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Cellular algebras and Cartan matrices

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Abstract

Let A be a finite dimensional algebra over a field k, and let C be the Cartan matrix of A. Usually, the eigenvalues of C being integers do not imply the semisimplicity of A. However, we prove that a cellular algebra A is semisimple if and only if $\det(C) = 1$ and all eigenvalues of C are integers. Moreover, we use Cartan matrices to classify the cellular algebras with the property that the determinant of the Cartan matrix equals a given prime p and all eigenvalues are integers. We also give a classification of cellular Nakayama algebras with integral eigenvalues of their Cartan matrices. Finally, we show that if A is a cellular algebra then its trivial extension $T(A)$ is also a cellular algebra. In particular, if a non-simple connected cellular algebra A is quasi-hereditary, then the Cartan matrix of $T(A)$ has at least one non-integral eigenvalue. The main tool used in this paper is the well-known Perron–Frobenius theory on non-negative matrices.

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1. Introduction

Semisimple algebras play an important role in many branches of mathematics and physics: the tower construction of semisimple algebras by Jones was used to study the index of subfactors and the polynomial invariant for knots (see [5,8]), generically semisimple Birman–Wenzl algebras were applied in the construction of quantum invariant, modular categories and topological quantum field theory [18,19],

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and semisimple Temperley–Lieb algebras had a significant application in the statistic mechanics [16,17]. Recently, a new class of algebras which are called cellular algebras was introduced in [9]. It includes lots of well-known algebras such as Hecke algebras of types A and B, Brauer algebras [9], Birman–Wenzl algebras of type A [21], partition algebras [22] and Temperley–Lieb algebras [20]. In general, cellular algebras are not always semisimple. So a natural question is: when is a cellular algebra semisimple? In [9] there is a local answer by verification of bilinear forms defined on standard modules. In this paper we provide a global criterion for a cellular algebra to be semisimple. Our result is the following theorem:

Theorem 1.1. *Let k be a field and A a cellular k-algebra* (*with respect to an involution i*). *Then the algebra A is semisimple if and only if all eigenvalues of the Cartan matrix of A are rational numbers and the Cartan determinant is* 1.

One should note that outside the class of cellular algebras this criterion may be false, even for the quasi-hereditary algebra of two-by-two triangular matrices over a field, the criterion cannot be used.

For cellular algebras, the Cartan determinant of value 1 implies the quasi-heredity [12]. From this point of view, the Cartan determinant measures how far a cellular algebra is from being quasi-hereditary. The above result suggests the following problems:

Problem 1. Given a prime p, classify cellular algebras with the properties that the Cartan determinant equals p and that all eigenvalues of the Cartan matrix are integers.

More generally, we have the following question:

Problem 2. Classify all cellular algebras with the property that all eigenvalues of the Cartan matrix are positive integers.

Of course, we cannot completely answer Problem 2. But for Problem 1, we reduce it to an inverse eigenvalue problem and a problem in elementary number theory. Our result is the following theorem.

Theorem 1.2. *Let $S(n)$ denote the set of partitions λ of n such that the product of any two parts of λ is a square. Let $M(n)$ be the set of all symmetric matrices C* (*modulo congruence*) *over the natural numbers such that the spectrum of C is $\{\mu_1 = n+1, \mu_2 = \cdots = \mu_m = 1\}$. Then the cardinalities of $S(n)$ and $M(n)$ are the same. In particular, if $s_1(n)$ stands for the number of partitions in $S(n)$ whose parts are coprime, then the number of the Cartan matrices* (*up to congruence*) *of indecomposable cellular algebras with the properties in Problem* 1 *is $\sum_{d|p-1} s_1((p-1)/d)$.*

To the Problem 2, first we have the following result.

Theorem 1.3. *Let A be an indecomposable basic Nakayama algebra. Then A is a cellular algebra such that all eigenvalues of its Cartan matrix are integers if and only if it is self-injective with at most two simple modules.*

The next result provides a more general way to construct cellular algebras with integral eigenvalues of Cartan matrices.

Theorem 1.4. *Let A be a indecomposable cellular algebra over a field k (with respect to an involution i). Then there is an involution on the trivial extension $T(A) := A \oplus DA$, which is an extension of i, such that $T(A)$ is a cellular algebra with respect to this involution. In particular, if A is a cellular algebra with integral eigenvalues of its Cartan matrix, then so is $T(A)$. Conversely, if all eigenvalues of the Cartan matrix of $T(A)$ are integers, then A is either a simple algebra or has infinite global dimension.*

The paper is organized as follows: after we recall some elementary facts on cellular algebras in Section 2, we prove Theorem 1.1 in Section 3. In Section 4 we give a full answer to Problem 1. In Section 5 we list all cellular algebras for the cases $2 \leqslant p \leqslant 7$, namely, we have a full list of cellular algebras with desired property. Of course, one could expect that the algebras in the list should not far away from semisimple algebras, however, all of them must be of infinite global dimension. In Section 6, we give a class of cellular algebras with properties in Problem 2, namely, we classify cellular Nakayama algebras and show that a large class of them has the properties mentioned in Problem 2. The last section is devoted to the proof of Theorem 1.4.

The method in the paper is a combination of the use of the theory of cellular algebras and the Perron–Frobenius theory on non-negative matrices.

2. Background

First we recall the two equivalent definitions of cellular algebras and the definition of quasi-hereditary algebras. Then we collect several facts on linear algebra and prove the criterion.

For simplicity we stick to the ground ring being an (arbitrary) field k. By algebra we always mean a finite dimensional associative algebra with unit.

Definition 2.1 [9]. An associative k-algebra A is called a *cellular algebra* with cell datum (I, M, C, i) if the following conditions are satisfied:

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has a k-basis $C^{\lambda}_{S,T}$, where (S, T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is a k-linear anti-automorphism of A with $i^2 = id$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^{\lambda}_{S,T}$ can be written as $(\sum_{U \in M(\lambda)} r_a(U, S)C^{\lambda}_{U,T}) + r'$ where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $r_a(U, S) \in k$ do not depend on T.

In the following we shall call a k-linear anti-automorphism i of A with $i^2 = id$ an *involution* of A. In [11] it has been shown that this definition is equivalent to the following one.

Definition 2.2. Let A be a k-algebra. Assume there is an antiautomorphism i on A with $i^2 = id$. A two-sided ideal J in A is called a *cell ideal* if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that Δ has finite k-dimension and that there is an isomorphism of A-bimodules $\alpha : J \simeq \Delta \otimes_k i(\Delta)$ (where $i(\Delta) \subset J$ is the i-image of Δ) making the following diagram commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_R i(\Delta) \\ i\Big\downarrow & & \Big\downarrow x \otimes y \mapsto i(y) \otimes i(x) \\ J & \xrightarrow{\alpha} & \Delta \otimes_R i(\Delta) \end{array}$$

The algebra A (with the involution i) is called *cellular* if and only if there is a vector space decomposition $A = J'_1 \oplus J'_2 \oplus \cdots \oplus J'_n$ (for some n) with $i(J'_j) = J'_j$ for each j and such that setting $J_j = \bigoplus_{l=1}^{j} J'_l$ gives a chain of two sided ideals of A: $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ (each of them fixed by i) and for each j ($j = 1, \ldots, n$) the quotient $J'_j = J_j/J_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1}.

The modules $\Delta(j)$, $1 \leqslant j \leqslant n$, are called *standard modules* of the cellular algebra A, and the above chain in A is called a *cell chain*. (Standard modules are called cell modules in [9].)

Let us also recall the definition of quasi-hereditary algebras introduced in [7].

Definition 2.3 [7]. Let A be a k-algebra. An ideal J in A is called a *heredity ideal* if J is idempotent, $J(\mathrm{rad}(A))J = 0$ and J is a projective left (or, right) A-module. The algebra A is called *quasi-hereditary* provided there is a finite chain $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ of ideals in A such that J_j/J_{j-1} is a heredity ideal in A/J_{j-1} for all j. Such a chain is then called a heredity chain of the quasi-hereditary algebra A.

We also need the notion of a *Cartan matrix* in the following abstract sense (which coincides with the one used in group theory if A is the group algebra of a finite group over a splitting field). Denote the simple A-modules by $L(1), \ldots, L(m)$ and their projective covers by $P(1), \ldots, P(m)$. The entries $c_{j,h}$ of the Cartan matrix

$C(A)$ are the composition multiplicities $[P(j) : L(h)]$. The determinant of $C(A)$ is called the *Cartan determinant*. In general this can be any integer.

Now we collect a number of auxiliary statements for later use.

Lemma 2.4. *Let A be a cellular algebra with involution i and cell chain $0 = J_0 \subset J_1 \subset \cdots \subset J_n = A$. Then*:

(1) *There is a natural bijection between isomorphism classes of simple A-modules and indices $l \in \{1, \ldots, n\}$ such that $J_l^2 \not\subset J_{l-1}$. The inverse of this bijection is given by sending such an l to $\Delta(l)/\mathrm{rad}(\Delta(l))$ (which in this case is simple).*
In the following we index the simple modules in this way by a subset of the set $\{1, \ldots, n\}$.

(2) *If l is the index of a simple module $L(l)$ as in (1), then the composition factors $L(j)$ of the standard module $\Delta(l)$ satisfy $j \geqslant l$ and $j = l$ occurs with multiplicity one (and this factor is the unique simple quotient $\Delta(l)/\mathrm{rad}(\Delta(l))$).*

(3) *Let d_{ij} denote the multiplicity of the simple module L_j in the standard module $\Delta(i)$ and let $D = (d_{ij})$. Then the Cartan matrix C of A is $D^{\mathrm{tr}}D$, where D^{tr} stands for the transpose matrix of D. If the Cartan determinant is 1, then D is an $n \times n$ square matrix, where n is the number of isomorphism classes of simple modules. Moreover, the Cartan matrix C of A is a positive definite matrix.*

Proof. (1) is implicit in [9, Theorem 3.4] and another proof is given in [11, Proposition 4.1]. (2) is the Proposition 3.6 in [9] and it also follows from Proposition 4.1 of [11]. (3) If $\det(C) = 1$, then D is an invertible square matrix by (2) and [12]. Hence C is a positive definite matrix. □

We also need the following elementary facts on symmetric matrices with real entries. The identity matrix is denoted by I.

Proposition 2.5. *Let C be an $n \times n$ positive definite real matrix. Then*

(1) *Any principal submatrix is positive definite.*
(2) *All eigenvalues of C are positive real numbers.*
(3) *Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of a positive definite matrix C and let $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_k$ be the eigenvalues of a principle submatrix of C of order k. Then $\lambda_m \geqslant \mu_m \geqslant \lambda_{n+m-k}$ for $m = 1, 2, \ldots, k$.*
(4) *If C is a symmetric matrix with non-negative integral entries such that all eigenvalues of C are $a, 1, \ldots, 1$, where a is a positive integer, then the eigenvalues of any principal submatrix of C are integers.*

Proof. (1) and (2) are well known from linear algebra. As to (3), one may see the book [15, Theorem 8.4.1, p. 294].

(4) follows from (3). □

3. The criterion

In this section we prove the criterion for a cellular algebra to be semisimple.

Theorem 3.1. *Let k be a field and A a cellular k-algebra* (*with respect to an involution i*). *Let C denote the Cartan matrix of A. Then the following are equivalent*:

(1) *The algebra A is semisimple.*
(2) *All eigenvalues of C are integers and* $\det(C) = 1$.
(3) *The Cartan matrix C is the identity matrix.*
(4) *All standard modules are simple and pairwise non-isomorphic.*

Before we start with the proof of Theorem 3.1, let us first prove the following lemma in the linear algebra.

Lemma 3.2. *Let C be a positive definite* (*symmetric*) *matrix with non-negative integers as its entries. If* $\det(C) = 1$ *and all eigenvalues of C are integers*, *then C is in fact the identity matrix.*

Proof. Since C is a positive definite matrix with non-negative entries and $\det(C) = 1$, we know that all eigenvalues of C are positive integers and their product is 1. This means that all eigenvalues λ_i of C are equal to 1. Let $C = (c_{ij})$ be of order n. Then $\sum_i^n c_{ii} = \sum_i \lambda_i = n$. Hence $c_{ii} = 1$ for all i. Since C is positive definite, every principal submatrix is positive definite, too. In particular, for any pair i and j, the principal submatrix

$$\begin{pmatrix} 1 & c_{ij} \\ c_{ji} & 1 \end{pmatrix}$$

is positive definite. This yields that $1 - c_{ij}^2 > 0$ and $c_{ij} = 0$. Thus the Cartan matrix C of A is an identity matrix. □

Proof of Theorem 3.1. It is obvious that (1) implies (2). The equivalence of (1) and (4) is proved in [9]. Clearly, (1) and (3) are equivalent. The implication from (2) to (1) follows now from Lemma 3.2 immediately. □

As a consequence, we have the following corollary.

Corollary 3.3. *Let A be a quasi-hereditary algebra with a duality which fixes all simple modules. Then A is semisimple if and only if all eigenvalues of the Cartan matrix C of A are integers.*

Proof. Under the assumption the Cartan matrix C of A is positive definite. By [6], $\det(C) = 1$. Thus the corollary follows immediately from Lemma 3.2. □

Usually, the tensor product of two semisimple algebras may not be semisimple, but for cellular algebras we have the following corollary.

Corollary 3.4. *Let A and B be two cellular algebras over a perfect field k. Then $A \otimes_k B$ is semisimple if and only if A and B are semisimple.*

Proof. Suppose A and B are two algebras over k. Let us denote by C_A the Cartan matrix of A. Then $C_{A\otimes B} = C_A \otimes C_B$ and $\det(C_{A\otimes B}) = \det(C_A)^n \det(C_B)^m$. Note that the eigenvalues of $C_{A\otimes B}$ are of the form $\lambda\mu$, where λ and μ are eigenvalues of C_A and C_B respectively. Now the corollary follows from Theorem 3.1 immediately since $A \otimes_k B$ is a cellular algebra. $\square$

As another consequence, we have the following corollary which tells us how to know whether a cellular algebra is simple.

Corollary 3.5. *Let A be an indecomposable cellular algebra over a field k. Suppose all eigenvalues of its Cartan matrix are integers. If* 1 *appears in the main diagonal of C, then A is a simple algebra.*

Proof. Since A is indecomposable as an algebra, the Cartan matrix is irreducible. By the theory of non-negative matrices (see, for example [4]), there is a positive real vector x such that $Cx = \rho(C)x$, where $\rho(C)$ is the spectral radius of C. If $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m > 0$ are the eigenvalues of C, then there is a unitary matrix $U = (u_{ij})$ over the real numbers such that $C = I_m + U \cdot \mathrm{diag}(\lambda_1 - 1, \ldots, \lambda_m - 1) \cdot U^{\mathrm{tr}}$. Since the first column u of U is the eigenvector corresponding to $\lambda_1 = \rho(C)$, we know that u is a scalar multiple of x, thus each component of u is not zero. Now it follows from $c_{ii} = 1 + \sum_j (\lambda_j - 1)u_{ij}^2 = 1$ that $\sum_j (\lambda_j - 1) u_{ij}^2 = 0$. This implies that $\lambda_j = 1$ for all j. Then, by Theorem 3.1, A is semisimple. But we know that A is indecomposable, thus A must be a simple algebra. $\square$

Finally, let us remark that the two conditions in (2) of Theorem 3.1 are necessary. First, there is a cellular algebra whose Cartan matrix has all integral eigenvalues, but it is not semisimple. For example, we consider the following algebra given by the quiver

$$\begin{tikzcd} 1 \bullet \arrow[r, shift right, "\beta"'] & \bullet\, 2 \arrow[l, shift right, "\alpha"'] \end{tikzcd}$$

with relations $\alpha\beta\alpha = \beta\alpha\beta = 0$. Clearly, the Cartan matrix is

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and has 1 and 3 as its eigenvalues, but the algebra is not semisimple. Note that the Cartan determinant is 3.

There is also a cellular algebra with $\det(C) = 1$, which is not semisimple. For example, let A be the algebra given by the quiver

$$\begin{tikzcd} 1 \; \bullet \arrow[r, shift right, "\beta"'] & \bullet \; 2 \arrow[l, shift right, "\alpha"'] \end{tikzcd}$$

with the relation $\alpha\beta = 0$. Clearly, the Cartan matrix of this algebra is

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and has non-integral eigenvalues. The algebra is a non-semisimple quasi-hereditary cellular algebra. This example shows also that the conditions on the eigenvalues of the Cartan matrix in Corollary 3.5 is necessary.

Recently, Kiyota, Murai and Wada consider in [14] the rationality of eigenvalues of Cartan matrices in finite groups, where they established some connections between the eigenvalues of the Cartan matrix and the order of a defect group.

4. Cellular algebras with prime Cartan determinant

Having known that for a cellular algebra, the determinant and the eigenvalues of its Cartan matrix play an important role in the study of cellular algebras, we investigate in this section the following problem proposed in Section 1.

Problem. Given a prime p, classify cellular algebras A such that all eigenvalues of the Cartan matrix C are integers and $\det(C) = p$ or p^2.

Throughout this section we assume that A is an indecomposable cellular algebra with the Cartan matrix $C = (c_{ij})$ such that all eigenvalues of C are integers and $\det(C) = s$, where $s = p$, or p^2. (Note that the assumption on the determinant is needed only to ensure that C has the spectrum of the form $\{a, 1, \ldots, 1\}$ with a a positive integer.)

We denote by $\mathbb{N}$ the natural numbers, and by $\mathbb{N}_0$ the non-negative integers.

Recall that a matrix M is called *irreducible* if there is no permutation matrix P such that PCP^{tr} is of the form

$$\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix},$$

where X and Z are square matrices.

The following lemma gives some informations on the Cartan matrix.

Lemma 4.1

(1) *C is an irreducible matrix. Moreover, C is positive, that is, $c_{ij} > 0$ for all i, j.*

(2) *$c_{ii} \geqslant 2$ for all i, and $c_{ij} < c_{ii}$ if $c_{ii} \geqslant c_{jj}$ and $i \neq j$.*

(3) $c_{ij} \leqslant s$ *for all* i, j. *Moreover, if there is an entry* $c_{ij} = s$, *then* C *is the* 1×1 *matrix* (s).
(4) *If* C *is an* m*-by-*m *matrix, then* $m \leqslant s - 1$, *that is, the number of non-isomorphic simple* A*-modules is bounded by* $s - 1$.

Proof. Since A is an indecomposable cellular algebra, we know that C is irreducible by [15, Theorem 1, p. 529]. Now we apply Theorem 8.4.4 in [10] to C: s is the spectral radius of C, and there is a positive vector x such that $Cx = sx$. If we require that $\sum_i x_i = 1$, then x is the unique Perron-vector, and any eigenvector of C corresponding to the eigenvalue s is a scalar multiple of x. By assumption, the eigenvalues of C are $s, 1, \ldots, 1$, where 1 appears $m - 1$ times. It follows from linear algebra that there is a unitary matrix U with real entries such that $C = I_m + U\mathrm{diag}(s - 1, 0, \ldots, 0)U^{\mathrm{tr}}$. Since $UU^{\mathrm{tr}} = U^{\mathrm{tr}}U = I$, the first column of U is an eigenvector of C corresponding to the eigenvalue s. It follows from Perron–Frobenius theory that we may assume that the first column u of U is a scalar multiple of the Perron-vector x, say $u = \lambda x$ with λ in $\mathbb{R}$. This implies that $c_{ii} = 1 + (s - 1)u_i^2$ and $c_{ij} = (s - 1)u_i u_j$. Since u is not 0 and x is positive, we see that c_{ij} is not 0 for all i, j. This shows that C is in fact a positive matrix, and therefore it is irreducible.

(2) is clear from the above proof. As to (3), we note that a principal submatrix of a positive definite matrix is positive definite and that the spectral radium of a principal submatrix of C is less than or equal to that of C. Hence $c_{ii} \leqslant s$. It follows from $c_{ii}c_{jj} - c_{ij}^2 > 0$ that s occurs only in the diagonal of C. We may assume that $c_{11} = s$. It follows from $\sum_{i=1}^m c_{ii} = m + s - 1$ that $\sum_{i=2}^m c_{ii} = m - 1$ and $c_{ii} = 1$ for all $i \geqslant 2$. Since C is irreducible, we have $C = (s)$.

We now prove (4). The irreducible matrix C has the spectral radium s and is positive. Thus, by the Perron–Frobenius theory (see, for example, Exercise 7, p. 537 in [10]), the spectral radius s satisfies either $s = \sum_{j=1}^m c_{ij}$ for all i, or there is an index i such that $\sum_{j=1}^m c_{ij} < s$. In the former case we have $m + 1 \leqslant \sum_j c_{ij} = s$. In the latter case we have $m + 1 < s$. Here we have used the fact that $c_{ii} \geqslant 2$. □

From the proof of Lemma 4.1 we can get some information on projective modules from the Cartan matrix.

Corollary 4.2. *Let* A *be an indecomposable cellular algebra whose Cartan matrix has eigenvalues* $n, 1, \ldots, 1$. (*Here* n *is an arbitrary natural number.*) *Then each indecomposable projective module* P *of* A *is sincere, that is,* P *has each simple module as composition factor, thus the dimension of* P *is at least* $m + 1$, *where* m *is the number of non-isomorphic simple modules.*

Lemma 4.3. *The Cartam matrix* C *has no principal submatrix of the form*

$$\begin{pmatrix} m+1 & c_{ij} \\ c_{ij} & m \end{pmatrix} \quad or \quad \begin{pmatrix} m+2 & c_{ij} \\ c_{ij} & m \end{pmatrix}.$$

Proof. Suppose such a principal submatrix

$$\begin{pmatrix} m+1 & c_{ij} \\ c_{ij} & m \end{pmatrix}$$

exists. Then, by Lemma 2.5(4), the eigenvalues of the principal submatrix of C are integers. Thus $4c_{ij}^2 + 1$ is the square of a natural number, which is impossible: If $4c_{ij}^2 + 1 = m^2$ for some m, then $m = 2b + 1$ for some natural number b. Thus $c_{ij}^2 = b(b + 1)$, a contradiction.

For the other case, the proof is similar to the above one. We need to note that there is no integer x such that $1 + c_{ij}^2$ is the product of two natural numbers $x + 1$ and $x - 1$. □

In the following we shall give a general description of which matrix C could have the properties mentioned at the beginning of this section.

We need the definition of partitions of n. Recall that a *partition* λ of n is a sequence of integers $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_m > 0$ such that $\sum_j n_j = n$. In this case we write $\lambda := (n_1, n_2, \ldots, n_m) \vdash n$.

We denote by $S(n)$ the set of all partitions λ of n with the property that for all i, j the product $n_i n_j$ is a square of a natural number.

Proposition 4.4. *There is a one-to-one correspondence between the set* $S(n)$ *and the set of all symmetric matrices* C *over* $\mathbb{N}$ *with* $\det(C) = n + 1$ *and* $c_{ii} \geqslant c_{i+1,i+1}$ *such that each eigenvalue of* C *is either* $n + 1$ *or* 1.

Proof. As in the proof of (1) in Lemma 4.1, we can write $C = I_m + nu^{\mathrm{tr}}u$, where $u^{\mathrm{tr}} = (u_1, \ldots, u_m) \in \mathbb{R}^m$ can be chosen to be a positive vector such that $\sum_j u_j^2 = 1$. So the matrix C is uniquely determined by u. Since C has natural numbers as its entries, the condition for u can be interpreted as follows:

1. $nu_iu_j \in \mathbb{N}$ for all i, j; and
2. $\sum_j u_j^2 = 1$.

Put $n_i = nu_i^2$ for all i. Then $n_i \in \mathbb{N}$ and $nu_iu_j = \sqrt{n_in_j} \in \mathbb{N}$. The second condition says that $\sum_j n_j = n$. Now it is clear that if we send u to $\lambda := (n_1, n_2, \ldots, n_m)$, then we have a desired correspondence. □

Proof of Theorem 1.2. The first statement of Theorem 1.2 follows from the above proposition. For the second statement, note that under the assumption of Theorem 1.2, the eigenvalues of Cartan matrix are $p, 1, \ldots, 1$ since the spectral radius of an irreducible non-negative matrix is a simple root of its characteristic polynomial. Thus the desired statement follows now from the above proposition and Proposition 4.5 below. □

For the later use, we write down the correspondence in Proposition 4.4 explicitly: If $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in S(n)$, then the corresponding matrix C is

$$\begin{pmatrix} \lambda_1 + 1 & \sqrt{\lambda_1\lambda_2} & \cdots & \sqrt{\lambda_1\lambda_m} \\ \sqrt{\lambda_1\lambda_2} & \lambda_2 + 1 & \cdots & \sqrt{\lambda_2\lambda_m} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\lambda_m\lambda_1} & \sqrt{\lambda_m\lambda_2} & \cdots & \lambda_m + 1 \end{pmatrix},$$

and C has m eigenvalues $n + 1, 1, \ldots, 1$. (Hence $\det(C) = n + 1$.)

Now we consider the set $S(n)$. We define $S_0(n)$ to be the subset of $S(n)$ consisting of partitions whose parts are squares of natural numbers. For each divisor d of n, we define

$$S_d(n) := \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in S(n) \mid \text{the greatest common divisor of the } \lambda_j\text{'s is } d\}.$$

Then we have the following proposition.

Proposition 4.5
(1) $S(n) = \dot{\bigcup}_{d|n} S_d(n)$.
(2) *If $d, c \in \mathbb{N}$ such that cd is a divisor of n, then $cS_d(n/c) = dS_c(n/d)$, where $cS_d(n) := \{(c\mu_1, c\mu_2, \ldots, c\mu_m) \mid (\mu_1, \mu_2, \ldots, \mu_m) \in S_d(n)\}$. In particular, $S_d(n) = dS_1(n/d)$.*
(3) *$S_1(n) \subseteq S_0(n)$ for any number n. In particular, $S(p) = S_0(p) \cup \{(p)\}$ for any prime p.*

Proof. (1) is clear. For (2), we note that if $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ is in $S_d(n/c)$, then $(\lambda'_1, \lambda'_2, \ldots, \lambda'_m)$ is in $S_c(n/d)$, where $\lambda'_j = c\lambda_j/d$. Conversely, if $(\lambda'_1, \lambda'_2, \ldots, \lambda'_m)$ is in $S_c(n/d)$, then $(d\lambda'_1/c, d\lambda'_2/c, \ldots, d\lambda'_m/c)$ is in $S_d(n/c)$. (3) follows from the following observation. $\square$

Lemma 4.6. *Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be natural numbers such that $\lambda_i\lambda_j$ is a square for all i, j. If $\lambda_1, \lambda_2, \ldots, \lambda_m$ are coprime, then each λ_i itself is a square.*

Proof. First we note the following two facts: (1) If two integers are coprime and their product is a square, then each of them is a square. (2) If a and b are two natural numbers such that ab^2 is a square, then a itself is a square.

We prove the lemma by induction on m. For $m = 2$, it is clear by the fact (1). Suppose the lemma is true for $m - 1$. Let $\lambda_i\lambda_j = x_{ij}^2$ for some $x_{ij} \in \mathbb{N}$. If d denotes the greatest common divisor of $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$, then $\lambda'_j := \lambda_j/d$, $1 \leqslant j \leqslant m - 1$, are coprime. Since $\lambda_i\lambda_j = d^2\lambda'_i\lambda'_j$ is a square, we know, by the fact (2), that $\lambda'_i\lambda'_j$ is a square. Now it follows by induction that λ'_j is a square for $j = 1, 2, \ldots, m - 1$. Since $\lambda_{m-1}\lambda_m = d\lambda'_{m-1}\lambda_m$ is a square, we know by the above two facts that d and

λ_m are squares since d and λ_m are coprime. Thus we finish the proof of the lemma and also the proof of the above proposition. □

Proposition 4.5 reduces the calculation of $S(n)$ to that of $S_1(n)$. The following result tells us how to compute $S_1(n)$ and $S_0(n)$ recursively.

Recall that an integer is called *square-free* if it cannot be divided by p^2 for any prime p.

Lemma 4.7. $S_0(n) = \dot{\bigcup}_{d^2|n} d^2 S_1(n/d^2)$. *In particular, the equality* $S_1(n) = S_0(n)$ *holds if and only if n is square-free.*

Proof. If $(\lambda_1, \ldots, \lambda_m) \in S_0(n)$, then $(\lambda_1/d, \ldots, \lambda_m/d) \in S_1(n/d)$, where d is the greatest common divisor of $\lambda_1, \ldots, \lambda_m$. By Lemma 4.5, $S_1(n/d) \subseteq S_0(n/d)$, and hence λ_j/d is a square. This yields that d is a square. Thus we have $S_0(n) \cap S_{d^2}(n) = d^2 S_1(n/d^2)$. This implies the lemma. □

For the calculation of the cardinality of $S_0(n)$, we may use the following results in [1,2].

Proposition 4.8. *Let H denote the set* $\{1, 2^2, 3^2, 4^2, \ldots, \}$ *and let* $s_0(n)$ *be the number of partitions in* $S_0(n)$. *Then,*

(1) *For all* $|q| < 1$,

$$\sum_{n \geqslant 0} s_0(n) q^n = \prod_{n \in H} (1 - q^n)^{-1}.$$

(2) *Let* $\sigma_2(n) = \sum_{d^2|n} d^2$. *Then*

$$n s_0(n) = \sum_{j=1}^{n} \sigma_2(j) s_0(n-j).$$

As we have seen, Problem 1 mentioned in Section 1 was reduced to a problem in the elementary number theory.

5. Cellular algebras with small Cartan determinant

Inside the class of cellular algebras the Cartan determinant of value 1 is a characterization of quasi-hereditary algebras. From this point of view, cellular algebras with smaller Cartan determinant are nearer to quasi-hereditary algebras than those with bigger Cartan determinant. In this section we classify cellular algebras such that

$2 \leqslant \det(C) \leqslant 7$ and all eigenvalues of C are integers. Of course, all of these algebras are of infinite global dimension and not quasi-hereditary. First we have a complete classification list of cellular alberas with $\det(C) = 2$ or 3.

Proposition 5.1. *Let k be an algebraically closed field and A a cellular k-algebra. Denote by C the Cartan matrix of A. Suppose that all eigenvalues of C are integers.*

(1) *If* $\det(C) = 2$, *then A is Morita equivalent to* $k[x]/(x^2) \oplus k \oplus \cdots \oplus k$.
(2) *If* $\det(C) = 3$, *then A is Morita equivalent to one of the following algebras*:
 1) $B \oplus k \oplus \cdots \oplus k$, *where B is a* 3*-dimensional local cellular k-algebra*.

 2) $$\begin{tikzcd} \bullet \arrow[r, bend right, "\beta"'] & \bullet \arrow[l, bend right, "\alpha"'] & \bullet \cdots \bullet \end{tikzcd} \qquad \alpha\beta\alpha = \beta\alpha\beta = 0.$$

 3) $$\begin{tikzcd} \bullet \arrow[loop left, "\delta"] \arrow[r, bend right, "\beta"'] & \bullet \arrow[l, bend right, "\alpha"'] & \bullet \cdots \bullet, \end{tikzcd} \qquad \beta\alpha = \alpha\delta = \delta^2 = \delta\beta = 0.$$

 4) $$\begin{tikzcd} \bullet \arrow[loop left, "\gamma"] \arrow[r, bend right, "\beta"'] & \bullet \arrow[loop right, "\delta"] \arrow[l, bend right, "\alpha"'] & \bullet \cdots \bullet \end{tikzcd}$$
 $$\alpha\beta = \alpha\gamma = \beta\alpha = \beta\delta = \gamma^2 = \gamma\beta = \delta^2 = \delta\alpha = 0.$$

In fact, Proposition 5.1 follows directly from the results in the previous section, here we omit its proof.

Proposition 5.2. *Let k be an algebraically closed field and A an indecomposable cellular k-algebra. Suppose all eigenvalues of the Cartan matrix C of A are integers.*

(1) *If* $\det(C) = 4$, *then A is Morita equivalent to either a* 4*-dimensional local cellular algebra*, *or a cellular algebra with* 3 *simple modules such that its Cartan matrix equals*
$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$
(2) *If* $\det(C) = 5$, *then A is Morita equivalent to either a* 5*-dimensional local cellular algebra*, *or a cellular algebra with* 2 *simple modules such that its Cartan matrix equals*
$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix},$$
or a cellular algebra with 4 *simple modules such that its Cartan matrix has* 2 *as the diagonal entries and* 1 *for all off-diagonal entries*.

(3) *If* $\det(C) = 6$, *then A is Morita equivalent to either a* 6*-dimensional local cellular algebra*, *or a cellular algebra with* 2 *simple modules such that its Cartan matrix equals*

$$\begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix},$$

or a cellular algebra with 5 *simple modules such that its Cartan matrix has* 2 *as the diagonal entries and* 1 *for all off-diagonal entries*.

(4) *If* $\det(C) = 7$, *then A is Morita equivalent to either a* 7*-dimensional local cellular algebra*, *or a cellular algebra with* 2 *simple modules such that its Cartan matrix equals*

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix},$$

or a cellular algebra with 3 *simple modules such that its Cartan matrix equals either*

$$\begin{pmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \quad or \quad \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

or a cellular algebra with 6 *simple modules such that its Cartan matrix has diagonal entries all equal to* 2 *and off diagonal entries all equal to* 1.

Proof. We prove the case (3). The other cases can be done by applying the general theory of the previous section. Here we exploit a different method to deduce the result in (3). The spectrum of C has two cases: All eigenvalues of C are (1) $\{6, 1, \dots, 1\}$, or (2) $\{3, 2, 1, \dots, 1\}$. In the first case we can use 4.4. As to the second case, we have to develop another method. The following fact will be useful (for a proof, one may use the results 2.1.5 and 2.1.10 in [4] and basic property of irreducible matrices):

Lemma 5.3. *If A is an arbitrary indecomposable cellular algebra with n non-isomorphic simple modules*, *then* (1) *its Cartan matrix C is irredeucible and positive definite*. (2) $1 \leqslant c_{ii} \leqslant \rho(C)$ *for all* i, *where* $\rho(C)$ *is the spectral radius of* C. *Moreover*, *if* $\rho(C)$ *appears as an entry in* C, *then* $C = (\rho(C))$.

Note that $c_{ii} \geqslant 2$ for all i by the proof of 3.5. Since the main diagonal of C corresponds to the partitions of $n - 2 + 3 + 2 = n + 3$, we have to consider a few cases indeed:

(a) The diagonal of C is $(3, 2)$ (and $n = 2$). Then, by the above lemma, this is impossible.

(b) The diagonal of C is $(2, 2, 2)$ (and $n = 3$). Then for $i \neq j$ the entries c_{ij} take values 0 or 1. An easy verification shows that we cannot get a desired matrix in this case. Thus (3) is proved. $\square$

Note that all matrices displayed in the proposition can be realized as a Cartan matrix of an indecomposable cellular algebras.

Let us end this section by the following question which is suggested by the above propositions:

Question. Given a prime p, which numbers could occur as the numbers of non-isomorphic simple modules over indecomposable cellular algebras such that their Cartan matrices have determinant equal to p and all eigenvalues are integers?

The above propositions suggest that all divisors of $p - 1$ would be some of the desired numbers.

6. Cellular Nakayama algebras

Recall that an artin algebra is said to be a *Nakayama algebra* if both the indecomposable projective and indecomposable injective modules are uniserial. The representation theory of this class of algebras was investigated in the book [3]. In this section we will discuss which Nakayama algebras are cellular and determine which cellular Nakayama algebras have the property that all eigenvalues of their Cartan matrices are integers. To this end, we first give a classification of cellular Nakayama algebras. The following are the main result in this section.

Theorem 6.1. *Let A be a connected finite dimensional algebra given by a quiver with relations. Then*

(1) *A is a cellular Nakayama algebra if and only if A is isomorphic with one of the following algebras*:
  1) $k[X]/(X^n)$ *for* $n \in \{1, 2, 3, \ldots\}$;
  2) *A Nakayama algebra with the Cartan matrix*
$$\begin{pmatrix} m+1 & m \\ m & m+1 \end{pmatrix}, \quad \textit{where } m \geqslant 1;$$
  3) *A Nakayama algebra with the Cartan matrix of the form*
$$\begin{pmatrix} m+1 & m \\ m & m \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} m & m \\ m & m+1 \end{pmatrix}, \quad \textit{where } m \geqslant 1.$$

(2) *The cellular algebras in* 1) *and* 2) *have the property that all eigenvalues of their Cartan matrices are integers*, *but the cellular algebras in* 3) *do not have this property*.

Proof. Let A be a connected cellular Nakayama algebra. Since a cellular algebra has an involution, one has a duality on the whole module category which fixes all simple modules. Thus it is transparent that a Nakayama algebra being cellular has at most two simple modules. If it has only one simple module, then 1) follows. Now assume that it has two simple modules, then the Cartan matrix determines the algebra uniquely. Suppose the Cartan matrix is

$$C = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Clearly, $a \geqslant b$ and $c \geqslant b$. By [12], $\det(C) \geqslant 1$. If we rewrite $a = b + x$ and $c = b + y$, then $\det(C) = b(x + y) - xy \geqslant 1$. Since A is a Nakayama algebra, we deduce that $x, y \in \{0, 1\}$. This yields that C is of the form displayed in 2) and 3).

To finish the proof of (1), we have to show that a Nakayama algebra with the Cartan matrix of the above form is cellular. For this we exploit a result in [13] to check whether a given basis is a cellular basis or not. Let us explain this by an example for $b = 2$. Then the algebra A is given by the quiver

$$\begin{tikzcd} 1 \bullet \arrow[r, shift right, "\beta"'] & \bullet\, 2 \arrow[l, shift right, "\alpha"'] \end{tikzcd}$$

with relations either $\alpha\beta\alpha\beta = 0$, or $\alpha\beta\alpha\beta\alpha = 0 = \beta\alpha\beta\alpha\beta$. In the latter, if we factor out a simple socle from A, we come to the former case. Let us now display a cellular basis for this case:

$$\begin{array}{llllc} \alpha\beta, & \alpha\beta\alpha, & e_2, & \alpha, & \\ & & & & e_1. \\ \beta\alpha\beta, & \beta\alpha\beta\alpha, & \beta, & \beta\alpha, & \end{array}$$

As to (2), one notes that there is no natural number n such that $n(n + 1)$ is a square of another natural number. Note also that the eigenvalues of the Cartan matrix in 3) being integers depend on whether $(2m + 1)^2 - 4m$ is a square of a natural number. If $4m^2 + 1 = n^2$ for some n, then $n = 2d + 1$ for some natural number d. Thus $m^2 = d(d + 1)$, a contradiction. This completes the proof. $\square$

We say that a cellular algebra has the *integral property* if all eigenvalues of its Cartan matrix are integers. We may construct a large family of cellular algebras with the integral property.

Proposition 6.2. *The tensor product of two basic cellular algebras with the integral property over an algebraically closed field is again a cellular algebra with the integral property. In particular, the tensor product of two cellular Nakayama algebras in* 1) *and* 2) *has the integral property.*

In the following section we shall provide another method to get cellular algebras with integral property.

7. Trivial extensions of cellular algebras

In this section we shall give a general construction of cellular algebras with the integral property. Here the notion of trivial extensions plays a central role.

Let A be a finite dimensional algebra over a field k. We denote by $T(A)$ the *trivial extension* of A, which has the underlying vector space $A \oplus DA$, where $DA = \mathrm{Hom}_k(A, k)$ is viewed as an A-A-bimodule, and $T(A)$ has multiplication given by

$$(a + f)(b + g) = ab + fb + ag, \quad a, b \in A, \quad f, g \in DA.$$

It is a symmetric algebra with non-degenerate bilinear form $\langle a + f, b + g\rangle = f(b) + g(a)$.

If there is an involution i on A, then we may define an involution ϵ on the trivial extension $T(A)$ by $a + f \mapsto i(a) + i \circ f$, where $i \circ f$ stands for the composition of the linear map i with the function f.

We have the following result.

Theorem 7.1. *Let A be a cellular algebra with respect to an involution i. Then $T(A)$ is a cellular algebra with respect to the involution ϵ defined above.*

Proof. If N is a subset of A, we denote by $N^{\perp}$ the set $\{f \in DA \mid f(N) = 0\}$. Since A is a cellular algebra with respect to i, there is an i-invariant decomposition: $A = J_1' \oplus J_2' \oplus \cdots \oplus J_m'$ such that if we define $J_j = \bigoplus_{l=1}^{j} J_l'$ for all j then the chain $0 \subset J_1 \subset J_2 \subset \cdots \subset J_m = A$ is a cell chain for A. Thus we have a decomposition of $T(A)$:

$$T(A) = D(J_m') \oplus D(J_{m-1}') \oplus \cdots \oplus D(J_1') \oplus J_1' \oplus J_2' \oplus \cdots \oplus J_m'.$$

Let $I_j := J_{j+1}' \oplus J_{j+2}' \oplus \cdots \oplus J_m'$. Then $A = J_j \oplus I_j$ and $J_j^{\perp} \simeq D(I_j)$. Note that $J_j^{\perp}$ and $D(I_j)$ are in fact isomorphic as $A-A$-bimodules. So the decomposition of $T(A)$ gives rise a chain of ideals of $T(A)$:

$$\begin{aligned} J_m^{\perp} = 0 \subset J_{m-1}^{\perp} = D(I_{m-1}) \subset J_{m-2}^{\perp} \\ = D(I_{m-2}) \subset \cdots \subset J_1^{\perp} = D(I_1) \subset DA. \end{aligned}$$

The subquotient $J_{j-1}^{\perp}/J_j^{\perp}$ is isomorphic to $D(J_j')$ as A-A-bimodules. Thus it is a cell ideal in $T(A)/J_j^{\perp}$. This implies that the decomposition of $T(A)$ satisfies all conditions in the definition of cellular algebras. Thus $T(A)$ is a cellular algebra with respect to the involution ϵ. □

As a consequence, we have the following result which reduces the study of the module categories of arbitrary cellular algebras to that of full subcategories of the module categories of symmetric cellular algebras.

Corollary 7.2. *Any cellular algebra is a homomorphic image of a symmetric cellular algebra.*

The following result tells us we can get cellular algebras with integral property by trivial extensions.

Proposition 7.3. *If A is a cellular algebra with integral property, then so is $T(A)$.*

Proof. Let C_A denote the Cartan matrix of A. Then it is easy to see that for any finite dimensional algebra A the Cartan matrix of $T(A)$ is $C_A + C_A^{\mathrm{tr}}$. Since A is a cellular algebra, the matrix C_A is symmetric. Thus $C_{T(A)} = 2C_A$, and the statement follows obviously. □

We note that the standard modules for the cellular algebra $T(A)$ with respect to the involution ϵ are the union of all standard A-modules and all costandard A-modules. Thus, for $T(A)$, the set of standard modules is the same as the set of costandard modules. A quasi-hereditary cellular algebra A has this property if and only if A is semisimple.

We remark that one may use $T(A)$ to study the original algebra A. For example, a cellular algebra A with n non-isomorphic simple modules is quasi-hereditary if and only if the determinant of $C_{T(A)} = 2^n$. This is a consequence of the above proposition and a result in [12]. On the other hand, the A-module category can be embedded in the stable category of $T(A)$.

Proposition 7.4. *Let A be a non-simple indecomposable cellular algebra with n non-isomorphic simple modules. If the trivial extension $T(A)$ of A has the integral property, then the global dimension of A is infinite.*

Proof. Suppose that the global dimension of A is finite. Then we know that A is a quasi-hereditary algebra by [12] and that the determinant $\det(C)$ of the Cartan matrix C of A is 1. Since $T(A)$ has the integral property, all eigenvalues of $2C$ are integers. Thus the spectrum of $2C$ must be of the form $\{2^{n_1}, \ldots, 2^{n_t}, 1, \ldots, 1\}$, where 1 appears $s = n - t$ times, and the n_i are natural numbers. This means also that the spectrum of C is of the form $\{2^{m_1}, \ldots, 2^{m_r}, 1, \ldots, 1, 1/2, \ldots, 1/2\}$, where $1/2$ occurs s times, and the m_i are natural numbers. By Theorem 3.1, we have $0 < s < n$. Moreover, since $\det(C) = 1$ and the trace of C is a positive integer, we know that s is even and equal to $\sum_{j=1}^{r} m_j$. Now let us calculate the coefficients of the characteristic polynomial $f(x)$ of C. Clearly, all coefficients are integers. In particular, the coefficient c of x in $f(x)$ is an integer. In the following we shall prove that this is impossible. If $r = 1$, then the spectrum of C is $\{2^s, 1, \ldots, 1, 1/2, \ldots, 1/2\}$, and we have $c = 1/2^s + 1 + \cdots + 1 + 2s$, which is not an integer, a contradiction. So we can assume that $r \geqslant 2$. Note that the coefficient is

$$c = 1/2^{m_1} + 1/2^{m_2} + \cdots + 1/2^{m_r} + 1 + \cdots + 1 + 2s.$$

We now consider the sum $a := 1/2^{m_1} + 1/2^{m_2} + \cdots + 1/2^{m_r}$. Since A is an indecomposable cellular algebra, the matrix C is irreducible and its spectral radius is a simple root of $f(x)$. This implies that we may assume that $m_1 > m_2 \geqslant m_3 \geqslant \cdots \geqslant m_r \geqslant 1$. In this case, we have $2^{m_1}a = 1 + 2^{m_1-m_2} + \cdots + 2^{m_1-m_r}$. If a is a natural number, then $0 \equiv 1 \pmod 2$. This is a contradiction. Thus the global dimension of A is infinite. □

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