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Quasi-hereditary algebras with a duality

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In their study of representations of complex Lie-algebras and algebraic groups, Cline, Parshall and Scott introduced in [CPS 1] the concept of quasi-hereditary algebras to describe the so-called highest weight categories. Of particular interest is the case when quasi-hereditary algebras have a duality on their module categories which fixes simple modules. Such a quasi-hereditary algebra is called in [I] a BGG-algebra. The name BGG-algebra is a dedication to the authors of [BGG] since Bernstein, Gelfand and Gelfand proved there the famous BGG reciprocity principle for the category $\mathcal{O}$ in the representation theory of complex semisimple Lie algebras. Motivated by this, an axiomatically defined class of algebras (or categories), the class of BGG-algebras (or BGG-categories) is studied (see [CPS 2] and [I]), for which the BGG reciprocity holds. As examples of BGG-algebras one may think of the Schur algebras [Gr] or q-Schur algebras [DJ], they are important in linking the representation theory of symmetric groups and general linear groups.

In the present paper we are going to develop some properties of BGG-algebras, especially the shape of their quivers and the relationship between the duality functor and the Auslander-Reiten translation. The main result, Theorem 3.1, is the description of the quivers of representation-finite BGG-algebras, namely, the basic graph of a connected BGG-algebra is the Dynkin diagram $\mathbb{A}_n$.

The paper is detailed as follows. In section one we recall some definitions and basic facts, including a construction of a class of BGG-algebras which will be discussed in detail in a further paper. In section two we give some properties of BGG-algebras needed later and in section three we prove the main result of the paper.

In this paper algebra always means a finite dimensional algebra and module always a finitely generated left module. For the terminology we refer to [R 1].

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1. Definition of BGG-algebras and basic facts

Throughout this paper we denote by A a finite-dimensional k-algebra over an algebraically closed field k, and by A-mod the category of all A-modules. If Θ is a class of A-modules (closed under isomorphisms), $\mathscr{F}(\Theta)$ stands for the class of all A-modules M which have a Θ-filtration, i.e., a filtration $M = M_0 \supset M_1 \supset \cdots \supset M_t \supset \cdots \supset M_m = 0$ such that all factor modules M_{t-1}/M_t, $1 \leqq t \leqq m$, belong to Θ.

Let $E(1), \ldots, E(n)$ be the simple A-modules (one from each isomorphism class), note that we fix here a particular ordering of simple modules. Let $P(i)$ be the projective cover of $E(i)$, and $Q(i)$ denote the injective envelop of $E(i)$. By $\Delta(i)$ we denote the maximal factor module of $P(i)$ with composition factors of the form $E(j)$, where $j \leqq i$; the modules $\Delta(i)$ are called the standard modules, and we set $\Delta = \{\Delta(i) \mid 1 \leqq i \leqq n\}$. Similarly, we denote by $\nabla(i)$ the maximal submodule of $Q(i)$ with composition factors of the form $E(j)$ with $j \leqq i$; in this way, we get a set $\nabla = \{\nabla(i) \mid 1 \leqq i \leqq n\}$ of costandard modules. Now let us recall the definition of a quasi-hereditary algebra.

1.1. Definition. Let A be an algebra with standard modules Δ. The algebra is called quasi-hereditary if

(1) $\mathrm{End}_A(\Delta(i)) \cong k$ for all i, and

(2) every projective module belongs to $\mathscr{F}(\Delta)$.

For other equivalent definitions of quasi-hereditary algebras one may consult [DR 2]. To define BGG-algebras, we need one more definition.

1.2. Definition. A duality on A-mod is a contravariant, exact, additive functor δ from A-mod to itself such that $\delta \cdot \delta$ is naturally equivalent to the identity functor on A-mod and that δ induces a k-linear map on the vector spaces $\mathrm{Hom}_A(M, N)$ for all $M, N \in A$-mod.

Note that this definition is different from the one in [I] and [CPS 2] and more restricted than that given in [I]. Following [I], we define BGG-algebras as follows.

1.3. Definition. Let A be a quasi-hereditary algebra with standard modules Δ. If there is a duality δ on A-mod such that $\delta E(i) \cong E(i)$ for all i, then A is called a BGG-algebra.

Clearly, BGG-algebras are invariant under Morita equivalences and the opposite algebra A^{op} of a BGG-algebra A is also a BGG-algebra.

Schur algebras are examples of BGG-algebras (see [Gr], p. 32 and p. 71).

1.4. Remark. Let δ be a duality on A-mod. Then the following are equivalent:

(1) $\delta E(i) \cong E(i)$ for all i,

(2) $\delta P(i) \cong Q(i)$ for all i,

(3) $\delta \Delta(i) \cong \nabla(i)$ for all i.

To see whether a given algebra is a BGG-algebra, the following result may be useful.

1.5. Theorem. *Suppose that A is a basic quasi-hereditary algebra with standard modules $\Delta(1), \ldots, \Delta(n)$ and that $P(i) = Ae_i$, $1 \leqq i \leqq n$, where all e_i form a complete system of pairwise orthogonal primitive idempotents of A. If there is an anti-automorphism $\varepsilon : A \to A$ of the algebra A such that $A\varepsilon(e_i) \cong Ae_i$ for all i, then A is a* BGG-*algebra.*

Recall that an anti-automorphism $\varepsilon : A \to A$ of an algebra A is a k-linear map such that

(1) $\varepsilon(a + b) = \varepsilon(a) + \varepsilon(b)$,

(2) $\varepsilon(ab) = \varepsilon(b)\varepsilon(a)$,

(3) $\varepsilon^2(a) = a$ for all $a, b \in A$.

For the proof of this theorem one may refer to [CPS 2]. Note that the duality is induced from the anti-automorphism as follows: Let M be an A-module, we denote by M^* the finite-dimensional k-space $\mathrm{Hom}_k(M, k)$. Now we define a module structure on M^* as follows (cf. [BGG]): For any $a \in A$ and $f \in M^*$, set

$$(af)(m) = f(\varepsilon(a)m).$$

Then M^* becomes an A-module. For $\alpha \in \mathrm{Hom}_A(M, N)$, we define

$$\alpha^* = \mathrm{Hom}_k(\alpha, k) : N^* \to M^*$$

by $f \mapsto \alpha f$ for all $f \in N^*$.

Now let us give an example of BGG-algebras to end this section.

1.6. Example. Suppose that a finite-dimensional algebra A is given by the quiver $Q = (Q_0, Q_1)$ with relations ϱ_i, $i \in I$. We define a new quiver Q_B whose vertex set is Q_0, and the set of arrows is $Q_1 \cup Q_1'$, where $Q_1' = \{\alpha' : i \to j \mid$ if there is an arrow $\alpha : j \to i$ in $Q_1\}$. If $\varrho = \alpha_1\alpha_2 \cdots \alpha_m$ is a path in Q, then we denote by ϱ' the path $\alpha_m' \cdots \alpha_2'\alpha_1'$ in Q_B. Now let B be the algebra over k given by the quiver Q_B with relations ϱ_i, ϱ_i', $i \in I$ and $\alpha\beta' = 0$, $\alpha \in Q_1$, $\beta' \in Q_1'$.

It is clear that B is a finite-dimensional k-algebra since A is finite-dimensional, and that A is a subalgebra of B and also a factor algebra of B. Moreover, we shall prove the following:

If the quiver of A does not contain any oriented cycle, then B is a BGG-algebra.

Proof. Let $\varepsilon : B \to B$ be the k-linear map induced by $\varepsilon(e_i) = e_i$, $\varepsilon(\alpha) = \alpha'$ and $\varepsilon(\alpha') = \alpha$ for $\alpha \in Q_1$. Then, by Theorem 1.5, it suffices to prove that B is a quasi-hereditary algebra. To this purpose, we use the following equivalent definition of quasi-hereditary algebras (see for example [DR 2]).

Recall that an ideal J of an algebra A is called a heredity ideal in A if $J^2 = J$, $J(\mathrm{rad}(A))J = 0$ and ${}_AJ$ is a projective module. The algebra A is quasi-hereditary if and only if there is a chain

$$0 = J_0 \subset J_1 \subset \cdots \subset J_m = A$$

of ideals in A such that J_{i+1}/J_i is a heredity ideal of A/J_i for each i.

Since the quiver of A has no oriented cycle, we may have an ordering of simple A-modules $E(i)$ such that $\mathrm{Hom}_A(P_A(j), P_A(i)) = 0$ for $j > i$. Let us consider the ideal Be_nB. We shall prove that Be_nB is a heredity ideal. (1) It is clear that $Be_n = Ae_n$ and $e_nBe_n = e_nAe_n \cong k$. (2) Note that $e_n\mathrm{rad}(A) = 0$ and $\omega(\mathrm{rad}(A)) = 0$ for each ω which is a linear combination of monomials in $(Q')^* := \{(\alpha')^* \mid \alpha' \in Q'\} \subset B$. Let M be the set of all ω which are linear combinations of monomials in $(Q')^*$. Then $\dim_k M = \dim_k \mathrm{rad}(A)$. According to the definition of B, we have that

$$e_nB = ke_n + e_n\mathrm{rad}(B) = ke_n + e_n\mathrm{rad}(A) + e_nM + e_n\mathrm{rad}(A)M = ke_n + e_nM .$$

Thus $\dim_k e_nB = \dim_k Ae_n$ and $\dim_k Be_n \otimes_k e_nB = (\dim_k Ae_n)^2$. (3) By the definition of a heredity ideal, it remains to prove that Be_nB is a projective B-module. To do this, it is equivalent to showing by [DR1] that the multiplication map

$$\mu : Be_n \otimes_k e_nB \rightarrow Be_nB$$

is bijective. Let us calculate the dimension of Be_nB. Since $Be_nB = Ae_nB = Ae_n + Ae_nM$, and by the definition of B there holds

$$\dim_k(\mathrm{rad}(A))e_nM = \dim_k(\mathrm{rad}(A))e_n \otimes_k e_nM = (\dim_k Ae_n - 1)^2 ,$$

we have

$$\begin{aligned}\dim_k Be_nB &= \dim_k Ae_n + \dim_k e_nM + \dim_k(\mathrm{rad}(A))e_nM \\ &= \dim_k Ae_n + \dim_k(\mathrm{rad}(A))e_n + (\dim_k Ae_n - 1)^2 \\ &= \dim_k Ae_n + \dim_k Ae_n - 1 + (\dim_k Ae_n)^2 - 2\dim_k Ae_n + 1 \\ &= (\dim_k Ae_n)^2 \\ &= \dim_k(Be_n \otimes_k e_nB) .\end{aligned}$$

Hence the surjective map μ is bijective and therefore Be_nB is a heredity ideal of B.

Since B/Be_nB can be obtained from A/Ae_nA by the construction, we know by induction on the number of simple modules over A that B/Be_nB is quasi-hereditary. Thus B is quasi-hereditary and thus a BGG-algebra.

Note that the BGG-algebra obtained in this way has exact Borel subalgebra in the sense of [K] and other nice properties. In a subsequent paper we will investigate this kind of BGG-algebras B in details, especially the finiteness of the category $\mathscr{F}(\Delta_B)$.

2. Some properties of BGG-algebras

In this section we study some properties of BGG-algebras. We begin with the following lemma.

2.1. Lemma ([I]). *Let* $A = P(1) \oplus \cdots \oplus P(n)$ *be a* BGG-*algebra and*

$$c_{ij} := \dim_k \operatorname{Hom}_A(P(i), P(j)).$$

Put $C_A = (c_{ij})$, *the Cartan matrix of* A. *Let* d_{ij} *be equal to* $\dim_k \operatorname{Hom}_A(P(j), \Delta(i))$, *and* $D = (d_{ij})$. *Then:*

(1) $C_A = D^t D$ *is a symmetric matrix.*

(2) $\dim M = \dim \delta M$.

(3) $[P(i):\Delta(j)] = [\Delta(j):E(i)]$, *where* $[P(i):\Delta(j)]$ *stands for the number of quotients in a* Δ*-filtration of* $P(i)$ *which are isomorphic with* $\Delta(j)$.

From 2.1 we have the following

2.2. Lemma. (1) $\dim_k A = \sum_{j=1}^{n} (\dim_k \Delta(j))^2$.

(2) *Let* χ_A *be the Euler characteristic form of* A *introduced by Ringel in* [R1], *namely*,

$$\chi_A(\underline{\dim}\, M) = \sum_{t \geqq 0} (-1)^t \dim_k \operatorname{Ext}_A^1(M, M).$$

Then χ_A *is positive-definite.*

Proof. (1) follows from Lemma 2.1 (1).

(2) Since C_A is positive-definite by [G], the matrix C_A^{-t} is also positive-definite. So it follows that

$$\chi_A(x) = \langle x, x \rangle = x C_A^{-t} x^t$$

is positive-definite.

From the above lemma, we have

2.3. Corollary. *If* X *is an* A*-module with* $\operatorname{End}_A(X) \cong k$ *and* proj. dim. $X \leqq 1$, *then* $\operatorname{Ext}_A^1(X, X) = 0$.

2.4. Lemma. *Suppose* A *is a quasi-hereditary algebra with standard modules* $\Delta(1), \ldots, \Delta(n)$ *and costandard modules* $\nabla(1), \ldots, \nabla(n)$. *Then* $\langle \underline{\dim}\, \Delta(i), \underline{\dim}\, \nabla(j) \rangle = \delta_{ij}$. *In particular, if* A *is a* BGG-*algebra, then* $\langle \underline{\dim}\, \Delta(i), \underline{\dim}\, \Delta(j) \rangle = \delta_{ij}$, *and the number of the positive roots of* χ_A *is* n, *where* δ_{ij} *is the Kronecker symbol.*

Recall that a vector $0 \neq x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ with $x_i \geqq 0$ for all i is called a positive root of χ_A if $\chi_A(x) = 1$.

2.5. Theorem. *Let A be a* BGG*-algebra with a duality δ. Then*

$$\dim_k \operatorname{Ext}_A^t(M, N) = \dim_k \operatorname{Ext}_A^t(\delta(N), \delta(M))$$

*for all $M, N \in A$-*mod *and $t \geqq 0$.*

Proof. We prove the theorem by induction on t. For $t = 0$ and $t = 1$ the assertion follows from the definition of δ. Let $t \geqq 2$, and suppose the result is true for $t - 1$. Let

$$0 \to K \to P(M) \to M \to 0$$

be an exact sequence such that $P(M) \to M$ is a projective cover of M. Then

$$0 \to \delta M \to \delta(P(M)) \to \delta K \to 0$$

is an exact sequence with $\delta M \to \delta(P(M))$ an injective envelop of δM. It follows from $\operatorname{Ext}_A^1(M, N) \cong \operatorname{Ext}_A^1(K, N)$ that

$$\dim_k \operatorname{Ext}_A^t(M, N) = \dim_k \operatorname{Ext}_A^{t-1}(K, N) = \dim_k \operatorname{Ext}_A^{t-1}(\delta N, \delta K) = \dim_k \operatorname{Ext}_A^t(\delta N, \delta M)$$

as desired.

As a consequence of 2.5 we have the following important corollary which describes the shape of the quiver of a BGG-algebra.

2.6. Corollary. *Assume that A is a* BGG*-algebra. Then*

$$\dim_k \operatorname{Ext}_A^1(E(i), E(j)) = \dim_k \operatorname{Ext}_A^1(E(j), E(i))$$

for all i, j.

Suppose a BGG-algebra is given by the quiver $Q = (Q_0, Q_1)$ with relations. Let $w = \alpha_1 \cdots \alpha_m$ be a non-zero path from 0 to m with $\alpha_i \in Q_1$. To each arrow α_i from $i - 1$ to i we have a non-zero map $P(\alpha^*): P(i) \to P(i-1)$ (see [R1], p. 46 for the details). Hence we have a non-zero map $P(w^*): P(m) \to P(0)$ which is a product of $P(\alpha_i^*)$. If we apply the duality δ to the map $P(w^*)$ then we have a non-zero map $Q(0) \to Q(m)$ which is a product of $\delta(P(\alpha_i^*))$. From the quiver point of view, this implies the following fact:

2.7. Lemma. *If w is a non-zero path from i to j in Q then there is a non-zero path from j to i in Q.*

2.8. Now let us consider the relationship of the duality δ and the Auslander-Reiten translation. Assume that the duality δ in the definition of a BGG-algebra is given by an anti-automorphism ε with $A\varepsilon(e_i) \cong Ae_i$, $1 \leqq i \leqq n$. Thus $\delta = *$ as defined in 1.5.

Recall that the Nakayama functor ν is given by $D\operatorname{Hom}_A(-,{}_AA)$, and it is an equivalence between the projective modules and injective modules. The inverse of ν is $\nu^-(=\operatorname{Hom}_A(D(A_A),-))$. The following result establishes a connection between these functors.

Theorem. *For any module M there holds $(\nu M)^* \cong \nu^-(M^*)$.*

Proof. For a module M we denote by ${}_AM'$ the k-space $\operatorname{Hom}_A(M,A)$ with the following left module structure:

$$a\cdot f : m \mapsto (mf)\varepsilon(a)\,, \quad a\in A, f\in M', \quad m\in M\,.$$

To prove the theorem, we shall show below that $M' \overset{\varphi}{\cong} (D\operatorname{Hom}_A(M,A))^*$ and $M' \overset{\psi}{\cong} \operatorname{Hom}_A(DA, M^*)$ as modules.

Let us first define the map φ. For each $f: M \to_A A$ we have a map

$$\varphi_f : D\operatorname{Hom}_A({}_AM, A) \to k$$

by sending each $\alpha \in D\operatorname{Hom}_A(M,A)$ to $\alpha(f)$, the image of f under the map α. One can check that φ is an A-homomorphism and injective. Thus it follows from comparing the dimensions of two spaces that φ is an isomorphism.

Now we turn to defining the second map ψ. Given an A-homomorphism $f: M \to A$, let ψ_f be the map from DA to M^* which maps $x\in DA$ to $f\varepsilon x$. We can verify that ψ_f is an A-homomorphism. Moreover, ψ is an injective A-homomorphism. This yields that ψ is even an isomorphism and finishes the proof of 2.8.

2.9. Lemma. *For any homomorphism $f: X \to Y$ the following diagram commutes:*

$$\begin{array}{ccc}
\nu^-(Y^*) & \xrightarrow{\nu^-(Df)} & \nu^-(X^*) \\
{\scriptstyle\psi_Y}\uparrow & & \uparrow{\scriptstyle\psi_X} \\
Y' & \xrightarrow[\operatorname{Hom}_A(f,A)]{} & X' \\
{\scriptstyle\varphi_Y}\downarrow & & \downarrow{\scriptstyle\varphi_X} \\
(\nu Y)^* & \xrightarrow[D(\nu f)]{} & (\nu X)^*\,.
\end{array}$$

The proof of this lemma is routine, we omit it. Let us denote by τ the Auslander-Reiten translation $D\operatorname{Tr}$. As a consequence of 2.8 we have

2.10. Theorem. *For any module M there holds:*

(1) $(\tau M)^* \cong \tau^{-1}(M^*)\,,$

(2) $(\tau^{-1}M)^* \cong \tau(M^*)\,.$

Proof. We prove only (1), the second statement follows dually. We may assume that M is indecomposable. We start with a minimal projective presentation of M, say

$$P_1 \xrightarrow{p} P_0 \longrightarrow M \longrightarrow 0\,.$$

Then τM is given by the kernel of νp. Note that the Nakayama functor is right exact, thus we obtain the following exact sequence

$$0 \to \tau M \to \nu P_1 \to \nu P_0 \to \nu M \to 0\,.$$

In case M is indecomposable and not projective, τM is indecomposable, and we obtain in this way a minimal injective presentation of τM (with cokernel added to the right). Now applying $*$ to the exact sequence, we get

$$0 \to (\nu M)^* \to (\nu P_0)^* \to (\nu P_1)^* \to (\tau M)^* \to 0\,.$$

On the other hand, we have an exact sequence from the minimal projective presentation of M by applying the duality $*$:

$$0 \to M^* \to P_0^* \to P_1^*\,.$$

By the construction of τ^{-1} we have the following exact sequence

$$0 \to \nu^-(M^*) \to \nu^-(P_0^*) \to \nu^-(P_1^*) \to \tau^{-1}(M^*) \to 0\,.$$

According to 2.8 and 2.9 we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & (\nu M)^* & \to & (\nu P_0)^* & \to & (\nu P_1)^* & \to & (\tau M)^* & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \nu^-(M^*) & \to & \nu^-(P_0^*) & \to & \nu^-(P_1^*) & \to & \tau^{-1}(M^*) & \to 0\,. \end{array}$$

This implies that $(\tau M)^* \cong \tau^-(M^*)$.

2.11. Remark. By induction, we can prove that for any positive integer m and any module M,

(1) $(\tau^m M)^* \cong \tau^{-m}(M^*)$,

(2) $(\tau^{-m} M)^* \cong \tau^m(M^*)$,

(3) if $M \cong M^*$, then $\tau^m M \cong (\tau^{-m} M)^*$.

2.12. For a BGG-algebra with a duality δ, we call an indecomposable module M with $\delta M \cong M$ self-dual. The important class of self-dual modules are $\mathscr{F}(\Delta) \cap \mathscr{F}(\nabla)$, as described in [R 2] (see also [AR]). The following result shows that we can get new self-dual modules from a given one. This may be helpful if one wants to construct the AR-quiver of the algebra.

Lemma. *Suppose A is an algebra with the duality δ, and let*

$$0 \to M \to \bigoplus_{i=1}^{m} X^{n_i} \to \delta M \to 0$$

be an Auslander-Reiten sequence with X_i indecomposable and $X_i \not\cong X_j$ for $i \neq j$. If $\mathrm{Hom}_A(X_i, X_j) = 0$ for all $i \neq j$, then X_i, $1 \leq i \leq m$, are self-dual.

3. Representation-finite BGG-algebras

In this section we always assume that A is a representation-finite BGG-algebra which is also connected and basic. Suppose $A = P(1) \oplus \cdots \oplus P(n)$ with standard modules $\Delta(i)$, $1 \leq i \leq n$, note that the ordering of the primitive idempotents $e_1, \ldots, e_n$ (or simple modules) is the usual order $1 < 2 < \cdots < n$. We always assume that $P(i) = Ae_i$ for all i and denote by N the Jacobson radical of A. For some basic properties on quasi-hereditary algebras we refer the reader to [R 2]. Our aim in this section is to determine the quiver of A, namely, we prove the following theorem (see 3.10 below for the definition of basic graphs).

3.1. Theorem. *Let A be a representation-finite connected basic* BGG*-algebra. Then the basic graph $G(A)$ of A is a Dynkin graph of type $\mathbb{A}_n$.*

In order to prove the theorem, we require some preparations.

I. General facts. The following lemma is well-known in the literature.

3.2. Lemma. (1) *If B is a representation-finite algebra then for every idempotent e the algebra eBe is representation-finite.*

(2) *A basic local algebra is representation-finite if and only if it is isomorphic to $k[X]/(X^m)$ for some positive integer m.*

3.3. Lemma. *Let B be a basic algebra and e_1, e_2 two idempotents. Assume that Be_1 and Be_2 are non-isomorphic indecomposable modules and $e_iBe_j \neq 0$ for all $i, j \in \{1, 2\}$. If* $\mathrm{End}_B(Be_2) \cong k$ *and* $\dim_k e_1Be_1 \geq 5$, *then* $\mathrm{End}_B(B(e_1 + e_2))$ *is representation-infinite.*

Proof. Put $\mathscr{C} = \mathrm{add}(Be_1 \oplus Be_2)$. We compute the quiver of the algebra $R := \mathrm{End}_B(Be_1 \oplus Be_2)$. Clearly, it has two vertices. Now let us denote by $\mathrm{Irr}_{\mathscr{C}}(Be_i, Be_j)$ the irreducible maps in $\mathscr{C}$. Thus, if $\dim_k \mathrm{Irr}_{\mathscr{C}}(Be_i, Be_j) \geq 2$ then R is representation-infinite. Hence we may assume that $\dim_k \mathrm{Irr}_{\mathscr{C}}(Be_i, Be_j) \leq 1$ for all $i, j \in \{1, 2\}$. Note that $\dim_k \mathrm{Irr}_{\mathscr{C}}(Be_2, Be_2) = 0$ and $e_2Be_1Be_2 = 0$ since $\mathrm{End}_B(Be_2) \cong k$, and that $\dim_k \mathrm{Irr}_{\mathscr{C}}(Be_i, Be_j) \neq 0$ for $i, j \in \{1, 2\}$ with $i \neq j$ since $\mathrm{Hom}_B(Be_i, Be_j) \neq 0$.

(1) Suppose $\mathrm{Irr}_{\mathscr{C}}(Be_1, Be_1) = 0$. In this case the opposite quiver of the quiver of R has the following form

$$1 \circ \underset{\alpha}{\overset{\beta}{\rightleftarrows}} \circ 2 .$$

Clearly, there is the relation $\alpha\beta = 0$. So the dimension of $e_1 Be_1$ which is the same as that of the endomorphism algebra of the projective R-module corresponding to the vertex 1 of the above quiver is smaller than 5, a contradiction. Hence we have

(2) $\mathrm{Irr}_{\mathscr{C}}(Be_1, Be_1) \neq 0$. In this case the opposite quiver of the quiver of R is of the following form

$$\gamma \circlearrowright 1 \underset{\alpha}{\overset{\beta}{\rightleftarrows}} 2 .$$

By [F], p. 97, the algebra with our dimension condition given by the above quiver is representation-infinite.

The proof of the following lemma is straightforward.

3.4. Lemma. *If B is the algebra given by the quiver*

$$1 \underset{\beta}{\overset{\alpha}{\leftrightarrows}} 3 \underset{\gamma}{\overset{\delta}{\leftrightarrows}} 2$$

with relations $\alpha\beta = \gamma\delta = 0$, then the algebra B is representation-infinite, in fact, the modules M_m given by the following Loewy diagram

$$\begin{array}{ccccccccccccc}
 & 1 & & 2 & & 1 & & & & 1 & & & \\
 & \beta\downarrow & & \delta\downarrow & & \beta\downarrow & & \cdots & & \downarrow\beta & & & \\
 & 3 & & 3 & & 3 & & & & 3 & & & \\
\swarrow\alpha & & \searrow\gamma\ \swarrow\alpha & & \searrow\gamma\ \swarrow\alpha & & \searrow\gamma & & \swarrow\alpha & & \searrow\gamma & & \\
1 & & 2 & & 1 & & 2 & 1 & & & 2 & &
\end{array}$$

where the number 3 occurs $2m+1$ times, are a family of non-isomorphic indecomposable modules.

II. Serality of standard modules.

3.5. Lemma. *Let $c_{ij} = \dim_k \mathrm{Hom}_A(P(i), P(j))$ be the (i,j)-th entry of the Cartan matrix C_A of A. Then $c_{in} = c_{ni} \leqq 1$ for all i.*

Proof. Since Ae_nA is a heredity ideal of A, the module $\bigoplus_{j=1}^{c_{ni}} P(n)$ can be embedded in $P(i)$. (Note that this fact is often used in [X] and will be used without reference in what follows.) If $c_{ni} \geqq 2$, then, by 2.1 (1) and 3.3, one can see that $(e_i + e_n)A(e_i + e_n)$ is representation-infinite. A contradiction to 3.2 (1).

3.6. Proposition. *There is an indecomposable projective module P which is a serial module, i.e. it has a unique composition series.*

Proof. Let P be the projective module $\Delta(n)$, we shall show it is a serial module. Suppose that P is not serial. Then we consider the following filtration

$$P = Ae_n \supset Ne_n \supset \cdots \supset N^l e_n \supset \cdots \supset 0$$

of P. Let l be the minimal number such that $N^l e_n / N^{l+1} e_n$ is not simple, say $N^l e_n / N^{l+1} e_n \cong E(i) \oplus E(j) \oplus U$, where U is a semisimple A-module. If

$$N^{l-1} e_n / N^l e_n \cong E(n'),$$

then $\mathrm{Ext}^1_A(E(n'), E(i)) \neq 0 \neq \mathrm{Ext}^1_A(E(n'), E(j))$. Since the Cartan matrix C_A is symmetric by 2.1, $\mathrm{Hom}_A(P(n), P(i)) \neq 0 \neq \mathrm{Hom}_A(P(n), P(j))$. Thus $c_{ni} = c_{nj} = 1$. From the heredity of the ideal of Ae_nA it follows that P can be embedded in $P(i)$ and $P(j)$, respectively. Note that $i \neq j$ by 3.5. In this case, we consider the factor algebra $\bar{A} := A / N^{l+1} e_n A$. By identifying n' with 3 and i, j with 1, 2 in 3.4, respectively, and using the family of indecomposable modules in 3.4, we can construct infinitely many indecomposable $\bar{A}$-modules (or using the list of [F], 2.6, 2.7, 2.1 (1) and the heredity of Ae_nA to prove that the algebra $(e_{n'} + e_i + e_j) A (e_{n'} + e_i + e_j)$ is representation-infinite). This is a contradiction and shows that P must be a serial module.

Since A/Ae_nA is again a representation-finite BGG-algebra, we have as a consequence of 3.6 the following result.

3.7. Corollary. *Every standard module $\Delta(i)$ of a representation-finite* BGG-*algebra is serial.*

If we suppose $c_{ij} \leqq 1$ for all $i \neq j$, then we can say something more on the standard modules.

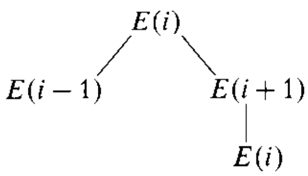
3.8. Lemma. *If $c_{ij} \leqq 1$ for all $i \neq j$, then each standard module of A is serial with Loewy length at most 2.*

Proof. To prove the lemma, it is sufficient to demonstrate that the projective module Ae_n has the Loewy length $\mathrm{LL}(Ae_n)$ smaller than 3. Since A is connected and C_A is symmetric, we see that if $\mathrm{LL}(Ae_n) = 1$, then A is a simple algebra. Thus the lemma is trivial. Now we suppose $\mathrm{LL}(Ae_n) > 2$. In this case, consider the series

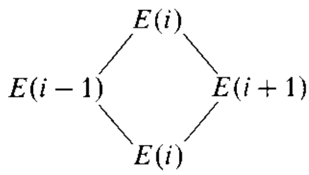
$$Ae_n \supset Ne_n \supset N^2 e_n \supset N^3 e_n \supset \cdots$$

with $Ne_n / N^2 e_n = E(i)$ and $N^2 e_n / N^3 e_n = E(j)$, where i, j, n are pairwise distinct. This means that $c_{ni} \neq 0 \neq c_{ij}$ and $E(i)$ appears in the top of Ne_j and $\mathrm{Ext}^1_A(E(i), E(j)) \neq 0$. Thus $\mathrm{Ext}^1_A(E(j), E(i)) \neq 0$ by 2.6. Since $c_{nj} \neq 0$ and Ae_nA is a heredity ideal in A, the projective module Ae_n can be regarded as a submodule of Ne_j. Hence $E(i)$ occurs at least two times as composition factors in a composition series of $P(j)$ and then $c_{ij} \geqq 2$. This yields a contradiction to our assumption.

3.9. Proposition. *Suppose $c_{ij} \leqq 1$ for all $i \neq j$. If $n \geqq 3$, then the indecomposable projective module $P(i)$ is of the form*



or

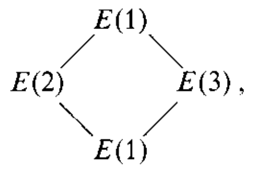


for $2 \leqq i \leqq n-1$.

Proof. If $n = 3$ then $B = A/Ae_nA$ is a representation-finite BGG-algebra and one can see immediately that B is isomorphic to the following algebra given by the quiver

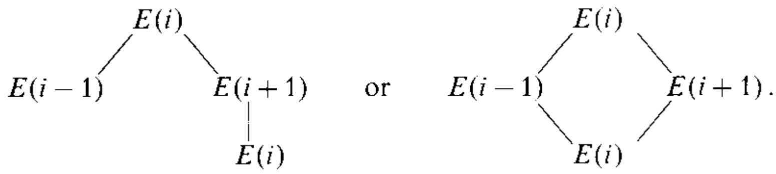
$$1 \circ \underset{\alpha}{\overset{\beta}{\rightleftarrows}} \circ 2$$

with relation $\alpha\beta = 0$. If $\mathrm{Hom}_A(P(1), P(3)) \neq 0$ then $P(1)$ must be of the form

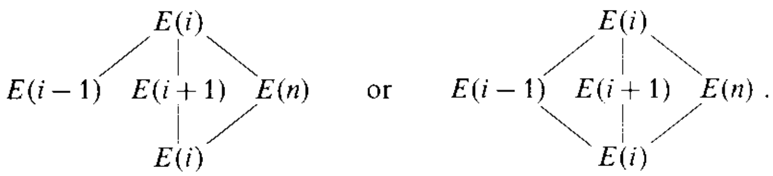


since e_1Ae_1 is a serial algebra and $P(3)$ is of Loewy length 2 by 3.8. This would imply that A is not a quasi-hereditary algebra. Thus we have $c_{13} = 0$ and $c_{23} \neq 0$. In this case, we have the wished form for $P(2)$.

Now suppose we have proved the proposition for $n-1$ with $n \geqq 4$. Then $B = A/Ae_nA$ is again a BGG-algebra with $c'_{ij} \leqq 1$, where $C_B = (c'_{ij})$ is the $(n-1)\times(n-1)$ Cartan matrix of B. Since there is only one i such that $\mathrm{Hom}_A(P(i), P(n)) \neq 0$ by 3.8, we know that the indecomposable A-module $P(j)$ for $j \neq i$ coincides with the projective B-module $P_B(j)$ corresponding to the vertex j in the quiver of B, in particular, the projective module $P(j)$, $2 \leqq j \leqq n-1$ and $j \neq i$, have the form in 3.9. Now let us consider the case $2 \leqq j = i$. If $i = n-1$ then the argument in case $n = 3$ shows that $P(i)$ is of the desired form. Now we may suppose that $i \leqq n-2$. Note that $P_B(i)$ is of the form



Since e_iAe_i is a serial local algebra, we have that $P(i)$ must be of the form



In both cases we have $P_B(i) \ncong P(i)/(Ae_nA)P(i)$. This is impossible. Hence the proof is completed.

3.10. Definition. Let B be a BGG-algebra with n non-isomorphic simple modules $E(1), \ldots, E(n)$. We define a graph $G(B)$ whose vertex set is $\{1, \ldots, n\}$, and there are d edges between i and j with $i \neq j$ if $d = \dim_k \mathrm{Ext}^1_B(E(i), E(j))$. We call this graph the basic graph of B. (Note that the quiver of B can be recovered from $G(B)$.)

3.11. Corollary. *If the Cartan matrix $C_A = (c_{ij})$ has the property that $c_{ij} \leqq 1$ for all $i \neq j$, then the basic graph $G(A)$ of A is a Dynkin graph $\mathbb{A}_n$.*

III. Small BGG-algebras. First we have the following lemma.

3.12. Lemma. *If A has three non-isomorphic simple modules, then A is isomorphic to one of the algebras given by the following quiver with different relations:*

$$1 \underset{\alpha'}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta'}{\overset{\beta}{\rightleftarrows}} 3$$

(I) $\alpha\beta = \beta'\alpha' = \beta'\beta = 0, \beta\beta' = \alpha'\alpha$,

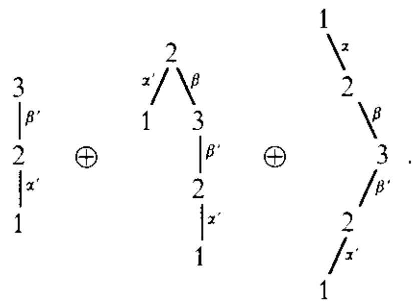
(II) $\beta'\beta = \alpha'\alpha = \beta'\alpha' = \alpha\beta = 0$,

(III) $\beta'\beta = 0, \beta\beta' = \alpha'\alpha$.

Proof. By 3.6 the projective module Ae_3 is a serial module. Since $c_{i3} \leqq 1$ for all i by 3.5, the Loewy length of Ae_3 is at most 3. Of course, $\mathrm{LL}(Ae_3) \neq 1$ since A is assumed to be connected with more than one simple modules.

1) $\mathrm{LL}(Ae_3) = 2$. In this case we may use the argument in the proof of Proposition 3.9 to get the first two groups of relations described in (I) and (III).

2) $\mathrm{LL}(Ae_3) = 3$. In this case one should consider the following two situations: a) $\alpha'\alpha \neq 0$ and b) $\alpha'\alpha = 0$. In case b) we have the following regular representation of ${}_AA$:



(Note that we should keep always in mind the fact that e_iAe_i is a local serial algebra.) This would show clearly that A is not quasi-hereditary. Thus b) is impossible. Now assume a). In this case we must have the relation $\alpha'\alpha = \beta\beta'$ since e_2Ae_2 is a representation-finite local algebra. Thus we arrive at the case (III) in 3.12 as desired.

3.13. Lemma. *If a representation-finite* BGG*-algebra B with* 4 *simple modules, then its quiver is not of the form*

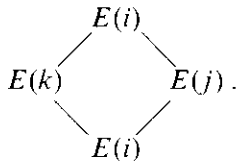
$$\begin{array}{ccccc} & & j & & \\ & & \beta'\downarrow\uparrow\beta & & \\ k & \underset{\alpha'}{\overset{\alpha}{\rightleftarrows}} & i & \underset{\gamma'}{\overset{\gamma}{\rightleftarrows}} & 4. \end{array}$$

Proof. Suppose the quiver of B is of the form. We shall show this yields a contradiction.

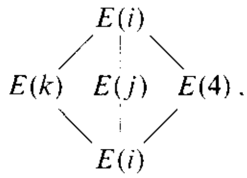
Note that $\mathrm{LL}(Be_4) \leqq 3$. Put $C = B/Be_4B$, then C is again a representation-finite BGG-algebra with 3 simple modules.

1. case: $\mathrm{LL}(Be_4) = 2$.

1.a) C is given by (I) in 3.12, and we may assume that $P_C(i)$ is of the shape



Since e_iBe_i is a local serial algebra, the shape of $Be_i/\mathrm{rad}(B)^3Be_i$ is of the form



but this would imply that $P_C(i) \not\cong Be_i/Be_4Be_i$, a contradiction.

1.b) C is given by (II) in 3.12. In this case, a similar argument to 1.a) shows that there is a contradiction.

1.c) C is given by (III) in 3.12. Since e_iBe_i is a serial local algebra, we must have $\alpha'\alpha = \beta\beta' = \gamma'\gamma$. This leads to the relation $\beta\beta'\alpha' = \gamma'\gamma\alpha' = \gamma' \cdot 0 = 0$ because $\mathrm{LL}(Be_4) = 2$. On the other hand, we have that $\beta\beta'\alpha'$ is not equal to zero in C by the definition in (III) of 3.12 and thus a contradiction.

2. case: $\mathrm{LL}(Be_4) = 3$.

If C is given by (I) or (II) in 3.12, one can argument as in 1.a) to obtain a contradiction that B is not quasi-hereditary.

If C is given by (III) of 3.12 then there are two possibilities for C to be considered. One possibility is that $P_C(k)$ is a projective standard module for C, and the other one is that $P_C(j)$ is a projective standard module for C. But if it was one of the both cases, one can show that C would not be a quasi-hereditary algebra. Hence the proof is finished.

IV. Proof of Theorem 3.1. (a) If A is representation-finite BGG-algebra, then $G(A)$ is one of the Dynkin graphs $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$.

Indeed, if a factor algebra of A is representation-infinite, then the algebra itself is representation-infinite. Let us consider the algebra A/N^2. Thus we can use Gabriel's theorem to determine the representation types. If the graph $G(A)$ contains a cycle, then the separated quiver associated to A/N^2 contains a full subquiver of type $\tilde{\mathbb{A}}_n$, and this implies that A is representation-infinite. Hence $G(A)$ is a tree. With a similar argument one sees that $G(A)$ must be a Dynkin graph.

(b) Suppose $G(A)$ is a graph in $\{\mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ and c the centre of the graph $G(A)$ (the vertex of degree 3). Since $\Delta(i)$ is a serial module for all i, we get, after finitely many steps, a factor algebra B of A which is a representation-finite BGG-algebra with basic graph $G(B)$ whose quiver is of the following form

$$\begin{array}{ccccc} & & j & & \\ & & {\scriptstyle\beta'}\downarrow\uparrow{\scriptstyle\beta} & & \\ k & \underset{\alpha'}{\overset{\alpha}{\rightleftarrows}} & i & \underset{\gamma'}{\overset{\gamma}{\rightleftarrows}} & 4. \end{array}$$

By Lemma 3.13, this is impossible. Hence we have the Theorem 3.1.

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After the paper was submitted, I learned that an explicit result is obtained by S. Donkin and I. Reiten independently, but proofs are different.

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