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On representation types of q-Schur algebras

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Abstract

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A sufficient condition for a q-Schur algebra to have finitely many finitely generated non-isomorphic indecomposable modules is provided. Moreover, if the condition is satisfied, then the q-Schur algebra is quadratic and its structure can be determined. It is also proved that if a Schur algebra is representation infinite then it is wild.

1. Introduction

Let k be an algebraically closed field of arbitrary characteristic, and let n,r be positive integers. We take an n-dimensional vector space E over k with a basis $\{e_1, \ldots, e_n\}$, and denote by $E^{\otimes r}$ the r-fold tensor product $E \otimes_k E \otimes_k \cdots \otimes_k E$. Of course, $E^{\otimes r}$ has a k-basis

$$\{e_{i_1} \otimes \cdots \otimes e_{i_r} \mid 1 \le i_j \le n \text{ for all } 1 \le j \le r\}\ .$$

The symmetric group $G(r)$ of degree r acts on $E^{\otimes r}$ by permutation in the following way:

$$\pi(e_{i_1} \otimes \cdots \otimes e_{i_r}) = e_{i_{\pi(1)}} \otimes \cdots \otimes e_{i_{\pi(r)}}\ ,$$

where π is an element of $G(r)$.

This action can be extended by linearity to the group algebra $kG(r)$ of $G(r)$ and we then have a left $kG(r)$-module $E^{\otimes r}$. We recall that the endomorphism ring

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$\mathrm{End}_{kG(r)}(E^{\otimes r})$ is said to be a Schur algebra and denoted by $S_k(n, r)$, or simply by $S(n, r)$ (see [10]). Recently, Dipper and James introduced the important notion of q-Schur algebras which are generalizations of Schur algebras [7]. For the definition we follow Jie Du quoted in [4].

Let $q^{1/2} \in k$ be an invertible element. For the Coxeter group $G(r)$, we will consider the corresponding Hecke algebra $H_q(r)$ which has distinguished basis $\{T_w \mid w \in G(r)\}$ satisfying the relations

$$T_w T_s = \begin{cases} T_{ws} & \text{if } l(ws) > l(w)\,, \\ (q-1)T_w + + qT_{ws} & \text{otherwise}\,. \end{cases}$$

Here $w, s \in G(r)$ with $s = (t, t+1)$ a simple reflection, and l is the usual length function on $G(r)$.

Using the left permutation action of $G(r)$ on $E^{\otimes r}$, one considers the action of the Hecke algebra $H_q(r)$ on $E^{\otimes r}$ which is defined by setting for $s = (t, t+1)$ and $e = e_{i_1} \otimes \cdots \otimes e_{i_r}$

$$T_s e = \begin{cases} -q^{1/2} se & \text{if } i_t < i_{t+1}\,, \\ -e & \text{if } i_t = i_{t+1}\,, \\ (q-1)e - q^{1/2} se & \text{if } i_t > i_{t+1}\,. \end{cases}$$

The q-Schur algebra $S_k(n, r, q)$ is defined as the corresponding centralizer ring:

$$S_k(n, r, q) = \mathrm{End}_{H_q(r)}(E^{\otimes r})\,.$$

Observe that if $q^{1/2} = 1$, clearly $S_k(n, r, q)$ is isomorphic to the Schur algebra $S_k(n, r)$ described above.

The objective of this paper is to determine the representation type of q-Schur algebras. Recall that a finite-dimensional k-algebra A is said to be representation finite provided there are only finitely many isomorphism classes of finitely generated indecomposable modules. Otherwise, we say that the algebra A is representation infinite. The algebra is said to be representation wild (or wild) provided there is a full and exact subcategory of the category of all finitely generated (left) A-modules which is equivalent to the category of all finitely generated $k\langle x, y\rangle$-modules, where $k\langle x, y\rangle$ is the free associative k-algebra with generators x and y. The algebra is said to be representation tame if it is neither representation finite, nor representation wild.

If $H_q(r)$ is semisimple, then $S_k(n, r, q)$ is a semisimple algebra and it is, of course, representation finite. Now we assume that $H_q(r)$ is not semisimple. In [5] it was shown that for $0 \neq q \in k$, the q-Schur algebra $S_k(n, r, q)$ is not semisimple if and only if either $q = 1$ and the characteristic of k is not bigger than r or $q \neq 1$ is an mth root of the unity $(2 \leq m \leq r)$. Hence our consideration in this paper is concentrated only on these two cases.

Our main results are the following:

1.1. Theorem. *Let k be an algebraically closed field of prime characteristic $p > 0$ and $n \geq r$. Then the Schur algebra $S_k(n, r)$ is representation finite if and only if $r < 2p$. Moreover, in this case, the Schur algebra is quadratic (i.e., the relations for the algebra are generated by elements of degree 2).*

Of course, this result is also a statement on polynomial representations of the general linear group $GL_n(k)$, it gives also information on $kG(n)$:

Corollary. *If $n = p + i$ with $0 \leq i < p$, then $kG(n)$ has only $p(i)$ blocks with non-zero radicals, where $p(i)$ denotes the number of all partitions of i.*

1.2. Theorem. *If the Schur algebra $S_k(n, r)(r \leq n)$ is representation infinite, then it is wild.*

1.3. Theorem. *Let k be an algebraically closed field of arbitrary characteristic and $q \neq 1$ an m'th root of the unity $(2 \leq m' \leq r)$. Suppose that m is the minimal number such that $1 + q + \cdots + q^{m-1}$ is equal to zero. If $r < 2m$, then the q-Schur algebra $S_k(n, r, q)$ with $r \leq n$ is representation finite and quadratic.*

Note that if the q-Schur algebra $S_k(n, r, q)$ with $r \leq n$ is representation finite, then one can easily write out the quiver of the basic algebra of $S_k(n, r, q)$ by using a result in [17].

Throughout this article, all algebras are finite-dimensional over an algebraically closed field and all modules are finitely generated left modules. Let A be an algebra. We denote by A-mod the category of all A-modules. For a module M we denote by $\mathrm{Soc}(M)$, $\mathrm{Top}(M)$ and $\mathrm{rad}(M)$ the socle, the top and the radical of M, respectively.

2. Representation finite q-Schur algebras

In this section we shall determine when a given q-Schur algebra $S_k(n, r, q)$ with $n \geq r$ is representation finite, where k is an algebraically closed field. Since the algebra $S_k(n, r, q)(n \geq r)$ is Morita equivalent to $S_k(r, r, q)$, it is enough to consider the Schur algebra $S_k(n,n, q)$ for n a positive integer.

Throughout this section we fix a positive integer n and assume that $0 \neq q \in k$ and that there is a minimal number $m \leq n$ such that $[m]_q := 1 + q + \cdots + q^{m-1}$ is equal to zero.

Let us first recall some definitions and known facts.

2.1. Definition [13]. Let $m \leq n$ be positive integers and $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition of n (i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ and $\sum \lambda_i = n$, denoted by $\lambda \vdash n$). If λ does not contain m parts which are equal, then λ is called *m-regular*. Otherwise λ

is called *m-singular*. The *m-core* $\tilde{\lambda}$ of λ is obtained by removing all m-rims in the Young diagram $[\lambda]$ of λ.

From the definition we have the following lemma.

2.2. Lemma. *Let* $1 \leq i < m$ *be a natural number and* $n = m + i$. *Suppose* $\lambda \vdash i$. *Then*:

(1) *The m-singular partitions of n are of the form* $(\lambda, 1^m)$, *where* λ *is a partition of* i.

(2) *If* $\lambda = (\lambda_1, \ldots, \lambda_i) \vdash i$, *then* $\mu = (\lambda_1 + m, \lambda_2, \ldots, \lambda_i)$ *is an m-regular partition of* n. □

2.3. It is well known that the simple $H_q(n)$-modules are indexed by m-regular partitions and the simple $S_k(n, n, q)$-modules are indexed by all partitions of n.

The proofs of the following results one can find in [6], [9] and [13].

Lemma (Nakayama conjecture). (1) *Let* λ, μ *be m-regular partitions of* n. *Then two simple* $H_q(n)$*-modules* D^λ *and* D^μ *corresponding to* λ *and* μ *are in the same block of* $H_q(n)$ *if and only if the m-cores of* λ *and* μ *coincide.*

(2) *Let* λ, μ *be partitions of* n. *The two simple* $S_k(n, n, q)$*-modules* F_λ *and* F_μ *corresponding to* λ *and* μ *are in the same block of* $S_k(n, n, q)$ *if and only if the m-cores of* λ *and* μ *coincide.* □

2.4. If k has prime characteristic p, then, by the classical result that a finite group is representation finite if and only if its p-Sylow groups are cyclic, the group algebra $kG(n)$ is representation finite if and only if $n < 2p$. Since $H_q(n) \cong eS_k(n, n, q)e$ for an idempotent $e \in S_k(n, n, q)$ (cf. [10, 6.1d] and [7, 2.12]) and the functor $S_k(n, n, q)e \otimes_{H_q(n)} - : H_q(n)\text{-mod} \to S_k(n, n, q)\text{-mod}$ is full and faithful, it follows that if $S_k(n, n)$ is representation finite then $n < 2p$. We shall show that this condition is also sufficient for $S_k(n, n)$ being representation finite.

2.5. Lemma ([7], [9] and [12]). *Let* $\{I^\lambda \mid \lambda \vdash n\}$ *be a complete set of indecomposable Young modules. Then the basic algebra of* $S_k(n, n, q)$ *is isomorphic to* $\mathrm{End}_{H_q(n)}(\bigoplus_{\lambda \vdash n} I^\lambda)$. □

Note that $H_q(r)$ is symmetric and $S_k(n, r, q)$ is of the form $\mathrm{End}_{H_q(r)}({}_{H_q(r)}H_q(r) \oplus M)$, where M is a module over $H_q(r)$ (cf. [5, 7]).

2.6. We recall that an ideal J of an algebra A is said to be hereditary provided (1) $J^2 = J$, (2) $JNJ = 0$, where N stands for the Jacobson radical of A, and (3) the left A-module J is projective. The algebra A is said to be quasi-hereditary if there exists a finite chain

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_t = A$$

of ideals of A such that J_i/J_{i-1} is a heredity ideal of A/J_{i-1} for $i \in \{1, \ldots, t\}$.

Lemma. *Let A be a connected basic symmetric algebra. Assume that there is an indecomposable module M such that $E := \mathrm{End}_A({}_AA \oplus M)$ is quasi-hereditary. Then E is representation finite.*

Proof. By [17], E is isomorphic to a factor algebra of the trivial extension $T(B)$ of a serial algebra B with square-zero radical and finite global dimension. Thus with $T(B)$ also E is representation finite. □

2.7. Theorem. *Let k be an algebraically closed field and n a positive integer. Let $0 \neq q \in k$ of finite multiplicative order and suppose that $\mathrm{char}(k) = p > 0$ if $q = 1$. Let m be the order of q if $q \neq 1$ and p otherwise. If $n < 2m$, then the q-Schur algebra $S_k(n, n, q)$ is representation finite.*

Note that m is the minimal natural number such that $1 + q + \cdots + q^{m-1} = 0$ and if $q \neq 1$, then q is a primitive mth root of unity.

Proof. In [7], it was shown that the Hecke algebra $H_q(n)$ is semisimple if $n < m$ (m as above). Thus the q-Schur algebra $S_k(n, n, q)$, as the endomorphism algebra of a semisimple $H_q(n)$-module, is representation finite. So we can always assume that $m \leq n < 2m$. Let $n = m + i$ with $0 \leq i < m$, and let $\lambda^1, \ldots, \lambda^l$ be a list of all partitions of i. Then $\mu^j := (\lambda_1^j + m, \lambda_2^j, \ldots, \lambda_i^j)$ is an m-regular partition of n for each $j \in \{1, \ldots, l\}$ by Lemma 2.2. Let B_j be the block of $H_q(n)$ which contains the simple module D^{μ^j} corresponding to the partition μ^j.

We want to show that the blocks B_j contain besides the projective only one more Young module and that the other blocks are all semisimple. In fact, the Young module I^λ is distinguished by the property that it contains the Specht module $S^{\lambda'}$ with multiplicity one and other Specht module S^μ only for partitions μ dominated by λ', where λ' is the conjugate partition to λ. Being indecomposable I^λ has to be in the same block as $S^{\lambda'}$. So the claim on B_j follows by direct inspection. Since there are precisely l m-singular partitions of n, and those are already distributed into the l blocks B_j. Consequently the other blocks can only contain projective Young modules. Those blocks contain only one Specht module as well (whose weight is an m-core) which is therefore the only Specht module which is a subfactor of the Young modules in that block. Since Young modules are in the Grothendieck group equivalent to a linear combination of Specht modules (this is true for the modules M^λ hence inductively for Young modules) and one Specht module comes up with exact multiplicity one we conclude that in those blocks the Specht module equals every Young module in it. Consequently there is only one Young module which is equal to the unique Specht module. Hence the Specht module is in particular projective and the decomposition matrix of that block is the 1×1-matrix (1). We conclude that the block must be semisimple. Since the q-Schur algebra $S_k(n, n, q)$ is quasi-hereditary (see [4]), we can apply Lemma 2.6 to the block B_j. Thus the q-Schur algebra $S_k(n, n, q)$ is representation finite. □

2.8. Corollary. *The Hecke algebra $H_q(n)$ has precisely as many blocks with non-zero radicals as there are partitions of i, where $n = m + i$ and $0 \leq i < m$. (This holds even for* $\mathrm{char}(k) = 0$). □

Recall that an algebra A is called quadratic if in the quiver of the basic algebra of A all relations are of degree 2. Theorem 2.8 in [17] says that if there is an indecomposable module M over a symmetric algebra A such that $E := \mathrm{End}_A({}_AA \oplus M)$ is quasi-hereditary, then the algebra E is quadratic. The above proof of Theorem 2.7 shows that the non-trivial blocks of $S_k(n, n, q)$ are of the form $\mathrm{End}(B_j \oplus I^\lambda)$, where I^λ is a non-projective Young module. This shows in particular the following theorem.

2.9. Theorem. *Let k, q, m, n be as in Theorem 2.7. If $n < 2m$, then the q-Schur algebra $S_k(n, n, q)$ is quadratic.* □

2.10. Remark. A quadratic algebra A is said to be formal if the Yoneda algebra $\mathrm{Ext}^*(A/\mathrm{rad}(A))$ is isomorphic to the Priddy dual $A^!$ of A [1]. In general, a Schur algebra may not be a formal algebra in the above sense of [1]. However, if a Schur algebra $S(n, n)$ is representation finite, then it is a formal algebra by the description of $S(n, n)$ in [17] and Theorem 1.1 in [1].

3. Quivers of Schur algebras $S_k(4, 4)$ and $S_k(5, 5)$

In this section we want to work out the quivers of $S_k(4, 4)$ and $S_k(5, 5)$ in case the field k has characteristic 2. Using these we can decide in the next section the representation type of $S(n, n)$ for $n \geq 2p$. Our method in this section is based on the determination of indecomposable Young modules.

We begin with the following useful lemma.

3.1. Lemma. *Let A be an algebra and $M_1, \ldots, M_n$ be pairwise non-isomorphic indecomposable modules. Let $\mathscr{C} = \mathrm{add}\{M_1, \ldots, M_m\}$ with $m < n$. Suppose $\mathrm{End}_A(\bigoplus_{i=1}^n M_i)$ is quasi-hereditary and $\dim \mathrm{Irr}_{\mathscr{C}}(M_1, M_1) = 1$ (see [16]). If $0 \neq f \in \mathrm{Irr}_{\mathscr{C}}(M_1, M_1)$, then there is a module M_j with $j > m$ and two submodules U and*

V of M_j *with* $U \supset V$ *such that the image* $\mathrm{Im}(f)$ *of* f *is isomorphic to* U/V. *In particular, if* $\dim M_j = \dim \mathrm{Im}(f)$, *then* $M_j \cong \mathrm{Im}(f)$.

Proof. Notice that if the quiver of an algebra has a loop then the algebra cannot be quasi-hereditary. So the homomorphism f factors through a module $X = X_1 \oplus X_2$, where $X_1 \in \mathscr{C}$ and $X_2 = \bigoplus_{j=1}^{l} X_{2j}$ with $X_{2j} \in \{M_{m+1}, \ldots, M_n\}$:

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_1 \\ & {\scriptstyle [f_1,\, f_2]} \searrow \quad \nearrow {\scriptstyle [g_1,\, g_2]^t} & \\ & X_1 \oplus X_2 & \end{array}$$

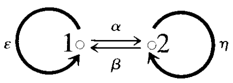
and $f_2 g_2$ cannot factor through a module in add $\mathscr{C}$. Let $f_2 = (f_{21}, \ldots, f_{2l})$ and $g_2 = (g_{21}, \ldots, g_{2l})$. Since $f_2 g_2 \neq 0$, there is one j such that the composition $f_{2j} g_{2j}$ is a non-zero map in $\mathrm{Irr}_{\mathscr{C}}(M_1, M_1)$. We may assume that $X_{2j} = M_j$ and $f = f_{2j} g_{2j}$. Consider the following canonical diagram:

$$\begin{array}{ccccc} M_1 & & \xrightarrow{\quad f \quad} & & M_1 \\ \downarrow & \searrow^{f_{2j}} & & & \| \\ \mathrm{Im}(f_{2j}) & \xrightarrow{\mu} & M_j & \xrightarrow{g_{2j}} & M_1 \end{array}$$

Since $\mathrm{Im}(f) = \mathrm{Im}(\mu g_{2j})$, one obtains a surjective map $\mu g_{2j} : \mathrm{Im}(f_{2j}) \to \mathrm{Im}(f)$. Put $U := \mathrm{Im}(f_{2j})$ and $V := \mathrm{Ker}(\mu g_{2j})$. Then the lemma follows. □

From now on, we assume in this section that the field considered is of characteristic 2.

3.2. Lemma [8]. *The basic algebra of* $kG(4)$ *is given by the following quiver with relations*:



$$\varepsilon^2 = \alpha\eta = \eta\beta = \beta\alpha = 0\,,$$

$$\eta^2 = \beta\varepsilon\alpha\,, \qquad \varepsilon\alpha\beta = \alpha\beta\varepsilon\,. \qquad \square$$

In the following we denote by I^{λ} the indecomposable Young module corresponding to the partition $\lambda \vdash n$. Following [11] we denote by M^{λ} the $kG(n)$-module induced from the trivial module on the Young subgroup S_{λ}.

3.3. Lemma. (1) $I^{(4)} = D^{(4)}$ *is a simple* $kG(4)$*-module.*

(2) $I^{(3,1)}$ *is a serial module with composition factors* $D^{(4)}$, $D^{(3,1)}$ *and* $D^{(4)}$, *reading from the top.*

(3) $I^{(2,2)}$ *has* $\mathrm{Top}(I^{(2,2)}) \cong \mathrm{Soc}(I^{(2,2)}) \cong D^{(4)} \oplus D^{(3,1)}$. *Moreover, it holds that* $\mathrm{rad}^2(I^{(2,2)}) = 0$.

Proof. (1) and (2) follow from [11, 17.1]. By [12], $M^{(2,2)}$ and $I^{(2,2)}$ have the same dimension over k and $I^{(2,2)}$ is a submodule of $M^{(2,2)}$. Thus we have $M^{(2,2)} = I^{(2,2)}$. From $\dim \mathrm{Hom}(D^{(4)}, M^{(2,2)}) \neq 0 \neq \dim \mathrm{Hom}(D^{(3,1)}, M^{(2,2)})$ and the self-duality of $M^{(2,2)}$ the statement (3) follows. □

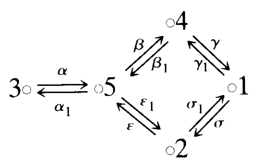
3.4. Lemma. (1) $I^{(3,1)}$ *is a string module* (*see* [3] *for a definition, note that a string module is a module of first kind in the term of* [15]).

(2) *The module* $I^{(2,2)}$ *is a string module corresponding to the string* $\alpha\eta^{-1}\beta$.

Proof. Let $P(i)$ be the projective modules corresponding to the vertex i. Note that $P(1) = I^{(1^4)}$ and $P(2) = I^{(2,1^2)}$ are Young modules. Let $M_1 = P(1)$, $M_2 = P(2)$, $M_3 = I^{(4)}$, $M_4 = I^{(3,1)}$, $M_5 = I^{(2,2)}$ and $\mathscr{C} = \mathrm{add}\{M_1, M_2\}$. Then $\dim \mathrm{Irr}_{\mathscr{C}}(M_1, M_1) = 1$. Take $0 \neq f \in \mathrm{Irr}_{\mathscr{C}}(M_1, M_1)$. Then by Lemma 3.1, f factors over some module M_j with $j \in \{3, 4, 5\}$. By Lemma 3.3, we must have $j = 4$. Then $M_4 = \mathrm{Im}(f)$, since $\dim M_4 = \dim \mathrm{Im}(f)$. This shows that $M_4 = I^{(3,1)} = \mathrm{Im}(f) \cong M(\alpha\beta)$, where $M(\alpha\beta)$ stands for the string module corresponding to $\alpha\beta$.

(2) Note that $E := \mathrm{End}(\oplus_{j=1}^{5} M_j)$ is quasi-hereditary. Let $\mathscr{C} = \mathrm{add}\{M_1, M_2\}$. Then $\dim \mathrm{Irr}_{\mathscr{C}}(M_2, M_2) = 1$. Let $0 \neq f$ be a map in $\mathrm{Irr}_{\mathscr{C}}(M_2, M_2)$. Then $\mathrm{Im}(f) \cong M(\eta)$ and $M(\eta)$ is a subquotient module of M_5 (that is, there are submodules $U \supset V$ of M_5 such that $M(\eta) \cong U/V$). This implies together with Lemma 3.3 that M_5 is isomorphic to $M(\alpha\eta^{-1}\beta)$ or M_5 is a band module corresponding to $\alpha\eta^{-1}\beta$ (i.e., a module of the second kind [15]). Since E is quasi-hereditary, a computation of the quiver of E shows that the latter is impossible. Thus $M_5 = M(\alpha\eta^{-1}\beta)$. □

3.5. Proposition. *The basic algebra of* $S_k(4,4)$ *is given by the following quiver with relations*:



$$\alpha\alpha_1 = \alpha\varepsilon_1 = \gamma\gamma_1 = \beta_1\beta = \sigma_1\sigma = \varepsilon\alpha_1 = 0\,,$$

$$\gamma\sigma = \beta_1\varepsilon_1\,, \quad \beta\beta_1 = \varepsilon_1\varepsilon\,, \quad \sigma_1\gamma_1 = \varepsilon\beta\,,$$

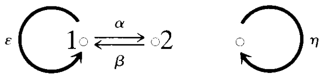
$$\alpha_1\alpha\beta\gamma = \varepsilon_1\sigma_1\,, \quad \gamma_1\beta_1\alpha_1\alpha = \sigma\varepsilon\,.$$

Proof. By Lemmas 3.3 and 3.4, we know all the structures of the modules I^λ,

$\lambda \vdash 4$. Note that $I^{(2,1^2)}$ and $I^{(1^4)}$ are projective and their structures are well known. Let i correspond to M_i defined in the proof of Lemma 3.4. Using Lemma 2.5 and making a computation of the quiver of $\mathrm{End}(\oplus_{\lambda \vdash 4} I^{\lambda})$, one gets the above quiver with the relations displayed. $\square$

Now we turn to working out the quiver of $S_k(5,5)$.

3.6. Lemma [2]. *The basic algebra of $kG(5)$ is given by the following quiver with relations*:



$$\eta^2 = \beta\alpha = \varepsilon^2 = 0\,,$$

$$\varepsilon\alpha\beta\varepsilon\alpha\beta = \alpha\beta\varepsilon\alpha\beta\varepsilon\,. \quad \square$$

3.7. Lemma. (1) $I^{(5)} = D^{(5)}$ *is a simple module with* $\dim D^{(5)} = 1$.

(2) $I^{(4,1)} = D^{(4,1)}$ *is a simple module.*

(3) $I^{(3,2)}$ *is a serial module with composition factors, reading from the top, isomorphic to* $D^{(5)}$, $D^{(3,2)}$ *and* $D^{(5)}$.

(4) $I^{(2,2,1)}$ *is isomorphic to* $M(\alpha\beta\varepsilon\alpha\beta)$.

(5) $I^{(3,2)}$ *is isomorphic to* $M(\alpha\beta)$.

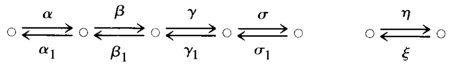
Proof. (1) and (2) are clear from [11].

(3) Since $\dim \mathrm{Hom}(D^{(5)}, I^{(3,2)}) \neq 0$ and $I^{(3,2)}$ is self-dual, we know that $\mathrm{Top}(I^{(3,2)}) \cong \mathrm{Soc}(I^{(3,2)}) \supseteq D^{(5)}$. By [11, 12.2], the Specht module $S^{(3,2)}$ is indecomposable with $\mathrm{Top}(S^{(3,2)}) \cong D^{(3,2)}$. It follows from $S^{(3,2)} \subset I^{(3,2)}$ that $I^{(3,2)}$ must be a serial module with composition factors $D^{(5)}$, $D^{(3,2)}$ and $D^{(5)}$, reading from the top.

(4) Put $M_1 = P(1)$, $M_2 = P(2)$, $M_3 = I^{(5)}$, $M_4 = I^{(4,1)}$, $M_5 = I^{(3,2)}$ and $M_6 = I^{(2,2,1)}$. Note that $P(1) = I^{(1^5)}$ and $P(2) = I^{(3,1^2)}$ are Young modules. Then $E := \mathrm{End}(\oplus M_j)$ is quasi-hereditary. Let $\mathscr{C} = \mathrm{add}\{M_1, M_2\}$. Then $\dim \mathrm{Irr}_{\mathscr{C}}(M_1, M_1) = 1$. We take a map $0 \neq f \in \mathrm{Irr}_{\mathscr{C}}(M_1, M_1)$. Then f factors over M_6 and $\mathrm{Im}(f) = M_6$, because $\dim M_6 = \dim \mathrm{Im}(f)$. Thus $M_6 = \mathrm{Im}(f) = M(\alpha\beta\varepsilon\alpha\beta)$.

(5) Suppose M_5 is a module of the second kind. Then we can calculate the quiver of E and find that in this case the algebra E is not quasi-hereditary. This contradiction means M_5 must be a string module isomorphic to $M(\alpha\beta)$. $\square$

3.8. Proposition. *The basic algebra of $S_k(5,5)$ is given by the following quiver with relations*:



$$\alpha\alpha_1 = \beta\beta_1 = \gamma\gamma_1 = \sigma_1\sigma = \xi\eta = 0\,,$$

$$\alpha\beta\gamma\sigma = \sigma_1\gamma_1\beta_1\alpha_1 = 0, \quad \gamma\sigma\sigma_1\gamma_1 = \beta_1\beta\,,$$

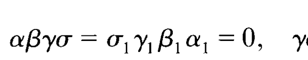
$$\beta\gamma\sigma\sigma_1 = \alpha_1\alpha\beta\gamma\,, \quad \gamma_1\beta_1\alpha_1\alpha = \sigma\sigma_1\gamma_1\beta_1\,. \qquad \square$$

3.9. Remark. Propositions 3.5 and 3.8 show that in general a Schur algebra may not be quadratic.

4. Representation infinite Schur algebras

In this section we shall discuss representation infinite Schur algebras and demonstrate that they are of representation wild type. Note that it is known that $S_k(n, n)$ is of representation infinite type if $n \geq 2p$ ($p = \mathrm{char}(k) = m$).

4.1. Lemma. *Let Q be the following quiver*:



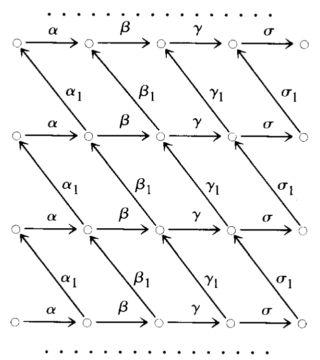
We consider the following two groups of relations:

$$(I_1) \quad \alpha\alpha_1 = \beta\beta_1 = \beta_1\beta = \gamma\gamma_1 = \sigma_1\sigma = 0\,,$$
$$\gamma\sigma = \sigma_1\gamma_1 = \alpha_1\alpha\beta\gamma = \gamma_1\beta_1\alpha_1\alpha = 0\,.$$

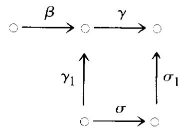
$$(I_2) \quad \alpha\alpha_1 = \beta\beta_1 = \gamma\gamma_1 = \sigma_1\sigma = \beta_1\beta = 0\,,$$
$$\alpha\beta\gamma\sigma = \sigma_1\gamma_1\beta_1\alpha_1 = \gamma\sigma\sigma_1\gamma_1 = \beta\gamma\sigma\sigma_1 = \alpha_1\alpha\beta\gamma = 0\,,$$
$$\gamma_1\beta_1\alpha_1\alpha = \sigma\sigma_1\gamma_1\beta_1 = 0\,.$$

Let A_i be the algebra given by the quiver with relations I_i. Then A_i is wild for $i = 1,2$.

Proof. One can easily construct a covering $(\tilde{Q}, \tilde{I})$ of (Q, I) as follows:



where the relations $\tilde{I}_i$ correspond to I_i. We can find a convex subquiver of the above covering which looks like the following



with no relations. This subquiver is wild, and by [14], the algebra A_i is wild. □

4.2. Theorem. *Let k be an algebraically closed field with characteristic $p > 0$. If the Schur algebra $S_k(n, n)$ is representation infinite, then it is wild.*

Proof. Let C_2 denote the cyclic group of order 2. Since $C_2 \times C_2 \times C_2 \subset G(n)$ for $n \geq 6$, the algebra $kG(n)$ is wild for $n \geq 6$. By 2.4, the Schur algebra $S_k(n, n)$ is wild for $n \geq 6$. Now let n be a positive number smaller than 6. Since we assume that $S_k(n, n)$ is representation infinite, it holds that $p = 2$ by Theorem 2.7. This yields that we have to consider the Schur algebras $S_k(4, 4)$ and $S_k(5, 5)$ in case $p = 2$. It is clear that the category of all finitely generated $S_k(4, 4)$-modules has a full subcategory which is equivalent to the module category of the algebra A_1 given in Lemma 4.1 if one requires in Proposition 3.5 that $\varepsilon = \varepsilon_1 = 0$. Thus $S_k(4, 4)$ is wild. In case $p = 2$ and $n = 4$ we observe that there exists a surjective homomorphism from $S_k(5, 5)$ onto the algebra A_2 given in Lemma 4.1. Hence the algebra $S_k(5, 5)$ is wild too. □

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After I finished this paper, I learned that K. Erdmann has independently obtained the result of Theorem 1 also, but our proof is very different from hers.

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