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HOMOLOGICAL SYSTEMS IN MODULE CATEGORIES OVER PRE-ORDERED SETS

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Abstract

We introduce the so-called homological systems in a module category over a pre-ordered set, which generalize the notion of a stratifying system over a linearly ordered set, and study both the corresponding modules filtrated by the systems and algebras stratified by the systems. In particular, we recover the tilting theory for pre-standardly stratified algebras, and get a general formula for computing the Cartan determinants of pre-standardly stratified algebras in terms of standard modules and simple modules. Also, the finitistic dimension of a given algebra, and the relative homological dimensions of full subcategories of the modules related to a homological system, are discussed. As an application, we get a new bound for the finitistic dimension of a pre-standardly stratified algebra.

1. Introduction

As a generalization of quasi-hereditary algebras, standardly stratified algebras were introduced in several papers [8, 9, 20], and studied in a large variety of the literature [1–3, 8, 10, 11, 20, 22]. One of the important ingredients of a standardly stratified algebra is the set of the standard modules parametrized by a linearly ordered set or a partially ordered set. If we extend this partially ordered set to a pre-ordered set, we get the notion of a pre-standardly stratified algebra; and so, a prescribed set of modules with suitable vanishing of homomorphism and extensions determines the ‘highest weight structure’. This point of view was first taken in [10] and subsequently followed in earlier papers [11, 13, 18]. The recent results in [19] show that the Alperin weight conjecture in the representation theory of finite groups can be connected with investigation of some problems on pre-standardly

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stratified algebras. Thus, a general formulation and systematic study of a collection of modules with certain homological properties, as the standard modules have, would be interesting. In this paper, we continue with this point of view. It allows more flexibility and new applications.

Let A be an Artin R-algebra over a commutative Artin ring R. The category of all finitely generated left A-modules is denoted by A-mod. We first generalize the notion of a stratifying system, in A-mod, over a linearly ordered set in [**11**] to a system in A-mod with certain homological properties over an arbitrary pre-ordered set, which is the so-called homological Θ-system over a pre-ordered set in the terminology of the present paper; and then, we characterize this system as a Θ-injective (or Θ-projective) system which reflects the intrinsic structure of the category of modules associated to the Θ-system. In this way, we re-obtain the tilting theory of a pre-standardly stratified algebra for the Θ-system under weak assumptions (comparing with [**18**, **19**]). Our main results can be stated as follows.

THEOREM 1.1 *Let* $(\Lambda, \leq)$ *be a pre-ordered set. If* $(\Theta; \Lambda, \leq)$ *is a finite* Θ*-system in* A-mod*, then there is a finite* Θ*-injective system* $(\Theta, Y; \Lambda, \leq)$ *in* A-mod*. If* $(\Theta, Y'; \Lambda, \leq)$ *is another one, then* $Y(\lambda) \simeq Y'(\lambda)$ *for all* $\lambda \in \Lambda$*. Moreover,* $\mathrm{End}(\bigoplus_A Y(\lambda))$ *is a right pre-standardly stratified algebra. Conversely, if* $(\Theta, Y; \Lambda, \leq)$ *is a finite* Θ*-injective system in* A-mod*, then* $(\Theta; \Lambda, \leq)$ *is a finite* Θ*-system in* A-mod*.*

For a pre-standardly stratified algebra A, the determinant $\det(C_A)$ of the Cartan matrix C_A of A is given by the following formula in terms of the standard modules and simple modules.

THEOREM 1.2 *Let* $(A; \Lambda, \leq)$ *be a pre-standardly stratified* R*-algebra with standard modules* $\Delta(\lambda), \lambda \in \Lambda$*. Suppose* $(\bar{\Lambda}, \leq)$ *is the partially ordered set induced by the pre-ordered set* $(\Lambda, \leq)$*. Then*

$$\det(C_A) = \frac{\prod_{[\lambda]\in\bar{\Lambda}} \det(C[\lambda])}{\prod_{\lambda\in\Lambda} \ell(\mathrm{End}(_A L(\lambda)))},$$

where ℓ *is the* R*-length function of* R*-modules and* $C[\lambda]$ *is the matrix with entries* $\ell(\mathrm{Hom}_A(\Delta(\gamma), \Delta(\mu)))$ *indexed by* $[\lambda] \in \bar{\Lambda}$*.*

Finally, as an application of our discussions on homological systems, we consider the finitistic dimension of a pre-standardly stratified algebra. Recall that the finitistic dimension conjecture says that every finite-dimensional algebra over a field has finite finitistic dimension. This conjecture is still open. Our main result on homological dimensions of a Θ-system in A-mod is Theorem 5.6, which gives, in particular, a bound of the finitistic dimension of an algebra A in terms of the resolution dimensions of modules related to the Θ-system. As a consequence of this result, we have the following corollary on the finitistic dimension of a pre-standardly stratified algebra. It seems that the bound here is new.

COROLLARY 1.3 *Let* $(A; \Lambda, \leq)$ *be a pre-standardly stratified algebra,* $T := \bigoplus_{\lambda\in\Lambda} T(\lambda)$ *the characteristic tilting module associated to A, and* $B := \mathrm{End}(_A T)$*. We define* $_B\delta(\lambda)$ *to be the quotient of the* B*-module* $\mathrm{Hom}_A(_A T_B, T(\lambda))$ *modulo the sum of images of all homomorphisms from* $\mathrm{Hom}_A(T, T(\mu))$

to $\mathrm{Hom}_A(T, T(\lambda))$ *with* $\mu < \lambda$, *and* $B(\rho) := \mathrm{End}(\bigoplus_{\lambda\in\rho}\ {}_B\delta(\lambda))$ *for* $\rho \in \bar{\Lambda}$. *Then*

$$\mathrm{fpd}(A) \leq \mathrm{pd}({}_AT) + |\bar{\Lambda}| - 1 + \sum_{\rho\in\bar{\Lambda}} \mathrm{fpd}\,(B(\rho)^{\mathrm{op}}) \leq 2\,|\bar{\Lambda}| - 2 + \sum_{\rho\in\bar{\Lambda}} \mathrm{fpd}\,(B(\rho)^{\mathrm{op}}),$$

where pd *and* fpd *stand for the projective dimension and finitistic projective dimension, respectively.*

The paper is organized as follows. In section 2, we recall some basic definitions and facts on pre-standardly stratified algebras and pre-ordered sets. In section 3, we introduce the notion of a Θ-system over a pre-ordered set, discuss its properties and prove the main result, Theorem 1.1. In section 4, we prove Theorem 1.2. In section 5, we study the homological dimensions related to a Θ-system. As an application of the results in section 5, we deduce Corollary 1.3 for pre-standardly stratified algebras in section 6.

Our main results in this paper extend some of the main ones in [**11**, **12**, **14**, **15**, **18**].

2. Preliminaries

In this section, we recall some definitions and facts. Also, new approaches to some well-known results will be given.

Throughout the paper, A denotes an Artin algebra, and A-mod (respectively, mod-A) the category of all finitely generated left (respectively, right) A-modules. By module, we mean a left module. The usual duality of an Artin algebra is denoted by D.

Given two morphisms $f : L \to M$ and $g : M \to N$ in A-mod, we denote the composition of f with g by fg, which is a morphism from L to N. So, we make the following convention: for a morphism f between two (right, or left) modules, we write f on the opposite side of the scalars of the modules.

2.1. *Pre-ordered sets and partially ordered sets*

In this subsection, we recall some basic facts on pre-ordered sets.

Let Λ be a non-empty set and $\leq$ be a relation on Λ. The pair $(\Lambda, \leq)$ is called a pre-ordered set provided that: (a) $\lambda \leq \lambda$ for all $\lambda \in \Lambda$ and (b) if $\lambda \leq \mu$ and $\mu \leq \gamma$ for $\lambda, \mu, \gamma \in \Lambda$ then $\lambda \leq \gamma$.

Let$(\Lambda, \leq)$ be a pre-ordered set. If the relation $\lambda \leq \mu$ does not hold, we write either $\lambda \not\leq \mu$ or $\mu \not\geq \lambda$; if $\lambda \leq \mu$ and $\mu \not\leq \lambda$, we write $\lambda < \mu$. If $\lambda \leq \mu$ and $\mu \leq \lambda$, we say that λ and μ are equivalent, and write $\lambda \sim \mu$. The equivalence class of λ will be denoted by $[\lambda]$. Clearly, the set $\bar{\Lambda}$ of all equivalence classes of Λ forms a partially ordered set, namely we define $[\lambda] \leq [\mu]$ in $\bar{\Lambda}$ if $\lambda \leq \mu$ in Λ.

If $\bar{\Lambda}$ is a finite set, we can write the elements in $\bar{\Lambda}$ as $\{\rho_1, \rho_2, \ldots, \rho_n\}$, where $\rho_i \leq \rho_j$ implies that $i \leq j$. A linearization $(\Lambda, \leq^L)$ of $(\Lambda, \leq)$ is defined as follows: we choose a fixed linear order for each ρ_i; and then, we set $\lambda <^L \mu$ for each λ in ρ_i and each μ in ρ_j if $i < j$. Clearly, if $[\lambda] < [\mu]$ then $\lambda \leq^L \mu$. Note that if $(\Lambda, \leq^L)$ is a linearization of $(\Lambda, \leq)$, it induces a linearization of $(\bar{\Lambda}, \leq)$.

The relationship between the pre-order and its linearization is as follows.

Lemma 2.1 *Let* $(\Lambda, \leq)$ *be a finite pre-ordered set and* $(\Lambda, \leq^L)$ *be a linearization of* $(\Lambda, \leq)$. *If* $\lambda \leq^L \mu$ *for* λ, μ *in* Λ, *then* $\mu \not< \lambda$.

Proof. If λ and μ belong to the same equivalence class, then $\mu \not< \lambda$ by the definition of $\leq$ in Λ. Suppose $\lambda \in \rho_i$ and $\mu \in \rho_j$ with $i < j$; so we have $\lambda \not\sim \mu$. If $\rho_i < \rho_j$ then $\lambda \leq \mu$ and $\lambda < \mu$. If ρ_i and ρ_j are incomparable in $\bar{\Lambda}$, then λ and μ are so in Λ. Hence, we have $\mu \not< \lambda$.

2.2. *Homologically finite subcategories*

In this subsection, we recall some definitions from [6].

A morphism $f : M \to N$ in A-mod is said to be *right minimal* if any endomorphism $g : M \to M$ is an automorphism whenever $f = gf$. It is well known that f is right minimal if and only if the restriction of f to any non-zero direct summand of M is non-zero. A subcategory $\mathcal{X}$ of A-mod is called *contravariantly finite* if for each module C there is a right $\mathcal{X}$-approximation, that is, there is a morphism $f : X \to C$, with $X \in \mathcal{X}$, such that the induced sequence $\mathrm{Hom}_A(X', X) \to \mathrm{Hom}_A(X', C) \to 0$ is exact for all X' in $\mathcal{X}$. A right $\mathcal{X}$-approximation $f : X \to C$ is said to be a minimal right $\mathcal{X}$-approximation if f is right minimal. Dually, we have the notions of left minimal morphisms, left $\mathcal{X}$-approximations and covariantly finite subcategory in A-mod. A subcategory $\mathcal{X}$ of A-mod is called *functorially finite* if it is contravariantly finite and covariantly finite in A-mod.

A subcategory $\mathcal{X}$ of A-mod is said to be *resolving* if it is (a) closed under extensions (that is, if $0 \to L \to M \to N \to 0$ is an exact sequence in A-mod with $L, N \in \mathcal{X}$, then $M \in \mathcal{X}$), (b) closed under kernels of surjective morphisms (that is, if $f : M \to N$ is a morphism in $\mathcal{X}$ and f is surjective, then $\mathrm{Ker}(f) \in \mathcal{X}$) and (c) contains all projective modules. Dually, a subcategory $\mathcal{X}$ of A-mod is said to be *coresolving* if it is closed under extensions and co-kernels of injective morphisms and contains all injective A-modules.

Given a set Θ of A-modules in A-mod, we denote by $\mathcal{F}(\Theta)$ to the full subcategory of A-mod whose objects are the modules M which have a Θ-filtration, namely there is a finite chain

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_t = M$$

of submodules of M such that M_i/M_{i-1} is isomorphic to a module in Θ for all i.

An A-module X is called Θ-*injective* if $\mathrm{Ext}^1_A(Y, X) = 0$ for all $Y \in \Theta$. We denote by $\mathcal{I}(\Theta)$ the full subcategory of all Θ-injective A-modules in A-mod. Dually, an A-module Y is called Θ-*projective* if $\mathrm{Ext}^1_A(Y, X) = 0$ for all $X \in \Theta$. We denote by $\mathcal{P}(\Theta)$ to the full subcategory of all Θ-projective A-modules in A-mod. If X is in A-mod, we simply write $\mathcal{I}(X)$ for $\mathcal{I}(\{X\})$.

For full subcategories $\mathcal{X}$ and $\mathcal{Y}$ in A-mod, we write $\mathrm{Ext}^1_A(\mathcal{X}, \mathcal{Y}) = 0$ if $\mathrm{Ext}^1_A(X, Y) = 0$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Thus, if Y is Θ-injective then $\mathrm{Ext}^1_A(\mathcal{F}(\Theta), Y) = 0$, and if X is Θ-projective then $\mathrm{Ext}^1_A(X, \mathcal{F}(\Theta)) = 0$.

2.3. *Pre-standardly stratified algebras*

Let A be an Artin algebra, and let $(\Lambda, \leq)$ be a pre-ordered set which indexes the non-isomorphic simple A-modules $L(\lambda)$, $\lambda \in \Lambda$. Denote by $P(\lambda)$ the projective cover of $L(\lambda)$. Let $P^{\not\leq\lambda} := \bigoplus_{\mu\not\leq\lambda} P(\mu)$ and $P^{>\lambda} := \bigoplus_{\mu>\lambda} P(\mu)$. For A-modules M and X, we denote by $\mathrm{tr}_M(X)$ the trace of M in X. That is, $\mathrm{tr}_M(X)$ is the sum of images of all homomorphisms from M to X. We also set $\Delta(\lambda) := P(\lambda)/\mathrm{tr}_{P^{\not\leq\lambda}}(P(\lambda))$, $\delta(\lambda) := P(\lambda)/\mathrm{tr}_{P^{>\lambda}}(P(\lambda))$ and $\overline{\delta}(\lambda) := P(\lambda)/\mathrm{tr}_{P^{\geq\lambda}}(\mathrm{rad}\, P(\lambda))$.

Dually, let $I(\lambda)$ be the injective envelope of $L(\lambda)$, we define $\nabla(\lambda)$ to be the intersection of kernels of all homomorphisms from $I(\lambda)$ to $I(\mu)$ with $\mu \not\leq \lambda$. So, $\overline{\nabla}(\lambda)$ will denote to the maximal submodule of the injective envelope $I(\lambda)$ of $L(\lambda)$ such that $[\overline{\nabla}(\lambda)/\mathrm{soc}\,(\overline{\nabla}(\lambda)) : L(\mu)] = 0$ for all $\lambda \leq \mu$. For convenience, we call $\Delta(\lambda)$ the *standard module* and $\nabla(\lambda)$ the *co-standard module*. Note that $\delta(\lambda)$ and $\Delta(\lambda)$ are related via the following exact sequence

$$0 \longrightarrow \mathrm{tr}_{P^{\not\leq\lambda}}(P(\lambda))/\mathrm{tr}_{P^{>\lambda}}(P(\lambda)) \longrightarrow \delta(\lambda) \longrightarrow \Delta(\lambda) \longrightarrow 0.$$

Hence, we may call the module $\delta(\lambda)$ the *big standard module* of A corresponding to λ. As in [9, 12], we call $\overline{\delta}(\lambda)$ the *big proper standard module*, and $\overline{\nabla}(\lambda)$ the *proper co-standard module*. Note that all these modules defined above depend upon the given pre-ordered set.

DEFINITION 2.2 An Artin algebra A, together with a pre-ordered set $(\Lambda, \leq)$, is called a pre-standardly stratified algebra if, for each $\lambda \in \Lambda$, there is an exact sequence

$$0 \longrightarrow Q(\lambda) \longrightarrow P(\lambda) \longrightarrow \Delta(\lambda) \longrightarrow 0$$

in A-mod such that $Q(\lambda)$ has a filtration with sections isomorphic to $\Delta(\mu)$ with $\lambda < \mu$. If $(A; \Lambda, \leq)$ is pre-standardly stratified and if $\Lambda = \bar{\Lambda}$, then we call the algebra A standardly stratified.

REMARKS (1) In some literature (see [1–3, 22]), the notion of standardly stratified algebras means usually those pre-standardly stratified algebras $(A; \Lambda, \leq)$ in which the pre-order is a linear order (or a partial order). We want to distinguish this special class of algebras from the much more general class of pre-standardly stratified algebras (in fact, somehow the whole class of Artin algebras), and introduce the new name 'pre-standardly stratified algebras' with the stress of the pre-ordered set $(\Lambda, \leq)$ in order to avoid confusion. We reserve the notion of standardly stratified algebras for the case of linear (or partial) orders. In fact, the pre-standardly stratified algebras behave very differently from the usual standardly stratified algebras.

(2) The pre-standardly stratified algebras are called 'standardly stratified algebras' in [8], and the terminology was used also in [12, 19]. In [12], the definition of a standard module $\Delta(\lambda)$ is different from the one in our paper. In fact, the standard module $\Delta(\lambda)$ in [12] is equal to $\delta(\lambda)$ in our notation. However, the definition of standardly stratified algebras in [12] coincides with the one in [8, 19]; it also coincides with the one of pre-standardly stratified algebras given in the present paper.

(3) Pre-standardly stratified algebras can also be defined either by using chains of strong idempotent ideals or stratified ideals (see [8, 9, 20]). For further information on stratified ideals, we refer to [4].

(4) We may use the category mod-A to define the notion of right pre-standardly stratified algebras.

In the following, we write $\Delta = \{\Delta(\lambda) \mid \lambda \in \Lambda\}$ for a given pre-standardly stratified algebra $(A; \lambda, \leq)$. The next result states some properties of standard modules. We include here a proof of the statement (c) below, which is different from [12].

PROPOSITION 2.3 *Let $(A; \Lambda, \leq)$ be a pre-standardly stratified algebra with standard modules $\Delta(\lambda)$, $\lambda \in \Lambda$. Then*

(a) $\mathrm{Hom}_A(\Delta(\lambda), \Delta(\mu)) = 0$ *if* $\lambda \not\leq \mu$*;*
(b) $\mathrm{Ext}^1_A(\Delta(\lambda), \Delta(\mu)) = 0$ *if* $\lambda \not< \mu$*;*
(c) $\mathcal{F}(\Delta)$ *is a resolving subcategory in* A-mod.

Proof. (a) is trivial since the composition factors $L(\lambda)$ of $\Delta(\mu)$ satisfy $\lambda \leq \mu$.

(b) Applying $\mathrm{Hom}_A(-, \Delta(\mu))$ to the exact sequence $0 \to Q(\lambda) \to P(\lambda) \to \Delta(\lambda) \to 0$, we get an exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_A(P(\lambda), \Delta(\mu)) \longrightarrow \mathrm{Hom}_A(Q(\lambda), \Delta(\mu)) \longrightarrow \mathrm{Ext}^1_A(\Delta(\lambda), \Delta(\mu)) \longrightarrow 0.$$

Since $Q(\lambda)$ is filtered by $\Delta(\rho)$ with $\lambda < \rho$, we see that if $\lambda \not< \mu$ then $\rho \not\leq \mu$. This implies that $\mathrm{Hom}_A(\Delta(\rho), \Delta(\mu)) = 0$. Thus $\mathrm{Hom}_A(Q(\lambda), \Delta(\mu)) = 0$ if $\lambda \not\leq \mu$. Hence $\mathrm{Ext}^1_A(\Delta(\lambda), \Delta(\mu)) = 0$ if $\lambda \not< \mu$.

(c) By (b) and Lemma 2.1, we may take a linearization $(\Lambda, \leq^L)$ of $(\Lambda, \leq)$ such that the set Δ of the standard modules $\Delta(\lambda)$ satisfies that $\mathrm{Ext}^1_A(\Delta(\lambda), \Delta(\mu)) = 0$ if $\mu \leq^L \lambda$. Thus, for each module $X \in \mathcal{F}(\Delta)$, the multiplicity $[X : \Delta(\lambda)]$ of $\Delta(\lambda)$ occurring as a section factors in a Δ-filtration of X is well defined. Let X be in $\mathcal{F}(\Delta)$, we define $S(X) := \{[\lambda] \in \bar{\Lambda} \mid [X : \Delta(\lambda)] \neq 0\}$ and $C(X) := \{[\lambda] \in \bar{\Lambda} \mid [X : L(\lambda)] \neq 0\}$, where $L(\lambda)$ is the simple A-module corresponding to λ. Clearly, if $X \in \mathcal{F}(\Delta)$, then $S(X) \subseteq C(X)$. If $\max(S(X))$ denotes the maximal element of the set $S(X)$ with respect to $\leq^L$, then $\max(S(X))= \max(C(X))$ since each $\Delta(\lambda)$ has composition factors $L(\mu)$ with $\mu \leq \lambda$.

To prove that $\mathcal{F}(\Delta)$ is a resolving subcategory, we need only to show that $\mathcal{F}(\Delta)$ is closed under kernels of surjective morphisms. Suppose $f : X \to Y$ is a surjective morphism between two modules X and Y in $\mathcal{F}(\Delta)$. We use induction on the cardinality $|S(X)|$ of $S(X)$ to show that $\mathrm{Ker}(f)$ lies in $\mathcal{F}(\Delta)$. If $|S(X)| = 1$, then $X = \bigoplus_{\mu\in[\lambda]} \Delta(\mu)^{n_\mu}$ for some non-negative integers n_μ. Since Y is in $\mathcal{F}(\Delta)$, there is a filtration $0 \subseteq Y_n \subseteq Y_{n-1} \subseteq \cdots \subseteq Y_2 \subseteq Y_1 = Y$ such that Y_i/Y_{i-1} is a direct sum of modules $\Delta(\gamma)$ with $\gamma \in [\lambda_i]$ and that $\lambda_1 \leq^L \lambda_2 \leq^L \cdots \leq^L \lambda_n$. Since we have a surjective map $X \to Y \to Y/Y_2$, this means that there must be an indecomposable direct summand $\Delta(\mu)$ of X which maps surjectively to an indecomposable direct summand of Y/Y_2, namely there is a surjective morphism $h : \Delta(\mu) \to \Delta(\rho)$ with $\mu \in [\lambda]$ and $\rho \in [\lambda_1]$. Since $\Delta(\lambda)$ is a local module, we get that $\mu = \rho$. This means that $[\lambda_1] = [\rho] = [\mu] = [\lambda]$. Thus $[\lambda] = [\lambda_1] = \min(S(Y)) \leq^L \max(S(Y)) = \max(C(Y)) \leq^L \max(C(X)) = \max(S(X)) = [\lambda]$, and all λ_i are in the same equivalence class $[\lambda]$. This implies that Y is a direct sum of modules $\Delta(\mu)$ with $\mu \in [\lambda]$. To see that the kernel of f is again a direct sum of modules $\Delta(\mu)$ with $\mu \in [\lambda]$, we note that a surjective homomorphism from $\Delta(\gamma)$ to $\Delta(\gamma')$ must be an isomorphism. Thus, we split all isomorphisms from f and deduce that $\mathrm{Ker}(f)$ is just a direct summand of X.

Suppose that the statement is true for X in $\mathcal{F}(\Delta)$ with $|S(X)| < n$. Let X be in $\mathcal{F}(\Delta)$ with $\mid S(X) \mid = n$. Suppose $\max(S(X)) = [\lambda]$. Then we have an exact sequence $0 \to X' \to X \to X/X' \to 0$ such that X' is a direct sum of modules $\Delta(\mu)$ with $\mu \in [\lambda]$ and that X/X' lies in $\mathcal{F}(\{\Delta(\mu) \mid \mu \in S(X), [\mu] \neq [\lambda]\})$. Similarly, there is an exact sequence $0 \to Y' \to Y \to Y/Y' \to 0$ such that Y' is a direct sum of modules $\Delta(\mu)$ with $\mu \in [\lambda]$ and that Y/Y' lies in $\mathcal{F}(\{\Delta(\mu) \mid \mu \in S(Y), [\mu] \neq [\lambda]\})$. Note that Y' may be zero. Now, we claim that $\mathrm{Hom}_A(\Delta(\lambda'), \Delta(\rho)) = 0$ for $\lambda' \in [\lambda]$ and $\rho \in S(Y)$ with $[\rho] \neq [\lambda]$. In fact, we have $[\rho] \leq^L \max(S(Y)) = \max(C(Y)) \leq^L \max(C(X)) = [\lambda]$. Thus $\lambda \not\leq \rho$ by Lemma 2.1. Hence our claim follows from (a). This yields that $\mathrm{Hom}_A(X', Y/Y') = 0$. From this, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X' & \longrightarrow & X & \xrightarrow{\pi} & X/X' & \longrightarrow & 0 \\
 & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle h}\downarrow & & \\
0 & \longrightarrow & Y' & \xrightarrow{\gamma} & Y & \longrightarrow & Y/Y' & \longrightarrow & 0
\end{array}$$

where g is the restriction of f.

Since f is surjective, the Snake lemma gives an exact sequence

$$0 \longrightarrow \mathrm{Ker}(g) \longrightarrow \mathrm{Ker}(f) \longrightarrow \mathrm{Ker}(h) \longrightarrow \mathrm{Coker}(g) \longrightarrow 0.$$

Assume (for a contradiction) that g is not surjective. Then, in particular, Y' is non-zero; and some top composition factor $L(\mu)$ of Y' occurs as a composition factor of $\mathrm{Coker}(g)$. Then it also occurs in $\mathrm{Ker}(h)$ and hence in X/X'. So we have $[\mu]$=$[\lambda]$ but also $\max C(X/X') \leq^L [\lambda]$, which is a contradiction. By induction, Ker(g) and Ker(h) are in $\mathcal{F}(\Theta)$ and so is Ker(f). This finishes the proof of (c).

REMARKS (1) It follows from the proof of Proposition 2.3 that if there is a surjective homomorphism $f : X \to Y$ for $X, Y \in \mathcal{F}(\Delta)$, then $S(Y) \subseteq S(X)$.

(2) Suppose we are given a pre-standardly stratified algebra A with a pre-ordered set $(\Lambda, \leq)$. Then we have a set of standard modules $\Delta := \{\Delta(\lambda) \mid \lambda \in \Lambda\}$. If we take a linearization $(\Lambda, \leq^L)$ of $(\Lambda, \leq)$ and define $\Delta'(\lambda)$ with respect to this linear order, then the set Δ' will be different from the given Δ in general, thus $\mathcal{F}(\Delta)$ may be different from $\mathcal{F}(\Delta')$. However, if we use this linear order to reorder the modules in Δ, we get the same $\mathcal{F}(\Delta)$ by Proposition 2.3(2).

(3) For a pre-standardly stratified algebra $(A, \Lambda, \leq)$, the following are true: (a) If $\lambda \in \Lambda$ with $[\lambda]$ maximal in $\bar{\Lambda}$, then $\Delta(\lambda)$ is projective by Definition 2.2. (b) By induction on the poset $\bar{\Lambda}$, we can see that the projective dimension of each module X in $\mathcal{F}(\Delta)$ is bounded above by $|\ \bar{\Lambda}\ | - 1$.

3. Homological systems over pre-ordered sets

In this section, we shall introduce several homological systems of modules over a pre-ordered set, which generalize the notion of a stratifying system over a linearly ordered set in [11], and develop some of their basic properties.

DEFINITION 3.1 A Θ-system $(\Theta; \Lambda, \leq)$, in A-mod, consists of the following:

(S1) $(\Lambda, \leq)$ is a pre-ordered set;
(S2) $\Theta = \{\Theta(\lambda) \mid \lambda \in \Lambda\}$ is a family of pairwise non-isomorphic indecomposable A-modules;
(S3) $\mathrm{Hom}_A(\Theta(\lambda), \Theta(\mu)) = 0$ if $\lambda \not\leq \mu$;
(S4) $\mathrm{Ext}^1_A(\Theta(\lambda), \Theta(\mu)) = 0$ if $\lambda \not< \mu$.

Note that, in Definition 3.1, we do not assume that $\mathcal{F}(\Theta)$ is closed under taking direct summands, while this is required in [18, Theorem 9.1].

A canonical example of a Θ-system is the standard modules Δ of a pre-standardly stratified algebra $(A, \Lambda, \leq)$ with $\Theta := \Delta$.

Clearly, if $(\Theta; \Lambda, \leq)$ is a Θ-system in A-mod and if Θ' is a subset of Θ, then Θ' defines a canonical Θ'-system in A-mod.

DEFINITION 3.2 A Θ-injective system $(\Theta, Y; \Lambda, \leq)$, in A-mod, consists of the following:

(IS1) $(\Lambda, \leq)$ is a pre-ordered set;
(IS2) $\Theta = \{\Theta(\lambda) \mid \lambda \in \Lambda\}$ is a family of pairwise non-isomorphic and non-zero A-modules;
(IS3) $\mathrm{Hom}_A(\Theta(\lambda), \Theta(\mu)) = 0$ if $\lambda \not\leq \mu$;
(IS4) $Y = \{Y(\lambda) \mid \lambda \in \Lambda\}$ is a family of indecomposable A-modules $Y(\lambda)$ such that $Y(\lambda)$ is Θ-injective and there is an exact sequence

$$0 \longrightarrow \Theta(\lambda) \xrightarrow{i_\lambda} Y(\lambda) \longrightarrow Z(\lambda) \longrightarrow 0$$

in $\mathcal{F}(\Theta)$ with $Z(\lambda) \in \mathcal{F}(\{\Theta(\mu) \mid \mu < \lambda\})$.

Dually, we define a Θ-projective system $(\Theta, X; \Lambda, \leq)$ as follows.

DEFINITION 3.3 A Θ-projective system $(\Theta, X; \Lambda, \leq)$, in A-mod, consists of the following:

(PS1) $(\Lambda, \leq)$ is a pre-ordered set;
(PS2) $\Theta = \{\Theta(\lambda) \mid \lambda \in \Lambda\}$ is a family of pairwise non-isomorphic and non-zero A-modules;
(PS3) $\mathrm{Hom}_A(\Theta(\lambda), \Theta(\mu)) = 0$ if $\lambda \not\leq \mu$;
(PS4) $X = \{X(\lambda) \mid \lambda \in \Lambda\}$ is a family of indecomposable A-modules $X(\lambda)$ such that $X(\lambda)$ is Θ-projective and there is an exact sequence

$$0 \longrightarrow U(\lambda) \longrightarrow X(\lambda) \longrightarrow \Theta(\lambda) \longrightarrow 0$$

in $\mathcal{F}(\Theta)$ with $U(\lambda) \in \mathcal{F}(\{\Theta(\mu) \mid \lambda < \mu\})$.

As an example of a Θ-projective system, we may consider a pre-standardly stratified algebra $(A, \Lambda, \leq)$. Then $(\Delta, X; \Lambda, \leq)$ is a Δ-projective system, where $X(\lambda) = P(\lambda)$ for $\lambda \in \Lambda$.

Clearly, in a Θ-injective system $(\Theta, Y; \Lambda, \leq)$, if $[\lambda]$ is minimal in $\bar{\Lambda}$, then $\Theta(\lambda) = Y(\lambda)$. In a Θ-projective system $(\Theta, X; \Lambda, \leq)$, if $[\lambda]$ is maximal in $\bar{\Lambda}$, then $\Theta(\lambda) = X(\lambda)$. Moreover, we have the following observation.

PROPOSITION 3.4 $(\Theta, X; \Lambda, \leq)$ *is a* Θ-*projective system in* A-mod *if and only if* $(D\Theta, DX; \Lambda, \leq^{\mathrm{op}})$ *is a* $D\Theta$-*injective system in* mod-A, *where* D *is the usual duality of an Artin algebra and* $\leq^{\mathrm{op}}$ *is the opposite order of* $\leq$ *in* Λ.

Thus, once a result is obtained for one system, we can transfer it to the other one. So, we may only deal with Θ-injective systems.

The maps between two homological systems, in A-mod, are defined in the following manner.

DEFINITION 3.5 Let $(\Theta, Y; \Lambda, \leq)$ and $(\Theta', Y'; \Lambda', \leq')$ be two Θ-injective systems. A morphism $f : (\Theta, Y; \Lambda, \leq) \to (\Theta', Y'; \Lambda', \leq')$ consists of a map $\varphi : (\Lambda, \leq) \to (\Lambda', \leq')$ of pre-ordered sets and two families of morphisms $f_\lambda : \Theta(\lambda) \to \Theta'(\varphi(\lambda))$ and $g_\lambda : Y(\lambda) \to Y(\varphi(\lambda))$, in A-mod, such that the following diagram commutes for each $\lambda \in \Lambda$.

$$\begin{array}{ccc} \Theta(\lambda) & \xrightarrow{i_\lambda} & Y(\lambda) \\ {\scriptstyle f_\lambda}\downarrow & & \downarrow{\scriptstyle g_\lambda} \\ \Theta'(\varphi(\lambda)) & \xrightarrow{i'_{\varphi(\lambda)}} & Y'(\varphi(\lambda)) \end{array}$$

Dually, we define the morphisms between two Θ-projective systems.

Now, we give some properties of a Θ-system over a pre-ordered set.

LEMMA 3.6 *Let* $(\Lambda, \leq)$ *be a pre-ordered set and* $\Theta = \{\Theta(\lambda) \mid \lambda \in \Lambda\}$ *be a family of* A-*modules satisfying the condition* (S4) *of Definition* 3.1.

(a) *If* $\lambda \not< \mu$ *for* $\lambda, \mu \in \Lambda$, *then* $\mathrm{Ext}^1_A(\Theta(\lambda'), \Theta(\mu')) = 0$ *for all* $\lambda' \sim \lambda$ *and* $\mu' \sim \mu$.

(b) *If* $\lambda \sim \mu$ *for* $\lambda, \mu \in \Lambda$, *then* $\mathrm{Ext}_A^1(\Theta(\lambda), \Theta(\mu)) = 0 = \mathrm{Ext}_A^1(\Theta(\mu), \Theta(\lambda)) = 0$.
(c) *For* $\lambda \in \Lambda$, $\Lambda_\lambda := \{\mu \in \Lambda \mid \mu \leq \lambda\}$ *and* $\Theta_\lambda := \{\Theta(\mu) \mid \mu \in \Lambda_\lambda\}$, *we have*

$$\mathrm{Ext}_A^1(\mathcal{F}(\Theta \setminus \Theta_\lambda), \mathcal{F}(\Theta_\lambda)) = 0.$$

Proof. (a) This follows from the fact that $\lambda' \not< \mu'$ if $\lambda \not< \mu$.
(b) By definition, $\lambda \not< \mu$ if and only if $\lambda \not\leq \mu$ or $\mu \leq \lambda$. If $\lambda \sim \mu$, we always have $\mu \leq \lambda$ and $\lambda \leq \mu$; and so $\lambda \not< \mu$ and $\mu \not< \lambda$. Thus (b) follows.
(c) If $\Theta(\mu) \in \Theta \setminus \Theta_\lambda$ then $\mu \not< \gamma$ for any $\Theta(\gamma) \in \Theta_\lambda$. Thus $\mathrm{Ext}_A^1(\Theta(\mu), \Theta(\gamma)) = 0$. This implies that $\mathrm{Ext}_A^1(\mathcal{F}(\Theta \setminus \Theta_\lambda), \mathcal{F}(\Theta_\lambda)) = 0$.

The following is a property of a Θ-injective system.

LEMMA 3.7 *Let* $(\Theta, Y; \Lambda, \leq)$ *be a* Θ*-injective system in* A-mod.

(a) *If* $\lambda \sim \mu$ *then* $\mathrm{Hom}_A(\Theta(\lambda), Z(\mu)) = 0$.
(b) *If* $\lambda \not\leq \mu$ *then* $\mathrm{Hom}_A(\Theta(\lambda), Z(\mu)) = 0$.
(c) $Y = \{Y(\lambda) \mid \lambda \in \Lambda\}$ *is a family of pairwise non-isomorphic* A*-modules.*
(d) *For each module* $X \in \mathcal{F}(\Theta)$, *there is an exact sequence*

$$0 \longrightarrow X \longrightarrow T' \longrightarrow Z \longrightarrow 0$$

such that T' *is in* $\mathrm{add}(\bigoplus_\lambda Y(\lambda))$ *and* $Z \in \mathcal{F}(\Theta)$. *In particular, we have*

$$\mathcal{I}(\Theta) \cap \mathcal{F}(\Theta) = \mathrm{add}\left(\bigoplus_{\lambda \in \Lambda} Y(\lambda)\right).$$

Proof. (b) Suppose $\lambda \not\leq \mu$. Note that $Z(\mu)$ has a Θ-filtration such that the section factors are $\Theta(\rho)$ with $\rho < \mu$. Since $\lambda \not\leq \mu$ and $\rho < \mu$, we have $\lambda \not\leq \rho$. This implies that $\mathrm{Hom}_A(\Theta(\lambda), Z(\mu)) = 0$ by (IS3). Hence (b) follows. (a) can be proved similarly.

(c) Suppose that there is an isomorphism $f : Y(\lambda) \to Y(\mu)$. We claim that $\lambda \sim \mu$. If $\lambda \not\leq \mu$, then $\mathrm{Hom}_A(\Theta(\lambda), Z(\mu)) = 0$ by (b). Thus, we may form the following exact commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Theta(\lambda) & \xrightarrow{\alpha} & Y(\lambda) & \xrightarrow{\beta} & Z(\lambda) & \longrightarrow & 0 \\
 & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle h}\downarrow & & \\
0 & \longrightarrow & \Theta(\mu) & \xrightarrow{\gamma} & Y(\mu) & \xrightarrow{\delta} & Z(\mu) & \longrightarrow & 0
\end{array}$$

By (IS3), we see that the map g must be zero. On the other hand, since f is injective, the Snake lemma shows that g is injective. It follows that $\Theta(\lambda) = 0$, which is a contradiction. Thus $\lambda \leq \mu$. Similarly, we show that $\mu \leq \lambda$, proving that $\lambda \sim \mu$.

Now, by (a), we have a map $g : \Theta(\lambda) \to \Theta(\mu)$ such that $\alpha f = g\gamma$, and a map $g' : \Theta(\mu) \to \Theta(\lambda)$ such that $g'\alpha = \gamma f^{-1}$. So $gg'\alpha = g\gamma f^{-1} = \alpha f f^{-1} = \alpha$; since α is injective, we have $gg' = 1$.

In the same way, we obtain $g'g = 1$. This implies that $\Theta(\lambda)$ and $\Theta(\mu)$ are isomorphic. By (IS2), we must have $\lambda = \mu$.

(d) We use induction on the length of a Θ-filtration of a module X in $\mathcal{F}(\Theta)$. If X has a Θ-filtration of length 1, then X is isomorphic to some $\Theta(\lambda)$. In this case, we have the desired sequence by (IS4). Suppose the statement is true for all modules X in $\mathcal{F}(\Theta)$ with a Θ-filtration of length less than n. Now, take a module X in $\mathcal{F}(\Theta)$ and suppose there is a Θ-filtration of length n. So, there is an exact sequence

$$0 \longrightarrow X' \xrightarrow{\gamma} X \longrightarrow \Theta(\lambda) \longrightarrow 0$$

with X' having Θ-length less than n. By induction, there is an exact sequence

$$0 \longrightarrow X' \xrightarrow{g} T_1 \longrightarrow Z_1 \longrightarrow 0$$

with $T_1 \in \mathrm{add}(\bigoplus_{\lambda\in\Lambda} Y(\lambda))$ and $Z_1 \in \mathcal{F}(\Theta)$. Since T_1 is Θ-injective, there is a homomorphism $f : X \to T_1$ such that $\gamma f = g$. Now, we may form the following commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X' \arrow[r, "\gamma"] \arrow[d, "g"'] & X \arrow[r] \arrow[d, "(f,\gamma'\alpha)"'] & \Theta(\lambda) \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & T_1 \arrow[r, "(1,0)"] \arrow[d] & T_1 \oplus Y(\lambda) \arrow[r, "(0,1)'"] \arrow[d] & Y(\lambda) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_1 \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & Z(\lambda) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Note that $(f, \gamma'\alpha)$ is injective. Since $\mathcal{F}(\Theta)$ is closed under extensions, we see that Z lies in $\mathcal{F}(\Theta)$. Thus, the exact sequence $0 \longrightarrow X \longrightarrow T_1 \bigoplus Y(\lambda) \longrightarrow Z \longrightarrow 0$ is a desired one; proving the result.

A Θ-system $(\Theta; \Lambda, \leq)$, in A-mod, with Λ a finite set will be called a *finite* Θ*-system*. Similarly, we have the notions of finite Θ-injective system and finite Θ-projective system.

From now on, we are interested exclusively in finite Θ-systems.

From Lemma 3.6, we see that, for any finite Θ-system $(\Theta; \Lambda, \leq)$ and any $N \in \mathcal{F}(\Theta)$, the multiplicity $[N : \Theta(\lambda)]$ of $\Theta(\lambda)$ in a Θ-filtration of N can be calculated by giving the modules in Θ a linear order. Thus $[N : \Theta(\lambda)]$ is well defined for all $\lambda \in \Lambda$. So, we may introduce the following notion of support for modules in $\mathcal{F}(\Theta)$.

DEFINITION 3.8 Let $(\Theta; \Lambda, \leq)$ be a finite Θ-system. For $X \in \mathcal{F}(\Theta)$, we define the Θ-support of X with respect to $\bar{\Lambda}$ as

$$\mathrm{Supp}_{(\Theta,\bar{\Lambda})}(X) := \{\rho \in \bar{\Lambda} \mid \text{ there exists } \lambda \in \rho \text{ with } [X : \Theta(\lambda)] \neq 0\}.$$

Let $(\bar{\Lambda}, \leq^L)$ be a linearization of $(\bar{\Lambda}, \leq)$. For $0 \neq X \in \mathcal{F}(\Theta)$, let $\max_{(\bar{\Lambda},\leq^L)}(X)$ denote the maximum of $\mathrm{Supp}_{(\Theta,\bar{\Lambda})}(X)$ with respect to the linear order $\leq^L$. Analogously, let $\min_{(\bar{\Lambda},\leq^L)}(X)$ denote the minimum of $\mathrm{Supp}_{(\Theta,\bar{\Lambda})}(X)$ with respect to the linear order $\leq^L$. Finally, we set $\max_{(\bar{\Lambda},\leq^L)}(0) := -\infty$ and $\min_{(\bar{\Lambda},\leq^L)}(0) := +\infty$.

In case of a finite Θ-injective system, Lemma 3.7 can be strengthened as follows.

PROPOSITION 3.9 *Let $(\Theta, Y; \Lambda, \leq)$ be a finite Θ-injective system in A-*mod *and $(\bar{\Lambda}, \leq^L)$ be a linearization of $(\Lambda, \leq)$. Then, for any $0 \neq X \in \mathcal{F}(\Theta)$, there exists an exact sequence $0 \to X \to E \to Z \to 0$ in $\mathcal{F}(\Theta)$ such that $E \in$* add $(\bigoplus_{\lambda\in\Lambda} Y(\lambda))$ *and* $\max_{(\bar{\Lambda},\leq^L)}(Z) <^L \max_{(\bar{\Lambda},\leq^L)}(X)$.

Proof. Let $\rho_0 := \max_{(\bar{\Lambda},\leq^L)}(X)$. We shall prove the proposition by induction on the cardinality of the set $\mathrm{Supp}_{(\Theta,\bar{\Lambda})}(X)$. If the cardinality of $\mathrm{Supp}_{(\Theta,\bar{\Lambda})}(X)$ is 1, then, by Lemma 3.6(b), we have $X \simeq \bigoplus_{\mu\in\rho_0} \Theta(\mu)^{[X:\Theta(\mu)]}$. Hence, by Definition 3.2, we get the following exact sequence in $\mathcal{F}(\Theta)$:

$$0 \longrightarrow X \longrightarrow \bigoplus_{\mu\in\rho_0} Y(\mu)^{[X:\Theta(\mu)]} \longrightarrow \bigoplus_{\mu\in\rho_0} Z(\mu)^{[X:\Theta(\mu)]} \longrightarrow 0.$$

Since $Z = \bigoplus_{\mu\in\rho_0} Z(\mu) \in \mathcal{F}(\Theta(\lambda) \mid \lambda < \mu)$ and since $\lambda < \mu$ implies that $[\lambda] <^L [\mu] = \rho_0$, we conclude that $\max_{(\bar{\Lambda},\leq^L)}(Z) <^L \rho_0$.

Now, assume that the cardinality of the set $\mathrm{Supp}_{(\Theta,\overline{\Lambda})}(X)$ is greater than 1. Then, we have an exact sequence in $\mathcal{F}(\Theta)$

$$0 \longrightarrow \bigoplus_{\mu\in\rho_0} \Theta(\mu)^{[X:\Theta(\mu)]} \longrightarrow X \longrightarrow X'' \longrightarrow 0,$$

where $0 \neq X''$ and $\max_{(\bar{\Lambda},\leq^L)}(X'') <^L \rho_0 = \max_{(\bar{\Lambda},\leq^L)}(X)$. By induction, the result is true for X''; therefore, we can construct the following exact commutative diagram in $\mathcal{F}(\Theta)$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{\mu\in\rho_0} \Theta(\mu)^{[X:\Theta(\mu)]} & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{\mu\in\rho_0} Y(\mu)^{[X:\Theta(\mu)]} & \longrightarrow & E & \longrightarrow & T_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{\mu\in\rho_0} Z(\mu)^{[X:\Theta(\mu)]} & \longrightarrow & Z & \longrightarrow & Z_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

such that

$$T_2 \in \mathrm{add}\left(\bigoplus_{\lambda\in\Lambda} Y(\lambda)\right) \quad \text{and} \quad \max_{(\bar{\Lambda},\leq^L)}(Z_2) <^L \max_{(\bar{\Lambda},\leq^L)}(X'') <^L \rho_0.$$

So we have $\max_{(\bar{\Lambda},\leq^L)}(Z) <^L \rho_0$ since

$$\max_{(\bar{\Lambda},\leq^L)}\left(\bigoplus_{\mu\in\rho_0} Z(\mu)^{[X:\Theta(\mu)]}\right) <^L \rho_0.$$

This finishes the proof.

The following lemma due to Ringel in [17], which was proved for the linear order, will be used in the paper.

LEMMA 3.10 *Let* $\Theta = \{\Theta(1), \Theta(2), \ldots, \Theta(n)\}$ *be a set of A-modules satisfying* (S4) *for the natural linear order* (*that is,* $\mathrm{Ext}^1_A(\Theta(j), \Theta(i)) = 0$ *if* $i \leq j$). *Then*

(a) *for any* $1 \leq t \leq n$ *and any A-module N with* $\mathrm{Ext}^1_A(\Theta(j), N) = 0$ *for all* $j > t$*, there is an exact sequence* $0 \to N \to Y \to Z \to 0$ *with Y* Θ*-injective and* $Z \in \mathcal{F}(\{\Theta(i) \mid i \leq t\})$*;*
(b) $\mathcal{F}(\Theta)$ *is functorially finite in* A-mod*.*

As a consequence of Lemmas 3.10 and 3.6, we have the following result.

COROLLARY 3.11 *Let* Θ *be a finite set of modules* $\Theta(\lambda)$, *which are parametrized by a pre-ordered set* $(\Lambda, \leq)$, *satisfying* (S4) *in Definition* 3.1. *Then*

(a) $\mathcal{F}(\Theta)$ *is functorially finite in* A-mod*;*
(b) *for any module* $X \in A$-mod*, there is an exact sequence* $0 \to X \to Y_X \to Z_X \to 0$ *such that* $Y_X \in \mathcal{I}(\Theta)$ *and* $Z_X \in \mathcal{F}(\Theta)$.

Proof. We take an arbitrary linearization $(\Lambda, \leq^L)$ of $(\Lambda, \leq)$. We shall check that the condition (S4) in Lemma 3.10 is satisfied.

By Lemma 3.6, we see that $\mathrm{Ext}^1_A(\Theta(\lambda), \Theta(\mu)) = 0$ for λ, μ in ρ_i. So, we are allowed to choose an arbitrary linear order for modules $\Theta(\lambda)$ with λ in the same equivalence class. Now, suppose $i < j$, $\lambda \in \rho_j$ and $\mu \in \rho_i$. Thus $\mu <^L \lambda$. By Lemma 2.1, we know that $\lambda \not< \mu$. Hence the condition (S4) means that $\mathrm{Ext}^1_A(\Theta(\lambda), \Theta(\mu)) = 0$ if $\mu \leq^L \lambda$. Then, (a) follows from Lemma 3.10. The statement (b) is a direct consequence of Lemma 3.10 for $t = n$.

THEOREM 3.12 *If* $(\Theta; \Lambda, \leq)$ *is a finite* Θ*-system in* A-mod*, then there is a finite* Θ*-injective system* $(\Theta, Y; \Lambda, \leq)$ *in* A-mod*. If* $(\Theta, Y'; \Lambda, \leq)$ *is another one, then* $Y(\lambda) \simeq Y'(\lambda)$ *for all* $\lambda \in \Lambda$.

Proof. To show the result, we have to prove that the condition (IS4), in Definition 3.2, is satisfied. Namely we construct an exact sequence

$$0 \longrightarrow \Theta(\lambda) \xrightarrow{i_\lambda} Y(\lambda) \longrightarrow Z(\lambda) \longrightarrow 0 \qquad (*)$$

in A-mod such that $Z(\lambda) \in \mathcal{F}(\{\Theta(\mu) \mid \mu < \lambda\})$ and $Y(\lambda)$ is Θ-injective and indecomposable. The idea of the construction is similar to that in [11], but more complicated.

We define $Y(\mu) = \Theta(\mu)$ for all $\mu \in \Lambda$ such that $[\mu]$ is a minimal element in $(\bar{\Lambda}, \leq)$, and $Z(\mu) = 0$. For any $\gamma \in \Lambda$, we have $\gamma \sim \mu$ or $\gamma \not< \mu$. This implies that $\mathrm{Ext}^1_A(\Theta(\gamma), Y(\mu)) = \mathrm{Ext}^1_A(\Theta(\gamma), \Theta(\mu)) = 0$ by Lemma 3.6 and (S4). Thus $Y(\mu)$ is Θ-injective and indecomposable.

Next, we take an element λ_0 in Λ such that $[\lambda_0]$ is not minimal in $\bar{\Lambda}$, and define $\Lambda_{\lambda_0} := \{\mu \in \Lambda \mid \mu \leq \lambda_0\}$. Then $(\Lambda_{\lambda_0}, \leq)$ is a pre-ordered subset of $(\Lambda, \leq)$. The set $\Theta_{\lambda_0} := \{\Theta(\mu) \mid \mu \in \Lambda_{\lambda_0}\}$ satisfies the condition (S4) in Definition 3.1. Thus, we can take a linearization of Θ_{λ_0} such that λ_0 is the smallest element in $[\lambda_0]$ with respect to the linear order $\leq^L$ on Λ_{λ_0}. Suppose $\bar{\Lambda}_{\lambda_0} = \{\rho_1, \rho_2, \dots, \rho_i\}$ and $\rho_1 <^L \rho_2 <^L \cdots <^L \rho_i = [\lambda_0]$. We construct, by induction on k with $1 \leq k < i$, non-split exact sequences

$$0 \longrightarrow \Theta(\lambda_0) \longrightarrow U_k \longrightarrow V_k \longrightarrow 0 \qquad (\xi_k)$$

in $\mathcal{F}(\Theta)$ such that

(i) U_k is indecomposable,
(ii) $\mathrm{Ext}^1_A(\Theta(\lambda_j), U_k) = 0$ for all $\lambda_j \in \rho_j$ and all $i - k \leq j \leq i - 1$,
(iii) $V_k \in \mathcal{F}(\{\Theta(\lambda_j) \mid \lambda_j \in \rho_j, i - k \leq j \leq i - 1\})$,
(iv) $\mathrm{Ext}^1_A(\Theta(\lambda_0), U_k) = 0$.

Suppose $k = 1$. We set $M := \bigoplus_{\lambda \in \rho_{i-1}} \Theta(\lambda)$. In case $\mathrm{Ext}^1_A(M, \Theta(\lambda_0)) = 0$, we put $U_1 := \Theta(\lambda_0)$ and $V_1 := 0$. Now, assume that $\mathrm{Ext}^1_A(M, \Theta(\lambda_0)) \neq 0$. Thus, we can construct the universal extension

$$0 \longrightarrow \Theta(\lambda_0) \longrightarrow U \longrightarrow M^n \longrightarrow 0 \qquad (\varepsilon)$$

in $\mathcal{F}(\Theta)$. We recall that the connecting map $\mathrm{Hom}_A(M, M^n) \longrightarrow \mathrm{Ext}^1_A(M, \Theta(\lambda_0))$, induced by (ε), is surjective. Hence, $\mathrm{Ext}^1_A(M, U) = 0$ since $\mathrm{Ext}^1_A(M, M) = 0$ by Lemma 3.6(b). It follows, from (S4) and Lemma 2.1, that $\mathrm{Ext}^1_A(\Theta(\lambda_0), \Theta(\mu)) = 0$ for all $\mu \in \rho_{i-1}$. This implies that $\mathrm{Ext}^1_A(\Theta(\lambda_0), U) = 0$. Note that if $\lambda \in \rho_{i-1}$ and $\lambda_0 \leq \lambda$, then $\lambda_0 <^L \lambda$. Thus $\mathrm{Hom}_A(\Theta(\lambda_0), M) = 0$; and therefore, we get a non-split exact sequence

$$0 \longrightarrow \Theta(\lambda_0) \longrightarrow U_1 \longrightarrow V_1 \longrightarrow 0$$

such that U_1 and V_1 are direct summands of U and M^n, respectively; and moreover, U_1 is indecomposable. Furthermore, we know that $\mathrm{Ext}^1_A(M, U_1) = 0 = \mathrm{Ext}^1_A(\Theta(\lambda_0), U_1)$, and that $U_1 \in \mathcal{F}(\Theta)$ since both V_1 and $\Theta(\lambda_0)$ lie in $\mathcal{F}(\Theta)$.

Suppose we have already constructed the exact sequence (ξ_k) for $k \geq 1$ with the required properties. Now, we construct the exact sequence (ξ_{k+1}). If $\mathrm{Ext}^1_A(\bigoplus_{\lambda \in \rho_{i-k-1}} \Theta(\lambda), U_k) = 0$, we set $U_{k+1} := U_k$ and $V_{k+1} := V_k$; in this case, the corresponding conditions (i)–(iv) are clearly satisfied. So, we may assume that $\mathrm{Ext}^1_A(\bigoplus_{\lambda \in \rho_{i-k-1}} \Theta(\lambda), U_k) \neq 0$. Then, by constructing the universal extension, we get a non-split exact sequence in $\mathcal{F}(\Theta)$

$$0 \longrightarrow U_k \longrightarrow U \longrightarrow \Big(\bigoplus_{\lambda \in \rho_{i-k-1}} \Theta(\lambda)\Big)^m \longrightarrow 0. \qquad (**)$$

Since $\mathrm{Ext}^1_A(\Theta(\lambda), \Theta(\mu)) = 0$ for $\lambda \sim \mu$, we obtain that $\mathrm{Ext}^1_A(\bigoplus_{\lambda \in \rho_{i-k-1}} \Theta(\lambda), U) = 0$. Let $j \geq i - k$ and $\lambda_j \in \rho_j$. Applying the functor $\mathrm{Hom}_A(\Theta(\lambda_j), -)$ to $(**)$, we get an exact sequence

$$\mathrm{Ext}^1_A(\Theta(\lambda_j), U_k) \longrightarrow \mathrm{Ext}^1_A(\Theta(\lambda_j), U) \longrightarrow \mathrm{Ext}^1_A\Big(\Theta(\lambda_j), \Big(\bigoplus_{\lambda \in \rho_{i-k-1}} \Theta(\lambda)\Big)^m\Big).$$

Note that the first term vanishes by induction and the last one by (S4). Thus $\mathrm{Ext}^1_A(\Theta(\lambda_j), U) = 0$ for all $\lambda_j \in \rho_j$ with $j \geq i - k$.

Since U_k is in $\mathcal{F}(\{\Theta(\lambda) \mid \lambda \in \rho_t, i-k \leq t \leq i\})$, we have $\mathrm{Hom}_A(U_k, \Theta(\mu)) = 0$ for $\mu \in \rho_{i-k-1}$ by (S3). Otherwise, if $\lambda \in \rho_t$ with $i-k \leq t \leq i$ and $\lambda \leq \mu$ with $\mu \in \rho_{i-k-1}$, then $t \leq i-k-1$, which is a contradiction. So, from $(**)$, we get a non-split exact sequence

$$0 \longrightarrow U_k \longrightarrow U_{k+1} \longrightarrow U' \longrightarrow 0$$

such that $U' \in \mathrm{add}(\bigoplus_{\lambda\in\rho_{i-k-1}} \Theta(\lambda))$ and U_{k+1} is an indecomposable direct summand of U. In particular, $U_{k+1} \in \mathcal{F}(\Theta)$. Now, we form the pushout diagram of the maps $V_k \leftarrow U_k \rightarrow U_{k+1}$.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Theta(\lambda_0) \arrow[r] \arrow[d, equal] & U_k \arrow[r] \arrow[d] & V_k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Theta(\lambda_0) \arrow[r] & U_{k+1} \arrow[r] \arrow[d] & V_{k+1} \arrow[r] \arrow[d] & 0 \\
 & & U' \arrow[r, equal] \arrow[d] & U' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

We define (ξ_{k+1}) to be the middle horizontal sequence in the above diagram. It remains to verify that this sequence satisfies all conditions (i)–(iv). Since U_{k+1} is a direct summand of U, we deduce that $\mathrm{Ext}^1_A(\bigoplus_{\lambda\in\rho_{i-k-1}} \Theta(\lambda), U_{k+1}) = 0 = \mathrm{Ext}^1_A(\Theta(\lambda), U_{k+1})$ for all $\lambda \in \rho_j$ with $j \geq i-k$. Applying the functor $\mathrm{Hom}_A(\Theta(\lambda_0), -)$ to $(**)$, we get the following exact sequence:

$$\mathrm{Ext}^1_A(\Theta(\lambda_0), U_k) \longrightarrow \mathrm{Ext}^1_A(\Theta(\lambda_0), U) \longrightarrow \mathrm{Ext}^1_A\left(\Theta(\lambda_0), \left(\bigoplus_{\lambda\in\rho_{i-k-1}} \Theta(\lambda)\right)^m\right). \qquad (***)$$

By using an argument which is similar to the one above, we show that the first and the last terms of $(***)$ vanish; and so $\mathrm{Ext}^1_A(\Theta(\lambda_0), U) = 0$.

Finally, we define $Y(\lambda_0) := U_{i-1}$. Hence, by setting $k := i-1$ in (ξ_k), we obtain the following exact sequence in $\mathcal{F}(\Theta)$:

$$0 \longrightarrow \Theta(\lambda_0) \longrightarrow Y(\lambda_0) \longrightarrow V_{i-1} \longrightarrow 0$$

with $V_{i-1} \in \mathcal{F}(\{\Theta(\lambda) \mid \lambda \in \rho_j, 1 \leq j \leq i-1\})$. Thus $V_{i-1} \in \mathcal{F}(\{\Theta(\mu) \mid \mu < \lambda_0)$. We assert that the preceding exact sequence is the desired one corresponding to $\lambda = \lambda_0$. We have to check that $Y(\lambda_0)$ is Θ-injective. In fact, since $\mathrm{Ext}^1_A(\Theta(\mu), Y(\lambda_0)) = 0$ for $\mu < \lambda_0$ and $\mathrm{Ext}^1_A(\Theta(\lambda_0), Y(\lambda_0)) = 0$ by construction, we have $\mathrm{Ext}^1_A(\Theta(\mu), Y(\lambda_0)) = 0$ for $\mu \in \Lambda_{\lambda_0}$. By Lemma 3.6, we see that $Y(\lambda_0) \in \mathcal{I}(\Theta)$.

The uniqueness of the $(\Theta, Y; \Lambda, \leq)$ follows easily since the map $i_\lambda : \Theta(\lambda) \to Y(\lambda)$ is a left minimal $\mathcal{I}(\Theta)$-approximation. Thus we finish the proof.

PROPOSITION 3.13 *Let* $(\Theta, Y; \Lambda, \leq)$ *be a finite* Θ*-injective system in* A-mod*, and let* $T = \bigoplus_{\lambda\in\Lambda} Y(\lambda)$. *Then* $B := \mathrm{End}({}_AT)$ *is a right pre-standardly stratified algebra with* $(\Lambda, \leq^{\mathrm{op}})$ *as the index set.*

Proof. We set $\Delta(\lambda) := \mathrm{Hom}_A(\Theta(\lambda), T)$, $P_B(\lambda) := \mathrm{Hom}_A(Y(\lambda), T)$ and $Q(\lambda) := \mathrm{Hom}_A(Z(\lambda), T)$. By Definition 3.2 (IS4), we have an exact sequence

$$0 \longrightarrow \Theta(\lambda) \longrightarrow Y(\lambda) \longrightarrow Z(\lambda) \longrightarrow 0$$

in $\mathcal{F}(\Theta)$ with $Z(\lambda) \in \mathcal{F}(\{\Theta(\mu) \mid \mu < \lambda\})$ and $Y(\lambda)$ indecomposable in $\mathcal{I}(\Theta) \cap \mathcal{F}(\Theta)$. From this sequence, we get an exact sequence

$$0 \longrightarrow Q(\lambda) \longrightarrow P_B(\lambda) \longrightarrow \Delta(\lambda) \longrightarrow 0$$

since T is Θ-injective and $\mathrm{Hom}_A(-, T)$ is an exact functor on $\mathcal{F}(\Theta)$. Note that $Z(\lambda)$ has a Θ-filtration with sections $\Theta(\mu)$ such that $\mu < \lambda$. This implies that $Q(\lambda)$ has a Δ-filtration with sections $\Delta(\mu)$ such that $\lambda <^{\mathrm{op}} \mu$.

To finish the proof, we have to show that $Q(\lambda)$ is equal to $U(\lambda) := \mathrm{tr}_{P_{\not\leq^{\mathrm{op}}\lambda}}(P_B(\lambda))$. First, we note that $Q(\lambda)$ is contained in $U(\lambda)$. This is because $Q(\lambda)$ has a filtration with section $\Delta(\rho)$ such that $\rho < \lambda$, and each $\Delta(\rho)$ is covered by $P_B(\rho)$ with $\rho \not\leq^{\mathrm{op}} \lambda$. Secondly, we show that if $\mu \not\leq^{\mathrm{op}} \lambda$, that is $\lambda \not\leq \mu$, then every morphism $f : P_B(\mu) \longrightarrow P_B(\lambda)$ factors through $Q(\lambda)$. Since $\mathrm{add}({}_AT)$ and $\mathrm{add}(B_B)$ are equivalent categories via the functor $\mathrm{Hom}_A(-, T)$, we see that there is a morphism $f' : Y(\lambda) \longrightarrow Y(\mu)$ such that $f = \mathrm{Hom}_A(f', T)$. By Lemma 3.7, $\mathrm{Hom}_A(\Theta(\lambda), Z(\mu)) = 0$ since $\lambda \not\leq \mu$. This yields a morphism $g : \Theta(\lambda) \longrightarrow \Theta(\mu)$ such that the following diagram is commutative.

$$\begin{tikzcd}
0 \arrow[r] & \Theta(\lambda) \arrow[r, "\alpha"] \arrow[d, "g"'] & Y(\lambda) \arrow[r, "\beta"] \arrow[d, "f'"'] & Z(\lambda) \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & \Theta(\mu) \arrow[r, "\gamma"] & Y(\mu) \arrow[r, "\delta"] & Z(\mu) \arrow[r] & 0
\end{tikzcd}$$

Note that the condition (IS3) implies that $g = 0$. Thus f' factors through $Z(\lambda)$, and f factors through $Q(\lambda)$. This means that $U(\lambda) \subseteq Q(\lambda)$; and so $Q(\lambda) = U(\lambda)$, proving the result.

REMARK In [18, Propositions 8.6 and 9.1], we find the statement '$\cdots T(\lambda)$ are precisely the indecomposable Ext-injective objects'. This statement seems to be too strong because the modules $\Theta(\lambda)'$ are not assumed to be indecomposable by [18, Hypothesis (8.1)].

Recall that a module $T \in A$-mod is called a *tilting module* if

(a) $\mathrm{Ext}^i_A(T, T) = 0$ for all $i > 0$,
(b) the projective dimension of T is finite, and
(c) there is an exact sequence

$$0 \longrightarrow {}_AA \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \ldots \longrightarrow T_s \longrightarrow 0$$

such that $T_i \in \mathrm{add}\,(T)$ for all i.

As a corollary, we get the following result.

COROLLARY 3.14 *Let $(A; \Lambda, \leq)$ be a pre-standardly stratified algebra with standard modules $\Delta(\lambda)$, $\lambda \in \Lambda$. Then, there is a tilting A-module T in $\mathcal{F}(\Delta)$ such that* $\mathrm{End}({}_AT)$ *is a right pre-standardly stratified algebra with $(\Lambda, \leq^{\mathrm{op}})$ as the index set. Moreover, we can take $T = \bigoplus_{\lambda\in\Lambda} T(\lambda)$ satisfying the following property: for each $\lambda \in \Lambda$, there is an exact sequence*

$$0 \longrightarrow \Delta(\lambda) \longrightarrow T(\lambda) \longrightarrow V(\lambda) \longrightarrow 0$$

in A-mod *such that $V(\lambda) \in \mathcal{F}(\{\Delta(\mu) \mid \mu < \lambda\})$ and $\mathcal{I}(\Delta) \cap \mathcal{F}(\Delta) = \mathrm{add}(T)$.*

Proof. We know that the set $\Delta = \{\Delta(\lambda),\ \lambda \in \Lambda\}$ form a Δ-system $(\Delta; \Lambda, \leq)$ by Proposition 2.2. So, Theorem 3.12 provides us a Δ-injective system $(\Delta, \{T(\lambda) \mid \lambda \in \Lambda\}; \Lambda, \leq)$. Let T be the direct sum of $T(\lambda)$ with $\lambda \in \Lambda$. Then T is the desired module by Proposition 3.13 (see also Proposition 3.9), Remark (3) at the end of section 2 and Lemma 3.7.

Note that the module T in Corollary 3.14 is usually called the characteristic tilting module for the pre-standardly stratified algebra A. It plays an important role for understanding homological properties of the category A-mod (see, for example, the last two sections).

PROPOSITION 3.15 *Let $(\Theta, Y; \Lambda, \leq)$ be a finite Θ-injective system in* A-mod*, $T := \bigoplus_{\lambda\in\Lambda} Y(\lambda)$, $B :=$* $\mathrm{End}(\bigoplus_{\lambda\in\Lambda} {}_AY(\lambda))$ *and $\Delta(\lambda) := \mathrm{Hom}_A(\Theta(\lambda), T)$. Then the subcategory $\mathcal{F}(\Theta)$ of* A-mod *and the subcategory $\mathcal{F}(\Delta)$ of* mod-B *are anti-equivalent via the exact functor $F := \mathrm{Hom}_A(-, {}_AT)$ and its inverse $G := \mathrm{Hom}_B(-, T_B)$.*

Proof. We know that F is exact and that its image lies in $\mathcal{F}(\Delta)$. To see that the composition of F with G is isomorphic to the identity functor on $\mathcal{F}(\Theta)$, we use the natural morphism $\alpha_X : X \longrightarrow \mathrm{Hom}_B(\mathrm{Hom}_A(X, {}_AT)_B, T_B)$. This map is an isomorphism if and only if there is an exact sequence

$$0 \longrightarrow X \xrightarrow{f} T_0 \longrightarrow T_1, \quad \text{with } T_i \in \mathrm{add}({}_AT),$$

and the map f is a left $\mathrm{add}({}_AT)$-approximation of X. (This fact is due to Auslander and Solberg [7]; for a proof, we may also see [21]). Note that, by Lemma 3.7(d), we do have such an exact sequence for each X in $\mathcal{F}(\Theta)$. Thus $(G \circ F)(X) \simeq X$ for all $X \in \mathcal{F}(\Theta)$.

Now, we show that the composition of G with F is isomorphic to the identity functor on $\mathcal{F}(\Delta)$. First, we see that $\mathrm{Ext}^1_B(-, T_B)$ vanishes on $\mathcal{F}(\Delta)$. By (IS4), we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Theta(\lambda) & \longrightarrow & Y(\lambda) & \longrightarrow & Z(\lambda) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & G(\Delta(\lambda)) & \longrightarrow & G(P_B(\lambda)) & \longrightarrow & GF(Z(\lambda)) & \longrightarrow & \mathrm{Ext}^1_B(\Delta(\lambda), T_B) & \longrightarrow & 0
\end{array}$$

where the vertical arrows are the canonical maps $\alpha_X : X \longrightarrow \mathrm{Hom}_B(\mathrm{Hom}_A(X, {}_AT), T_B)$, $x \mapsto (x)f$ for ${}_AX \in A$-mod. Since all vertical maps are bijective, we get $\mathrm{Ext}^1_B(-, T_B) = 0$ on $\mathcal{F}(\Delta)$. This implies that $\mathrm{Hom}_B(-, T_B)$ is exact on $\mathcal{F}(\Delta)$. Thus, the image of G on $\mathcal{F}(\Delta)$ is contained in $\mathcal{F}(\Theta)$.

We know that $FG(\Delta(\lambda)) = FG(F(\Theta(\lambda))) \simeq F(GF(\Theta(\lambda))) \simeq F(\Theta(\lambda)) = \Delta(\lambda)$. By induction on the length of a Δ-filtration of modules in $\mathcal{F}(\Delta)$, we prove that the natural map $M \rightarrow$

$\mathrm{Hom}_A({}_A\mathrm{Hom}_B(M, T_B), {}_AT)$ is an isomorphism for each module M in $\mathcal{F}(\Delta)$. To do so, we choose a module M with a Δ-filtration of length n. Then, there is an exact sequence $0 \to M' \to M \to \Delta(\lambda) \to 0$ in which M' has Δ-length less than n. By induction, we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "f"'] & M \arrow[r] \arrow[d] & \Delta(\lambda) \arrow[r] \arrow[d, "g"'] & 0 \\ 0 \arrow[r] & FG(M') \arrow[r] & FG(M) \arrow[r] & FG(\Delta(\lambda)) \arrow[r] & 0 \end{tikzcd}$$

such that f is an isomorphism. Note that g is also an isomorphism. Hence $M \simeq FG(M)$. The proof is completed.

Corollary 3.16 *If* $(\Theta; \Lambda, \leq)$ *is a finite* Θ*-system in* A-mod*, then* $\mathcal{F}(\Theta)$ *is closed under direct summands.*

Proof. By Theorem 3.12, a finite Θ-system $(\Theta; \Lambda, \leq)$ gives rise to a finite Θ-injective system $(\Theta, Y; \Lambda, \leq)$. By Proposition 3.15, we know that $\mathcal{F}(\Theta)$ is equivalent to the category $\mathcal{F}(\Delta)$ of a right pre-standardly stratified algebra B with standard modules $\Delta(\lambda)$, $\lambda \in \Lambda$. By [12, Theorem 3], the category $\mathcal{F}(\Delta)$ is closed under direct summands; and so, $\mathcal{F}(\Theta)$ is closed under direct summands. This proves the corollary.

Corollary 3.17 *Let* $(\Theta, Y; \Lambda, \leq)$ *be a finite* Θ*-injective system in* A-mod *and* $f : M \longrightarrow N$ *be an injective morphism in* $\mathcal{F}(\Theta)$*. Then,* $\mathrm{Coker}(f) \in \mathcal{F}(\Theta)$ *if and only if* $\mathrm{Ext}^1_A(\mathrm{Coker}(f), Y(\lambda)) = 0$ *for all* $\lambda \in \Lambda$*. Dually, let* $(\Theta, X; \Lambda, \leq)$ *be a finite* Θ*-projective system in* A-mod *and* $f : M \longrightarrow N$ *be a surjective morphism in* $\mathcal{F}(\Theta)$*. Then,* $\mathrm{Ker}(f) \in \mathcal{F}(\Theta)$ *if and only if* $\mathrm{Ext}^1_A(X(\lambda), \mathrm{Ker}(f)) = 0$ *for all* $\lambda \in \Lambda$.

Proof. We keep the notation used in the proof of Proposition 3.15. We know that $\mathrm{Ext}^1_B(-, T_B)$ vanishes on $\mathcal{F}(\Delta)$. Suppose that $\mathrm{Ext}^1_A(\mathrm{Coker}(f), {}_AY(\lambda)) = 0$ for all λ. Since F is exact and $\mathcal{F}(\Delta)$ is closed under kernels of surjective homomorphisms, the kernel of $F(f)$, which is isomorphic to $F(\mathrm{Coker}(f))$, belongs to $\mathcal{F}(\Delta)$ since $\mathrm{Ext}^1_A(\mathrm{Coker}(f), {}_AY(\lambda)) = 0$. Then, we have the following exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\simeq"'] & N \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Coker}(f) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & GF(M) \arrow[r] & GF(N) \arrow[r] & G(\mathrm{Ker}(F(f))) \arrow[r] & \mathrm{Ext}^1_B(F(M), T_B) = 0 \end{tikzcd}$$

It follows, from the above diagram, that $\mathrm{Coker}(f)$ lies in $\mathcal{F}(\Theta)$. The converse is trivial.

The following result is more general than the corresponding one in [13, Theorem 2.6].

Corollary 3.18 *Let* $(\Theta, Y; \Lambda, \leq)$ *be a finite* Θ*-injective system in* A-mod*,* $T = \bigoplus_{\lambda\in\Lambda} Y(\lambda)$*,* $B = \mathrm{End}({}_AT)$ *and* $s(A)$ *the number of non-isomorphic simple* A*-modules. If* ${}_AA \in \mathcal{F}(\Theta)$*, then*

(a) $s(A) \leq |\Lambda|$; *and if the equality holds, then* T_B *is a tilting right* B*-module;*
(b) $A \simeq \mathrm{End}(T_B)$.

Proof. Since $A \in \mathcal{F}(\Theta)$, we know, from Lemma 3.7(d), that there is an exact sequence

$$0 \longrightarrow A \xrightarrow{f} T' \longrightarrow T''$$

such that $T', T'' \in \operatorname{add}({}_AT)$ and f is an $\operatorname{add}({}_AT)$-approximation. This means that $A \simeq \operatorname{End}(T_B)$. Since $A \in \mathcal{F}(\Theta)$, we see that $\mathcal{F}(\Delta)$ contains $F(A) = T_B$, which is in $\mathcal{I}(\Delta) \cap \mathcal{F}(\Delta) = \operatorname{add}(K_B)$, where K is the tilting right B-module determined by (the right version of) Proposition 3.14. Note that the number of non-isomorphic indecomposable direct summands of T_B is equal to the one of ${}_AA$ by using the functor G in Proposition 3.15. Thus, we have $s(A) \leq |\Lambda|$. If $s(A) = |\Lambda|$, then $\operatorname{add}(T_B) = \operatorname{add}(K_B)$ and therefore T_B is a tilting module.

The following is a way to construct Θ-projective systems, in A-mod, from Θ-injective systems.

COROLLARY 3.19 *If* $(\Theta, Y; \Lambda, \leq)$ *is a finite* Θ*-injective system in* A-mod*, then there is a finite* Θ*-projective system* $(\Theta, X; \Lambda, \leq)$ *in* A-mod*. Moreover,* $\mathcal{F}(\Theta) \cap \mathcal{P}(\Theta) = \operatorname{add}(\bigoplus_{\lambda\in\Lambda} X(\lambda))$.

Proof. Since $B = \operatorname{End}(\bigoplus_{\lambda\in\Lambda} Y(\lambda))$ is a right pre-standardly stratified algebra, we use the right version of Proposition 3.14 to get the characteristic tilting right B-module $K = \bigoplus_{\lambda\in\Lambda} K(\lambda)$. Then, we have a Δ-injective system $(\Delta, \{K(\lambda)\ \lambda \in \Lambda\}, \leq^{\mathrm{op}})$ in mod-B. Now, we apply the equivalence (contravariant) functor G in Proposition 3.15 to this system; and so, we get a Θ-projective system $(\Theta, X; \Lambda, \leq)$ with $X(\lambda) = G(K(\lambda))$ for $\lambda \in \Lambda$. The equality of the two categories is just the dual statement of Lemma 3.7(4).

THEOREM 3.20 *If* $(\Theta, Y; \Lambda, \leq)$ *is a finite* Θ*-injective system in* A-mod*, then* $(\Theta; \Lambda, \leq)$ *is a* Θ*-system in* A-mod.

Proof. In accord with Definition 3.1, we have to check all the conditions from (S1) to (S4). However, (S1) and (S3) are trivial by Definition 3.2. So, we have to show that (S2) and (S4) hold. We first show that (S4) holds true.

Suppose $\lambda \not< \mu$. Applying the functor $\operatorname{Hom}_A(\Theta(\lambda), -)$ to the exact sequence $0 \longrightarrow \Theta(\mu) \longrightarrow Y(\mu) \longrightarrow Z(\mu) \longrightarrow 0$, we get the exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_A(\Theta(\lambda), \Theta(\mu)) &\longrightarrow \operatorname{Hom}_A(\Theta(\lambda), Y(\mu)) \longrightarrow \operatorname{Hom}_A(\Theta(\lambda), Z(\mu)) \\ &\longrightarrow \operatorname{Ext}^1_A(\Theta(\lambda), \Theta(\mu)) \longrightarrow \operatorname{Ext}^1_A(\Theta(\lambda), Y(\mu)). \end{aligned}$$

Since $Y(\mu)$ is Θ-injective, the last term in the above exact sequence vanishes. On the other hand, by Lemma 3.7, $\operatorname{Hom}_A(\Theta(\lambda), Z(\mu)) = 0$. Hence, we have $\operatorname{Ext}^1_A(\Theta(\lambda), \Theta(\mu)) = 0$ if $\lambda \not< \mu$.

Now we prove (S2). To see that $\Theta(\lambda)$ is indecomposable, we show that the endomorphism algebra $\operatorname{End}(\Theta(\lambda))$ is a local one. Let f be in $\operatorname{End}(\Theta(\lambda))$. Since $Y(\lambda)$ is Θ-injective, there is a homomorphism $g : Y(\lambda) \longrightarrow Y(\lambda)$ such that the following diagram is commutative.

$$\begin{tikzcd} 0 \arrow[r] & \Theta(\lambda) \arrow[r, "\alpha"] \arrow[d, "f"'] & Y(\lambda) \arrow[r, "\beta"] \arrow[d, "g"'] & Z(\lambda) \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & \Theta(\lambda) \arrow[r, "\gamma"] & Y(\lambda) \arrow[r, "\delta"] & Z(\lambda) \arrow[r] & 0 \end{tikzcd}$$

Since $Y(\lambda)$ is indecomposable, we know that g is either nilpotent or an automorphism. In the former case, we may say that $g^m = 0$. It follows from $f^m\gamma = \alpha g^m$ that $f^m = 0$ since γ is injective. Now, suppose that g is an automorphism. Then $f\gamma = \alpha g$ is injective. This shows that f is injective. Thus, f is also surjective since $\Theta(\lambda)$ is a module of finite length. Hence f is an automorphism. This completes the proof of (S2).

REMARK It is clear, by the dual results of Corollary 3.19, Theorems 3.20 and 3.12, that a Θ-system determines, in a unique way (up to isomorphisms), both a Θ-injective system and a Θ-projective system; and conversely, a Θ-injective (or Θ-projective) system determines a Θ-system uniquely (up to isomorphisms). Hence, given a Θ-system, we may speak of the Θ-injective (or Θ-projective) system associated to it.

Finally, we point out that one can construct many standardly stratified algebras from a given Θ-injective system.

PROPOSITION 3.21 *Let $(\Theta; \Lambda, \leq)$ be a finite Θ-system in A-mod, and let $\bar{\Lambda} = \{[\lambda_1], [\lambda_2], \dots, [\lambda_n]\}$. Then, for each $\Theta' = \{\Theta(\lambda_i) \mid i = 1, 2, \dots, n\}$, there is a standardly stratified algebra $(A(\Theta'); \Lambda', \leq)$ with standard modules $\Delta'(\lambda_j)$, $1 \leq j \leq n$, such that $\mathcal{F}(\Delta')$ is equivalent to $\mathcal{F}(\Theta')$, where $\Lambda' = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is a poset with the partial order induced from the pre-ordered set $(\Lambda, \leq)$.*

Proof. By Theorem 3.12 and Proposition 3.13, it is sufficient to prove that $(\Theta'; \Lambda', \leq)$ is a Θ'-system. However, this is clear by Definition 3.1.

4. Cartan determinants of pre-standardly stratified algebras

In this section, we give a formula to calculate the Cartan determinants of pre-standardly stratified algebras $(A; \Lambda, \leq)$ by using only the information of standard modules and simple modules. If A is a standardly stratified algebra, namely the pre-order $\leq$ is a linear order, a formula for the Cartan determinant of A was known in [11]. In that case, it is the product of dimensions of the endomorphism algebras of the standard modules. But this formula is no longer true when we come to pre-orders. For example, let A be the k-algebra over a field k given by the following regular representation.

$$ {}_AA = \begin{array}{c} 1 \\ 2 \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ 1 \\ 2 \end{array} $$

Then this algebra is not standardly stratified over any linearly ordered set. However, it is pre-standardly stratified algebra over the pre-order: $1 \leq 2 \leq 1$. In this case, we have $\Delta(i) = P(i)$, which is an indecomposable projective module. Clearly, the product of the dimension of the endomorphism algebras of two standard modules is 4 and the Cartan determinant of the algebra is 3 if $\mathrm{End}(L(i)) = k$. This shows that the Cartan determinant of a pre-standardly stratified algebra in general may have a different formula. Also, we should note that the Cartan determinants of standardly stratified algebras are always non-zero; but for pre-standardly stratified algebras, their Cartan determinants could be any integers.

For convenience of the reader, we recall the definition of the Cartan matrix of an Artin algebra.

Let A be an Artin R-algebra with R a commutative Artin ring. Let $\{L(1), L(2), \dots, L(n)\}$ be a complete set of non-isomorphic simple A-modules, and let $P(i)$ be the projective cover of $L(i)$. Denote by $c_{i,j}$ the multiplicity $[P(j) : L(i)]$ of the simple module $L(i)$ in the projective module $P(j)$. Then, the matrix $C_A := (c_{ij})$ is the *Cartan matrix* of A, and $\det(C_A)$ is the *Cartan determinant* of A.

For a module M over an Artin R-algebra A, we denote by $\ell(M)$ the length of M as an R-module. Let $(A; \Lambda, \leq)$ be a pre-standardly stratified algebra with standard modules $\Delta(\lambda)$, $\lambda \in \Lambda$. We denote by $C[\lambda]$ the matrix indexed by $[\lambda]$, in which the (γ, μ)-entry is $\ell(\mathrm{Hom}_A(\Delta(\gamma), \Delta(\mu)))$. Our result concerning Cartan determinants is the following theorem, which generalizes a result in [11].

THEOREM 4.1 *Let $(A; \Lambda, \leq)$ be a pre-standardly stratified algebra with standard modules $\Delta(\lambda)$, $\lambda \in \Lambda$; and let $\bar{\Lambda}$ be the partially ordered set induced by Λ. Then*

$$\det(C_A) = \frac{\prod_{[\lambda]\in\bar{\Lambda}} \det(C[\lambda])}{\prod_{\lambda\in\Lambda} \ell(\mathrm{End}({}_A L(\lambda)))}.$$

Proof. First, we recall the following well-known fact: $\ell(\mathrm{Hom}_A(P(\lambda), M)) = \ell(\mathrm{End}({}_A L(\lambda)))[M : L(\lambda)]$ for any A-module M. In particular, $\ell(\mathrm{End}({}_A L(\lambda)))[\Delta(\lambda) : L(\mu)] = \ell(\mathrm{Hom}_A(P(\lambda), \Delta(\mu)))$. Second, we assert that $\mathrm{Hom}_A(P(\lambda), \Delta(\mu)) \simeq \mathrm{Hom}_A(\Delta(\lambda), \Delta(\mu))$ if $\lambda \not< \mu$. Indeed, applying the functor $\mathrm{Hom}_A(-, \Delta(\mu))$ to the exact sequence in Definition 2.2, we get the following exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_A(\Delta(\lambda), \Delta(\mu)) \longrightarrow \mathrm{Hom}_A(P(\lambda), \Delta(\mu)) &\longrightarrow \mathrm{Hom}_A(Q(\lambda), \Delta(\mu)) \\ &\longrightarrow \mathrm{Ext}^1_A(\Delta(\lambda), \Delta(\mu)). \end{aligned}$$

Note that the last term, in the above exact sequence, vanishes by Lemma 2.2(b) because $\lambda \not< \mu$. Since $Q(\lambda)$ is filtered by $\Delta(\rho)$ with $\lambda < \rho$, we see that if $\lambda \not< \mu$ then $\mathrm{Hom}_A(\Delta(\rho), \Delta(\mu)) = 0$ by Lemma 2.2(a). Thus $\mathrm{Hom}_A(Q(\lambda), \Delta(\mu)) = 0$; and therefore, $\mathrm{Hom}_A(P(\lambda), \Delta(\mu)) \simeq \mathrm{Hom}_A(\Delta(\lambda), \Delta(\mu))$ if $\lambda \not< \mu$.

Let Γ be the matrix with entries $g_{\mu\lambda} = [P(\lambda) : \Delta(\mu)]$, where $[P(\lambda) : \Delta(\mu)]$ is the multiplicity of $\Delta(\mu)$ in a Δ-filtration of $P(\lambda)$. Note that this multiplicity is well defined. Let D be the matrix with (μ, λ)-entry $d_{\mu,\lambda} := [\Delta(\lambda) : L(\mu)]$, which is the multiplicity of $L(\mu)$ as a composition factor in $\Delta(\lambda)$. Thus

$$c_{\mu,\lambda} = [P(\lambda) : L(\mu)] = \sum_{\gamma\in\Lambda} [P(\lambda) : \Delta(\gamma)][\Delta(\gamma) : L(\mu)] = \sum_{\gamma\in\Lambda} g_{\gamma,\lambda} d_{\mu,\gamma}.$$

Then we get the equality $C_A = D\Gamma$.

It follows, from the exact sequence in Definition 2.2, that $g_{\lambda,\lambda} = 1$ and $g_{\gamma,\lambda} = 0$ if $[\gamma] \not> [\lambda]$. As we know, the composition factors of $\Delta(\gamma)$ are of the form $L(\rho)$ with $\rho \leq \gamma$. This implies that $d_{\mu,\gamma} = 0$ if $\mu \not\leq \gamma$. Let $\bar{\Lambda} = \{\rho_1, \rho_2, \ldots, \rho_s\}$ such that $\rho_i \leq \rho_j$ implies $i \leq j$. We partition all matrices involved here by the equivalence classes in such a way that the blocks, in the main diagonal, are indexed by $\rho_1, \rho_2, \ldots, \rho_s$, respectively. Let $l_\mu = \ell(\mathrm{End}({}_A L(\mu)))$ and $l_{\mu,\gamma} = \ell(\mathrm{Hom}_A(\Delta(\mu), \Delta(\gamma)))$. If $\rho_i = \{\lambda_1, \ldots, \lambda_n\}$, then the ith block D_i, in the main diagonal of D, is of the form

$$\begin{pmatrix} \frac{l_{\lambda_1,\lambda_1}}{l_{\lambda_1}} & \frac{l_{\lambda_1,\lambda_2}}{l_{\lambda_1}} & \cdots & \frac{l_{\lambda_1,\lambda_n}}{l_{\lambda_1}} \\ \frac{l_{\lambda_2,\lambda_1}}{\lambda_2} & \frac{l_{\lambda_2,\lambda_2}}{\lambda_2} & \cdots & \frac{l_{\lambda_2,\lambda_n}}{\lambda_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{l_{\lambda_n,\lambda_1}}{\lambda_n} & \frac{l_{\lambda_n,\lambda_2}}{\lambda_n} & \cdots & \frac{l_{\lambda_n,\lambda_n}}{\lambda_n} \end{pmatrix}$$

since $\lambda_i \sim \lambda_j$ implies $\lambda_i \not< \lambda_j$.

The matrix D is then an upper triangular block matrix with $D_1, \dots, D_s$ in the main diagonal. It is not difficult to see that the matrix Γ is a lower triangular matrix such that the diagonal blocks are identity matrices. Hence the determinant of Γ is 1. Clearly, the determinant of D is

$$\det(D) = \frac{\prod_{i=1}^{s} \det(C[\rho_i])}{\prod_{\lambda\in\Lambda} l_\lambda}.$$

This is just the conclusion of Theorem 4.1.

We obtain the following corollaries as a direct consequence of Theorem 4.1.

COROLLARY 4.2 *Let $(A; \Lambda, \leq)$ be a pre-standardly stratified algebra. If $\Lambda = \bar{\Lambda}$ then*

$$\det(C_A) = \prod_{\lambda\in\Lambda} \frac{\ell(\mathrm{End}({}_A\Delta(\lambda)))}{\ell(\mathrm{End}({}_A L(\lambda)))}.$$

In particular, $\det(C_A) > 0$.

Proof. Assume that $\Lambda = \bar{\Lambda}$. In this case, each equivalence class in Λ has exactly one element. Thus $\det(C[\lambda])$ is just the length of the R-module $\mathrm{End}({}_A\Delta(\lambda))$; and therefore $\det(C_A) > 0$.

COROLLARY 4.3 *Let A be a finite-dimensional algebra over an algebraically closed field k. If $(\Theta, Y; \Lambda, \leq)$ is a finite Θ-injective system in A-mod and $B = \mathrm{End}(\bigoplus_{\lambda\in\Lambda} Y(\lambda))$, then*

$$\det(C_B) = \prod_{[t]\in\bar{\Lambda}} \det\Big(\dim_k \mathrm{Hom}_A\big(\Theta(\lambda), \Theta(\mu)\big)_{\lambda,\mu\in[t]}\Big).$$

Proof. This follows easily from Theorem 4.1, Proposition 3.15 and the fact that $\mathrm{End}({}_B L(\lambda)) \simeq k$ for all simple B-modules $L(\lambda)$ if k is algebraically closed.

5. Homological dimensions related to finite Θ-systems

In this section, we shall investigate some homological dimensions of finite Θ-systems over a pre-ordered set. In particular, we are interested in the finitistic dimensions of algebras or categories arising from finite Θ-systems over pre-ordered sets. The results in this section generalize some of the corresponding results in [15].

Before we start our discussion, we recall some definitions and introduce some notation.

Let $\mathcal{X}$ be a class of objects in A-mod. We denote by $\mathcal{X}^\wedge$ the full subcategory of A-mod whose objects are those A-modules X for which there exists a finite $\mathcal{X}$-resolution; that is, there is a long exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to X \to 0$ with $X_i \in \mathcal{X}$ for all $0 \leq i \leq n$. Dually, $\mathcal{X}^\vee$ is the full subcategory of A-modules whose objects have a finite $\mathcal{X}$-coresolution.

The projective dimension of the class $\mathcal{X}$ is defined to be $\mathrm{pd}(\mathcal{X}) := \sup\{\mathrm{pd}({}_A X) \mid X \in \mathcal{X}\}$, where $\mathrm{pd}\,({}_A X)$ is the projective dimension of ${}_A X$. Dually, the injective dimension of $\mathcal{X}$ is define to be $\mathrm{id}(\mathcal{X}) := \sup\{\mathrm{id}({}_A X) \mid X \in \mathcal{X}\}$, where $\mathrm{id}\,({}_A X)$ is the injective dimension of ${}_A X$. Furthermore, we denote by $\mathrm{resdim}_{\mathcal{X}}(M)$ the *$\mathcal{X}$-resolution dimension* of an A-module M. It is defined similarly by $\mathrm{resdim}_{\mathcal{X}}(M) := \min\{r \geq 0 \mid$ there is an exact sequence $0 \to X_r \to \cdots \to X_0 \to M \to$

0 with $X_i \in \mathcal{X}\}$ if $M \in \mathcal{X}^\wedge$, and $\operatorname{resdim}_{\mathcal{X}}(M) := \infty$ in case $M \notin \mathcal{X}^\wedge$. Dually, we have the notion of *$\mathcal{X}$-coresolution dimension* of M, which will be denoted by $\operatorname{coresdim}_{\mathcal{X}}(M)$. For a class $\mathcal{C}$ of A-modules, we set $\operatorname{resdim}_{\mathcal{X}}(\mathcal{C}) := \sup\{\operatorname{resdim}_{\mathcal{X}}(M) \mid M \in \mathcal{C}\}$. Dually, we define $\operatorname{coresdim}_{\mathcal{X}}(\mathcal{C})$.

The following result generalizes [15, Theorem 6.4].

LEMMA 5.1 *Let $(\Theta, Y; \Lambda, \leq)$ be a finite Θ-injective system in A-mod, and let $(\Theta, X; \Lambda, \leq)$ be the Θ-projective system associated to the Θ-injective one, $T := \bigoplus_{\lambda\in\Lambda} Y(\lambda)$ and $C := \bigoplus_{\lambda\in\Lambda} X(\lambda)$. Then*

(a) *$\mathcal{F}(\Theta)$ is functorially finite in A-mod and closed under extensions and direct summands;*
(b) *$\mathcal{F}(\Theta) \cap \mathcal{I}(\Theta) = \operatorname{add}(T)$ and $\mathcal{F}(\Theta) \cap \mathcal{P}(\Theta) = \operatorname{add}(C)$;*
(c) *$\operatorname{resdim}_{\operatorname{add}(C)}(\mathcal{F}(\Theta)) \leq \mid \bar{\Lambda} \mid -1$ and $\operatorname{coresdim}_{\operatorname{add}(T)}(\mathcal{F}(\Theta)) \leq \mid \bar{\Lambda} \mid -1$;*
(d) *$\operatorname{pd}({}_AT) = \operatorname{pd}(\mathcal{F}(\Theta)) \leq \operatorname{pd}({}_AC) + |\bar{\Lambda}| - 1$ and $\operatorname{id}({}_AC) = \operatorname{id}(\mathcal{F}(\Theta)) \leq \operatorname{id}({}_AT) + |\bar{\Lambda}| - 1$.*

Proof. (a) follows from Corollaries 3.11 and 3.16. (b) follows from Lemma 3.7(d) and Corollary 3.19. (c) follows from Proposition 3.9 and its dual. To prove (d), we shall prove that $\operatorname{pd}({}_AM) \leq \operatorname{pd}({}_AT) \leq \operatorname{pd}({}_AC) + |\bar{\Lambda}| - 1$ for any $M \in \mathcal{F}(\Theta)$. Note that the other inequality in (d) follows by duality.

First, we prove that $\operatorname{pd}({}_AM) \leq \operatorname{pd}({}_AT)$. To do so, we proceed by induction on $\min_{(\bar{\Lambda}, \leq^L)}(M)$. Thus, let $0 \neq M \in \mathcal{F}(\Theta)$ be not projective (otherwise we have nothing to prove).

If $\max_{(\bar{\Lambda}, \leq^L)}(M) = \min(\bar{\Lambda}, \leq^L)$, then $\operatorname{Supp}_{(\Theta, \overline{\Lambda})}(M)$ consists of a single element, say $\operatorname{Supp}_{(\Theta, \overline{\Lambda})}(M) = \{\rho_1\}$. By Lemma 3.6(b), we get that $M \simeq \bigoplus_{\lambda\in\rho_1} \Theta(\lambda)^{[M:\Theta(\lambda)]}$. Furthermore, ρ_1 is also minimal in $(\bar{\Lambda}, \leq)$. By Definition 3.2, we conclude that $M \simeq \bigoplus_{\lambda\in\rho_1} Y(\lambda)^{[M:\Theta(\lambda)]}$. This shows that $\operatorname{pd}({}_AM) \leq \operatorname{pd}({}_AT)$.

Now, assume that $\max_{(\bar{\Lambda}, \leq^L)}(M) >^L \min(\bar{\Lambda}, \leq^L) = \rho_1$. Then, by Proposition 3.9, we get an exact sequence in $\mathcal{F}(\Theta)$

$$0 \longrightarrow M \longrightarrow Y_0 \longrightarrow M' \longrightarrow 0$$

such that $\max_{(\bar{\Lambda}, \leq^L)}(M') <^L \max_{(\bar{\Lambda}, \leq^L)}(M)$ and $Y_0 \in \operatorname{add}(T)$. Hence $\operatorname{pd}({}_AM') \leq \operatorname{pd}({}_AT)$ by induction. It follows from the preceding exact sequence that

$$\operatorname{pd}({}_AM) \leq \max\{\operatorname{pd}({}_AY_0), \operatorname{pd}({}_AM') - 1\} \leq \operatorname{pd}({}_AT).$$

Next, we prove that $\operatorname{pd}({}_AT) \leq \operatorname{pd}({}_AC) + |\bar{\Lambda}| - 1$. Since $T \in \mathcal{F}(\Theta)$, we have, by the dual of Proposition 3.9, an exact sequence in $\mathcal{F}(\Theta)$

$$0 \longrightarrow M_1 \longrightarrow Q_1 \longrightarrow T \longrightarrow 0$$

such that $Q_1 \in \operatorname{add}(C)$ and $\min_{(\bar{\Lambda}, \leq^L)}(T) <^L \min_{(\bar{\Lambda}, \leq^L)}(M_1)$. Therefore,

$$\operatorname{pd}({}_AT) \leq \max\{\operatorname{pd}({}_AC), \operatorname{pd}({}_AM_1) + 1\}.$$

If $M_1 = 0$, then $Q_1 \simeq T$ and $\operatorname{pd}({}_AT) \leq \operatorname{pd}({}_AC)$. If $M_1 \neq 0$ then, by the dual of Proposition 3.9, we have an exact sequence

$$0 \longrightarrow M_2 \longrightarrow Q_2 \longrightarrow M_1 \longrightarrow 0$$

in $\mathcal{F}(\Theta)$ such that $Q_2 \in \operatorname{add}(C)$ and $\min_{(\bar{\Lambda}, \leq^L)}(M_1) <^L \min_{(\bar{\Lambda}, \leq^L)}(M_2)$. As a consequence, we obtain that

$$\operatorname{pd}({}_AT) \leq \max\{\operatorname{pd}({}_AC), \operatorname{pd}({}_AM_1) + 1\} \leq \max\{\operatorname{pd}({}_AC), \operatorname{pd}({}_AM_2) + 2\}.$$

If $M_2 = 0$, then $M_1 \simeq Q_2$ and $\mathrm{pd}({}_A T) \leq \mathrm{pd}({}_A C) + 1 \leq \mathrm{pd}({}_A C) + |\bar{\Lambda}| - 1$. In case $M_2 \neq 0$, we proceed as we did for $M_1 \neq 0$. This process will end in a finite number m of steps with $m \leq |\ \bar{\Lambda}\ | -1$; and finally, we infer that

$$\mathrm{pd}({}_A T) \leq \max\{\mathrm{pd}({}_A C), \mathrm{pd}({}_A M_m) + m\} \quad \text{and} \quad M_m \simeq Q_{m+1} \in \mathrm{add}(C).$$

Thus $\mathrm{pd}({}_A T) \leq \mathrm{pd}({}_A C) + |\ \bar{\Lambda}\ | -1$, as desired.

Similar to the case of a linearly ordered set in [14], we have the following general result for Θ-injective systems over a pre-ordered set. Recall that, for an A-module M, we define $M^{\perp} := \{X \in A\text{-mod} \mid \mathrm{Ext}_A^i(M, X) = 0 \text{ for all } i \geq 1\}$ and $\mathcal{X}^{\perp} := \bigcap_{X \in \mathcal{X}} X^{\perp}$.

Proposition 5.2 *Let* $(\Theta, Y; \Lambda, \leq)$ *be a finite* Θ*-injective system in* A-mod, *and let* $T := \bigoplus_{\lambda \in \Lambda} Y(\lambda)$.

(a) *If there is a tilting* A*-module* $\overline{T}$ *such that* $\mathcal{I}(\Theta) = \overline{T}^{\perp}$, *then* T *is a direct summand of* $\overline{T}$.
(b) *There exists a tilting* A*-module* $\overline{T}$ *such that* $\mathcal{I}(\Theta) = \overline{T}^{\perp}$ *if and only if* $\mathrm{pd}({}_A T) < \infty$ *and* $\mathrm{Ext}_A^2(\mathcal{F}(\Theta), \mathcal{I}(\Theta)) = 0$.

Proof. (a) Let $\overline{T}$ be a tilting A-module such that $\mathcal{I}(\Theta) = \overline{T}^{\perp}$. By Lemma 5.1(a), we know that $T \in \mathcal{I}(\Theta) = \overline{T}^{\perp}$. Therefore, by [14, Lemma 3.1(a)], we get an exact sequence $0 \to K \to \overline{T}_0 \to T \to 0$ with $\overline{T}_0 \in \mathrm{add}(\overline{T})$ and $K \in \overline{T}^{\perp} = \mathcal{I}(\Theta)$. Since $T \in \mathcal{F}(\Theta)$, this exact sequence splits by Lemma 5.1(b). Hence $T \in \mathrm{add}(\overline{T})$. On the other hand, by Lemma 3.7(c), T is a basic A-module. Hence T has to be a direct summand of $\overline{T}$.

(b) Suppose $\mathcal{I}(\Theta) = \overline{T}^{\perp}$ for some tilting A-module $\overline{T}$. Then $\mathrm{pd}({}_A T) \leq \mathrm{pd}(\overline{{}_A T}) < \infty$ by (a). On the other hand, by the dual result of [5, Theorem 5.5(a)], we conclude that $\mathcal{I}(\Theta)$ is a coresolving subcategory of A-mod. Hence, $\mathrm{Ext}_A^2\,(\mathcal{F}(\Theta), \mathcal{I}(\Theta)) = 0$ by [15, Lemma 2.5].

Now, assume that $\mathrm{pd}({}_A T) < \infty$ and that $\mathrm{Ext}_A^2\,(\mathcal{F}(\Theta), \mathcal{I}(\Theta)) = 0$. In particular, $\mathcal{I}(\Theta)$ is a coresolving subcategory of A-mod (see [15, Lemma 2.5]). Moreover, from Corollary 3.11 (b), we see that $\mathcal{I}(\Theta)$ is also a covariantly finite subcategory of A-mod. In order to prove that $\mathcal{I}(\Theta) = \overline{T}^{\perp}$ for some tilting A-module $\overline{T}$, it is enough to prove that $\mathcal{I}(\Theta)^{\vee} = A$-mod (see the dual result of [5, Theorem 5.5]). Let M be an A-module. Since $\mathcal{I}(\Theta)$ is coresolving, we have, for each $d > 0$, a long exact sequence in A-mod

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \ldots \longrightarrow I_{d-1} \longrightarrow \Omega^{-d}(M) \longrightarrow 0$$

with I_i injective for all $i = 0, 1, \ldots, d-1$, where $\Omega^{-n}(M)$ denotes the nth co-syzygy of M. If $\mathrm{pd}(T) = 0$, we set $d := 1$; otherwise, we put $d := \mathrm{pd}(T)$. We claim that $\Omega^{-d}(M) \in \mathcal{I}(\Theta)$. Indeed, by Lemma 5.1(d), we have $\mathrm{pd}(\mathcal{F}(\Theta)) = \mathrm{pd}(T)$. Hence $\mathrm{Ext}_A^1\,(N, \Omega^{-d}(M)) \simeq \mathrm{Ext}_A^{d+1}\,(N, M) = 0$ for any $N \in \mathcal{F}(\Theta)$; proving that $\Omega^{-d}(M) \in \mathcal{I}(\Theta)$, and so $M \in \mathcal{I}(\Theta)^{\vee}$.

Let $\mathcal{X}$ and $\mathcal{C}$ be two classes of A-modules. For an A-module M, we denote by $\mathrm{pd}_{\mathcal{X}}(M)$ the *relative projective dimension* of M with respect to $\mathcal{X}$, that is,

$$\mathrm{pd}_{\mathcal{X}}(M) := \min\{n \mid \mathrm{Ext}_A^j(M, -)|_{\mathcal{X}} = 0 \text{ for any } j > n \geq 0\}.$$

Furthermore, we define $\mathrm{pd}_{\mathcal{X}}(\mathcal{C}) := \sup\{\mathrm{pd}_{\mathcal{X}}(M) \mid M \in \mathcal{C}\}$. Dually, we denote by $\mathrm{id}_{\mathcal{X}}(M)$ the *relative injective dimension* of M with respect to $\mathcal{X}$, and define $\mathrm{id}_{\mathcal{X}}(\mathcal{C}) := \sup\{\mathrm{id}_{\mathcal{X}}(M) \mid M \in \mathcal{C}\}$.

Of particular interest are the following two subcategories: $\mathcal{P}^{<\infty}(\mathcal{C}) := \{X \in \mathcal{C} \mid \mathrm{pd}({}_AX) < \infty\}$ and $\mathcal{I}^{<\infty}(\mathcal{C}) := \{X \in \mathcal{C} \mid \mathrm{id}({}_AX) < \infty\}$. The *finitistic projective dimension* of the category $\mathcal{C}$ is defined by $\mathrm{pd}(\mathcal{P}^{<\infty}(\mathcal{C}))$, and is denoted by $\mathrm{fpd}(\mathcal{C})$. Dually, $\mathrm{fid}(\mathcal{C})$ denotes the *finitistic injective dimension* $\mathrm{id}(\mathcal{I}^{<\infty}(\mathcal{C}))$ of $\mathcal{C}$. By abuse of notation, we write $\mathrm{fid}(A)$ and $\mathrm{fpd}(A)$ for fid (A-mod) and fpd (A-mod), respectively. Also, for simplicity, we denote the class $\mathcal{P}^{<\infty}(A\text{-mod})$ by $\mathcal{P}^{<\infty}(A)$, and $\mathcal{I}^{<\infty}(A\text{-mod})$ by $\mathcal{I}^{<\infty}(A)$.

With this notation, we have the following extension of [15, Proposition 6.6].

PROPOSITION 5.3 *Let $(\Theta, Y; \Lambda, \leq)$ be a finite Θ-injective system in A-mod, and let $(\Theta, X; \Lambda, \leq)$ be the Θ-projective system associated to the Θ-injective one. Put $T := \bigoplus_{\lambda\in\Lambda} Y(\lambda)$ and $C := \bigoplus_{\lambda\in\Lambda} X(\lambda)$. Then*

(a) $\mathrm{pd}(\mathcal{F}(\Theta)) \leq \mathrm{id}_{\mathcal{F}(\Theta)}(\mathcal{I}(\Theta)) + |\bar{\Lambda}|$ *and* $\mathrm{id}(\mathcal{F}(\Theta)) \leq \mathrm{pd}_{\mathcal{F}(\Theta)}(\mathcal{P}(\Theta)) + |\bar{\Lambda}|$*;*
(b) $\mathrm{fid}(A) \leq \max\{\mathrm{fid}(\mathcal{I}(\Theta)), \mathrm{id}({}_AC) + 1\}$ *and* $\mathrm{fpd}(A) \leq \max\{\mathrm{fpd}(\mathcal{P}(\Theta)), \mathrm{pd}({}_AT) + 1\}$*;*
(c) $\mathrm{gl.dim}(A) = \mathrm{pd}(\mathcal{I}(\Theta)) = \mathrm{id}(\mathcal{P}(\Theta))$*;*
(d) $\mathrm{pd}({}_AT)$ *finite implies that* $\mathrm{fpd}(A) = \mathrm{fpd}(\mathcal{I}(\Theta))$. *Dually, if* $\mathrm{id}({}_AC)$ *is finite, then* $\mathrm{fid}(A) = \mathrm{fid}(\mathcal{P}(\Theta))$.

Proof. By Lemma 5.1(a), $\mathcal{F}(\Theta)$ is a functorially finite subcategory of A-mod, which is closed under extensions and direct summands. So, the result follows from Lemma 5.1(b), (c) and (d), together with [15, Theorem 2.15] and its dual.

PROPOSITION 5.4 *Let $(\Theta, Y; \Lambda, \leq)$ be a finite Θ-injective system in A-mod, $T := \bigoplus_{\lambda\in\Lambda} Y(\lambda)$ and $s(A)$ the number of isomorphism classes of simple A-modules. If $\mathcal{I}(\Theta)$ is coresolving, then*

(a) $\mathrm{pd}(\mathcal{F}(\Theta)) \leq |\bar{\Lambda}| \leq s(A)$ *and* $\mathrm{id}(\mathcal{F}(\Theta)) \leq \mathrm{id}(\mathcal{I}(\Theta)) + |\bar{\Lambda}|$*;*
(b) *if ${}_AA \in (\mathrm{add}(T))^{\vee}$, T is a tilting R-module and $\mathcal{I}(\Theta) = T^{\perp}$.*

Proof. (a) By Lemma 5.1 (d), we get $\mathrm{pd}({}_AT) = \mathrm{pd}(\mathcal{F}(\Theta))$. On the other hand, the fact that $\mathcal{I}(\Theta)$ is coresolving and also using that the injective modules belong to $\mathcal{I}(\Theta)$, imply that $\mathrm{id}_{\mathcal{F}(\Theta)}(\mathcal{I}(\Theta)) = 0$. So, from Proposition 5.3 (a), it follows that $\mathrm{pd}(\mathcal{F}(\Theta)) \leq |\bar{\Lambda}|$. Moreover, by Proposition 5.2, we see that T is a direct summand of a tilting A-module; and so $|\bar{\Lambda}| \leq s(A)$.

(b) By (a) and Lemma 5.1 (d), we conclude that $\mathrm{pd}({}_AT)$ is finite. Then the statement (b) follows from Proposition 5.2.

LEMMA 5.5 *Let $(\Theta, Y; \Lambda, \leq)$ be a finite Θ-injective system in A-mod, $(\Theta, X; \Lambda, \leq)$ the Θ-projective system associated to the Θ-injective one, $T := \bigoplus_{\lambda\in\Lambda} Y(\lambda)$ and $C := \bigoplus_{\lambda\in\Lambda} X(\lambda)$. Then*

$$\mathrm{id}_{\mathrm{add}(T)}(\mathcal{F}(\Theta)) = \mathrm{id}_{\mathrm{add}(T)}(C) = \mathrm{pd}_{\mathrm{add}(C)}(T) = \mathrm{pd}_{\mathrm{add}(C)}(\mathcal{F}(\Theta)).$$

Proof. It is clear that $\mathrm{id}_{\mathrm{add}(T)}(C) = \mathrm{pd}_{\mathrm{add}(C)}(T)$. We shall prove that $\mathrm{id}_{\mathrm{add}(T)}(\mathcal{F}(\Theta)) = \mathrm{id}_{\mathrm{add}(T)}(C)$. To do this, it is enough to show that $\mathrm{id}_{\mathrm{add}(T)}(M) \leq d$ for any $M \in \mathcal{F}(\Theta)$, where $d := \mathrm{id}_{\mathrm{add}(T)}(C)$. We proceed by reverse induction on $\min_{(\bar{\Lambda}, \leq^L)}(M)$.

Let $\rho_n := \max(\bar{\Lambda}, \leq^L)$. If $\min_{(\bar{\Lambda}, \leq^L)}(M) = \rho_n$ then, by Lemma 3.6(b) and Definition 3.3, we get that $M \simeq \bigoplus_{\lambda\in\rho_n} X(\lambda)^{[M:\Theta(\lambda)]} \in \mathrm{add}(C)$; and so, $\mathrm{id}_{\mathrm{add}(T)}(M) \leq d$.

Suppose that $\min_{(\bar{\Lambda},\leq^L)}(M) <^L \rho_n$. Then, by the dual result of Proposition 3.9, we have an exact sequence $0 \to M' \to C_0 \to M \to 0$ in $\mathcal{F}(\Theta)$ such that $\min_{(\bar{\Lambda},\leq^L)}(M) <^L \min_{(\bar{\Lambda},\leq^L)}(M')$ and $C_0 \in \operatorname{add}(C)$. Applying the functor $\operatorname{Hom}_A(T,-)$ to this sequence, we get an exact sequence

$$\operatorname{Ext}^j_A(T, C_0) \longrightarrow \operatorname{Ext}^j_A(T, M) \longrightarrow \operatorname{Ext}^{j+1}_A(T, M').$$

Since $d = \operatorname{id}_{\operatorname{add}(T)}(C)$ and $\operatorname{id}_{\operatorname{add}(T)}(M') \leq d$ (by induction), we conclude that $\operatorname{id}_{\operatorname{add}(T)}(M) \leq d$, proving that $\operatorname{id}_{\operatorname{add}(T)}(\mathcal{F}(\Theta)) = \operatorname{id}_{\operatorname{add}(T)}(C)$.

The equality $\operatorname{pd}_{\operatorname{add}(C)}(\mathcal{F}(\Theta)) = \operatorname{pd}_{\operatorname{add}(C)}(T)$ can be proved in a similar way.

Now, we have the following result which was proved in [15] for the special case of Λ being a linear ordered set.

THEOREM 5.6 *Let $(\Theta, Y; \Lambda, \leq)$ a finite Θ-injective system in A-mod, $(\Theta, X; \Lambda, \leq)$ the Θ-projective system associated to the Θ-injective one, $T := \bigoplus_{\lambda\in\Lambda} Y(\lambda)$, $C := \bigoplus_{\lambda\in\Lambda} X(\lambda)$ and $s(A)$ the number of iso-classes of simple A-modules. If $\mathcal{I}(\Theta)$ is coresolving, then*

(a) $\operatorname{id}_{\mathcal{F}(\Theta)}(M) = \operatorname{coresdim}_{\operatorname{add}(T)}(M) = \operatorname{id}_{\operatorname{add}(T)}(M)$ *for any* $M \in (\operatorname{add}(T))^{\vee}$*;*
(b) $\operatorname{id}_{\mathcal{F}(\Theta)}(M) = \operatorname{coresdim}_{\mathcal{I}(\Theta)}(M) \leq \operatorname{pd}({}_AC) + |\bar{\Lambda}| - 1$ *for all* $M \in A$-mod*;*
(c) $\operatorname{coresdim}_{\mathcal{I}(\Theta)}(\mathcal{P}(\mathcal{I}(\Theta))) = \operatorname{pd}(\mathcal{P}(\mathcal{I}(\Theta))) = \operatorname{coresdim}_{\mathcal{I}(\Theta)}(A\text{-mod}) = \operatorname{pd}(\mathcal{F}(\Theta)) = \operatorname{pd}({}_AT)$ $= \operatorname{pd}_{\mathcal{P}(\Theta)}(\mathcal{F}(\Theta)) = \operatorname{coresdim}_{\mathcal{I}(\Theta)}(\mathcal{P}(\Theta)) \leq |\bar{\Lambda}| \leq s(A)$*;*
(d) $\operatorname{coresdim}_{\operatorname{add}(T)}(C) = \operatorname{id}_{\operatorname{add}(T)}(C) = \operatorname{pd}_{\operatorname{add}(C)}(T) = \operatorname{pd}_{\operatorname{add}(C)}(\mathcal{F}(\Theta)) = \operatorname{id}_{\operatorname{add}(T)}(\mathcal{F}(\Theta))$ $= \operatorname{coresdim}_{\operatorname{add}(T)}(\mathcal{F}(\Theta)) = \operatorname{pd}_{\mathcal{F}(\Theta)}(\mathcal{F}(\Theta)) = \operatorname{coresdim}_{\mathcal{I}(\Theta)}(\mathcal{F}(\Theta)) \leq |\bar{\Lambda}| - 1$*;*
(e) $\operatorname{pd}_{\mathcal{I}(\Theta)}(M) = \operatorname{resdim}_{\mathcal{P}(\mathcal{I}(\Theta))}(M)$ *for any* $M \in \mathcal{P}(\mathcal{I}(\Theta))^{\wedge}$*;*
(f) $\mathcal{P}(\mathcal{I}(\Theta))^{\wedge} = \{M \in A\text{-mod} \mid \operatorname{pd}_{\mathcal{I}(\Theta)}(M) < \infty\} = \mathcal{P}^{<\infty}(A)$*;*
(g) $\operatorname{fpd}(A) = \operatorname{fpd}(\mathcal{I}(\Theta)) \leq \operatorname{pd}({}_AT) + \operatorname{resdim}_{\mathcal{P}(\mathcal{I}(\Theta))}(\mathcal{P}^{<\infty}(\mathcal{I}(\Theta))) \leq \operatorname{pd}({}_AT)$ $+ \operatorname{resdim}_{\mathcal{P}(\mathcal{I}(\Theta))}(\mathcal{P}(\mathcal{I}(\Theta))^{\wedge})$, *where $\mathcal{P}(\mathcal{I}(\Theta))$ is the full subcategory of* A-mod *consisting of all $\mathcal{I}(\Theta)$-projective A-modules.*

Proof. By Lemma 5.1, $\mathcal{F}(\Theta)$ is a functorially finite subcategory of A-mod, and $\operatorname{pd}({}_AT) = \operatorname{pd}(\mathcal{F}(\Theta))$. Moreover, $\operatorname{pd}(\mathcal{F}(\Theta))$ is finite by Proposition 5.4 (a). Hence, by [15, Lemma 3.2], we have $\mathcal{F}(\Theta)$ is a partial tilting subcategory of A-mod. Moreover, it follows from [15, Lemma 2.5] that $\operatorname{id}_{\mathcal{F}(\Theta)}(\mathcal{I}(\Theta)) = 0$ since $\mathcal{I}(\Theta)$ is a coresolving subcategory in A-mod.

The statements (a), (b) and (c) follow from [15, Theorem 3.7] since $\mathcal{F}(\Theta)$ is a partial tilting, contravariantly finite subcategory of A-mod and since $\operatorname{id}_{\mathcal{F}(\Theta)}(\mathcal{I}(\Theta)) = 0$.

By Lemma 5.1(c), we have $\operatorname{coresdim}_{\operatorname{add} T}(\mathcal{F}(\Theta)) \leq |\bar{\Lambda}| - 1$. Hence, (d) follows from the equalities given in Lemma 5.5 and [15, Theorem 3.7(c)].

The statements (e) and (f) follow from [15, Theorem 3.10]. The statement (g) is a consequence of [15, Theorem 3.10(f)] since $\mathcal{F}(\Theta) \cap \mathcal{I}(\Theta) = \operatorname{add}({}_AT)$ by Lemma 5.1.

6. Applications to pre-standardly stratified algebras

In this section, we use the results of section 5 to describe a bound for the finitistic projective dimension of pre-standardly stratified algebras.

Let A be an Artin algebra, and let $(\Lambda, \leq)$ be a pre-ordered set which indexes the non-isomorphic simple A-modules $L(\lambda)$. Then, with respect to this pre-ordered set, we have the set Δ of standard

A-modules $\Delta(\lambda)$, the set δ of big standard A-modules $\delta(\lambda)$ and the set $\overline{\delta}$ of proper standard modules $\overline{\delta}(\lambda)$, with $\lambda \in \Lambda$. Similarly, we have the set ∇ of co-standard modules $\nabla(\lambda)$ and the set $\overline{\nabla}$ of proper co-standard modules $\overline{\nabla}(\lambda)$. Moreover, if $(A; \Lambda, \leq)$ is a pre-standardly stratified algebra, then $\delta(\lambda) = \Delta(\lambda)$ for any $\lambda \in \Lambda$, as mentioned in Remark (2) to Definition 2.2 in subsection 2.3.

Since several algebras will be involved in our discussions below, we shall write a lower index to indicate with which algebra we work. For example, we write ${}_A\overline{\delta}$ for the set $\{{}_A\overline{\delta}(\lambda) \mid \lambda \in \Lambda\}$ and ${}_A\overline{\nabla}$ for $\{{}_A\overline{\nabla}(\lambda) \mid \lambda \in \Lambda\}$. Note that the given pre-ordered set $(\Lambda, \leq)$, for the algebra A, induces an indexing of simple modules for the opposite algebra A^{op} of A, namely the set of non-isomorphic simple A^{op}-modules $\{DL(\lambda) \mid \lambda \in \Lambda\}$ can be indexed by $(\Lambda, \leq)$, where D is the usual duality of an Artin algebra. Thus, we may define ${}_{A^{\mathrm{op}}}\delta$, ${}_{A^{\mathrm{op}}}\overline{\delta}$ and ${}_{A^{\mathrm{op}}}\overline{\nabla}$, with respect to $(\Lambda, \leq)$. Note that $D({}_A\overline{\delta}(\lambda)) = {}_{A^{\mathrm{op}}}\overline{\nabla}(\lambda)$ for any $\lambda \in \Lambda$.

Recall that, for a pre-standardly stratified algebra $(A; \Lambda, \leq)$, we have the so called 'characteristic tilting module' T which belongs to $\mathcal{F}({}_A\Delta) \cap \mathcal{I}({}_A\Delta)$ and has the property that $(B^{\mathrm{op}}; \Lambda, \leq^{\mathrm{op}})$ is a pre-standardly stratified algebra, where $B := \mathrm{End}({}_AT)$ (see Corollary 3.14). Therefore, we may use the pre-ordered set $(\Lambda, \leq^{\mathrm{op}})$ to index the simple B-modules. Thus, with respect to this pre-order, we can define ${}_B\delta(\lambda)$ and ${}_B\overline{\delta}(\lambda)$ for the algebra B. Note that B may not be pre-standardly stratified with respect to $(\Lambda, \leq^{\mathrm{op}})$.

Lemma 6.1 [12] *Let* $(A; \Lambda, \leq)$ *be a pre-standardly stratified algebra with standard modules* ${}_A\Delta(\lambda)$, $\lambda \in \Lambda$, *and let* T *be the characteristic tilting* A*-module associated to* A *and* $B = \mathrm{End}({}_AT)$. *Then*

(a) *the functor* $F := \mathrm{Hom}_A({}_AT_B, -) : A\text{-mod} \longrightarrow B\text{-mod}$ *restricts to an exact equivalence from* $\mathcal{F}({}_A\overline{\nabla})$ *to* $\mathcal{F}({}_B\overline{\delta})$;
(b) $\mathrm{fpd}\,(\mathcal{F}({}_{A^{\mathrm{op}}}\overline{\delta})) \leq \sum_{\rho \in \bar{\Lambda}} \mathrm{fpd}\,(\mathrm{End}^{\mathrm{op}}(\bigoplus_{\lambda \in \rho}\ {}_{A^{\mathrm{op}}}\delta(\lambda))) + |\bar{\Lambda}| - 1$.

Proof. (a) and (b) are taken from [12, Theorem 5(ii), p. 20; Lemma 11, p. 23].

Lemma 6.2 *Let* $(A; \Lambda, \leq)$ *be a pre-standardly stratified algebra, and let* T *be the characteristic tilting module and* $B := \mathrm{End}({}_AT)$ *with the index set* $(\Lambda, \leq^{\mathrm{op}})$. *Then*

(a) $\mathcal{I}({}_A\Delta)$ *is coresolving and* $\mathcal{I}({}_A\Delta) = T^{\perp}$;
(b) $\mathrm{resdim}_{\mathrm{add}(T)}\,((\mathrm{add}(T))^{\wedge}) \leq \sum_{\rho \in \bar{\Lambda}} \mathrm{fpd}(\mathrm{End}^{\mathrm{op}}(\bigoplus_{\lambda \in \rho}\ {}_B\delta(\lambda))) + |\bar{\Lambda}| - 1$.

Proof. (a) By Proposition 2.3, we know that $\mathcal{F}({}_A\Delta)$ is a resolving subcategory in A-mod. Hence, by [5], we get that $\mathcal{I}({}_A\Delta)$ is coresolving in A-mod; and therefore, from Proposition 5.4 (b), we conclude that $\mathcal{I}({}_A\Delta) = T^{\perp}$.

(b) Consider the functor $F := \mathrm{Hom}_A({}_AT_B, -) : A\text{-mod} \to B\text{-mod}$. We claim that $\mathrm{Im}(F|_{T^{\perp}}) \simeq \mathcal{F}({}_B\overline{\delta})$. Indeed, by [12, Lemma 7, p. 15], we have $\mathcal{F}({}_A\overline{\nabla}) = \mathcal{I}({}_A\Delta)$; and so, $\mathcal{F}({}_A\overline{\nabla}) = T^{\perp}$ by Lemma 6.2(a). Now the claim follows from Lemma 6.1(a). By [14, Proposition 3.2(e)], we have $\mathrm{resdim}_{\mathrm{add}(T)}\,(\mathrm{add}(T))^{\wedge} \leq \mathrm{fpd}\,(\mathrm{Im}(F|_{T^{\perp}}))$. This means that $\mathrm{resdim}_{\mathrm{add}(T)}\,(\mathrm{add}(T))^{\wedge} \leq \mathrm{fpd}(\mathcal{F}({}_B\overline{\delta}))$. Hence Lemma 6.2(b) follows from Lemma 6.1(b) since $(B^{\mathrm{op}}; \Lambda, \leq^{\mathrm{op}})$ is a pre-standardly stratified algebra.

Corollary 6.3 *Let* $(A; \Lambda, \leq)$ *be a pre-standardly stratified algebra and* T *be the corresponding characteristic tilting module. Then*

(a) $\mathrm{id}_{\mathcal{F}({}_A\Delta)}\,(M) = \mathrm{coresdim}_{\mathrm{add}(T)}\,(M) = \mathrm{id}_{\mathrm{add}(T)}\,(M)$ *for any* $M \in (\mathrm{add}(T))^{\vee}$;

(b) $\mathrm{id}_{\mathcal{F}({}_A\Delta)}(M) = \mathrm{coresdim}_{\mathcal{I}({}_A\Delta)}(M) \leq |\bar{\Lambda}| - 1$ *for all* $M \in A$-mod*;*
(c) $\mathrm{coresdim}_{\mathcal{I}({}_A\Delta)}(\mathcal{F}({}_A\Delta)) = \mathrm{pd}(\mathcal{F}({}_A\Delta)) = \mathrm{coresdim}_{\mathcal{I}({}_A\Delta)}(A\text{-mod}) = \mathrm{pd}({}_A T) = \mathrm{pd}_{\mathcal{P}({}_A\Delta)}(\mathcal{F}({}_A\Delta)) = \mathrm{coresdim}_{\mathcal{I}({}_A\Delta)}(\mathcal{P}({}_A\Delta)) = \mathrm{coresdim}_{\mathrm{add}(T)}({}_A A) = \mathrm{id}_{\mathrm{add}(T)}({}_A A) = \mathrm{id}_{\mathrm{add}(T)}(\mathcal{F}({}_A\Delta)) = \mathrm{coresdim}_{\mathrm{add}(T)}(\mathcal{F}({}_A\Delta)) = \mathrm{pd}_{\mathcal{F}({}_A\Delta)}(\mathcal{F}({}_A\Delta)) = \mathrm{coresdim}_{\mathcal{I}({}_A\Delta)}(\mathcal{F}({}_A\Delta)) \leq |\bar{\Lambda}| - 1$*;*
(d) $\mathrm{pd}_{\mathcal{I}({}_A\Delta)}(M) = \mathrm{resdim}_{\mathcal{F}({}_A\Delta)}(M)$ *for any* $M \in \mathcal{F}({}_A\Delta)^{\wedge}$*;*
(e) $\mathcal{F}({}_A\Delta)^{\wedge} = \{M \in A\text{-mod} \mid \mathrm{pd}_{\mathcal{I}({}_A\Delta)}(M) < \infty\} = \mathcal{P}^{<\infty}(A)$*;*
(f) $\mathrm{fpd}(A) = \mathrm{fpd}(\mathcal{I}({}_A\Delta)) \leq \mathrm{pd}({}_A T) + \mathrm{resdim}_{\mathcal{F}({}_A\Delta)}(\mathcal{P}^{<\infty}(\mathcal{I}({}_A\Delta))) \leq \mathrm{pd}({}_A T) + \mathrm{resdim}_{\mathcal{F}({}_A\Delta)}(\mathcal{F}({}_A\Delta)^{\wedge})$.

Proof. Since $(A; \Lambda, \leq)$ is a pre-standardly stratified algebra, we get, on one hand, that $({}_A\Delta, X; \Lambda, \leq)$ is a ${}_A\Delta$-projective system with $X(\lambda) := P(\lambda)$ for $\lambda \in \Lambda$, and on the other hand, that $({}_A\Delta, Y; \Lambda, \leq)$ is a ${}_A\Delta$-injective system with $Y(\lambda) := T(\lambda)$ for $\lambda \in \Lambda$. Consider $C := \bigoplus_{\lambda\in\Lambda} X(\lambda)$ and $T := \bigoplus_{\lambda\in\Lambda} Y(\lambda)$. Note that $\mathrm{add}(C) = \mathrm{add}({}_A A)$. By [5, Proposition 1.10], we have $\mathcal{P}(\mathcal{I}({}_A\Delta)) = \mathcal{F}({}_A\Delta)$ since ${}_A A \in \mathcal{F}({}_A\Delta)$ and $\mathcal{F}({}_A\Delta)$ is contravariantly finite in A-mod. Hence, we can apply Theorem 5.6 since $\mathcal{I}({}_A\Delta)$ is coresolving in A-mod. This finishes the proof.

COROLLARY 6.4 *Let $(A; \Lambda, \leq)$ be a pre-standardly stratified algebra and T be the characteristic tilting module associated to A. We set $B := \mathrm{End}({}_A T)$ with simple modules indexed by $(\Lambda, \leq^{\mathrm{op}})$, and $B(\rho) := \mathrm{End}(\bigoplus_{\lambda\in\rho} {}_B\delta(\lambda))$ for $\rho \in \bar{\Lambda}$. Then*

$$\mathrm{fpd}(A) \leq \mathrm{pd}({}_A T) + \mathrm{resdim}_{\mathcal{F}({}_A\Delta)}((\mathrm{add}(T))^{\wedge}) \leq \mathrm{pd}({}_A T) + |\bar{\Lambda}| - 1 + \sum_{\rho\in\bar{\Lambda}} \mathrm{fpd}\,(B(\rho)^{\mathrm{op}})$$
$$\leq 2\,|\bar{\Lambda}| - 2 + \sum_{\rho\in\bar{\Lambda}} \mathrm{fpd}\,(B(\rho)^{\mathrm{op}}).$$

Proof. Since $\mathrm{resdim}_{\mathcal{F}({}_A\Delta)}((\mathrm{add}(T))^{\wedge}) \leq \mathrm{resdim}_{\mathrm{add}(T)}((\mathrm{add}(T))^{\wedge})$, Corollary 6.4 will be deduced from Corollary 6.3 and Lemma 6.2(b) if we can prove that $\mathcal{P}^{<\infty}(\mathcal{I}({}_A\Delta)) \subseteq (\mathrm{add}(T))^{\wedge}$. Let $X \in \mathcal{P}^{<\infty}(\mathcal{I}({}_A\Delta))$. Then, by Corollary 6.3(e), we know that $t := \mathrm{resdim}_{\mathcal{F}({}_A\Delta)}(X)$ is finite. On the other hand, by Wakamatsu's lemma (see, for example, [5]) together with Lemma 5.1 (a) and the fact that ${}_A A \in \mathcal{F}({}_A\Delta)$, we conclude, for any $M \in A$-mod, that there is an exact sequence

$$0 \longrightarrow K_0 \longrightarrow M_0 \xrightarrow{f} M \longrightarrow 0 \quad \text{with } K_0 \in \mathcal{I}({}_A\Delta)$$

and f a right-minimal $\mathcal{F}({}_A\Delta)$-approximation of M. In particular, since $\mathrm{resdim}_{\mathcal{F}({}_A\Delta)}(X) = t$, we get the following long exact sequence for X:

$$0 \longrightarrow X_t \xrightarrow{f_t} X_{t-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} X \longrightarrow 0,$$

where $X_i \in \mathcal{F}({}_A\Delta)$ and $K_i := \mathrm{Ker}\,(f_i) \in \mathcal{I}({}_A\Delta)$ for $0 \leq i \leq t$. Note that all X_i, $0 \leq i \leq t$, are in $\mathcal{F}({}_A\Delta) \cap \mathcal{I}({}_A\Delta)$ which is $\mathrm{add}(T)$ by Corollary 3.14. Thus $X \in (\mathrm{add}(T))^{\wedge}$.

COROLLARY 6.5 *Let $(A; \Lambda, \leq)$ be a pre-standardly stratified algebra and $s(A)$ be the number of isomorphism classes of simple A-modules. If $(\Lambda, \leq)$ is a partially ordered set, then*

$$\mathrm{fpd}(A) \leq 2\,s(A) - 2.$$

Proof. If $(\Lambda, \leq)$ is a partially ordered set, then each ρ in $\bar{\Lambda}$ consists of exactly one element. Thus $B(\rho)$ is a local algebra. The corollary follows immediately from Corollary 6.4 since the finitistic projective dimension of a local algebra is equal to zero.

REMARKS (1) Corollary 6.2 says that, for the class of standardly stratified algebras, the finitistic dimension conjecture holds true. Moreover, we have obtained the same bound, for this class of algebras, as for the class of standardly stratified algebras in [2], where the pre-order is a linear order.

(2) In [12, Theorem 7, p. 25], the following bound was shown for a pre-standardly stratified algebra A:

$$\mathrm{fpd}(A) \leq |\Lambda| + |\bar{\Lambda}| - 1 + \sum_{\rho \in \bar{\Lambda}} \mathrm{fpd}\,(B(\rho)^{\mathrm{op}}).$$

It seems that the bound in Corollary 6.4 is better than this one. Indeed, consider the example at the beginning of section 4. For this algebra, the bound given by Corollary 6.4 is $2 - 2 + 0 = 0$, while the bound given by [12, Theorem 7, p. 25] is $2 + 1 - 1 + 0 = 2$. It is easy to see that the finitistic dimension of the algebra is 0.

(3) In [12, Theorem 8, p. 29], it was proved that

$$\mathrm{fpd}(A) \leq \mathrm{coresdim}_{\mathcal{I}({}_A\Delta)}\,(A\text{-mod}) + \mathrm{resdim}_{\mathcal{F}({}_A\Delta)}\,(\mathcal{F}({}_A\Delta)^{\wedge})$$

for a very special subclass of pre-standardly stratified algebras, namely the so-called ‘weakly properly stratified algebras’. Corollary 6.3(c) and (f) extend this result to the whole class of pre-standardly stratified algebras.

(4) For a properly standardly stratified algebra A, there is a relation between the finitistic dimension and filtration dimension in [16]. In particular, the finitistic dimension of A can be bounded above by the projective dimension of the characteristic tilting module plus the injective dimension of the cotilting module associated to A. For details we refer the reader to [16].

For further results and new information on finitistic dimension, we refer to [23] and the references therein.

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