\partial\}$.

It follows from [**38**, Theorems 4.9, 5.1, and 5.3] that R_V is hereditary, λ is injective, and $R_V \oplus R_V/R$ is a tilting R-module. Moreover, by [**6**, Proposition 1.8], we get $R_V/R \simeq V[\infty]^2$ as R-modules. Note that $\mathrm{Hom}_R(R_V/R, R_V) = 0$ by Lemma 6.4(1).

For simplicity, we write $T := R_V \oplus V[\infty]^2$ and $B := \mathrm{End}_R(T)$.

Since $R_V e_1 \simeq R_V e_2$ as R_V-modules, we obtain the following ring isomorphisms:

$$ (*) \quad B \simeq M_2(\mathrm{End}_R(R_V e_1 \oplus V[\infty])) \simeq M_2\left(\left(\begin{array}{cc} e_1 R_V e_1 & \mathrm{Hom}_R(R_V e_1, V[\infty]) \\ 0 & \mathrm{End}_R(V[\infty]) \end{array} \right) \right). $$

Now, applying Corollary 6.6(1) to the tilting module T, we can get the following recollement of derived module categories:

$$ (**) \quad \mathscr{D}(R_V \sqcup_R S') \longrightarrow \mathscr{D}(B) \longrightarrow \mathscr{D}(R) $$

where $S' := M_2(\mathrm{End}_R(V[\infty]))$ and $R_V \sqcup_R S'$ is the coproduct of R_V and S' over R.

In the following, we shall describe the rings B, S' and $R_V \sqcup_R S'$ explicitly.

First, we claim that the universal localization of R at ∂ is given by $\lambda : R \to M_2(k[x])$, $\left(\begin{smallmatrix} a & (c,d) \\ 0 & b \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & c+dx \\ 0 & b \end{smallmatrix}\right)$, for any $a,b,c,d \in k$, and therefore $e_1 R_V e_1 \simeq k[x]$ as rings.

In fact, it suffices to check that λ satisfies the conditions (1) and (2) in Theorem 3.1. Put $\Lambda := M_2(k[x])$. Since λ sends $f := \left(\begin{smallmatrix} 0 & (1,0) \\ 0 & 0 \end{smallmatrix}\right)$ to $e_{12} := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, the homomorphism $\Lambda \otimes_R \partial : \Lambda e_1 \to \Lambda e_2$ is an isomorphism with the inverse map given by the right multiplication with $e_{21} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. This verifies the condition (1) in Theorem 3.1.

Now suppose that $\varphi : R \to \Gamma$ is a ring homomorphism such that $\Gamma \otimes_R \partial : \Gamma \otimes_R Re_1 \to \Gamma \otimes_R Re_2$ is an isomorphism. Let $\epsilon_i := (e_i)\varphi$ and $\xi := (f)\varphi$. Then $\Gamma \otimes_R \partial$ can be regarded as $\partial' : \Gamma \epsilon_1 \to \Gamma \epsilon_2$ which sends ϵ_1 to $\xi \in \epsilon_1 \Gamma \epsilon_2$. Since ∂' is an isomorphism, there is an element $\eta \in \epsilon_2 \Gamma \epsilon_1$ such that $\xi \eta = \epsilon_1$ and $\eta \xi = \epsilon_2$. Now we write $\Gamma = \left(\begin{smallmatrix} \epsilon_1 \Gamma \epsilon_1 & \epsilon_1 \Gamma \epsilon_2 \\ \epsilon_2 \Gamma \epsilon_1 & \epsilon_2 \Gamma \epsilon_2 \end{smallmatrix}\right)$, define a ring isomorphism $\rho : \Gamma \longrightarrow M_2(\epsilon_1 \Gamma \epsilon_1)$ by $\left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right) \in \Gamma \mapsto \left(\begin{smallmatrix} a_1 & a_2 \eta \\ \xi a_3 & \xi a_4 \eta \end{smallmatrix}\right) \in M_2(\epsilon_1 \Gamma \epsilon_1)$, and consider the composition $\varphi\rho : R \to M_2(\epsilon_1 \Gamma \epsilon_1)$. We can check that the map $\varphi\rho$ sends e_1 and e_2 and f in R to $e_1' := \left(\begin{smallmatrix} \epsilon_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $e_2' := \left(\begin{smallmatrix} 0 & 0 \\ 0 & \epsilon_1 \end{smallmatrix}\right)$ and $e_{12}' := \left(\begin{smallmatrix} 0 & \epsilon_1 \\ 0 & 0 \end{smallmatrix}\right)$ in $M_2(\epsilon_1 \Gamma \epsilon_1)$, respectively. Since $\left(\begin{smallmatrix} r & 0 \\ 0 & 0 \end{smallmatrix}\right) f = \left(\begin{smallmatrix} 0 & (r,0) \\ 0 & 0 \end{smallmatrix}\right) = f \left(\begin{smallmatrix} 0 & 0 \\ 0 & r \end{smallmatrix}\right) \in R$ for $r \in k$, we can write $\varphi\rho = \left(\begin{smallmatrix} \varphi_1 & \varphi_{12} \\ 0 & \varphi_1 \end{smallmatrix}\right)$, where $\varphi_1 : k \to \epsilon_1 \Gamma \epsilon_1$ is a ring homomorphism, and $\varphi_{12} : k^2 \to \epsilon_1 \Gamma \epsilon_1$ is an additive map. Let $y := ((0,1))\varphi_{12}$. One can verify that $((1,0))\varphi_{12} = \epsilon_1$ and $y(d)\varphi_1 = (d)\varphi_1 y$ for $d \in k$. This implies that φ_1 can be extended to a ring homomorphism $\psi_1 : k[x] \to \epsilon_1 \Gamma \epsilon_1$ such that $(x)\psi_1 = y$. Clearly, such a extension is unique. Now, we define $\psi = \left(\begin{smallmatrix} \psi_1 & \psi_1 \\ \psi_1 & \psi_1 \end{smallmatrix}\right) : \Lambda \to M_2(\epsilon_1 \Gamma \epsilon_1)$ by $(a_{ij})_{1 \leqslant i,j \leqslant 2} \mapsto ((a_{ij})\psi_1)_{1 \leqslant i,j \leqslant 2}$ for $a_{ij} \in k[x]$. One can check that ψ is a ring homomorphism such that $\varphi\rho = \lambda\psi$. It remains to show that such a ψ is unique. Let $\psi' : \Lambda \to M_2(\epsilon_1 \Gamma \epsilon_1)$ be another ring homomorphism satisfying $\varphi\rho = \lambda\psi'$. Then ψ' sends e_1 and e_2 and e_{12} in Λ to e_1', e_2' and e_{12}' in $M_2(\epsilon_1 \Gamma \epsilon_1)$, respectively. It follows that ψ' is of the form $\left(\begin{smallmatrix} \psi_1' & \psi_1' \\ \psi_1' & \psi_1' \end{smallmatrix}\right)$, where $\psi_1' : k[x] \to \epsilon_1 \Gamma \epsilon_1$ is a ring homomorphism. Since $\varphi\rho = \lambda\psi'$, the restriction of ψ_1' to k coincides with φ_1, and $(x)\psi_1' = y$. This implies $\psi_1' = \psi_1$ and $\psi' = \psi$. Thus, the condition (2) in Theorem 3.1 is satisfied, and the proof of the claim is completed.

Second, we claim that $\mathrm{End}_R(V[\infty])$ is isomorphic to $k[[x]]$. In fact, this follows from the following isomorphisms of abelian groups:

$$\mathrm{End}_R(V[\infty]) \simeq \varprojlim \mathrm{Hom}_R(V[m], V[\infty]) \simeq \varprojlim \mathrm{Hom}_R(V[m], V[m])$$
$$\simeq \varprojlim k[x]/(x^m) \simeq k[[x]],$$

where the composition of the above isomorphisms gives rise to a ring isomorphism $\omega : \mathrm{End}_R(V[\infty]) \to k[[x]]$. Thus, $S' \simeq M_2(k[[x]])$ as rings. In this sense, we can identify S' with $M_2(k[[x]])$ under the isomorphism ω.

Third, a direct calculation shows that the ring homomorphism $\mu : R \to S'$, which appears in Lemma 6.5(1), is given by $\begin{pmatrix} a' & (c',d') \\ 0 & b' \end{pmatrix} \mapsto \begin{pmatrix} a' & d'+c'x \\ 0 & b' \end{pmatrix}$ for any $a', b', c', d' \in k$.

Finally, we claim $R_V \sqcup_R S' \simeq M_2(k((x)))$ as rings.

In fact, we recall that R_V is the universal localization of R at $\Sigma := \{\partial\}$. Define $\varphi := S' \otimes_R \partial : S'e_1 \to S'e_2$. Then it follows from Lemma 6.2 that $R_V \sqcup_R S'$ is isomorphic to the universal localization S'_φ of S' at φ. Since $\mathrm{Hom}_{S'}(S'e_1, S'e_2) \simeq e_1S'e_2 \simeq k[[x]]$, the map φ corresponds to the matrix element $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ in S'. Now, let $\rho_x : k[[x]] \to k[[x]]$ be the right multiplication map by x. Since S' is Morita equivalent to $k[[x]]$, we conclude from Lemma 3.5 that $S'_\varphi = M_2(k[[x]]_{\rho_x})$, where $k[[x]]_{\rho_x}$ is the universal localization of $k[[x]]$ at ρ_x. Since $k[[x]]$ is commutative, the ring $k[[x]]_{\rho_x}$ is isomorphic to the localization $\Theta^{-1}k[[x]]$ of $k[[x]]$ at the multiplicative subset $\Theta := \{x^m \mid m \in \mathbb{N}\}$. Thus, $\Theta^{-1}k[[x]]$ is the Laurent power series ring $k((x))$. Therefore, we get the following isomorphisms of rings:

$$R_V \sqcup_R S' \simeq S'_\varphi \simeq M_2(k[[x]]_{\rho_x}) \simeq M_2(\Theta^{-1}k[[x]]) \simeq M_2(k((x))).$$

On the one hand, by setting $C := \mathrm{End}_R(R_Ve_1 \oplus V[\infty])$ and using Morita equivalences, the recollement $(**)$ can be rewritten as

$$\mathscr{D}(k((x))) \longrightarrow \mathscr{D}(C) \longrightarrow \mathscr{D}(R)$$

On the other hand, since $e_1R_Ve_1 \simeq k[x]$ and $\mathrm{End}_R(V[\infty]) \simeq k[[x]]$, it follows from $(*)$ that the ring C admits another recollement

$$\mathscr{D}(k[x]) \longrightarrow \mathscr{D}(C) \longrightarrow \mathscr{D}(k[[x]])$$

Since derived equivalences preserve the centers of rings, all the rings $k, k[x], k[[x]]$ and $k((x))$ are pairwise not derived equivalent. But, they are derived simple. Clearly, $\mathscr{D}(R)$ has a stratification of length 2 with composition factors $\mathscr{D}(k)$ and $\mathscr{D}(k)$. Thus, C admits two stratifications, one of which is of length 3 with three composition factors $k((x))$, k and k, and the other is of length 2 with composition factors $k[x]$ and $k[[x]]$. As a result, we have shown that the two stratifications of $\mathscr{D}(C)$ by derived categories of rings are of different lengths and without any common composition factors.

REMARKS. (1) For any simple regular R-module V', we can choose an automorphism $\sigma : R \to R$, such that the induced functor $\sigma_* : R\text{-Mod} \to R\text{-Mod}$ by σ is an equivalence and satisfies $\sigma_*(V') \simeq V$. Hence, instead of V, we may use V' to proceed the above procedure, but we will then get the same recollements, up to derived equivalence of each term.

(2) Let $K_0(R)$ be the Grothendieck group of R, that is, the abelian group generated by isomorphism classes $[P]$ of finitely generated projective R-modules P subject to the relation $[P] + [Q] = [P \oplus Q]$, where P and Q are finitely generated projective R-modules. One can check that $K_0(k((x))) \simeq \mathbb{Z}$ and $K_0(C) \simeq \mathbb{Z} \oplus \mathbb{Z}$. The above example shows that, even if $\mathscr{D}(A_2)$ is a

recollement of $\mathscr{D}(A_1)$ and $\mathscr{D}(A_3)$, where A_i are rings for $i = 1, 2, 3$, we cannot get $K_0(A_2) \simeq K_0(A_1) \oplus K_0(A_3)$ in general.

For a general consideration of stratifications of the endomorphism algebras of tilting modules over tame hereditary algebras, we shall discuss it in a forthcoming paper.

Finally, we remark that Theorem 1.1(2) can be extended to n-tilting modules. However, since there is not defined any reasonable torsion theory in module categories for general n-tilting modules, we are not able to extend Theorem 1.1(1) to n-tilting modules. So we mention the following open question.

QUESTION 1. Is Theorem 1.1(1) true for n-good tilting modules?

Another question related to our examples is:

QUESTION 2. Is there a ring R such that $\mathscr{D}(R)$ has two stratifications by derived module categories of rings, one of which is of finite length, and the other is of infinite length?

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