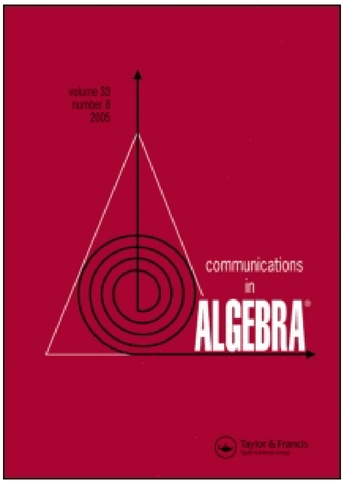

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QUADRATIC FORMS FOR Δ–GOOD MODULE CATEGORIES OF QUASI–HEREDITARY ALGEBRAS

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1. INTRODUCTION

Let A be a finite–dimensional algebra over an algebraically closed field k. We consider in this note only finite–dimensional left A–modules. Given a class Θ of A–modules, we denote by $\mathcal{F}(\Theta)$ the class of all A–modules M which have a filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = 0$ with $M_{i-1}/M_i \in \Theta$ for all $i = 1, \cdots, n$. The modules in $\mathcal{F}(\Theta)$ are called Θ–good modules.

Let $E(1), \cdots, E(n)$ be a complete list of non-isomorphic simple A–modules, note that here we fix a particular ordering for labelling the simple A–modules. For any i, let $P(i)$ be the projective cover of $E(i)$ and $\Delta(i)$ the maximal factor module of $P(i)$ in $\mathcal{F}(E(1), \cdots, E(i))$ which is called a standard module. Dually, we have the costandard module $\nabla(i)$ which is the maximal submodule of the injective hull $Q(i)$ of $E(i)$ and lies in $\mathcal{F}(E(1), \cdots, E(i))$. We denote by Δ the class of all standard modules $\Delta(i), 1 \leq i \leq n$.

Recall that an algebra A with an order E of the simple modules is called quasi–hereditary if $\mathrm{End}(\Delta(i)) \cong k$ for any $1 \leq i \leq n$ and the module ${}_AA \in \mathcal{F}(\Delta)$. Quasi–hereditary algebras are introduced by Cline, Parshall and Scott in their investigation of representations of complex Lie algebras and algebraic groups.

It is proved by Dlab and Ringel in [**DR**] that an algebra is hereditary if and only if it is quasi-hereditary algebra for any order of the simple modules. In this note we are going to consider hereditary algebras as quasi-hereditary algebras and mainly interested in the subcategory $\mathcal{F}(\Delta)$, especially, when it

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is finite (i.e. there are only finitely many indecomposable modules in $\mathcal{F}(\Delta)$). We introduce a quadratic form for arbitrary quasi-hereditary algebras and prove that for a hereditary algebra the subcategory $\mathcal{F}(\Delta)$ is finite if and only if the quadratic form is weakly positive. This result on $\mathcal{F}(\Delta)$ has the similarity to that for the whole module category (see [**BGP**], for example).

Throughout the note, we assume that k denotes an algebraically closed field and that an algebra means always a finite dimensional k-algebra. For a module M over a k-algebra A, we denote by $\underline{dim}M$ the usual dimension verctor of M. All notion and notation appeared in the note are standard.

2.THE QUADRATIC FORMS AND MAIN RESULTS

Let A be a quasi-hereditary algebra with standard modules $\Delta = \{\Delta(1), \cdots, \Delta(n)\}$. In this section we shall define a quadratic form associated with the standard modules and use the quadratic form to investigate the subcategory $\mathcal{F}(\Delta)$.

Note that for a quasi-hereditary algebra, we always have

(1) $Hom_A(\Delta(i), \Delta(j)) = 0$ for $i > j$,

(2) $Ext^t_A(\Delta(i), \Delta(j)) = 0$ for $i \geq j$ and $t \geq 1$,

(3) $\mathcal{F}(\Delta)$ is closed under direct summands.

Definition 1. For a quasi-hereditary algebra A with standard modules Δ we define an integral bilinear form $b_{(A.\Delta)} : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$ by

$$b_{(A.\Delta)}(x,y) = \sum_{t\geq 0} \sum_{i\leq j} (-1)^t dim_k Ext^t(\Delta(i), \Delta(j)) x_i y_j$$

for all $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{Z}^n$.

We define an integral quadratic form $q_{(A.\Delta)}(x)$ by

$$q_{(A.\Delta)} : \mathbb{Z}^n \longrightarrow \mathbb{Z}, \quad x \longmapsto b_{(A.\Delta)}(x,x)$$

for all $x \in \mathbb{Z}^n$.

Recall that an integral quadratic form $q : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ is called positive provided $q(x) > 0$ for all $0 \neq x \in \mathbb{Z}^n$, and weakly positive if $q(x) > 0$ for all $x > 0$($i.e., x \neq 0$ and $x_i \geq 0$ for $i = 1, \cdots, n$). Finally, an element $x \in \mathbb{Z}^n$ satisfying $q(x) = 1$ is called a root of q.

For a module M in $\mathcal{F}(\Delta)$ let us denote by $[M : \Delta]$ the vector $([M : \Delta(1)], \cdots, [M : \Delta(n)]) \in \mathbb{Z}^n$, where $[M : \Delta(i)]$ is the number of $\Delta(i)$ occurring as composition factors of a Δ-filtration.

The main results of this note are two theorems.

Theorem A. *Let A be a hereditary algebra with standard modules Δ. If $q_{(A.\Delta)}$ is weakly positive, then $\mathcal{F}(\Delta)$ is finite, and moreover, the number of* *isomorphic classes of indecomposable modules in $\mathcal{F}(\Delta)$ is just the number of positive roots of $q_{(A,\Delta)}$.*

To prove the above theorem, we need some preparations.

Lemma 2. *If A is quasi-hereditary algebra, then $q_{(A,\Delta)}([M:\Delta]) = \chi_A(\underline{dim}M)$ for all $M \in \mathcal{F}(\Delta)$, where χ_A is the Euler form of A introduced by Ringel in* [**R1**].

Proof. Since $\underline{dim}M = \sum_{i=1}^{n}[M:\Delta(i)]\underline{dim}\Delta(i)$, we obtain

$$\begin{aligned}
&\chi_A(\underline{dim}M) = \chi_A\left(\sum[M:\Delta(i)]\underline{dim}\Delta(i)\right) = \chi_A(\underline{dim}(\oplus\Delta(i)^{[M:\Delta(i)]}))\\
&=\sum_{t\geq 0}(-1)^t dim_k Ext^t(\oplus_i\Delta(i)^{[M:\Delta(i)]}, \oplus_j\Delta(j)^{[M:\Delta(j)]})\\
&=\sum_{t\geq 0}\sum i\leq j(-1)^t dim_k Ext^t(\Delta(i),\Delta(j))[M:\Delta(i)][M:\Delta(j)]\\
&=q_{(A,\Delta)}([M:\Delta]).
\end{aligned}$$

Here we use the property that $Ext^t(\Delta(j),\Delta(i)) = 0$ for $j \geq i$ and $t \geq 1$.)

The following lemma is taken from [DR].

Lemma 3. *Let A be a quasi-hereditary algebra for which the injective dimension of each costandard module is at most 1. Then $\mathcal{F}(\Delta)$ is closed under submodules .*

Lemma 4. *Let A be a hereditary algebra. If $q_{(A,\Delta)}$ is weakly positive, then $End(M) \cong k$ for every indecomposable module $M \in \mathcal{F}(\Delta)$.*

Proof. The method of the proof is analog to [**H**, p.164]. Let $M \in \mathcal{F}(\Delta)$ be a counterexample of minimal dimension and f a non-zero nilpotent endomorphism of M whose image I has minimal dimension. Thus I is indecomposable and $End(I) \cong k$ since $I \in \mathcal{F}(\Delta)$ by Lemma 3. This implies that $f^2 = 0$. Let K=ker(f)=$\oplus_{i=1}^{m}K_i$ with K_i indecomposable, then $I \subseteq K$. Let $\pi_j : K \longrightarrow K_j$ be the canonical projection, we consider the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\pi_j"] & M \arrow[r] \arrow[d, dashed] & I \arrow[r] \arrow[d, dashed, no head] & 0\\
0 \arrow[r] & K_j \arrow[r] & X \arrow[r] & I \arrow[r] & 0
\end{tikzcd}$$

Since M is indecomposable, we know that the lower exact sequence is not split and therefore $Ext^1(I, K_j) \neq 0$ for every j.

Take a non-zero projection $p_j : I \longrightarrow K_j$. Since I has minimal dimension, p_j is injective. Let Q=Cok(p_j), the cokernel of p_j. If we apply $Hom(-, K_j)$ to $0 \longrightarrow I \xrightarrow{p_j} K_j \longrightarrow Q \longrightarrow 0$, then we get $Ext^1(K_j, K_j) \neq 0$ since $Ext^2(-, K_j) = 0$. Note that dim $K_i <$ dim M and $K_j \in \mathcal{F}(\Delta)$. Hence

$dim\ End(K_j) = 1$, and by Lemma 2,

$$\begin{aligned} 0 < q_{(A,\Delta)}([K_j : \Delta]) &= \chi_A(\underline{dim}K_j) \\ &= dim\ End(K_j) - dim\ Ext^1(K_j, K_j) \\ &= 1 - dim\ Ext^1(K_j, K_j) \le 0. \end{aligned}$$

This is a contradiction.

Proof of Theorem A. We use the well-known Tits argument. Let A be given by the quiver $Q = (Q_0, Q_1)$ and $X \in \mathcal{F}(\Delta)$ be indecomposable with $\underline{dim}X = z$. Choosing a basis for X we may identify X with $(X(\alpha))_{\alpha \in Q_1} \in \prod_{(x \xrightarrow{\alpha} y) \in Q_1} k^{z_x \times z_y}$. This product is denoted by $V(z)$. It is an affine variety. The affine group $G(z) := \prod_{x \in Q_0} GL_{z_x}(k)$ acts on $V(z)$ in the following way: If $X = (X(\alpha))_{\alpha \in Q_1} \in V(z)$ and $g = (g_x)_{x \in Q_0} \in G(z)$, then $X^g = (g_y X(\alpha) g_x^{-1})_{(x \xrightarrow{\alpha} y) \in Q_1}$. Two modules X, Y are isomorphic if and only if they lie in the same $G(z)$–orbit in $V(z)$. Moreover, the stabilizer of X is $\mathrm{Aut}_A(X)$. Denote by $G(z)X := \{X^g | \ g \in G(z)\}$ the orbit of X in $V(z)$. Then, as in [**KR**], one has

$$\begin{aligned} &dimHom(X,X) - dimExt^1(X,X) = \chi_A(\underline{dim}X) \\ &= dim\ End(X) - (dimV(z) - dim\ G(z)X). \end{aligned}$$

Since $1 = q_{(A,\Delta)}([X : \Delta]) = \chi_A(\underline{dim}X)$, one gets $dimV(z) = dimG(z)X$ because $dim\,End(X) = 1$ by Lemma 4. It follows that the $G(z)$– orbit of X in $V(z)$ is Zariski–open and dense. Therefore, it coincides with the orbit of any other indecomposable module in $V(z)$, and the map $V \longmapsto [V : \Delta]$ provides an injective map into the set of positive roots of $q_{(A,\Delta)}$. Since a weakly positive quadratic form has only finitely many positive roots, the first part of Theorem A is proved.

To prove the second part of Theorem A, it suffices to show that for any positive root x of $q_{(A,\Delta)}$ there is an indecomposable module V in $\mathcal{F}(\Delta)$ such that $[V : \Delta] = x$. This follows from the following observation.

Lemma 5. *Let A be a hereditary algebra, and assume that $q_{(A,\Delta)}$ is weakly positive. Let x be a positive root of $q_{(A,\Delta)}$. If we take a module V in $\mathcal{F}(\Delta)$ with $[V : \Delta] = x$ such that the dimension of End(V) is minimal, then this module is indecomposable.*

For the proof of this lemma we need the following result due to Ringel [**R1**].

Lemma 6. *Let B be an algebra and $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ a non-split exact sequence in B-mod. Then* dim $End(X) <$ dim $End(X' \oplus X'')$.

Now we prove Lemma 5. Suppose we have a decomposition $V = \bigoplus_{i=1}^{s} V_i$ of V into indecomposable modules with $s \geq 2$. Then $V_i \in \mathcal{F}(\Delta)$ and $End(V_i) \cong k$ by Lemma 4. Note that we also have $Ext^1(V_i, V_i) = 0$. In the following we show that $Ext^1(V_i, V_j) = 0$ for all i, j.

Suppose we have $Ext^1(V_1, V_2) \neq 0$. Let $0 \longrightarrow V_2 \xrightarrow{\alpha} W \xrightarrow{\beta} V_1 \longrightarrow 0$ be a non-split exact sequence. Then with $V_i \in \mathcal{F}(\Delta)$ also W belongs to $\mathcal{F}(\Delta)$. It is very easy to verify that the short exact sequence

$$0 \longrightarrow V_2 \xrightarrow{(\alpha,0)} W \oplus \bigoplus_{j>2} V_j \xrightarrow{\begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}} V_1 \oplus \bigoplus_{j>2} V_j \longrightarrow 0$$

is not split. Put $V' = W \oplus \bigoplus_{j>2} V_j$, then $[V' : \Delta] = [V_2 : \Delta] + [V_1 \oplus \bigoplus_{j>2} V_j : \Delta] = [\oplus_{j=1}^{s} V_j : \Delta] = [V : \Delta]$. On the other hand, according to Lemma 6, we have $dim\ End(V') < dim\ End(V)$. This contradicts the choice of V and shows that $Ext^1(V_1, V_2) = 0$.

Since $s \geq 2$, one has $dim\ End(V) = \dim Hom(\oplus_{j=1}^{s} V_j, \oplus_{j=1}^{s} V_j) \geq 2$. However, from

$$\begin{aligned} 1 &= q_{(A,\Delta)}([V : \Delta]) = \chi_A(\underline{dim} V) \\ &= dimEnd(V) - dimExt^1(V, V) = dimEnd(V) \geq 2 \end{aligned}$$

one obtains a contradiction. Thus V must be indecomposable. This finishes the proof of the lemma and also the proof of Theorem A.

In the following we shall discuss the converse of Theorem A.

Lemma 7 [**R1**]. *Let B be an algebra and X a module satisfying $End(X) \cong k$, $Ext^1(X, X) \neq 0$ and $Ext^2(X, X) = 0$. Then $\mathcal{F}(X)$ is infinite.*

Lemma 8. *Suppose that A is a hereditary algebra. If $\mathcal{F}(\Delta)$ is finite, then for every indecomposable module $M \in \mathcal{F}(\Delta)$, there holds $End(M) \cong k$.*

Proof. Let $V \in \mathcal{F}(\Delta)$ be a counterexample of minimal dimension and f a non-zero endomorphism of V. We set $I = Im(f)$, the image of f, and $K = Ker(f) = \bigoplus K_i$, where K_i are indecomposable. As in the proof of Lemma 4, we get an indecomposable module $K_i \in \mathcal{F}(\Delta)$ with $\mathrm{End}(K_i) \cong k$ and $Ext^1(K_i, K_i) \neq 0$. Since proj.dim.$K_i \leq 1$ by assumption, we know from Lemma 7 that $\mathcal{F}(\Delta)$ is infinite. A contradiction.

Now we show the converse of Theorem A.

Theorem B. *Let A be a hereditary algebra. If $\mathcal{F}(\Delta)$ is finite, then $q_{(A,\Delta)}$ is weakly positive.*

Proof. Suppose that there is an element $0 < x \in \mathbb{N}^n$ such that $q_{(A,\Delta)}(x) \leq 0$. Let V be a module in $\mathcal{F}(\Delta)$ with $[V : \Delta] = x$ such that $\dim End(V)$ is minimal. Then we can prove that V is indecomposable. By Lemma 8 one has

$$\begin{aligned} q_{(A,\Delta)}(x) = \chi_A(\underline{dim}\ V) &= dimEnd(V) - dimExt^1(V,V) \\ &= 1 - dimExt^1(V,V) \leq 0 \end{aligned}$$

Thus $Ext^1(V,V) \neq 0$. By Lemma 7 we see that $\mathcal{F}(V)$ is infinite and therefore $\mathcal{F}(\Delta)$ is infinite. This contradicts our assumption and shows that $q_{(A,\Delta)}$ is weakly positive. Thus the proof is finished.

In the following we give several examples to explain our results.

Example . (1) Let A be a hereditary algebra given by the following quiver

$$\begin{tikzcd} 5 & & & & 2 \arrow[dl] & 1 \arrow[l] \\ & 7 \arrow[ul] \arrow[dl] & & 4 \arrow[ll] & & \\ 6 & & & & 3 \arrow[ul] & \end{tikzcd}$$

with $\Delta = \{\Delta(i) \mid i = 1, \cdots, 7\}$. Then A is representation wild and χ_A is not weakly positive. Since $q(y) = y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_7^2 - y_1y_2 - y_2y_4 - y_3y_4 - y_4y_7$ is positive and $q_{(A,\Delta)}(x) = \sum x_i^2 + x_5x_7 + x_6x_7 - x_1x_2 - x_2x_4 - x_3x_4 - x_4x_7$ $= q(x) + x_5^2 + x_6^2 + x_5x_7 + x_6x_7$, we see that $q_{(A,\Delta)}(x)$ is weakly positive (but not positive). Hence $\mathcal{F}(\Delta)$ is finite by Theorem A.

(2) Let A be the Auslander algebra of $k[T]/ < T^6 >$. Then A is given by the following quiver

$$\begin{tikzcd} 1 \arrow[r, shift right, "\alpha_1"'] & 2 \arrow[l, shift right, "\beta_2"'] \arrow[r, shift right, "\alpha_2"'] & 3 \arrow[l, shift right, "\beta_3"'] \arrow[r, shift right, "\alpha_3"'] & 4 \arrow[l, shift right, "\beta_4"'] \arrow[r, shift right, "\alpha_4"'] & 5 \arrow[l, shift right, "\beta_5"'] \arrow[r, shift right, "\alpha_5"'] & 6 \arrow[l, shift right, "\beta_6"'] \end{tikzcd}$$

with relations $\beta_6\alpha_5 = 0$ and $\beta_i\alpha_{i-1} = \alpha_i\beta_{i+1}$, $i = 2, ..., 5$. (Here we write $\alpha\beta$ to mean that α comes first and then β follows). One can check that $\dim Hom(\Delta(i),\Delta(j)) = \dim Ext^1(\Delta(i),\Delta(j)) = 1$ for $i < j$. Thus $q_{(A,\Delta)}$ is positive. For this algebra there hold proj.dim.$\Delta(i) \leq 1$ and inj.dim.$\nabla(i) \leq 1$, $1 \leq i \leq 6$. However, as shown in [**DR**], the subcategory $\mathcal{F}(\Delta)$ is infinite. This shows that the converse of the above theorem may be false. (Note that this algebra is not hereditary, so the dimension caculation of $Ext^1(X, X)$ in the Tits argument does not work). For other examples one may see the dual extensions of hereditary algebras defined in [**X**, 1.6]; for this class of quasi-hereditary algebras, the quadratic form $q_{(A,\Delta)}$ is always positive.

(3) Consider the Auslander algebra A of $k[X]/(X^2)$. Then A is representation finite and hence $\mathcal{F}(\Delta)$ is finite. On can also see that the projective

dimension of standard modules and the injective dimension of costandard modules are at most one. Moreover, the quadratic form $q_{(A,\Delta)}(x)$ is positive. But there is an indecomposable module $M \in \mathcal{F}(\Delta)$ with $\mathrm{End}_A(M) \cong k^2$, this implies that for non-hereditary algebra the lemma 4 and lemma 8 are not true in general.

References

[BGP] I.N.Bernstein, I.M. Gelfand and V.A.Ponomarev, Coxeter functors and Gabriel's theorem, Russian Math. Survey 28(1973),17–32.

[CPS] E.Cline, B.Parshall and L.Scott, Finite dimensional algebras and highest weight categories, J. reine angew. Math. 391 (1988), 85–99.

[DR] V.Dlab, C.M.Ringel, The module theoretic approach to quasi–hereditary algberas. London Math.Soc. Lect. Note Series 168 (1992), 200-224.

[H] D.Happel, Triangulated categories in the representation theory of finite dimensional algebras. London Math. Soc. Lecture Note Series 119 . Cambridge University Press, 1988.

[KR] H.Kraft, Ch.Riedtmann, Geometry of representations of quivers. London Math. Soc. Lecture Note Series 116(1986), 109–146.

[R1] C.M.Ringel, Tame algebras and integral quadratic forms. Springer LNM 1099(1984).

[R2] C.M.Ringel, The category of modules with good filtration over a quasi–hereditary algebra has almost split sequence. Math. Z. 208(1991), 209–223.

[X] C.C.Xi, Quasi-hereditary algebras with a duality, J. reine angew. Math. 449(1994), 201–215.

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