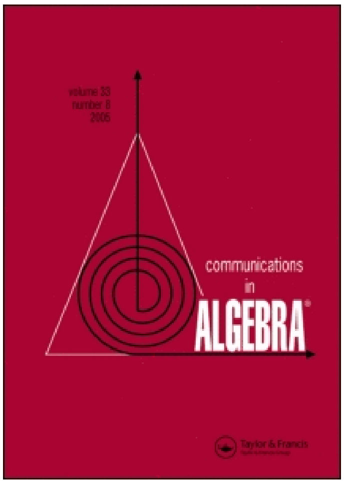

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RINGEL DUALS OF QUASI-HEREDITARY ALGEBRAS

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Quasi-hereditary algebras are introduced by Cline, Parshall and Scott in order to study highest weight categories in the representation theory of Lie algebras and algebraic groups (see [CPS]). These algebras seem to become a very interesting class of algebras.

Let A be a quasi-hereditary k-algebras over an algebraically closed field k. C.M.Ringel constructed in [R2] a new quasi-hereditary algebra $\mathcal{R}(A)$ from the given quasi-hereditary algebra A, his construction of $\mathcal{R}(A)$ from A has the property $\mathcal{R}(\mathcal{R}(A)) \cong A$ for any basic quasi-hereditary algebra A. The module used in his construction is in fact a generalized tilting module and of special interest in the representation theory of algebaic groups (see [D] and [E]). Following [E] we call the algebra $\mathcal{R}(A)$ the Ringel dual of A.

Suppose a quasi-hereditary algebra A is given by a quiver with relations. How to determine the quiver of $\mathcal{R}(A)$ and its relations seems to be a very difficult question. However, in this paper we try to study the Ringel duals for dual extension algebras, defined in [X], of incidence algebras of posets of tree type and to describe quivers of their Ringel duals. The main result of this paper is an explicite description of the quivers of the Ringel duals of these algebras and implies in particular that the quivers of $\mathcal{R}(A)$ are bipartite.

1. PRELIMINARIES

1.1 Let A be a finite dimensional algebra over an algebraically closed field k. We will consider finitely generated left A-modules, maps between A-modules will be written on the right side of the argument, thus the composition of maps

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$f : M_1 \longrightarrow M_2$ and $g : M_2 \longrightarrow M_3$ will be denoted by fg. The category of all finitely generated modules will be denoted by A–mod. Given a class Θ of A–modules, we denote by $\mathcal{F}(\Theta)$ the class of all A–modules which have a Θ–filtration, that is, a filtration

$$0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M$$

such that each factor M_{i-1}/M_i is isomorphic to an object in Θ for $1 \leq i \leq t$. For a module $M \in A$–mod, we denote by add(M) the full additive subcategory of A–mod consisting of all finite direct sums of direct summands of M.

Let X be a finite poset in bijective correspondence with the isomorphism classes of simple A–modules. For each $\lambda \in X$, let $E(\lambda)$ be a simple module in the isomorphism class corresponding to λ and $P(\lambda)$ (or $P_A(\lambda)$) a projective cover of $E(\lambda)$ and denote by $\Delta(\lambda)$ the maximal factor modules of $P(\lambda)$ with composition factors of the form $E(\mu)$, $\mu \leq \lambda$. Dually, let $Q(\lambda)$ (or $Q_A(\lambda)$) be an injective hull of $E(\lambda)$ and denote by $\nabla(\lambda)$ the maximal submodule of $Q(\lambda)$ with the composition factors of the form $E(\mu)$, $\mu \leq \lambda$. Let Δ (respectively, ∇) be the full subcategory of all $\Delta(\lambda)$, $\lambda \in X$ (respectively, all $\nabla(\lambda)$, $\lambda \in X$). We call the modules in Δ the standard modules and the ones in ∇ the costandard modules.

The algebra A is said to be quasi-hereditary with respect to $(X, \leq)$ if for each $\lambda \in X$ we have

(i) $\text{End}_A(\Delta(\lambda)) \cong k$;

(ii) $P(\lambda) \in \mathcal{F}(\Delta)$, and moreover, $P(\lambda)$ has a Δ–filtration with quotient $\Delta(\mu)$ for $\mu \geq \lambda$ in which $\Delta(\lambda)$ occurs exactly once.

For a quasi-hereditary algebra A with respect to a poset X we call the elements in X weights and X the weight poset of A. By (A, X) we denote a quasi-hereditary algebra A with the weight poset X.

If a quasi-hereditary algebra has a duality δ which fixes simple modules, we call it a BGG-algebra (see [I], [CPS]).

1.2 Let A be a quasi-hereditary algebra with the weight poset X. Then we have the following properties

(1) The intersection $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ contains exactly $s(A)$ isomorphism classes of indecomposable modules, where $s(A)$ is the cardinality of X. They may be parametrized as $T(\lambda)$, $\lambda \in X$ such that the following holds. There are exact sequences

$$(a) \qquad 0 \longrightarrow \Delta(\lambda) \longrightarrow T(\lambda) \longrightarrow X(\lambda) \longrightarrow 0$$

$$(b) \qquad 0 \longrightarrow Y(\lambda) \longrightarrow T(\lambda) \longrightarrow \nabla(\lambda) \longrightarrow 0$$

where $X(\lambda)$ is filtered by $\Delta(\mu)'s$ for certain $\mu < \lambda$ and $Y(\lambda)$ by $\nabla(\mu)$'s for certain $\mu < \lambda$. In particular, $T(\lambda)$ has a unique composition factor isomorphic to $E(\lambda)$ and all other composition factors are of the form $E(\mu)$ with $\mu < \lambda$, where $E(x)$ denotes the simple A-module corresponding to the weight $x \in X$. The modules $T(\lambda)$ are called canonical modules.

(2) The module $T := \bigoplus_{\lambda \in X} T(\lambda)$ which is a tilting–cotilting module is called the characteristic module for (A, X) and $\mathcal{R}(A) = \text{End}_A(T)$ is called the Ringel dual of

A, it is also quasi-hereditary, with standard modules $\Delta_{\mathcal{R}(A)}(\lambda) = \mathrm{Hom}_A(T, \nabla(\lambda))$, where the weight poset of $\mathcal{R}(A)$ is X^{op}.

(3)

$$\mathrm{Ext}^n_A(\Delta(\lambda), \nabla(\mu)) = \begin{cases} k & \text{if } n = 0 \text{ and } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

For the proofs of the above facts one may see [R2].

1.3 A special class of quasi-hereditary algebras is constructed in [X]. Let us now recall the construction.

Let B be a finite-dimensional basic algebra over k. As usual, we say that B is given by a quiver $Q_B = (Q_0, Q_1)$ with relations $\{\rho_i | \, i \in I_B\}$, that is, we consider the algebra $kQ^*/ < \{\rho_i^* \mid i \in I_B\} >$, where Q^* is the opposite quiver of Q and the multiplication $\alpha\beta$ of two arrows α and β means that α comes first and then β follows (for the notation see [R1, 2] for details). For each α from i to j in Q_1, let α' be an arrow from j to i. We denote by Q_1' the set of all such α' with $\alpha \in Q_1$. For a path $\alpha_1 \cdots \alpha_m$ we denote by $(\alpha_1 \cdots \alpha_m)'$ the path $\alpha_m' \cdots \alpha_1'$ in (Q_0, Q_1'). With this notation we may define a BGG-algebra.

Definition. Suppose that B is an algebra given by the quiver $Q_B = (Q_0, Q_1)$ with relations $\{\rho_i | \, i \in I_B\}$. Let A be the algebra given by the quiver $(Q_0, Q_1 \cup Q_1')$ with relations $\{\rho_i | \, i \in I_B\} \cup \{\rho_i' | \, i \in I_B\} \cup \{\alpha\beta' | \, \alpha, \beta \in Q_1\}$. Then it is a finite-dimensional algebra over k.

If B has no oriented cycle in its quiver, we may assume that $Q_0 = \{1, \cdots, n\}$ such that $\mathrm{Hom}_B(P_B(i), P_B(j)) = 0$ for $i > j$, then A is quasi-hereditary [X,1.6]. Furthermore, the standard A-modules are $\Delta_A(i) = P_B(i)$ for $i \in \{1, \cdots, n\}$. We say that A is the dual extension of B, denoted by $\mathcal{A}(B)$.

If B is the incidence algebra of a poset X, then the algebra $\mathcal{A}(B)$ is just the twisted double incidence algebra of X with all labelling matrices 0 (see [DX]). Note that if X is a tree and all labelling matrices are non-zero, the Ringel duals are completely described in [DX].

The following lemma and its dual are taken from [DR].

1.4 Lemma. Let A be a quasi-hereditary algebra. Then the following are equivalent:

(i) The projective dimension of any standard module is at most 1.

(ii) The projective dimension of the characteristic module T is at most 1.

(iii) $\mathcal{F}(\nabla)$ is closed under factor modules.

1.4* Lemma. Let A be a quasi-hereditary algebra. Then the following are equivalent:

(i) The injective dimension of any costandard module is at most 1.

(ii) The injective dimension of the characteristic module T is at most 1.

(iii) $\mathcal{F}(\Delta)$ is closed under submodules.

2. The main results

Let X be a poset. From now on, we assume that X is a tree, that is, the Hasse diagram of X is a tree. We consider the dual extension $\mathcal{A}(\mathcal{I}(X))$ of the incidence algebra $\mathcal{I}(X)$. For the simplicity, we write $A = \mathcal{A}(\mathcal{I}(X))$. Note that each standard (respectively, costandard) A-module has projective (respectively, injective) dimension at most 1, and therefore, by 1.4 and 1.4*, $\mathcal{F}(\Delta)$ (respectively, $\mathcal{F}(\nabla)$) is closed under submodules (respectively, factor modules). For the canonical modules of A, we have the following

2.1 Lemma. For each $\lambda \in X$, there is an exact sequence

$$0 \longrightarrow \Delta(\lambda) \xrightarrow{\iota_\lambda} T(\lambda) \xrightarrow{p_\lambda} \oplus_{\nu<\lambda} T(\nu) \longrightarrow 0.$$

Moreover, there is an indecomposable submodule $\wedge(\lambda)$ of $T(\lambda)$ with the following exact sequences

$$0 \longrightarrow \wedge(\lambda) \longrightarrow T(\lambda) \longrightarrow \oplus_{\mu\lessdot\lambda} T(\mu) \longrightarrow 0,$$

$$0 \longrightarrow \oplus_{\mu\lessdot\lambda} T(\mu) \longrightarrow \wedge(\lambda) \longrightarrow E(\lambda) \longrightarrow 0,$$

where the symbol $\mu \lessdot \lambda$ signifies that $\mu < \lambda$ and that there is no $\gamma \in X$ satisfying $\mu < \gamma < \lambda$.

Dually, for each $\lambda \in X$, there is an exact sequence

$$0 \longrightarrow \oplus_{\nu<\lambda} T(\nu) \xrightarrow{l_\lambda} T(\lambda) \xrightarrow{\pi_\lambda} \nabla(\lambda) \longrightarrow 0,$$

and there is an indecomposable factor module $V(\lambda)$ of $T(\lambda)$ with the following exact sequences

$$0 \longrightarrow \oplus_{\mu\lessdot\lambda} T(\mu) \longrightarrow T(\lambda) \longrightarrow V(\lambda) \longrightarrow 0,$$

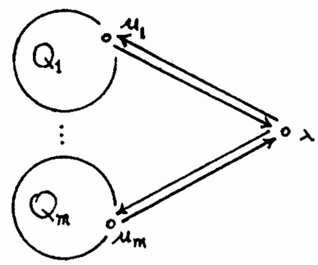
$$0 \longrightarrow E(\lambda) \longrightarrow V(\lambda) \longrightarrow \oplus_{\mu\lessdot\lambda} T(\mu) \longrightarrow 0.$$

Proof. We use induction on the number $|X|$ of elements in X to prove the lemma. If $|X| = 1$, there is nothing to prove. Suppose the statement is true for all trees X with $|X| \leq n-1$. Now let X be a tree with $|X| = n$ and $\lambda \in X$. If λ is not a maximal element in X then we choose a maximal element $s \in X$ such that $\lambda < s$. Set $X_1 = X \backslash \{s\}$ and denote by $A_1 = \mathcal{A}(\mathcal{I}(X_1))$ the dual extension algebra of the incidence algebra of the full subposet X_1 of X. Note that A_1 is isomorphic to the factor algebra of A by the ideal Ae_sA, where e_s denotes the idempotent of A corresponding to $s \in X$. Since $|X_1| = n-1$, by induction hypothesis, the lemma holds for $\lambda \in X_1$ in A_1-mod.

Since $T_A(\lambda) \in \mathcal{F}(\{\Delta_A(\gamma) | \gamma \neq s\}) = \mathcal{F}(\Delta_{A_1})$, the lemma is true for $\lambda \in X$ in A-mod. Hence we may assume that λ is a maximal element in X. Moreover, with a similar argument as above, we may assume that λ is the largest element in X and that the lemma holds for each $x < \lambda$.

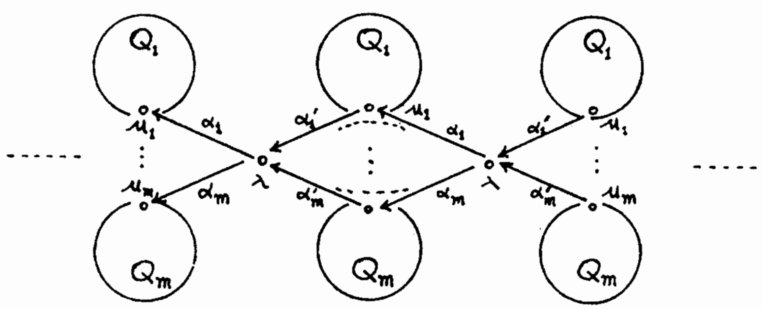
Let $\{\mu_1, ..., \mu_m\}$ be all elements x in X with $x \lessdot \lambda$. Then $X \backslash \{\lambda\}$ is the disjoint union of full subposets $X_i = \{x \in X | x \leq \mu_i\}$, $1 \leq i \leq m$.

By the definition of $A = \mathcal{A}(\mathcal{I}(X))$, the quiver Q of A has the form



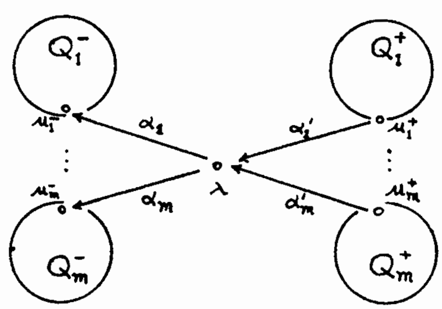
where Q_i denotes the quiver of the incidence algebra $A_i := \mathcal{A}(\mathcal{I}(X_i))$ for $1 \leq i \leq m$.

In order to use the covering technique, in the following we consider algebras as k-categories. Clearly, the algebra A admits a Galois covering $\hat{A}$ (see [G]) given by the following quiver $\hat{Q}$ and the relations $\hat{I}$, where $\hat{I}$ is induced by the relations I of A.



Note that the corresponding Galois group is the free group $\mathbb{Z}$. By $F : \hat{A} \to A$ we denotes the covering functor. From [BG], F induces a push-down functor F_λ : mod $\hat{A} \longrightarrow$ mod A which is exact. By Lemma 3.5 in [G], F_λ then preserves indecomposability.

By $\bar{A}$ we denote a full subcategory of $\hat{A}$ given by a subquiver $\bar{Q}$ of $\hat{Q}$ of the following form



where Q_i^- and Q_i^+ are identified with Q_i for all $1 \le i \le m$. We then denote by $\bar{F}$ the restriction of F_λ on $\bar{A}$-mod. Obviously, $\bar{F}$ is still exact and preserves indecomposability. Further, the corresponding algebra of $\bar{A}$, again denoted by $\bar{A}$, is quasi-hereditary with a weight poset $\bar{X} = \{x^-, x^+ | x \in X\backslash\{\lambda\}\} \cup \{\lambda\}$ whose order relation is such that 1) $x^- < y^-$, $x^+ < y^+$ if $x, y \in X\backslash\{\lambda\}$ and $x < y$ in X, 2) $x^- < \lambda$, $x^+ < \lambda$ for all $x \in X\backslash\{\lambda\}$. Moreover, there holds that

$$\bar{F}(\Delta_{\bar{A}}(x^{\mp})) = \Delta_A(x) \qquad \text{and} \qquad \bar{F}(\nabla_{\bar{A}}(x^{\mp})) = \nabla_A(x)$$

for $x \in X$, where $\lambda^- = \lambda^+ = \lambda$. Thus $\bar{F}$ induces a functor from $\mathcal{F}(\Delta_{\bar{A}})$ to $\mathcal{F}(\Delta_A)$ and a functor from $\mathcal{F}(\nabla_{\bar{A}})$ to $\mathcal{F}(\nabla_A)$. Because $\bar{F}$ preserves indecomposability, one gets that $\bar{F}(T_{\bar{A}}(x^{\mp})) = T_A(x)$ for all $x \in X$.

In the following we characterize the module $T_{\bar{A}}(\lambda)$. First, for each $\omega < \lambda$ in $\bar{X}$, the module $T_{\bar{A}}(\omega)$ can be considered as a module over $\mathcal{A}(\mathcal{I}(Y))$ for a poset Y with $|Y| < n$, then, by induction hypothesis, one obtains in $\bar{A}$-mod the following exact sequences

$$0 \longrightarrow \Delta_{\bar{A}}(\omega) \xrightarrow{\alpha_\omega} T_{\bar{A}}(\omega) \longrightarrow \oplus_{\delta<\omega} T_{\bar{A}}(\omega) \longrightarrow 0$$

and

$$0 \longrightarrow \oplus_{\delta<\omega} T_{\bar{A}}(\delta) \longrightarrow T_{\bar{A}}(\omega) \xrightarrow{\beta_\omega} \nabla_{\bar{A}}(\omega) \longrightarrow 0.$$

Further, one has the following exact sequence

$$0 \longrightarrow \oplus_{i=1}^m \Delta_{\bar{A}}(\mu_i^-) \longrightarrow \Delta_{\bar{A}}(\lambda) \longrightarrow E_{\bar{A}}(\lambda) \longrightarrow 0$$

and can form the pushout diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \oplus_{i=1}^m \Delta_{\bar{A}}(\mu_i^-) & \longrightarrow & \Delta_{\bar{A}}(\lambda) & \longrightarrow & E_{\bar{A}}(\lambda) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle f} & & \| & & \\
0 & \longrightarrow & \oplus_{i=1}^m T_{\bar{A}}(\mu_i^-) & \longrightarrow & \wedge_{\bar{A}}(\lambda) & \xrightarrow{g} & E_{\bar{A}}(\lambda) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & M_{\bar{A}}(\lambda) & = & M_{\bar{A}}(\lambda) & & & &
\end{array}$$

where $M_{\bar{A}}(\lambda) = \bigoplus_{i=1}^m \oplus_{\gamma<\mu_i^-} T_{\bar{A}}(\gamma)$ and $\alpha = \oplus_{i=1}^m \alpha_{\mu_i^-}$.

Clearly, the module $\wedge_{\bar{A}}(\lambda)$ is in $\mathcal{F}(\Delta_{\bar{A}})$.

Dually, we can form the following pullback diagram

$$\begin{array}{ccccccccc}
 & & & & N_{\bar{A}}(\lambda) & = & N_{\bar{A}}(\lambda) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_{\bar{A}}(\lambda) & \longrightarrow & V_{\bar{A}}(\lambda) & \longrightarrow & \oplus_{i=1}^m T_{\bar{A}}(\mu_i^+) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_{\bar{A}}(\lambda) & \longrightarrow & \nabla_{\bar{A}}(\lambda) & \longrightarrow & \oplus_{i=1}^m \nabla_{\bar{A}}(\mu_i^+) & \longrightarrow & 0
\end{array}$$

where $N_{\bar{A}}(\lambda) = \bigoplus_{i=1}^m \oplus_{\gamma<\mu_i^+} T_{\bar{A}}(\gamma)$.

Now if we apply $\mathrm{Hom}_A(\oplus_{i=1}^m T(\mu_i^+), -)$ to the exact sequence

$$0 \longrightarrow \oplus_{i=1}^m T(\mu_i^-) \longrightarrow \wedge_{\bar{A}}(\lambda) \xrightarrow{g} E_{\bar{A}}(\lambda) \longrightarrow 0$$

then we have the following exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^1_{\bar{A}}(\oplus_{i=1}^m T_{\bar{A}}(\mu_i^+), \oplus_{i=1}^m T_{\bar{A}}(\mu_i^-)) \longrightarrow \mathrm{Ext}^1_{\bar{A}}(\oplus_{i=1}^m T_{\bar{A}}(\mu_i^+), \wedge_{\bar{A}}(\lambda))$$
$$\longrightarrow \mathrm{Ext}^1_{\bar{A}}(\oplus_{i=1}^m T_{\bar{A}}(\mu_i^+), E_{\bar{A}}(\lambda)) \longrightarrow \mathrm{Ext}^2_{\bar{A}}(\oplus_{i=1}^m T_{\bar{A}}(\mu_i^+), \oplus_{i=1}^m T_{\bar{A}}(\mu_i^-)) \longrightarrow \cdots$$

It follows from $\mathrm{Ext}^j_{\bar{A}}(\oplus_{i=1}^m T_{\bar{A}}(\mu_i^+), \oplus_{i=1}^m T_{\bar{A}}(\mu_i^-)) = 0$ for $j \geq 1$ that

$$\mathrm{Ext}^1_A(\oplus_{j=1}^m T(\mu_j), \wedge_{\bar{A}}(\lambda)) \cong \mathrm{Ext}^1_A(\oplus_{j=1}^m T(\mu_j), E_{\bar{A}}(\lambda)).$$

Thus we get the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \wedge_{\bar{A}}(\lambda) & \longrightarrow & T_\lambda & \longrightarrow & \oplus_{i=1}^m T(\mu_i^+) & \longrightarrow & 0 \\
 & & \downarrow g & & \downarrow & & \| & & \\
0 & \longrightarrow & E_{\bar{A}}(\lambda) & \longrightarrow & V_{\bar{A}}(\lambda) & \longrightarrow & \oplus_{i=1}^m T(\mu_i^+) & \longrightarrow & 0.
\end{array}$$

¿From the above exact sequences, we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
 & & \Delta_{\bar{A}}(\lambda) & = & \Delta_{\bar{A}}(\lambda) & & & & \\
 & & \downarrow & & \downarrow \iota_\lambda & & & & \\
0 & \longrightarrow & \wedge_{\bar{A}}(\lambda) & \longrightarrow & T_\lambda & \longrightarrow & \oplus_{i=1}^m T_{\bar{A}}(\mu_i^+) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M_{\bar{A}}(\lambda) & \longrightarrow & \mathrm{Coker}(i_\lambda) & \longrightarrow & \oplus_{i=1}^m T_{\bar{A}}(\mu_i^+) & \longrightarrow & 0.
\end{array}$$

Since $\mathrm{Ext}^1_{\bar{A}}(\oplus_{i=1}^m T_{\bar{A}}(\mu_i^+), M_{\bar{A}}(\lambda)) = 0$, the lower exact sequence splits, and we have an exact sequence

$$0 \longrightarrow \Delta_{\bar{A}}(\lambda) \xrightarrow{\iota_\lambda} T_\lambda \xrightarrow{p_\lambda} M_{\bar{A}}(\lambda) \oplus (\bigoplus_{i=1}^m T_{\bar{A}}(\mu_i^+)) \longrightarrow 0.$$

This implies that T_λ lies in $\mathcal{F}(\Delta_{\bar{A}})$.

Dually, th re is an exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^m T(\mu_i^-) \oplus N_{\bar{A}}(\lambda) \xrightarrow{l_\lambda} T_\lambda \xrightarrow{\pi_\lambda} \nabla_{\bar{A}}(\lambda) \longrightarrow 0.$$

Thus T_λ is in $\mathcal{F}(\nabla_{\bar{A}})$. Therefore, T_λ lies in $\mathcal{F}(\Delta_{\bar{A}}) \cap \mathcal{F}(\nabla_{\bar{A}})$.

We now prove that T_λ is indecomposable. Take an arbitrary $\phi \in \mathrm{Hom}_{\bar{A}}(T_\lambda, T_\lambda)$ and denote by $\phi|_{\Delta_{\bar{A}}(\lambda)}$ the restriction of ϕ on $\Delta_{\bar{A}}(\lambda)$. If $\phi|_{\Delta_{\bar{A}}(\lambda)}$ is zero, then ϕ

factors through p_λ. Since $\mathrm{Hom}_{\bar{A}}(M_{\bar{A}}(\lambda)\bigoplus(\oplus_{i=1}^m T_{\bar{A}}(\mu_i^+)), \nabla_{\bar{A}}(\lambda)) = 0$, the map ϕ is the composition of the following maps

$$T_\lambda \xrightarrow{p_\lambda} M_{\bar{A}}(\lambda) \oplus (\textstyle\bigoplus_{i=1}^m T(\mu_i^+)) \xrightarrow{\bar{\phi}} \textstyle\bigoplus_{i=1}^m T(\mu_i^-) \oplus N_{\bar{A}}(\lambda) \xrightarrow{l_\lambda} T_\lambda$$

where $\bar{\phi}$ is induced by ϕ. The composition $l_\lambda p_\lambda$ is in the radical of $\mathrm{add}(T_{\bar{A}}) =: \mathcal{T}_{\bar{A}}$ since each indecomposable summand of $\bigoplus_{i=1}^m T_{\bar{A}}(\mu_i^-)\oplus N_{\bar{A}}(\lambda)$ and that of $M_{\bar{A}}(\lambda)\oplus(\bigoplus_{i=1}^m T_{\bar{A}}(\mu_i^+))$ are not isomorphic. Thus $\phi^2 = p_\lambda\bar{\phi}l_\lambda p_\lambda\bar{\phi}l_\lambda \in \mathrm{rad}_{\mathcal{T}_{\bar{A}}}(T_\lambda, T_\lambda)$ is nilpotent, i.e. ϕ is nilpotent. If $\phi|_{\Delta_{\bar{A}}(\lambda)}$ is not zero, since $\dim_k\mathrm{Hom}_{\bar{A}}(\Delta_{\bar{A}}(\lambda), T_\lambda) = 1$, there is an $0 \neq a \in k$ such that $\phi|_{\Delta_{\bar{A}}(\lambda)} = a \cdot \mathrm{id}_{T_\lambda}|_{\Delta_{\bar{A}}(\lambda)}$, where $\mathrm{id}_{T_\lambda}|_{\Delta_{\bar{A}}(\lambda)}$ denotes the restriction of the identity map id_{T_λ} of T_λ on $\Delta_{\bar{A}}(\lambda)$. Set $\psi = a\cdot\mathrm{id}_{T_\lambda} - \phi$, then $\psi|_{\Delta_{\bar{A}}(\lambda)} = 0$. From the above discussion, ψ is then nilpotent, that is, ϕ is an isomorphism. As a result, T_λ is indecomposable. From $T_\lambda \in \mathcal{F}(\Delta_{\bar{A}}) \cap \mathcal{F}(\nabla_{\bar{A}})$ it follows that $T_\lambda = T_{\bar{A}}(\lambda)$.

By applying the functor $\bar{F}$ to the module $T_{\bar{A}}(\lambda)$, one gets the wanted properties of $T_A(\lambda)$ in the lemma. This finishes the proof.

2.2 The following lemma, proved by Xi for an arbitrary quasi-hereditary algebra A with a weight poset X since a long time, provides us almost all information on the quiver of the Ringel dual in our discussed case.

Lemma. Let A be an arbitrary quasi-hereditary algebras with a weight poset X. Suppose λ is a maximal element in X. Then

1) The map p_λ in the exact sequence

$$0 \longrightarrow \Delta(\lambda) \xrightarrow{i_\lambda} T(\lambda) \xrightarrow{p_\lambda} X(\lambda) \longrightarrow 0$$

is a relative source map for $T(\lambda)$ in $\mathcal{F}(\Delta)$ if $X(\lambda) \neq 0$.

2) Dually, the map l_λ in the exact sequence

$$0 \longrightarrow Y(\lambda) \xrightarrow{l_\lambda} T(\lambda) \xrightarrow{\pi_\lambda} \nabla(\lambda) \longrightarrow 0$$

is a relative sink map for $T(\lambda)$ in $\mathcal{F}(\nabla)$ if $Y(\lambda) \neq 0$.

Proof. We prove only the statement 1). The statement 2) is the dual of the statement 1).

(a) p_λ is not a split monomorphism since $\Delta(\lambda) \neq 0$.

(b) Suppose that α is in $\mathrm{End}_A(X(\lambda))$ with $p_\lambda = p_\lambda\alpha$. It follows from the surjective map p_λ that α is surjective, and therefore α is an automorphism.

(c) Let $f : T(\lambda) \longrightarrow Y$ be a non-split monomorphism in $\mathcal{F}(\Delta)$. We show that there is a homomorphism $h : X(\lambda) \longrightarrow Y$ such that $p_\lambda h = f$. In fact, if $i_\lambda f = 0$ then there exists a homomorphism $h : X(\lambda) \longrightarrow Y$ with $p_\lambda h = f$ as desired. Suppose now $i_\lambda f \neq 0$. Then $i_\lambda f$ is injective since each non-zero homomorphism from $\Delta(\lambda)$ to a module $M \in \mathcal{F}(\Delta)$ is injective. Let U be the largest submodule of

Y such that $U \in \mathrm{add}\Delta(\lambda)$ and $Y/U \in \mathcal{F}(\{\Delta(\mu) \mid \mu \in X - \{\lambda\}\})$.Then the image of $i_\lambda f$ lies in U. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta(\lambda) & \xrightarrow{i_\lambda f} & Y & \longrightarrow & \mathrm{Coker}(i_\lambda f) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i_\lambda f} & & \| & & \downarrow & & \\ 0 & \longrightarrow & U & \longrightarrow & Y & \longrightarrow & Y/U & \longrightarrow & 0 \end{array}$$

Since $\mathrm{add}(\Delta(\lambda))$ is an abelian category, $U/\mathrm{Im}(i_\lambda f)$ belongs to $\mathrm{add}(\Delta(\lambda))$. Hence it follows from the above diagram that $Y/\mathrm{Im}(i_\lambda f)$ lies in $\mathcal{F}(\Delta)$:

Consider now the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta(\lambda) & \longrightarrow & T(\lambda) & \longrightarrow & X(\lambda) & \longrightarrow & 0 \\ & & \| & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & \Delta(\lambda) & \longrightarrow & Y & \xrightarrow{\pi} & Y/\mathrm{Im}(i_\lambda f) & \longrightarrow & 0 \end{array}$$

(Note that h exists). This yields the exact sequence

$$0 \longrightarrow T(\lambda) \xrightarrow{(f,p_\lambda)} Y \oplus X(\lambda) \xrightarrow{(\pi,-h)^t} Y/\mathrm{Im}(i_\lambda f) \longrightarrow 0$$

Since $\mathrm{Ext}^1_A(\mathcal{F}(\Delta), T(\lambda)) = 0$, this exact sequence splits and then $T(\lambda)$ is a direct summand of Y (the module $X(\lambda)$ has no composition factor of the form $E(\lambda)$). Thus f is split mono, which contradicts to our hypothesis and finishes the proof.

We now return to the dual extension algebra $A = \mathcal{A}(\mathcal{I}(X))$ of the incidence algebra of a tree X. With the help of Lemmas 2.1 and 2.2, we now can prove our main result of this paper,which describes the quiver of the Ringel dual $\mathcal{R}(A)$ of A.

2.3 Theorem. Let X be a tree. Then for every $x,\ y \in X$, there holds that

$$\dim_k \mathrm{irr}_{\mathcal{T}}(T(x), T(y)) = \begin{cases} 1, & \text{if } x < y \text{ and } y \text{ is maximal, or} \\ & \text{if } x > y \text{ and } x \text{ is maximal,} \\ 0, & \text{otherwise.} \end{cases}$$

where $\mathcal{T} = \mathrm{add}(T)$ and $\mathrm{irr}_{\mathcal{T}}(T(x), T(y)) := \mathrm{rad}_{\mathcal{T}}(T(x), T(y))/\mathrm{rad}^2_{\mathcal{T}}(T(x), T(y))$ is the bimodule of irreducible maps from $T(x)$ to $T(y)$ in $\mathcal{T}$.

Proof. By Lemma 2.2, it remains to prove that $\mathrm{irr}_{\mathcal{T}}(T(x), T(y)) = 0$ if neither x nor y is maximal. Since the Ringel dual does not contain a loop, we have to examine the following different cases.

i) **Case** $x \not\asymp y$ (i.e. x and y are not comparable). Then every map $f \in \mathrm{Hom}_A(T(x), T(y))$ factors through p_x since $T(y)$ has no composition factor $E(x)$ and $f|_{\Delta(x)} = 0$.

ii) **Case** $x < y$. Since y is not maximal, there is a $z \in X$ with $y \lessdot z \in X$. By Lemma 2.1, there is a submodule $\wedge(z)$ of $T(z)$ with the following exact sequences

$$0 \longrightarrow \wedge(z) \longrightarrow T(z) \xrightarrow{\alpha} \oplus_{u \lessdot z} T(u) \longrightarrow 0,$$

$$0 \longrightarrow \oplus_{u<z} T(u) \longrightarrow \wedge(z) \longrightarrow E(z) \longrightarrow 0.$$

Applying $\mathrm{Hom}_A(T(x), -)$ to the above exact sequences, one has the exact sequences

$$0 \longrightarrow \mathrm{Hom}_A(T(x), \wedge(z)) \longrightarrow \mathrm{Hom}_A(T(x), T(z)) \longrightarrow \mathrm{Hom}_A(T(x), \oplus_{u<z} T(u)) \\ \longrightarrow \mathrm{Ext}^1_A(T(x), \wedge(z))$$

and

$$\mathrm{Ext}^1_A(T(x), \oplus_{u<z} T(u)) \longrightarrow \mathrm{Ext}^1_A(T(x), \wedge(z)) \longrightarrow \mathrm{Ext}^1_A(T(x), E(z)).$$

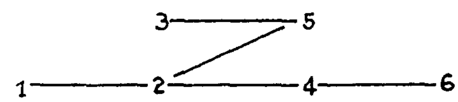
From $\mathrm{Ext}^1_A(T(x), \oplus_{u<z} T(u)) = 0$ and $\mathrm{Ext}^1_A(T(x), E(z)) = 0$ it then follows that $\mathrm{Ext}^1_A(T(x), \wedge(z)) = 0$. Therefore, each morphism $f \in \mathrm{Hom}_A(T(x), \oplus_{u<z} T(u))$ factors through α, that is, $\mathrm{irr}_{\mathcal{T}}(T(x), \oplus_{u<z} T(u)) = 0$. Since $y < z$, $\mathrm{irr}_{\mathcal{T}}(T(x), T(y))$ is a summand of $\mathrm{irr}_{\mathcal{T}}(T(x), \oplus_{u<z} T(u))$, so $\mathrm{irr}_{\mathcal{T}}(T(x), T(y)) = 0$.

iii) **Case** $x > y$. This is dual of ii).

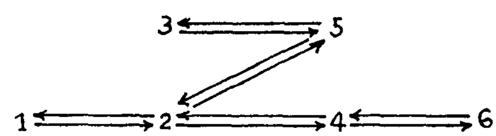
This finishes the proof of Theorem 2.3.

It follows from 2.3 that the quiver of $\mathcal{R}(A)$ is bipartite, that is, the vertex set is a disjoint union of $Q_0' = \{\lambda | \lambda \in X \text{ is maximal}\}$ and $Q_0'' = X \backslash Q_0'$, and the arrows are between Q_0' and Q_0''.

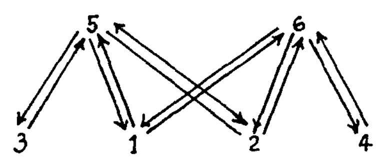
2.4 Example. Let X be a tree with the following Hasse diagram



Then $A = \mathcal{A}(\mathcal{I}(X))$ is given by the following quiver



with relations all $ab' = 0$. By 2.3, the quiver of the Ringel dual $\mathcal{R}(A)$ of A has the following form



Now we develop some properties which are helpful to calculate dimensions of Ringel duals. Let X be a tree. Then for every $x \leq y$ in X, there is a unique chain

$x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{l(x,y)} = y$ from x to y in X. We call $l(x,y)$ the length of this chain.

2.5 Proposition. For every $x < y$, there holds that

$$\dim_k \mathrm{Hom}_A(\Delta(x), T(y)) = \dim_k \mathrm{Hom}_A(T(y), \nabla(x)) = 2^{l(x,y)-1}$$

where $l(x,y)$ is the length of the unique maximal chain form x to y.

Proof. First note that $\dim_k \mathrm{Hom}_A(\Delta(x), T(x)) = 1$ for each $x \in X$. For $x < y$, let $x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_l = y$ be the unique maximal chain from x to y in X, where $l = l(x,y)$.

From Lemma 2.1, there is an exact sequence

$$0 \longrightarrow \oplus_{u \lessdot y} T(u) \longrightarrow T(y) \longrightarrow \nabla(y) \longrightarrow 0.$$

By applying $\mathrm{Hom}_A(\Delta(x), -)$ to the above exact sequence, one gets the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(\Delta(x), \oplus_{u \lessdot y} T(u)) \longrightarrow \mathrm{Hom}_A(\Delta(x), T(y)) \longrightarrow \mathrm{Hom}_A(\Delta(x), \nabla(y)) = 0.$$

Therefore,

$$\begin{aligned} \dim_k \mathrm{Hom}_A(\Delta(x), T(y)) &= \dim_k \mathrm{Hom}_A(\Delta(x), \oplus_{u \lessdot y} T(u)) \\ &= \sum_{i=0}^{l-1} \dim_k \mathrm{Hom}_A(\Delta(x), T(x_i)). \end{aligned}$$

If $l = 1$, then $\dim_k \mathrm{Hom}_A(\Delta(x), T(y)) = \dim_k \mathrm{Hom}_A(\Delta(x), T(x)) = 1$. If $l \geq 2$, then by induction, one obtains the formula.

2.6 Proposition. (1) For each $x \in X$, there holds

$$\dim_k \mathrm{Hom}_A(T(x), T(x)) = 1 + \sum_{z < x} 2^{2l(z,x)-2}.$$

(2) For every $x < y$ in X, there holds

$$\dim_k \mathrm{Hom}_A(T(x), T(y)) = 2^{l(x,y)-1}(1 + \sum_{z < x} 2^{2l(z,x)-1}).$$

Proof. (1) Apply $\mathrm{Hom}_A(-, T(x))$ to the exact sequence

$$0 \longrightarrow \Delta(x) \longrightarrow T(x) \longrightarrow \oplus_{z \lessdot x} T(z) \longrightarrow 0$$

we obtain the following exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_A(\oplus_{z \lessdot x} T(z), T(x)) \longrightarrow \mathrm{Hom}_A(T(x), T(x)) \longrightarrow \mathrm{Hom}_A(\Delta(x), T(x)) \\ &\longrightarrow \mathrm{Ext}^1_A(\oplus_{z \lessdot x} T(z), T(x)) = 0. \end{aligned}$$

Then one has that

$$\begin{aligned}\dim_k\mathrm{Hom}_A(T(x),T(x)) &= \dim_k\mathrm{Hom}_A(\Delta(x),T(x)) + \dim_k\mathrm{Hom}_A(\oplus_{z<x}T(z),T(x)) \\ &= 1+\sum_{z<x}\dim_k\mathrm{Hom}_A(T(z),T(x)).\end{aligned}$$

For each z with $l(z,x)=1$, by applying $\mathrm{Hom}_A(-,T(x))$ to the sequence

$$0 \longrightarrow \Delta(z) \longrightarrow T(z) \longrightarrow \oplus_{u<z}T(u) \longrightarrow 0$$

one gets that

$$\dim_k\mathrm{Hom}_A(T(z),T(x)) = 1+\sum_{u<z}\dim_k\mathrm{Hom}_A(T(u),T(x)).$$

Hence

$$\dim_k\mathrm{Hom}_A(T(x),T(x)) = 1+\sum_{z<x,l(z,x)=1} 1 + 2\sum_{\substack{z<x\\ l(z,x)\geq 2}}\dim_k\mathrm{Hom}_A(T(z),T(x)).$$

In general, for each $i>0$, one can show that

$$\begin{aligned}\sum_{\substack{z<x\\ l(z,x)\geq i}}\dim_k\mathrm{Hom}_A(T(z),T(x)) = &\sum_{\substack{z<x\\ l(z,x)=i}} 2^{i-1} \\ &+2\sum_{\substack{z<x\\ l(z,x)\geq i+1}}\dim_k\mathrm{Hom}_A(T(z),T(x)).\end{aligned}$$

Inductively, one thus obtains that

$$\begin{aligned}\dim_k\mathrm{Hom}_A(T(x),T(x)) =&1+\sum_{\substack{z<x\\ l(z,x)=1}} 1 + 2\sum_{z<x,l(z,x)=2} 2 \\ &+2^2\sum_{\substack{z<x\\ l(z,x)\geq 3}}\dim_k\mathrm{Hom}_A(T(z),T(x)) \\ =&1+\sum_{j=1}^{i}2^{j-1}\sum_{\substack{z<x\\ l(z,x)=j}}2^{j-1} \\ &+2^i\sum_{\substack{z<x\\ l(z,x)\geq i+1}}\dim_k\mathrm{Hom}_A(T(z),T(x)) \\ =&\cdots\cdots = 1+\sum_{z<x}2^{2l(z,x)-2}.\end{aligned}$$

(2) Suppose $x<y$ in X. Using Proposition 2.5 and a similar argument as in (1), we have that

$$
\begin{aligned}
\dim_k\mathrm{Hom}_A(T(x),T(y)) &= \dim_k\mathrm{Hom}_A(\Delta(x),T(y)) + \dim_k\mathrm{Hom}_A(\oplus_{z<x}T(z),T(y))\\
&=2^{l(x,y)-1} + \sum_{\substack{z<x\\ l(z,x)=1}} \dim_k\mathrm{Hom}_A(T(z),T(y))\\
&\quad + \sum_{\substack{z<x\\ l(z,x)\geq 2}} \dim_k\mathrm{Hom}_A(T(z),T(y))\\
&=2^{l(x,y)-1} + \sum_{\substack{z<x\\ l(z,x)=1}} (2^{l(z,y)-1} + \sum_{u<z}\dim_k\mathrm{Hom}_A(T(u),T(y)))\\
&\quad + \sum_{\substack{z<x\\ l(z,x)\geq 2}} \dim_k\mathrm{Hom}_A(T(z),T(y))\\
&=2^{l(x,y)-1} + \sum_{\substack{z<x\\ l(z,x)=1}} 2^{l(z,y)-1} + 2\sum_{\substack{z<x\\ l(z,x)\geq 2}} \dim_k\mathrm{Hom}_A(T(z),T(y))\\
&=2^{l(x,y)-1} + \sum_{\substack{z<x\\ l(z,x)=1}} 2^{l(z,y)-1} + 2\sum_{\substack{z<x\\ l(z,x)=2}} 2^{l(z,y)-1}\\
&\quad + 2^2\sum_{\substack{z<x\\ l(z,x)\geq 3}} \dim_k\mathrm{Hom}_A(T(z),T(y))\\
&=\cdots\cdots = 2^{l(x,y)-1} + 2^{l(z,x)-1}\sum_{z<x} 2^{l(z,y)-1}\\
&=2^{l(x,y)-1}(1+\sum_{z<x} 2^{2l(z,x)-1}).
\end{aligned}
$$

2.7 Remark. (1) If X is given by the linearly ordered set $\{1 < 2 < \cdots < n\}$. Then the dimension of $A = \mathcal{A}(\mathcal{I}(X))$ over k is

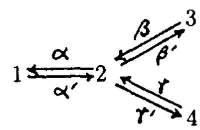
$$\sum_{i=1}^{n}(\dim_k\Delta(i))^2 = \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1).$$

By 2.5, one gets that $\dim_k\Delta_{\mathcal{R}(A)}(i) = \dim_k\mathrm{Hom}_A(T,\nabla(i)) = 2^{n-i}$. Note that with A also $\mathcal{R}(A)$ is a BGG–algebra. Then the dimension of $\mathcal{R}(A)$ is

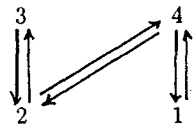
$$\dim_k\mathcal{R}(A) = \sum_{i=1}^{n}(\dim_k\mathrm{Hom}_A(T,\nabla(i)))^2 = \sum_{i=1}^{n}\left(2^{n-i}\right)^2 = \frac{1}{3}(4^n-1).$$

For example, if $n = 6$, then $\dim_k A = 91$ and $\dim_k\mathcal{R}(A) = 1365$. Therefore, in general, Ringel duals may be very complicated.

(2) If the labelling matrices are not all zero (see [DX]), then Theorem 2.3 may be false. For example, let $X = \{1,2,3,4\}$ be a poset with the order relation $1 < 2 < 3$ and $2 < 4$, and with the labelling matrices $M(1,3) = 1$ and $M(1,4) = 0$, that is, we consider the algebra A given by the quiver



with relations $\alpha\alpha' = \beta'\beta$, $\beta\beta' = 0 = \gamma\gamma'$ and $\beta\gamma' = 0 = \gamma\beta'$. Then the quiver of $\mathcal{R}(A)$ has the following form



which admits no arrows between vertices 1 and 3.

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