The number of simple modules of a cellular algebra

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1 Introduction

One of the central problems in the representation theory of finite groups and finite dimensional algebras is to determine the number of non-isomorphic simple modules. Recently, Graham and Lehrer introduced in ref.[1] the notion of a cellular algebra, which provides a systematic treatment of reducing the problem to that in linear algebra. Cellular algebras embrace many important algebras appeared in mathematics and physics: the Temperley-Lieb algebras, the Hecke algebras of type A, the Brauer algebras, the partition algebras, the Birman-Wenzl algebras, and so on. Theoretically, the number of non-isomorphic simple modules over a given cellular algebra A is determined by a subset of a finite poset associated to the algebra A, but what the number precisely is seems to be unknown. The recent results in ref.[2] and ref.[3] show that the eigenvalues and the determinant of the Cartan matrix of A may play an important role in the study of cellular algebras and shed light on determining which numbers could be the number of non-isomorphic simple modules over indecomposable cellular algebras with the property that their Cartan matrices have all rational eigenvalues. The rationality of the eigenvalues of the Cartan matrix measures how far a cellular algebra from being semi-simple ref.[2], and plays a role in the representation theory of finite groups ref.[4]. Our main interest of this paper stems from ref.[2].

Let n be a natural number, and let A be an indecomposable cellular algebra such that the spectrum of its Cartan matrix C is of the form $\{n, 1, \ldots, 1\}$. In general, not every natural number could be the number of non-isomorphic simple modules over such a cellular algebra. For example, if $n = 11$, it can be verified that the number of non-isomorphic simple modules over A is one of $\{1, 2, 4, 5, 7, 10\}$. This means that there is no indecomposable cellular algebra which has 3 or 6 simple modules such that the spectrum of its Cartan matrix C is of the form $\{11, 1, \ldots, 1\}$. Thus, two natural questions arise: (1) which numbers could be the number of non-isomorphic simple modules over such a cellular algebra A ? (2) Given such a number, is there a cellular algebra such that its Cartan matrix has the desired property ? In this paper, we shall completely answer the first question, and give a partial answer to the second question by constructing cellular algebras with the pre-described Cartan matrix.

The contents of this paper are as follows. In Section 2, we briefly recall the definitions of cellular algebras and Cartan matrices. In Section 3, we collect some results from the number theory for the later proofs. In Section 4, we prove our main results, Theorem 1 and Corollary 1. In the last section, we realize several classes of matrices as Cartan matrices of cellular algebras with the required property.

2 Cellular algebras and Cartan matrices

We first recall the definition of cellular algebras from ref.[1] which is given by the existence of a basis with certain properties, and then collect some basic facts on the Cartan matrix of a cellular algebra in ref.[2]. Note that there is also a basis-free definition of cellular algebras in terms of a chain of ideals (see ref.[5]), and the two definitions are equivalent.

For simplicity we assume that we work with the ground ring being an (arbitrary) field k. By algebra we always mean a finite dimensional associative k-algebra with unit.

Definition 1.(ref.[1]) An associative k–algebra A is called a **cellular algebra** with cell datum (I, M, C, i) if the following conditions are satisfied:

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has a k–basis $C^{\lambda}_{S,T}$ where (S, T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is a k–linear anti–automorphism of A with $i^2 = id$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^{\lambda}_{S,T}$ can be written as $(\sum_{U\in M(\lambda)} r_a(U, S)C^{\lambda}_{U,T}) + r'$ where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $r_a(U, S) \in k$ do not depend on T.

Typical examples of cellular algebras include the group algebras of symmetric groups, Brauer algebras (ref.[1]), Birman-Wenzl algebras (ref.[6]), and others.

We also need the notion of a **Cartan matrix** in the following abstract sense (which coincides with the one used in group theory if A is the group algebra of a finite group over a splitting field). Let us denote the simple A–modules by $L(1), \ldots, L(m)$ and their projective covers by $P(1), \ldots, P(m)$. The entries $c_{j,h}$ of the Cartan matrix $C(A)$ are the composition multiplicities $[P(j) : L(h)]$ of $L(h)$ in $P(j)$. The determinant of $C(A)$ is called the **Cartan determinant**. From the definition of Cartan matrix $C(A)$, we know that the number of non-isomorphic simple A–modules is the same as the degree of the matrix $C(A)$.

Let $\mathbb{N}$ denote the natural numbers. Now we assume that cellular algebras A are indecomposable and their Cartan matrices $C(A)$ have the spectrum of the form $\{a, 1, ..., 1\}$ with $a \in \mathbb{N}$. Note that $C(A)$ is a positive definite symmetric matrix over $\mathbb{N}$ proved in ref.[2].

Let n be a natural number. Recall that a **partition** λ of n is a sequence of integers $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_k > 0$ such that $\sum_j n_j = n$. In this case, we write $\lambda := (n_1, n_2, \ldots, n_k) \vdash n$ and call each n_i a **part** of λ. We define

$$S(n) := \{\lambda \vdash n \mid \text{the product of any two parts of } \lambda \text{ is the square of a natural number}\},$$

$$\begin{aligned} M(n) := \{C \quad & \mid C = (c_{ij}) \text{is a symmetric matrix over } \mathbb{N} \text{ of degree } m \text{ with } c_{ii} \geqslant c_{i+1,i+1} \\ & \text{and the spectrum} \{\mu_1 = n+1, \mu_2 = \ldots = \mu_m = 1\}, \text{where } m \text{ is an} \\ & \text{arbitrary natural number}\}. \end{aligned}$$

The relationship between $S(n)$ and $M(n)$ is described in ref.[2]. We restate this in the following lemma.

Lemma 1. There is a one-to-one correspondence between the set $S(n)$ and the set $M(n)$.

Let us write down the correspondence in Lemma 1 in details: if $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in S(n)$, then the corresponding matrix C is

$$\begin{pmatrix} \lambda_1 + 1 & \sqrt{\lambda_1\lambda_2} & \ldots & \sqrt{\lambda_1\lambda_m} \\ \sqrt{\lambda_2\lambda_1} & \lambda_2 + 1 & \ldots & \sqrt{\lambda_2\lambda_m} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\lambda_m\lambda_1} & \sqrt{\lambda_m\lambda_2} & \ldots & \lambda_m + 1 \end{pmatrix}.$$

Lemma 1 reduces the consideration of $M(n)$ to that of $S(n)$. Since the degree of a matrix in $M(n)$ is the number of the parts of the corresponding partition, we introduce a new subset of $\mathbb{N}$:

$$K(n) = \{m \in \mathbb{N} \mid \text{there is a partition } \lambda \in S(n-1) \text{ such that } \lambda \text{ has } m \text{ parts}\},$$

where $2 \leqslant n \in \mathbb{N}$. Note that if $m \in K(n)$, then $1 \leqslant m \leqslant n-1$ and $|K(n)| \leqslant n-1$.

Lemma 2. (ref.[2]) Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be natural numbers such that $\lambda_i\lambda_j$ is a square for all i, j. If $\lambda_1, \lambda_2, \ldots, \lambda_m$ are relatively prime, then each λ_i itself is a square.

Let $S_m := \{n \in \mathbb{N} \mid n \text{ is the sum of } m \text{ squares of natural numbers}\}$. Then we have the following proposition.

Proposition 1. m is in $K(n)$ if and only if there exists some divisor d of $n-1$ such that $d \in S_m$.

Proof. If $m \in K(n)$, then there exists $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in S(n-1)$. Let d be the greatest common divisor of $\lambda_1, \lambda_2, \ldots, \lambda_m$. Then $(\frac{\lambda_1}{d}, \frac{\lambda_2}{d}, \ldots, \frac{\lambda_m}{d}) \in S(\frac{n-1}{d})$. Since $\frac{\lambda_1}{d}, \frac{\lambda_2}{d}, \ldots, \frac{\lambda_m}{d}$ are relatively prime, each $\frac{\lambda_i}{d}$ is a square by Lemma 2. So $\frac{n-1}{d} \in S_m$. Conversely, if $d \mid n-1$ and $d = n_1^2 + n_2^2 + \ldots + n_m^2$ with $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_m > 0$, then $(\frac{n-1}{d}n_1^2, \frac{n-1}{d}n_2^2, \ldots, \frac{n-1}{d}n_m^2) \in S(n-1)$. So $m \in K(n)$.

Thus our first question in the introduction is reduced to a problem on the decomposition of a given number into sums of squares of some natural numbers.

3 Decomposition of numbers

In this section we collect some basic results, used very often in the next section, on the decomposition of numbers into sums of squares in number theory. One can find these facts in ref.[7] (from p.378 to p.409).

The following lemma can be used to see whether a number belongs to S_3.

Lemma 3. (1) A natural number n is the sum of two squares of integers if and only if the factorization of n into prime factors does not contain any prime divisor of the form $4k+3$ that has an odd exponent.

(2) A natural number n is the sum of three squares of integers if and only if it is not of the form $4^l(8k+7)$, where k, l are non-negative integers.

In particular, for the numbers of the form $8k+1$, we have the following result.

Lemma 4. The numbers of the form $8k+1$ except 1 and 25 are in S_3.

We also need the following result of which the last statement is due to G.Pall.

Lemma 5. (1) $n \in S_2$ if and only if all prime factors of the form $4k+3$ of the number n have even exponents in the standard factorization of n into primes and either the prime 2 has an odd exponent (in the factorization of n) or n has at least one prime divisor of the form $4k+1$.

(2) $n \in S_4$ if and only if it does not belong to$\{1, 3, 5, 9, 11, 17, 29, 41, 4^h \cdot 2, 4^h \cdot 6, 4^h \cdot 14$, where h runs over all non-negative integers$\}$.

(3) $n \in S_5$ if and only if it does not belong to $\{1, 2, 3, 4, 6, 7, 9, 10, 12, 15, 18, 33\}$.

(4) If m is a natural number $\geqslant 6$, then $n \in S_m$ if and only if it does not belong to $\{1, 2, 3, ..., m-1, m+1, m+2, m+4, m+5, m+7, m+10, m+13\}$.

4 Main results and proofs

In this section we shall prove our main results. The strategy of the proofs runs as follows: first we describe when a natural number m is in $K(n)$ for $2 \leqslant m \leqslant 5$. This will be done case by case. After that we treat the general case of $m \geqslant 6$.

Throughout this section we keep all notations introduced in the previous sections. Recall that we denote by $K(n)$ ($n \geqslant 2$) the set of all numbers m such that there is a partition in $S(n-1)$ with m parts.

The following is an easy observation.

Lemma 6. If d is a divisor of $n-1$, then $d \in K(n)$.

Now let us show when the number 2 is in $K(n)$.

Proposition 2. The number 2 lies in $K(n)$ if and only if the factorization of $n-1$ into prime factors contains either the prime divisor 2 or at least one prime divisor of the form $4k+1$.

Proof. If the factorization of $n-1$ into prime factors contains either the prime divisor 2 or at least one prime divisor of the form $4k+1$, then 2 is in $K(n)$ by Lemma 6 and Lemma 5(1). Otherwise, it follows from Lemma 5(1) that any divisor of $n-1$ is not in S_2. So 2 is not in $K(n)$.

The condition for the number 3 to be in $K(n)$ is as follows.

Proposition 3. Suppose $n-1$ is not a prime which is of the form $8k+5$ and bigger than $5 \cdot 10^{10}$. Then 3 is in $K(n)$ if and only if $n-1$ is not the following numbers: $13, 25, 37, 2^k 5$, 2^k, and prime numbers of the form $8l+7$, where k, l are non-negative integers.

Proof. We consider two cases: (1) $n-1$ is even, (2) $n-1$ is odd.

(1) Suppose that $n-1$ is even. Let q be an arbitrary prime divisor of $n-1$. Then either $q=2$ or q is one of the forms $8k+1, 8k+3, 8k+5, 8k+7$.

Suppose that q is of the form $8k+1$. By Lemma 4, the prime q is in S_3 since q can not be 1 or 25.

Now suppose that q is of the form $8k+3$. We infer from Lemma 3(2) that every natural number of the form $8k+3$ is the sum of three squares of integers, which must all be odd integers. So q is in S_3.

Further, if $q = 8k+7$ for some non-negative integer k, then it follows from Lemma 3(2) that q is not in S_3. However, we shall prove that $2q$ is in S_3. In fact, $2(8k+7)(\equiv 6(\text{mod } 8))$ is the sum of three squares of integers by Lemma 3(2). But $2(8k+7)$ can not be the sum of two squares of integers by Lemma 3(1). So $2(8k+7)$ is in S_3.

Suppose that $q = 8k+5$ for some non-negative integer k. We consider two cases: $q > 130$ and $q \leqslant 130$.

(i) Suppose that $q > 130$. We shall prove that q or $2q$ is in S_3. In fact, at most one number of the form $8h+2$ or $8h+5$ greater than 130 is not in S_3 (see ref.[7], p.393). If this number is not equal to q , then q is in S_3. If this number is just q, then q is not in S_3.

However, $130 < 2q = 2(8k+5) \equiv 2 (\mathrm{mod}\, 8)$, so $2q$ belongs to S_3.

(ii) Suppose that $q \leqslant 130$. Among the numbers, which are less than $5 \cdot 10^{10}$ and of the form $8h+5$, are only the numbers $5, 13, 37$, and 85 not in S_3 (see ref.[7], p.392). If q is not 5, 13 or 37, then q is in S_3. Note that $37 \times 2 = 74 = 8^2 + 3^2 + 1^2$ and $13 \times 2 = 26 = 4^2 + 3^2 + 1^2$. If $q = 37$ or $q = 13$, then $2q$ is in S_3. If $q = 5$ and $q^2 \mid (n-1)$, then $2q^2 \mid (n-1)$. Since $5 \times 5 \times 2 = 50 = 5^2 + 4^2 + 3^2$, we have that $2q^2$ is in S_3.

Now it remains to consider $n - 1 = 2^l 5$, where l is a positive integer. First, we claim that a natural number of the form $4k$ is in S_3 if and only if k itself is in S_3. In fact, if $4k = a^2 + b^2 + c^2$, where a, b, c are natural numbers, then a, b, c must be even numbers. Put $a = 2a_1, b = 2b_1$ and $c = 2c_1$ with a_1, b_1 and c_1 natural numbers. Then $k = a_1^2 + b_1^2 + c_1^2$, as desired. Conversely, the last equality implies that $4k = (2a_1)^2 + (2b_1)^2 + (2c_1)^2$. Now we use this claim to finish the case (ii): since 2, 4, 10 and 20 are not in S_3, all divisors of $n-1 = 2^l 5$ are not in S_3. Therefore, if $n - 1 = 2^l 5$ with l a positive integer, then 3 is not in $K(n)$.

If q is 2, then we may assume that $n - 1 = 2^l$ for some positive integer l. In this case we can use the claim in the above proof to show that 3 is not in $K(n)$.

Thus we have proved that, for $n-1$ an even number, 3 is in $K(n)$ if and only if $n-1$ is not equal to $2^k 5$ or 2^k, where k is a natural number.

(2) Suppose that $n - 1$ is odd. If $n - 1 = 1$, then it is clear that 3 is not in $K(n)$. Assume that $n \geqslant 3$. From the proof of the first case (1), we see that the prime divisors of $n-1$ not in S_3 may be $5, 13, 37, 8k+7 (k \geqslant 0)$ or a number $8k_0 + 5 \geqslant 5 \cdot 10^{10}$. Suppose that q_1, q_2 are such two (possibly the same) prime divisors. Then $q_1 q_2 \equiv 1$ or $3 (\mathrm{mod}\, 8)$. So $q_1 q_2$ is in S_3 except $q_1 q_2 = 5^2$. However, 5^3 is in S_3 since $5^3 = 10^2 + 4^2 + 3^2$. Thus, since $n - 1$ is not a prime which is of the form $8k + 5$ and bigger than $5 \cdot 10^{10}$, the number 3 is in $K(n)$ if and only if $n - 1$ is not one of the numbers: 1, 5, 13, 25, 37, prime numbers of the form $8k + 7 (k \geqslant 0)$.

Now summarizing the two cases (1) and (2) together, we have that 3 is in $K(n)$ if and only if $n-1$ is not equal to $13, 25, 37, 2^k 5, 2^k$, or the prime numbers of the form $8l + 7$, where k, l are non-negative integers. This finishes the proof.

Now let us consider the case when the number 4 is in $K(n)$.

Proposition 4. The number 4 is in $K(n)$ if and only if $n - 1$ does not lie in $\{1, 2, 3, 5, 6, 9, 11, 17, 29, 41\}$.

Proof. (1) Suppose that $n - 1$ is even. If $4 \mid (n-1)$, then 4 is in $K(n)$ by Lemma 6. If $4 \nmid (n-1)$ and $n - 1$ is not $2, 6$ or 14, then $n - 1$ is in S_4 by Lemma 5(2). Hence 4 is in $K(n)$. Clearly, the number 4 is not in $K(3)$ or $K(7)$. Since 7 is in S_4, the number 4 is in $K(15)$. Thus we have proved that 4 is in $K(n)$ if and only if $n - 1$ is not in $\{2, 6\}$.

(2) Suppose that $n-1$ is odd. In this case, an argument similar to (1) shows that 4 is in $K(n)$ if and only if $n-1$ is not in $\{1,3,5,9,11,17,29,41\}$.

Now we summarize the two cases together and know that 4 is in $K(n)$ if and only if $n-1$ is not in $\{1, 2, 3, 5, 6, 9, 11, 17, 29, 41\}$. This finishes the proof.

The following proposition handles the case when the number 5 is in $K(n)$.

Proposition 5. The number 5 is in $K(n)$ if and only if $n-1$ is not in $\{1,2,3,4,6,7,9,12,18\}$.

Proof. This is similar to the proof of Proposition 4 by Lemma 5(3).

For the general case of $m \geqslant 6$ we have the following proposition.

Proposition 6. Let $K_6(n) := \{m \in K(n) \mid m \geqslant 6\}$. Then

(1) If $n < 7$, then $K_6(n) = \emptyset$.

(2) $K_6(15) = \{14, 11, 8, 7, 6\}, K_6(21) = \{20, 17, 14, 12\ \ldots, 6\}$,

$K_6(27) = \{26, 23, 20, 18, 17, 15, \ldots, 6\}$.

(3) If $n \geqslant 7$ and $n \neq 15, 21, 27$, then

$K_6(n) = \{n-k \geqslant 6 \mid k \in \{1,4,7,9,10,12,13,15,16,17,\ldots\}$.

Proof. (1) is clear. Now we prove (2) and (3). Lemma 5(4) implies that only the numbers $m, m+3, m+6, m+8, m+9, m+11, m+12, m+14, \ldots$ are in S_m. Hence $n-1$ is in S_k, where $k \geqslant 6$ and lies in $\{n-1, n-4, n-7, n-9, n-10, n-12, n-13, n-15, n-16, n-17, \ldots\}$. Moreover, if $n-15 \geqslant 6$ and $n-15 \geqslant \frac{n-1}{2}$, that is, $n \geqslant 29$, then any divisor of $n-1$ is not more than $n-15$, whence $K_6(n) = \{n-1, n-4, n-7, n-9, n-10, n-12, n-13, n-15, \ldots, 6\}$.

For $7 \leqslant n \leqslant 28$, one can check that (3) holds except 15, 21 and 27. This finishes the proof.

From the discussion in this section, we know that for a natural number n, the set $K(n)$ can be described by the the following theorem, which is the main result of this paper.

Theorem 1. Let n be a fixed natural number such that $n-1$ is not a prime which is of the form $8k+5$ and bigger than $5\cdot 10^{10}$. If A is an indecomposable cellular algebra over a field such that its Cartan matrix has determinant n and each eigenvalue is either n or 1, then the number of non-isomorphic simple A-modules must be a number in $K(n)$, where $K(n)$ is given as follows:

n	$K(n)$	$\|K(n)\|$
2	1	1
3	$1,2$	2
4	$1,3$	2
5	$1,2,4$	3
6	$1,2,5$	3
7	$1,2,3,6$	4
8	$1,4,7$	3
10	$1,3,6,9$	4
12	$1,3,5,8,11$	5
13	$1,2,3,4,6,9,12$	7
14	$1,2,4,5,7,10,13$	7
15	$1,\dots,8,11,14$	10
18	$1,2,3,5,6,8,9,11,14,17$	10
19	$1,2,3,4,6,7,9,10,12,15,18$	11
21	$1,2,4,\dots,12,14,17,20$	14
26	$1,2,4,\dots,11,13,14,16,17,19,22,25$	17
27	$1,\dots,15,17,18,20,23,26$	20
30	$1,2,3,5,\dots,15,17,18,20,21,23,26,29$	21
38	$1,2,4,\dots,23,25,26,28,29,31,34,37$	29
42	$1,2,3,5,...,27,29,30,32,33,35,38,41$	33

Suppose n is not displayed in the above list. Then $K(n) = \{1,4,5\} \cup \{n-k \geqslant 6 \mid k \in \{1,4,7,9,10,12,13,15,16,17,\dots\}\} \cup X$, where X is a subset of $\{2,3\}$, and satisfies:

(1) 2 lies in X if and only if the factorization of $n-1$ into prime factors contains either the prime divisor 2 or at least one prime divisor of the form $4k+1$;

(2) 3 lies in X if and only if $n-1$ is not in $\{2^k5, 2^k$, prime numbers of the form $8l+7$, where k and l are positive integers$\}$.

As a consequence of Theorem 1, we have the following corollary which gives an answer to the question posed in ref.[2].

Corollary 1. Let p be a prime. If A is an indecomposable cellular algebra with the property that all eigenvalues of its Cartan matrix C are rational and $\det(C) = p$, then the number of non-isomorphic simple A-modules is a number in $K(p)$, where $K(p)$ is given as follows:

p	$K(p)$	$\|K(p)\|$
2	1	1
3	$1, 2$	2
5	$1, 2, 4$	3
7	$1, 2, 3, 6$	4
11	$1, 2, 4, 5, 7, 10$	6
13	$1, 2, 3, 4, 6, 9, 12$	7
17	$1, 2, 4, 5, 7, 8, 10, 13, 16$	9
19	$1, 2, 3, 4, 6, 7, 9, 10, 12, 15, 18$	11

Suppose $p \geqslant 23$. Then $K(p) = \{1, 2, 4, \ldots, p-15, p-13, p-12, p-10, p-9, p-7, p-4, p-1\} \cup Y$, where $Y = \emptyset$ if $p-1$ is either of the form $2^k 5$ or of the form 2^k for positive integers k, and $Y = \{3\}$ otherwise.

Let us remark that if the spectrum of a Cartan matrix is $\{n, m, ..., m\}$, our method still works, in this case we need to consider the partitions of $n-m$ instead.

The following proposition indicates that the set $K(n)$ is quite big when n is sufficiently large.

Proposition 7.

$$\lim_{n\to\infty} \frac{|K(n)|}{n} = 1.$$

Proof. If n is sufficiently large, then, $1, 4, 5, 6, \ldots, n-15, n-13, n-12, n-10, n-9, n-7, n-4, n-1$ must be in $K(n)$. So we have that $|K(n)| \geqslant n-1-2-7$. On the other hand, for a sufficiently large n, the numbers $n-2, n-3, n-5, n-6, n-8, n-11$ and $n-14$ are not in $K(n)$ by Theorem 1. So we have that $|K(n)| \leqslant n-1-7$. Thus $(n-10)/n \leqslant |K(n)|/n \leqslant (n-8)/n$. This implies that $\lim_{n\to\infty} \frac{|K(n)|}{n} = 1$.

Finally, let us mention the following question: is there a formula or recursive formula for $|K(n)|$?

5 Realization of cellular algebras

In this section we consider whether the matrix corresponding to a partition in $S(n)$ can be realized as the Cartan matrix of an indecomposable cellular algebra, namely, the following general question:

Question. Given a number m in $K(n)$, can we construct an indecomposable cellular algebra A such that the spectrum of the Cartan matrix of A is $\{\lambda_1 = n, \lambda_2 = 1, ..., \lambda_m = 1\}$?

Although we cannot answer this question in general, we do have information along this line in some special situations as we now proceed to show.

Proposition 8. Let C be an $m \times m$ matrix of the form

$$\begin{pmatrix} a+b_1 & a & \dots & a \\ a & a+b_2 & \dots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \dots & a+b_m \end{pmatrix}$$

with a a positive integer and b_i non-negative integers. Then there is an indecomposable cellular algebra A such that its Cartan matrix equals to C.

Proof. We shall define an algebra A by quiver and relations. The quiver Q of A has the vertex set $Q_0 = \{1, 2, ..., m\}$. The arrows are defined as follows:

For two vertices i, j, there are a arrows from i to j, which are labelled as $\alpha_{ij}(1)$, $\alpha_{ij}(2), ...,\ \alpha_{ij}(a)$. For each vertex i, we attach additionally b_i loops at i, which are labelled as $\beta_{ii}(1), \beta_{ii}(2), ..., \beta_{ii}(b_i)$. For relations we put $\mathrm{rad}^2(A) = 0$.

To make A a cellular algebra, we have to define an involution, which sends $\alpha_{ij}(l)$ to $\alpha_{ji}(l)$. So the involution fixes each vertex and each loops at each vertex.

To prove that A is a cellular algebra, we need to display a basis and check the conditions of ref.[8, Proposition 3.4].

$$\begin{matrix} \alpha_{11}(1) & \alpha_{12}(1) & \dots & \alpha_{1m}(1) \\ \alpha_{21}(1) & \alpha_{22}(1) & \dots & \alpha_{2m}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1}(1) & \alpha_{m2}(1) & \dots & \alpha_{mm}(1) \end{matrix} \quad ; \quad ... \quad ; \quad \begin{matrix} \alpha_{11}(a) & \alpha_{12}(a) & \dots & \alpha_{1m}(a) \\ \alpha_{21}(a) & \alpha_{22}(a) & \dots & \alpha_{2m}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1}(a) & \alpha_{m2}(a) & \dots & \alpha_{mm}(a) \end{matrix} \quad ;$$

$\beta_{11}(1);\quad ...;\quad \beta_{11}(b_1);\quad ...;\quad \beta_{mm}(1);\quad ...;\quad \beta_{mm}(b_m);\quad e_1;\quad ...;\quad e_m.$

Since the square of the radical of A is zero, one can check easily that this is a cellular basis for A with respect to the given involution.

From this proposition we know that if m is a divisor of $n-1$, then there is an indecomposable cellular algebra A with m simple modules such that the Cartan matrix of A has the spectrum $\{\lambda_1 = n, \lambda_2 = 1, ..., \lambda_m = 1\}$. This means that the corresponding partition has all equal parts. In the following we consider partitions with different parts.

Proposition 9. Let C be an $m \times m$ matrix of the form

$$\begin{pmatrix} x & a & a & \dots & a \\ a & 2 & 1 & \dots & 1 \\ a & 1 & 2 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & 1 & 1 & \dots & 2 \end{pmatrix}$$

with x a positive integer. If $x \geqslant (a-1)^2(m-1)+2$, then there is an indecomposable cellular algebra A such that its Cartan matrix equals to C.

Proof. The algebra A will be given by quiver and relations. The quiver Q of A has the vertex set $Q_0 = \{1, 2, ..., m\}$. The arrows are defined as follows:

There are a arrows from 1 to j with $j \geqslant 2$, which are labelled as $\alpha_{1j}(1),\ \alpha_{1j}(2)$, ..., $\alpha_{1j}(a)$. Of course, there are a arrows from j to 1 with $j \geqslant 2$, and they are labelled as $\alpha_{j1}(1),\ \alpha_{j1}(2), ...,\ \alpha_{j1}(a)$. For the vertex 1, we attach additionally $b := x-(a-1)^2(m-1)-2$ loops at 1, which are labelled as $\alpha_{11}(1), \alpha_{11}(2), ..., \alpha_{11}(b)$. There is only one arrow from i to j for all $2 \leqslant i, j \leqslant m$. For relations we take all paths of length 2 except the paths $\alpha_{1j}(i)\alpha_{j1}(l)$ with $2 \leqslant i, j, l \leqslant m$.

We define an involution on A by fixing all vertices and the loops at the vertex 1, and sending $\alpha_{ij}(l)$ to $\alpha_{ji}(l)$. To show that A is a cellular algebra with respect to this involution, we display a cell chain of ideals in A as follows:

Let J_{m+1} be an ideal generated by all $\alpha_{ij}(1)$ with $1 \leqslant i, j \leqslant m$, and let J_i be the ideal generated by e_i and J_{i+1} for $i = 2, ..., m$. If $b \geqslant 0$, we define I_1 to be the ideal generated by J_2 and $\alpha_{11}(1)$, and I_t to be the ideal generated by I_{t-1} and $\alpha_{11}(t)$ for $t \leqslant b$. Finally, we define J_1 to be the ideal generated by I_b and e_1. Then we have a chain

$$0 \subset J_{m+1} \subset J_m \subset ... \subset J_2 \subseteq I_1 \subset I_2 \subset ... \subset I_b \subset J_1 = A$$

of ideals in A. Using the proposition 3.4 in ref.[8], we can verify that this is a cell chain. Thus A is a cellular algebra.

From this result we see that if $(a-1)m-2a \leqslant 0$, then there is an indecomposable cellular algebra A such that its Cartan matrix corresponds to the partition $(a^2, 1, ..., 1)$ of m parts and has spectrum $\{\lambda_1 = a^2+m, \lambda_2 = 1, ..., \lambda_m = 1\}$.

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