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Standardly stratified algebras and cellular algebras

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1. *Introduction*

Let A be an Artin algebra. Then there are finitely many non-isomorphic simple A-modules. Suppose $S_1, S_2, \ldots, S_n$ form a complete list of all non-isomorphic simple A-modules and we fix this ordering of simple modules. Let P_i and Q_i be the projective cover and the injective envelope of S_i respectively. With this order of simple modules we define for each i the *standard module* $\Delta(i)$ to be the maximal quotient of P_i with composition factors S_j with $j \leqslant i$. Let Δ be the set of all these standard modules $\Delta(i)$. We denote by $\mathscr{F}(\Delta)$ the subcategory of A-mod whose objects are the modules M which have a Δ-filtration, namely there is a finite chain

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_t = M$$

of submodules of M such that M_i/M_{i-1} is isomorphic to a module in Δ for all i. The modules in $\mathscr{F}(\Delta)$ are called Δ-good modules. Dually, we define the *costandard module* $\nabla(i)$ to be the maximal submodule of Q_i with composition factors S_j with $j \leqslant i$ and denote by ∇ the collection of all costandard modules. In this way, we have also the subcategory $\mathscr{F}(\nabla)$ of A-mod whose objects are these modules which have a ∇-filtration. Of course, we have the notion of ∇-good modules. Note that $\Delta(n)$ is always projective and $\nabla(n)$ is always injective.

From the definition, we have the following properties of standard modules:

(1) $\mathrm{Hom}_A(\Delta(i), \Delta(j)) = 0$ if $i > j$,

(2) $\mathrm{Ext}^1_A(\Delta(i), \Delta(j)) = 0$ if $i \geqslant j$.

For the fixed order of simple modules, beside the subcategories $\mathscr{F}(\Delta)$ and $\mathscr{F}(\nabla)$, we shall investigate the following subcategories of A-mod:

(i) $\mathscr{Y}(\Delta) = \{Y \in A - \mathrm{mod} | \mathrm{Ext}^1(\mathscr{F}(\Delta), Y) = 0\}$,

(ii) $\mathscr{F}(\Delta) \bigcap \mathscr{Y}(\Delta)$,

(iii) $\mathscr{W}(\nabla) = \{W \in A - \mathrm{mod} | \mathrm{Ext}^1(W, \mathscr{F}(\nabla)) = 0\}$,

(iv) $\mathscr{W}(\nabla) \bigcap \mathscr{F}(\nabla)$.

If ${}_AA \in \mathscr{F}(\Delta)$ then A is said to be (left) *standardly stratified* (see [**1**, **7**, **8** or **15**]). A standardly stratified algebra is called a *quasi-hereditary algebra* if the endomorphism ring of each standard module is semisimple (here the definitions are always

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with respect to the ordering of simple modules). Quasi-hereditary algebras were introduced in [**6**] to deal with certain categories in the representation theory of Lie algebras and algebraic groups. They appear also in knot theory (see [**16**]). For a further generalization of standardly stratified algebras see [**7**].

In this paper we are mainly interested in studying these subcategories. First, we retreat the materials of [**3**] in our context and add some new results, here the point is that we want to give a more direct proof of the results in [**3**] only by working on the category $\mathscr{F}(\Delta)$ without knowing the modules in $\mathscr{Y}(\Delta)$. In fact, this paper comes from an understanding of standardly stratified algebras, especially the results in [**3**]. Second, we discuss some subcategories arising from cellular algebras. The paper is organized as follows: in Section 2 we collect some elementary facts and reprove that $\mathscr{F}(\Delta)$ is closed under kernels of surjections. This implies that if A is a standardly stratified algebra then $\mathscr{F}(\Delta)$ is a resolving subcategory. In Section 3 we consider the subcategory $\omega(\Delta) = \mathscr{F}(\Delta) \bigcap \mathscr{Y}(\Delta)$. Note that for a standardly stratified algebra, the subcategory $\omega(\Delta)$ is completely determined by a tilting module [**3**]. In general, this tilting module is not a cotilting module. We prove that it is a cotilting module if and only if the algebra A is a Gorenstein algebra. The endomorphism algebra of the tilting module is discussed in Section 4, here some more direct proofs are presented. In the last section we make some applications of our methods to cellular algebras. Our main interest is in the study of the cohomology of cell modules. The main results in this section are two new homological characterizations of quasi-hereditary algebras in terms of the cohomology groups of cell modules.

2. *Elementary facts on* $\mathscr{F}(\Delta)$

Throughout the paper we denote by A an Artin algebra and by A-mod (respectively, mod-A) the category of all finitely generated left (respectively, right) A-modules. By a module we mean usually a left module. Given two homomorphisms $f\colon L \to M$ and $g\colon M \to N$, we denote the composition of f and g by fg which is a homomorphism from L to N.

In this section, we collect some preliminary facts needed later on in the paper and give a more direct proof of the known fact that $\mathscr{F}(\Delta)$ is closed under kernels of surjective homomorphisms. In particular, if ${}_AA \in \mathscr{F}(\Delta)$, then $\mathscr{F}(\Delta)$ is a resolving subcategory in A-mod.

Now let us recall some definitions from [**5**].

A morphism $f\colon M \to N$ in A-mod is said to be *right minimal* if an endomorphism $g\colon M \to M$ is an automorphism whenever $f = gf$. Note that the morphism f is right minimal if and only if the restriction of f to any direct summand of M is non-zero. A subcategory $\mathscr{X}$ of A-mod is called *contravariantly finite* in A-mod if for each module C there is a right $\mathscr{X}$-approximation, that is, there is a morphism $f\colon X \to C$ with $X \in \mathscr{X}$ such that the induced sequence $\mathrm{Hom}_A(X', X) \to \mathrm{Hom}_A(X', C) \to 0$ is exact for all X' in $\mathscr{X}$. A right $\mathscr{X}$-approximation $f\colon X \to C$ is said to be a minimal right $\mathscr{X}$-approximation if f is right minimal. Dually, one has the notions of left minimal morphisms, left $\mathscr{X}$-approximations and covariantly finite subcategory in A-mod. A subcategory $\mathscr{X}$ in A-mod is called *functorially finite* in A-mod provided it is both contravariantly finite and covariantly finite in A-mod.

Note that the subcategory $\mathscr{F}(\Delta)$ is always functorially finite in A-mod and $\mathscr{Y}(\Delta)$ is covariantly finite in A-mod. Thus, as was proved in [**14**], the subcategory $\mathscr{F}(\Delta)$ is closed under direct summands and has (relative) almost split sequences.

A subcategory $\mathscr{X}$ of A-mod is said to be a *resolving subcategory* if it is closed under extensions, kernels of epimorphisms and it contains all projectives. Dually, a subcategory $\mathscr{X}$ of A-mod is said to be *coresolving* if it is closed under extensions, cokernels of monomorphisms and it contains all injectives.

Given a module M in $\mathscr{F}(\Delta)$, we denote by $[M : \Delta(i)]$ the multiplicity of $\Delta(i)$ in a Δ-filtration of M. Note that this number is independent of the choice of the filtration. By $S_\Delta(M)$ we denote the set of all numbers i in $\{1, 2, \ldots, n\}$ with $[M : \Delta(i)] \neq 0$. This set is called the Δ-*support* of M.

The following result is proved in [**10**, lemma 1.5]. We give here a more elementary proof.

PROPOSITION 2·1. *$\mathscr{F}(\Delta)$ is closed under kernels of surjections.*

Proof. First we have the following fact.

Let $1 \leqslant t \leqslant n$. Then each module M in $\mathscr{F}(\Delta)$ has a unique maximal submodule $M_1 \in \mathscr{F}(\Delta(t), \Delta(t+1), \ldots, \Delta(n))$ such that $M/M_1 \in \mathscr{F}(\Delta(1), \Delta(2), \ldots, \Delta(t-1))$. (This is the consequence of the Ext^1-property of standard modules.)

Now suppose M and N are in $\mathscr{F}(\Delta)$ and f is a surjection from M to N. We shall use induction on the cardinality of $S_\Delta(M)$ to prove that the kernel K of f is still in $\mathscr{F}(\Delta)$.

Suppose $|S_\Delta(M)| = 1$. In this case, $M = \Delta(i)^m$ and $N = \Delta(i)^s$ with $m \geqslant s$. We shall show that $\mathrm{Ext}^1(\Delta(i), K) = 0$ and then K is in $\mathscr{F}(\Delta)$. By definition of the $\Delta(i)$, we have the following exact sequence

$$0 \longrightarrow U(i) \longrightarrow P_i \longrightarrow \Delta(i) \longrightarrow 0,$$

where $U(i)$ is the sum of all images of the homomorphisms $g\colon P_j \to P_i$ with $j > i$. Suppose that h is a non-zero homomorphism from $U(i)$ to K. Then there is an element $x \in U(i)$ and a homomorphism $g\colon P_j \to P_i$ with $j > i$ such that $x \in \mathrm{Im}(g)$ and the image of x under h is not zero. Hence the composition $P_j \xrightarrow{g} \mathrm{Im}(g) \subseteq U(i) \xrightarrow{h} K$ is not zero. This means that K has a composition factor S_j with $j > i$, and thus M has a composition factor isomorphic to S_j, a contradiction. Hence we have proved that $\mathrm{Hom}_A(U(i), K) = 0$. Now applying $\mathrm{Hom}_A(-, K)$ to the above exact sequence, we get the following exact sequence

$$\cdots \longrightarrow \mathrm{Hom}(P_i, K) \longrightarrow \mathrm{Hom}(U(i), K) \longrightarrow \mathrm{Ext}^1(\Delta(i), K) \longrightarrow 0.$$

This shows that $\mathrm{Ext}^1(\Delta(i), K) = 0$, as we desired.

Suppose the theorem is proved for M with the cardinality of $S_\Delta(M)$ smaller than s. Now consider the case that the cardinality of $S_\Delta(M)$ is s. Let t be the largest number in $S_\Delta(M)$. Then, by the fact stated at the beginning of the proof, there is a submodule M_1 of M and a submodule N_1 of N such that M/M_1, $N/N_1 \in \mathscr{F}(\Delta(1), \Delta(2), \ldots, \Delta(t-1))$ and both M_1 and N_1 are in $\mathrm{add}(\Delta(t))$.

Let us consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M_1 & \xrightarrow{f'} & N_1 & & \\
 & & & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} & & \\
0 & \longrightarrow & \mathrm{Ker}(f) & \longrightarrow & M & \xrightarrow{f} & N & \longrightarrow & 0\\
 & & & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} & & \\
0 & \longrightarrow & K' & \longrightarrow & M/M_1 & \longrightarrow & N/N_1 & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Note that f' exists since $\mu f \pi = 0$ and that f' is surjective since f is surjective and t is the maximal number in the Δ-support of M. The kernel of f' lies in $\mathscr{F}(\Delta)$ because $|S_\Delta(M_1)| = 1$. Since the modules M/M_1 and N/N_1 have smaller Δ-supports than M, the kernel K' of the canonical homomorphism $M/M_1 \to N/N_1$ is already in $\mathscr{F}(\Delta)$ by induction. Then the induced exact sequence

$$0 \longrightarrow \mathrm{Ker}(f') \longrightarrow \mathrm{Ker}(f) \longrightarrow K' \longrightarrow 0$$

shows that $\mathrm{Ker}(f)$ is in $\mathscr{F}(\Delta)$ since $\mathscr{F}(\Delta)$ is closed under extensions. This finishes the proof.

Remark. If ${}_AA \in \mathscr{F}(\Delta)$, then we may use induction on the number of simple modules to prove Proposition 2·1. The argument in this case is based on the following simple fact.

Lemma 2·2. *If ${}_AA \in \mathscr{F}(\Delta)$, then for the module ${}_AA$ we have $A_1 = AeA$, where A_1 is the maximal submodule of ${}_AA$ such that A_1 lies in* $\mathrm{add}(\Delta(n))$ *with $\Delta(n) = Ae$ for a primitive idempotent e in A and A/A_1 is in* $\mathscr{F}(\Delta(1), \Delta(2), \ldots, \Delta(n-1))$.

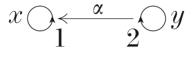
Dually, we have the following result on the subcategory $\mathscr{F}(\nabla)$.

Proposition 2·3. $\mathscr{F}(\nabla)$ *is closed under cokernels of injections.*

As an immediate consequence, we have the following result.

Proposition 2·4. (1) *If ${}_AA \in \mathscr{F}(\Delta)$, then $\mathscr{F}(\Delta)$ is a resolving subcategory in A*-mod.
(2) *If $DA \in \mathscr{F}(\nabla)$, then $\mathscr{F}(\nabla)$ is a coresolving subcategory of A*-mod.

Note that ${}_AA \in \mathscr{F}(\Delta)$ does not imply that $DA \in \mathscr{F}(\nabla)$, in general. Thus A being standardly stratified does not imply that the opposite algebra of A is also standardly stratified. An easy example is the following one: the algebra is given by the quiver



with relations $x^2 = \alpha x = y\alpha = y^2 = 0$. (The composition of two arrows α and x is written in the way that α comes first and then x follows.) Here the right regular module is Δ-filtered, while the left regular module is not.

Given a subcategory $\mathscr{X}$ in A-mod, we define the subcategory $\mathscr{Y}$ (related to $\mathscr{X}$) to be the full subcategory whose objects are the A-modules Y with $\mathrm{Ext}^1(X, Y) = 0$ for all $X \in \mathscr{X}$. We denote by $\mathscr{Y}(\Delta)$ the corresponding subcategory $\mathscr{Y}$ for the category $\mathscr{X} = \mathscr{F}(\Delta)$.

As a consequence, we have the following fact in [**1**, theorem 3·1(iii)].

Corollary 2·5. *If* ${}_AA \in \mathscr{F}(\Delta)$, *then* $\mathrm{Ext}^i_A(X, Y) = 0$ *for all* $X \in \mathscr{F}(\Delta)$ *and* $Y \in \mathscr{Y}(\Delta)$ *and* $i \geqslant 1$.

The following proposition shows the relation of $\mathscr{F}(\nabla)$ and $\mathscr{Y}(\Delta)$.

Proposition 2·6. (1) *If* ${}_AA \in \mathscr{F}(\Delta)$, *then* $\mathscr{Y}(\Delta)$ *is a coresolving subcategory of* A-mod.

(2) $\mathscr{F}(\nabla) \subseteq \mathscr{Y}(\Delta)$.

Proof. Though (1) follows from [**1**, theorem 3·1] by using the description of the modules in $\mathscr{Y}(\Delta)$, we prefer to have a direct proof of (1). $\mathscr{Y}(\Delta)$ is closed under extensions and contains all injective modules. Suppose $f\colon M \to N$ is an injective morphism with M and N in $\mathscr{Y}(\Delta)$. Denote the cokernel of f by C. Applying $\mathrm{Hom}_A(\Delta(i), -)$ to the exact sequence $0 \to M \to N \to C \to 0$, we get then

$$\mathrm{Ext}^1_A(\Delta(i), N) \longrightarrow \mathrm{Ext}^1_A(\Delta(i), C) \longrightarrow \mathrm{Ext}^2_A(\Delta(i), M).$$

By 2·5, both end terms vanish, thus the middle term vanishes, too. This means that C is in $\mathscr{Y}(\Delta)$.

(2) follows from [**10**, lemma 1·3].

The dual statement of the above result is the following.

Proposition 2·7. (1) *If* $DA \in \mathscr{F}(\nabla)$, *then* $\mathscr{W}(\nabla)$ *is resolving subcategory in* A-mod.

(2) $\mathscr{F}(\Delta) \subseteq \mathscr{W}(\nabla)$.

Remark. The above inclusion can be proper. Let us illustrate this by examples. We denote by k a field in the following examples.

(1) Consider the algebra $A = k[x]/(x^2)$. This is a local algebra with only one simple module. We have $\mathscr{F}(\Delta) = \mathscr{F}(\nabla) = \mathrm{add}(A)$. But $\mathscr{Y}(\Delta)$ is the whole module category which contains properly the subcategory $\mathscr{F}(\nabla)$.

(2) In the quasi-hereditary case, the intersection of $\mathscr{F}(\Delta)$ with $\mathscr{F}(\nabla)$ is not zero, but for the standardly stratified algebras the situation is different. Consider the algebra $A = k[x, y]/(x, y)^2$. It is clear that $\mathscr{F}(\Delta)$ is just the projective modules and that $\mathscr{F}(\nabla)$ is just the injective modules. Thus the intersection of $\mathscr{F}(\Delta)$ with $\mathscr{F}(\nabla)$ is zero. Of course, we also have a proper inclusion of $\mathscr{F}(\nabla)$ into $\mathscr{Y}(\Delta)$.

For quasi-hereditary algebras, we know that $\mathscr{F}(\nabla) = \mathscr{Y}(\Delta)$, as was shown in [**14**]. The following result in [**1**] shows when this is true for standardly stratified algebras.

Proposition 2·8. *Let* A *be a standardly stratified algebra. Then* $\mathscr{F}(\nabla) = \mathscr{Y}(\Delta)$ *if and only if* $\mathrm{gl.dim}(A) < \infty$ *if and only if* A *is quasi-hereditary.*

Note that we can give a more direct proof of this proposition without using the fact that $\mathscr{Y}(\Delta) = \mathscr{F}(\bar{\nabla})$ for standardly stratified algebras proved in [**1**].

3. *Tilting modules and cotilting modules*

In this section we consider the intersection of the subcategory $\mathscr{F}(\Delta)$ with $\mathscr{Y}(\Delta)$, which is denoted by $\omega(\Delta)$. This subcategory is determined by a tilting module. We shall prove that this tilting module is cotilting if and only if the algebra A is Gorenstein.

Definition 3·1. Let A be an Artin algebra. A module T in A-mod is called a *tilting module* if

(1) $\mathrm{Ext}_A^i(T,T) = 0$ for all $i > 0$, and

(2) the projective dimension of T is finite, and

(3) there is an exact sequence

$$0 \longrightarrow {}_AA \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_s \longrightarrow 0,$$

such that T_i belongs to add (T) for all i.

Dually, we have also the concept of cotilting module: an A-module T in A-mod is said to be a *cotilting module* if it satisfies (1), and

(2′) the injective dimension of T is finite, and

(3′) there is an exact sequence

$$0 \longrightarrow T_s \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow DA \longrightarrow 0,$$

such that T_i belongs to add (T) for all i. Here we use D to denote the usual duality for Artin algebras.

Now let us begin with the following general result proved in [**4**, proposition 3·4].

Lemma 3·2. *Let $\mathscr{X}$ be a resolving contravariantly finite subcategory of A*-mod. *Denote by $\mathscr{Y}$ the subcategory whose objects are the modules C with* $\mathrm{Ext}_A^1(\mathscr{X}, C) = 0$. *Then $\omega = \mathscr{X} \cap \mathscr{Y}$ has the following properties*:

(1) *ω is self-orthogonal, that is,* $\mathrm{Ext}_A^i(X,Y) = 0$ *for all $X, Y \in \omega$ and all $i > 0$;*

(2) *for each X in $\mathscr{X}$ there is an exact sequence $0 \to X \to W \to X' \to 0$ with W in ω and X' in $\mathscr{X}$;*

(3) *for each Y in $\mathscr{Y}$ there is an exact sequence $0 \to Y' \to W \to Y \to 0$ with W in ω and Y' in $\mathscr{Y}$.*

The following theorem is proved in [**3**]. We include here a short proof.

Theorem 3·3. *Let A be a standardly stratified algebra. Then there is a tilting module T* (*unique up to multiplicity of indecomposable direct summands*) *such that* add $(T) = \omega(\Delta)$.

Proof. It is easy to see that proj.dim $X \leqslant n-1$ for all $X \in \mathscr{F}(\Delta)$ (see also [**2**]). Let $X_{-1} = X$. By Lemma 3·2, we can construct exact sequences ϵ_i: $0 \to X_{i-1} \to W_i \to X_i \to 0$ with $W_i \in \omega(\Delta)$ and $X_i \in \mathscr{F}(\Delta)$. Applying $\mathrm{Hom}_A(X_{n-1}, -)$ to ϵ_i, we get that $\mathrm{Ext}_A^j(X_{n-1}, X_i) \cong \mathrm{Ext}_A^{j+1}(X_{n-1}, X_{i-1})$. This yields that $\mathrm{Ext}_A^1(X_{n-1}, X_{n-2}) = \mathrm{Ext}_A^2(X_{n-1}, X_{n-3}) = \cdots = \mathrm{Ext}_A^n(X_{n-1}, X_{-1}) = 0$ since proj.dim $X_{n-1} \leqslant n-1$. Hence the exact sequence ϵ_{n-1} splits and therefore the module X_{n-2} is a direct summand of W_{n-1} and in $\omega(\Delta)$. So we have an exact sequence

$$0 \longrightarrow X \longrightarrow W_0 \longrightarrow W_1 \longrightarrow \cdots \longrightarrow W_{n-2} \longrightarrow W_{n-1} \longrightarrow 0,$$

with $W_{n-1} = X_{n-2}$.

Since $A \in \mathscr{F}(\Delta)$, we have an exact sequence

$$0 \longrightarrow {}_A A \longrightarrow W_0 \longrightarrow W_1 \longrightarrow \cdots \longrightarrow W_{n-2} \longrightarrow W_{n-1} \longrightarrow 0,$$

with $W_i \in \omega(\Delta)$. Put $T = \oplus_j W_j$. Then T is a tilting module. Moreover, if $M \in \omega(\Delta)$, then $T \oplus M$ is a tilting module. By tilting theory, we must have $M \in \text{add}(T)$. Therefore $\omega(\Delta) = \text{add}(T)$. This finishes the proof.

Comparing this result with that for quasi-hereditary algebras, however, one cannot hope that the module T with $\text{add}(T) = \omega(\Delta)$ would be always a cotilting module. Let us see an example. Put $A = k[x, y]/(x, y)^2$. Then $\mathscr{F}(\Delta) = \text{add}(A)$ and $\mathscr{Y}(\Delta) = A$-mod. Hence $\mathscr{F}(\Delta) = \mathscr{Y}(\Delta) \bigcap \mathscr{F}(\Delta)$. Since A is a local algebra, we know that the A-modules of finite injective dimension are just the injective modules. This shows that the injective dimension of ${}_A A$ is infinite and therefore there is no cotilting module T with $\omega(\Delta) = \text{add}(T)$.

For a quasi-hereditary algebra, we know from [**14**] that there is a cotilting module T such that $\omega(\Delta) = \text{add}(T)$. The following easy corollary indicates that in some other cases we can obtain a cotilting module T, too.

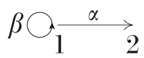
COROLLARY 3·4. *Let A be a standardly stratified algebra. Then the following are equivalent*:

(1) $\omega(\Delta) = \text{add}(DA)$,

(2) $DA \in \mathscr{F}(\Delta)$.

Proof. If $DA \in \mathscr{F}(\Delta)$, then $DA \in \omega(\Delta)$. Since the number of non-isomorphic indecomposable modules in $\omega(\Delta)$ is the number of non-isomorphic simple modules, we must have that $\text{add}(DA) = \omega(\Delta)$.

Now let us give an algebra satisfying the conditions in the above corollary. Consider the algebra A given by the following quiver with relations:



$\beta^2 = 0.$

Then $\Delta(2) = S_2$ and $\Delta(1)$ is the uniserial module with two composition factors which are isomorphic to S_1. Clearly, A is standardly stratified. It is obvious that the injective module Q_1 is isomorphic to $\Delta(1)$, thus it lies in $\mathscr{F}(\Delta)$. One can also check that the injective module Q_2 possesses a Δ-filtration. Hence $DA \in \mathscr{F}(\Delta)$. Note that this algebra is neither self-injective, nor quasi-hereditary.

The condition that $\omega(\Delta) = \text{add}(DA)$ gives us an interesting class of algebras. Recall that an Artin algebra is called a *Gorenstein algebra* if $\text{inj.dim}\, {}_A A < \infty$ and $\text{inj.dim}\, A_A < \infty$. We have the following easy observation.

COROLLARY 3·5. *Let A be a standardly stratified algebra. If $\omega(\Delta) = \text{add}(DA)$, then A is a Gorenstein algebra.*

Proof. We know from $\omega(\Delta) = \text{add}(DA)$ that $DA \in \mathscr{F}(\Delta)$, this implies that $\text{proj.dim}\, DA < \infty$, that is, $\text{inj.dim}\, A_A < \infty$. By [**2**], the finitistic dimension of A is finite. (Recall that the finitistic dimension of A is by definition the supremum of projective dimensions of all modules with finite projective dimension.) This yields together with [**4**, proposition 6·10] that $\text{inj.dim}\, {}_A A < \infty$. Thus A is Gorenstein.

It would be interesting to know how the algebra A looks when the module T with $\omega(\Delta) = \mathrm{add}(T)$ is cotilting. The rest of this section is devoted to a discussion of this and provides an answer to the question.

Theorem 3·6. *Let A be a standardly stratified algebra and $\omega(\Delta) = \mathrm{add}(T)$. The following are equivalent*:

(1) *T is a cotilting module*;

(2) $\mathrm{inj.dim}\, A_A < \infty$;

(3) *A is a Gorenstein algebra.*

Proof. If T is a cotilting module, then there is an exact sequence

$$0 \longrightarrow T_s \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow DA \longrightarrow 0,$$

with $T_i \in \omega(\Delta)$. Since each T_i in $\mathscr{F}(\Delta)$ has finite projective dimension, we see that DA has finite projective dimension, and this is equivalent to saying that A_A has finite injective dimension.

Conversely, if $\mathrm{inj.dim}\, A_A < \infty$, then $\mathrm{proj.dim}\, DA < \infty$. By Lemma 3·2, we can construct a series of short exact sequences

$$\epsilon_i\colon\ 0 \longrightarrow Y_i \longrightarrow W_i \longrightarrow Y_{i-1} \longrightarrow 0,$$

with $Y_{-1} = DA$, $Y_i \in \mathscr{Y}(\Delta)$ and $W_i \in \omega(\Delta)$. Assume that $\mathrm{proj.dim}\, DA = m < \infty$. We apply $\mathrm{Hom}_A(-, Y_m)$ to the exact sequence ϵ_i and get the following exact sequence

$$\mathrm{Ext}^j_A(W_i, Y_m) \longrightarrow \mathrm{Ext}^j_A(Y_i, Y_m) \longrightarrow \mathrm{Ext}^{j+1}_A(Y_{i-1}, Y_m) \longrightarrow \mathrm{Ext}^{j+1}_A(W_i, Y_m).$$

For $j \geqslant 1$ we have that $\mathrm{Ext}^j_A(W_i, Y_m) = 0 = \mathrm{Ext}^{j+1}_A(W_i, Y_m)$ since $W_i \in \mathscr{F}(\Delta)$ and $Y_i \in \mathscr{Y}(\Delta)$. This yields that $\mathrm{Ext}^j_A(Y_i, Y_m) \cong \mathrm{Ext}^{j+1}_A(Y_{i-1}, Y_m)$ for $j \geqslant 1$ and all i. In particular, $\mathrm{Ext}^1_A(Y_{m-1}, Y_m) = \mathrm{Ext}^2_A(Y_{m-2}, Y_m) = \cdots = \mathrm{Ext}^{m+1}_A(Y_{-1}, Y_m) = 0$ since $\mathrm{proj.dim}\, DA = m$. Thus the sequence ϵ_m splits and $Y_{m-1} \in \omega(\Delta)$. This implies that for DA we have an exact sequence

$$0 \longrightarrow W_m \longrightarrow \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow DA \longrightarrow 0,$$

with $W_i \in \omega(\Delta)$. Now we show that $T' := \oplus_j W_j$ is a cotilting module. It is sufficient to prove that the injective dimension of T' is finite. Since $\mathrm{inj.dim}\, A_A < \infty$ and the finitistic dimension of A is finite, we see that $\mathrm{inj.dim}\, {}_AA < \infty$ by [**4**, proposition 6·10]. This shows that A is Gorenstein. It follows now from [**4**, proposition 6·9] that each module with finite projective dimension also has finite injective dimension. Thus T' has finite injective dimension. By Theorem 3·3, the number of non-isomorphic indecomposable modules in $\omega(\Delta)$ is the number of non-isomorphic simple A-modules, hence $\omega(\Delta) = \mathrm{add}(T') = \mathrm{add}(T)$. This finishes the proof of the equivalence of (1) and (2). It follows from the above proof and the definition of Gorenstein algebras that (2) and (3) are also equivalent. Thus the proof is completed.

Clearly, $\mathscr{F}(\Delta) \subseteq {}^{\perp}\omega(\Delta)$ for any standardly stratified algebra, where ${}^{\perp}\omega(\Delta)$ is the subcategory of A-mod whose objects are the X with $\mathrm{Ext}^i_A(X, \omega(\Delta)) = 0$ for all $i > 0$. In general, this inclusion is proper, one can know this from an example like $A = k[x]/(x^2)$. It would be interesting to know under which conditions on the standardly stratified algebra A we could have $\mathscr{F}(\Delta) = {}^{\perp}\omega(\Delta)$. Of course, quasi-hereditary algebras have this property. The following is another example of this kind of algebra.

Let A be a local algebra which is not self-injective, then A is automatically a standardly stratified algebra with $\mathscr{F}(\Delta) = \mathrm{add}(A) = {}^{\perp}\omega(\Delta)$.

Remark. We remark that a special case is considered in [**3**], namely it was shown in [**3**] that $\mathscr{F}(\Delta) \bigcap \mathscr{Y}(\Delta) = \mathrm{add}(T)$ for a cotilting module T with $\mathscr{F}(\Delta) = {}^{\perp}(\mathrm{add}(T))$ if and only if the algebra is quasi-hereditary.

4. *Endomorphism algebras of tilting modules*

We have seen that if A is a standardly stratified algebra then there is a tilting module T such that $\omega(\Delta) = \mathrm{add}(T)$. In this section we consider the endomorphism algebra of this module T. Though the results in this section are basically known in [**3**], we prefer to give more direct proofs by just working with the category $\mathscr{F}(\Delta)$.

First let us describe the indecomposable modules in $\omega(\Delta)$. The following proposition is basically contained in [**3**, lemma 2·5].

PROPOSITION 4·1. *Let A be a standardly stratified algebra. Then, for each i, there is an exact sequence*

$$0 \longrightarrow \Delta(i) \stackrel{\beta}{\longrightarrow} T(i) \longrightarrow X(i) \longrightarrow 0,$$

with β a left minimal $\mathscr{Y}(\Delta)$-approximation and $X(i) \in \mathscr{F}(\Delta(1), \ldots, \Delta(i-1))$. Moreover, the module $T(i)$ is indecomposable and $\omega(\Delta) = \mathrm{add}(\oplus_j T(j))$.

Proof. By Lemma 3·2, or [**14**, lemma 4], there is an exact sequence

$$0 \longrightarrow \Delta(i) \stackrel{\beta'}{\longrightarrow} T'(i) \longrightarrow X'(i) \longrightarrow 0,$$

with $T'(i) \in \omega(\Delta)$ and $X'(i) \in \mathscr{F}(\Delta(1), \ldots, \Delta(i-1))$. Clearly, the β' is a left $\mathscr{Y}(\Delta)$-approximation for $\Delta(i)$. Thus we may choose a minimal left $\mathscr{Y}(\Delta)$-approximation for $\Delta(i)$:

$$0 \longrightarrow \Delta(i) \stackrel{\beta}{\longrightarrow} T(i) \stackrel{\pi}{\longrightarrow} X(i) \longrightarrow 0,$$

where $T(i)$ lies in $\mathscr{Y}(\Delta)$ and $X(i)$ in $\mathscr{F}(\Delta(1), \ldots, \Delta(i-1))$. Now we show that $T(i)$ is indecomposable.

Suppose that $T(i) = T_1 \oplus T_2$, $\beta = (\beta_1, \beta_2)$ and $\pi = (\pi_1, \pi_2)^t$, with $\beta_j\colon \Delta(i) \to T_j$ and $\pi_j\colon T_j \to X(i)$ for $j = 1, 2$. Since β is minimal, both β_1 and β_2 are non-zero. Now we consider the following pullback diagram:

$$\begin{array}{ccc} \Delta(i) & \xrightarrow{\beta_1} & T_2 \\ {\scriptstyle \beta_2}\downarrow & & \downarrow{\scriptstyle -\pi_1} \\ T_2 & \xrightarrow{\pi_2} & X(i) \end{array}$$

Since $X(i)$ has no composition factor isomorphic to S_i, we see that $\mathrm{Hom}_A(\Delta(i), X(i)) = 0$. This implies that $\beta_1\pi_1 = \beta_2\pi_2 = 0$. It follows then from the property of pullback diagrams and from $\beta_1(-\pi_1) = 0\pi_2 = 0$ that there is a homomorphism $\alpha\colon \Delta(i) \to \Delta(i)$ such that $\beta_1 = \alpha\beta_1$ and $0 = \alpha\beta_2$. Since $\mathrm{End}\Delta(i)$ is a local algebra, we know from $0 = \alpha\beta_2$ that α is nilpotent, and therefore $\beta_1 = \alpha\beta_1 = \alpha^m\beta_1$ for all m, thus β_1 is zero, a contradiction. Hence $T(i)$ is indecomposable. Since the composition factors of $T(i)$ are of the form S_j with $j \leqslant i$ and S_i occurs at least once, the modules $T(i)$, $1 \leqslant i \leqslant n$, are pairwise non-isomorphic, and therefore $\omega(\Delta) = \mathrm{add}(\oplus_j\, T(j))$.

We also need the following fact:

Lemma 4·2. *Suppose A is standardly stratified. Let $T = \oplus_j T(j)$ and $B = \mathrm{End}({}_AT)$. Then*

(1) *for each module $X \in \mathscr{F}(\Delta)$, the evaluation map*

$$e_X\colon {}_AX \cong {}_A\mathrm{Hom}_B(\mathrm{Hom}_A(X,T)_B, T)$$

is an isomorphism of A-modules.

(2) *The contravariant functor $\mathrm{Hom}_A(-,T)$ is an equivalence between the category $\mathscr{F}(\Delta)$ and its image which is a subcategory of* mod-B.

Proof. (1) The isomorphism is true for $X = T$, so it is true for direct summands of T and therefore for all modules X in $\mathrm{add}(T)$. By 3·2, for a general X in $\mathscr{F}(\Delta)$ we have an exact sequence

$$0 \longrightarrow X \longrightarrow T_0 \xrightarrow{f_0} T_1 \longrightarrow \cdots \xrightarrow{f_{s-1}} T_s \longrightarrow 0$$

such that the image X_i of f_i is in $\mathscr{F}(\Delta)$ and T_i in $\mathrm{add}(T)$. So we have the following exact sequences

$$\epsilon_i\colon\ 0 \longrightarrow X_{i-1} \longrightarrow T_i \longrightarrow X_i \longrightarrow 0$$

for $1 \leqslant i \leqslant s-1$ with $X_{-1} = X$ and $X_s = T_s$, and the following commutative diagram (for simplicity, we denote $\mathrm{Hom}(X,T)$ just by X^*):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_{s-2} & \longrightarrow & T_{s-1} & \longrightarrow & T_s & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle e_{X_{s-2}}} & & \downarrow{\scriptstyle e_{T_{s-1}}} & & \downarrow{\scriptstyle e_{T_s}} & & \\
0 & \longrightarrow & X^{**}_{s-2} & \xrightarrow{\pi_2} & T^{**}_{s-1} & \longrightarrow & T^{**}_s & &
\end{array}$$

Note that $\mathrm{Hom}_A(-,T)$ is exact on $\mathscr{F}(\Delta)$. Since $e_{T_{s-1}}$ and e_{T_s} are isomorphisms, we see that $e_{X_{s-2}}$ is an isomorphism. Inductively, we can show that e_X is an isomorphism. This proves (1).

Lemma 4·2(2) is essentially in [**3**, theorem 2·6(iv)]. It also follows from (1) and the canonical isomorphism

$$\mathrm{Hom}_R({}_RM, \mathrm{Hom}_S(N_S, {}_RU_S)) \cong \mathrm{Hom}_S(N_S, \mathrm{Hom}_R({}_RM, {}_RU_S))$$

for two rings R and S, and the modules M, N and U. Here we take $R = A$, $S = B$, $U = T$, $M = Y$ and $N = \mathrm{Hom}_A(X,T)$. This finishes the proof.

Now we reprove the following theorem 2·6(v) in [**3**].

Theorem 4·3. *Let A be a standardly stratified algebra. Then the endomorphism algebra of T is right standardly stratified algebra* (*the order is just the opposite order of that for the algebra A*).

Proof. We define $i' = n - i + 1$ and denote the projective right B-module $\mathrm{Hom}_A(T(i'),T)$ by $P'(i)$. Now we define $\Delta'(i) = \mathrm{Hom}_A(\Delta(i'),T)$. Let Δ' be the collection of $\Delta'(1), \Delta'(2), \ldots, \Delta'(n)$. Since $\Delta(1)$ is in $\omega(\Delta)$, we see that $\Delta'(n)$ is a projective right B-module. It is clear that each $P'(i)$ has a Δ'-filtration since the functor $F = \mathrm{Hom}_A(-, T)$ is exact on $\mathscr{F}(\Delta)$. To finish the proof, we need to show that $\Delta'(i)$ is obtained from $P'(i)$ by factoring out all homomorphic images $\mathrm{Im}(g)$ with g: $P'(j) \to P'(i)$ with $j > i$. It follows from Proposition 4·1 that we have the following exact sequence of right B-modules:

$$0 \longrightarrow F(X(n-i+1)) \longrightarrow F(T(n-i+1)) \longrightarrow F(\Delta(n-i+1)) \longrightarrow 0.$$

If g is a right B-module homomorphism from $P'(j)$ to $P'(i)$ with $j > i$, then, by Lemma 4·2, there is a left A-module homomorphism h from $T(i')$ to $T(j')$ such that $g = F(h)$. Let $\beta_{i'}$ denote the minimal morphism in Lemma 4·1, then $\beta_{i'}h = 0$ since $i' > j'$ and the composition factors of $T(j')$ are of the form S_t with $t \leqslant j'$. Thus h factors through the morphism $T(i') \to X(i')$ and then the image of g belongs to $FX(i')$. This shows that $\Delta'(i)$ is a maximal quotient module of $P'(i)$ with composition factors of indices at most i. This completes the proof.

If we start from the right standardly stratified algebra B and make the similar construction as we did for the left standardly stratified algebra A, then we get a tilting right B-module T'_B for the corresponding orthogonal category $\omega(\Delta')$. The following question now arises. Is the endomorphism algebra of T'_B isomorphic to the algebra A? Before we answer this question, we deduce the following lemma from 4·2 which is needed.

Lemma 4·4. *Suppose that A is left standardly stratified. Then, for modules X, Y in $\mathscr{F}(\Delta)$, the following holds*: $\mathrm{Ext}^1_B(FY, FX) \cong \mathrm{Ext}^1_A(X, Y)$, *where F is the contravariant functor* $\mathrm{Hom}_A(-, T)$.

Proof. By Lemma 3·2, we have an exact sequence

$$0 \longrightarrow Y \longrightarrow T_0 \longrightarrow Y_0 \longrightarrow 0,$$

with $T_0 \in \omega(\Delta)$ and $Y_0 \in \mathscr{F}(\Delta)$. This provides the following exact sequences:

$$0 \longrightarrow FY_0 \longrightarrow FT_0 \longrightarrow FY \longrightarrow 0$$

and

$$\mathrm{Hom}_B(FT_0, FX) \longrightarrow \mathrm{Hom}_B(FY_0, FX) \longrightarrow \mathrm{Ext}^1_B(FY, FX) \longrightarrow \mathrm{Ext}^1_B(FT_0, FX).$$

Note that the last term in the last sequence vanishes because FT_0 is a projective right B-module. Now it follows from Lemma 4·2 and the following commutative diagram

$$\begin{array}{ccccccc} \mathrm{Hom}_B(FT_0, FX) & \longrightarrow & \mathrm{Hom}_B(FY_0, FX) & \longrightarrow & \mathrm{Ext}^1_B(FY, FX) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Hom}_A(X, T_0) & \longrightarrow & \mathrm{Hom}_A(X, Y_0) & \longrightarrow & \mathrm{Ext}^1_A(X, Y) & \longrightarrow & 0, \end{array}$$

that the lemma is true since the first two vertical maps are bijective by Lemma 4·2.

The following theorem essentially proved in [**3**, theorem 2·6(vii)] is an answer to the above question.

Theorem 4·5. *Let A be a standardly stratified algebra and T the tilting module such that* $\mathrm{add}(T) = \omega(\Delta)$. *Let B be the endomorphism algebra of T. For the right standardly* *stratified algebra B, we denote by T' the right tilting B-module with* $\mathrm{add}(T') = \omega(\Delta')$. *Then the endomorphism algebra of T' is Morita equivalent to the opposite algebra of A.*

Proof. We show that FP_i is in $\omega(\Delta')$. Once we have achieved this, then the theorem follows immediately from Lemma 4·2. Since P_i has a Δ-filtration, FP_i is in $\mathscr{F}(\Delta')$. But the above lemma says that FP_i belongs also to $\mathscr{Y}(\Delta')$. Thus it is in $\omega(\Delta')$, as was desired.

5. *Subcategories from cellular algebras*

In this section we study some subcategories arising from the cell modules over a cellular algebra. We first recall the definitions of cellular algebras given in [**11**] and [**12**], and then we give new homological characterizations of quasi-hereditary algebras inside the class of cellular algebras in terms of the cell modules. For further information on the study of cellular algebras one may refer to the survey paper [**13**].

Let k be an arbitrary field.

Definition 5·1 ([**11**]). An associative k-algebra A is called a *cellular algebra* with cell datum (I, M, C, i) if the following conditions are satisfied.

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has a k-basis $C^{\lambda}_{S,T}$ where (S,T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is a k-linear anti-automorphism of A with $i^2 = id$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^{\lambda}_{S,T}$ can be written as $(\sum_{U \in M(\lambda)} r_a(U,S)C^{\lambda}_{U,T}) + r'$ where r' is a linear combination of basis elements with upper index μ strictly greater than λ, and where the coefficients $r_a(U,S) \in k$ do not depend on T.

Note that our partial order on Λ is just the opposite one used in [**11**]. Typical examples of cellular algebras are Hecke algebras of type A_n, Brauer algebras, partition algebras and many others (see [**11**, **13** and **16**]).

In the following we shall call a k-linear anti-automorphism i of A with $i^2 = id$ an *involution* of A.

For each $\lambda \in \Lambda$, there is a *cell module* $W(\lambda)$ with a k-basis $\{C_S | S \in M(\lambda)\}$, the module structure is given by

$$aC_S = \sum_{T \in M(\lambda)} r_a(T,S)C_T,$$

where the coefficients $r_a(T,S)$ are the same as in Definition 5·1. We also have a right cell module $i(W(\lambda))$ which is defined dually. For a cell module one can also define a bilinear form Φ_λ: $W(\lambda) \times W(\lambda) \to k$ by

$$C^{\lambda}_{S,S}C^{\lambda}_{T,T} \equiv \Phi_\lambda(C_S, C_T)C^{\lambda}_{S,T}$$

modulo the ideal generated by all basis elements with upper index greater than λ. We denote this ideal by $J^{>\lambda}$.

Let $\Lambda_0 = \{\lambda \in \Lambda | \Phi_\lambda \neq 0\}$.

We also need the following equivalent definition of cellular algebras in [**12**].

Definition 5·2 ([**12**]). Let A be a k-algebra. Assume that there is an involution i on A. A two-sided ideal J in A is called a *cell ideal* if and only if $i(J) = J$ and there exists a left ideal $W \subset J$ such that W has finite k-dimension and that there is an isomorphism of A-bimodules α: $J \simeq W \otimes_k i(W)$ (where $i(W) \subset J$ is the i-image of W) making the following diagram commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & W\otimes_k i(W) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle x\otimes y \mapsto i(y)\otimes i(x)} \\ J & \xrightarrow{\alpha} & W\otimes_k i(W) \end{array}$$

The algebra A (with the involution i) is called *cellular* if and only if there is a vector space decomposition $A = J_1' \oplus J_2' \oplus \cdots \oplus J_n'$ (for some n) with $i(J_j') = J_j'$ for each j and such that setting $J_j = \oplus_{l=1}^{j} J_l'$ gives a chain of two-sided ideals of A: $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ (each of them fixed by i) and for each j ($j = 1, \ldots, n$) the quotient $J_j' = J_j/J_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1}.

The modules $W(j)$, $1 \leqslant j \leqslant n$, which are obtained from the sections J_j/J_{j-1} of the chain, are the cell modules. The above chain in A is called a *cell chain*. It is proved ([**12**]) that a cell ideal J is either a heredity ideal or $J^2 = 0$. Note that the cell modules are called standard modules in [**12**, **13**].

The following result is shown in [**11**].

Lemma 5·3. *Let A be a cellular algebra with cell datum (I, M, C, i). Then*:

(1) *the simple modules are parameterized by $\Lambda_0 := \{\lambda \in \Lambda | \Phi_\lambda \neq 0\}$. For $\lambda \in \Lambda_0$, we denote by S_λ the simple module corresponding to λ, which is isomorphic to the top of the cell module $W(\lambda)$*;

(2) *the following holds for the multiplicity $[W(\lambda) : S_\mu]$ of a simple module $S_\mu (\mu \in \Lambda_0)$ in a cell module $W(\lambda)$*:

$$[W(\lambda) : S_\mu] = \begin{cases} 0, & \text{unless } \mu \leqslant \lambda, \\ 1, & \lambda = \mu. \end{cases}$$

In particular, for $\lambda, \mu \in \Lambda_0$, $\mathrm{Hom}_A(W(\lambda), W(\mu)) = 0$ unless $\lambda \leqslant \mu$, and $\mathrm{End}_A(W(\lambda)) = k$.

Let A be a cellular algebra with respect to an involution i. Then we have a natural duality i from A-mod to mod-A: given $X \in A$-mod, define $i(X) = X$ with the right module structure $x \cdot a = i(a)x$ for all $x \in X$ and $a \in A$. Furthermore, we define $X^* = \mathrm{Hom}_k(i(X), k)$. Clearly, the functor * is a self-dual functor and fixes isomorphism classes of simple modules by [**12**].

For a subset $\Phi \subseteq \Lambda$, we put $W(\Phi) = \{W(\mu) | \mu \in \Phi\}$ and $W(\Phi)^* = \{W(\mu)^* | \mu \in \Phi\}$.

Note also that given a cellular algebra A with a cell chain

$$0 \subset J_1 \subset J_2 \subset \cdots \subset J_m = A,$$

the length m of this chain is the cardinality of the poset Λ, and this number is usually larger than the number of non-isomorphic simple modules. It is known that the cell chain is a heredity chain if and only if J_j^2 is not contained in J_{j-1} for all j, and this is equivalent to that the poset Λ coincides with Λ_0 (see [**13**]).

Before we give our criteria for quasi-hereditary algebras, we first point out the following fact.

PROPOSITION 5·4. *Let A be a cellular algebra. If* $\mathrm{gl.dim}(A) = \infty$, *then there is a cell module $W(\lambda)$ with $\lambda \in \Lambda_0$ such that the projective dimension of $W(\lambda)$ is infinite.*

Proof. We prove the following statement: if $\mathrm{proj.dim}\, W(\mu) < \infty$ for all $\mu \in \Lambda_0$, then $\mathrm{proj.dim}\, S_\mu < \infty$ for all $\mu \in \Lambda_0$ and $\mathrm{gl.dim}\,(A) < \infty$.

Indeed, if μ is a minimal element in Λ_0, then $W(\mu)$ is a simple module since the composition factors of $W(\mu)$ are of the form S_λ with $\lambda \leqslant \mu$ and $[W(\mu) : S_\mu] = 1$. Hence $\mathrm{proj.dim}\, S_\mu = \mathrm{proj.dim}\, W(\mu) < \infty$. Suppose that μ is not a minimal element in Λ_0. Then the radical of $W(\mu)$ has composition factors of the form S_λ with $\lambda < \mu$ by Lemma 5·3, and by induction, for those λ we have that $\mathrm{proj.dim}\, S_\lambda < \infty$. It follows now from the exact sequence $0 \to \mathrm{rad}(W(\mu)) \to W(\mu) \to S_\mu \to 0$ that $\mathrm{proj.dim}\, S_\mu < \infty$. Hence $\mathrm{gl.dim}(A) < \infty$.

The next result gives a criterion for quasi-heredity in terms of first cohomology groups of cell modules.

THEOREM 5·5. *For a cellular algebra A the following are equivalent*:

(1) *A is quasi-hereditary*;

(2) $\mathrm{Ext}^1_A(W(\lambda), (W(\mu))^*) = 0$ *for all* $\lambda, \mu \in \Lambda$.

Proof. If A is quasi-hereditary, then we know that the cardinality of Λ is the number of the non-isomorphic simple A-modules and that the cell modules are just the standard modules. Hence the statement (2) holds true.

Conversely, assume that (2) holds. For the given cell datum, we have a cell chain

$$0 \subset J_1 \subset J_2 \subset \cdots \subset J_m = A.$$

Note that the cell modules are obtained from the sections of this chain. If J_1 is a heredity ideal in A, then, by induction on the length of the cell chain, we can show that A is a quasi-hereditary algebra. Since J_1 is either a heredity ideal or $J_1^2 = 0$ by [**12**], the remaining case to be considered is the latter one, i.e. when $J_1^2 = 0$. We shall prove that this is impossible unless $J_1 = 0$. Let $J = J_1$ and $B = A/J$. Then $J \cong W \otimes_k i(W)$, where W is a left cell module and $i(W)$ is the right cell module. This means that J_A is a direct sum of copies of $i(W)$. The canonical exact sequence

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

shows that $J \otimes_A J \cong \mathrm{Tor}_2^A(B, B) \cong \mathrm{Tor}_1^A(\oplus i(W), B)$ by [**12**, proposition 6·1]. From the definition of the cell chain, we may assume that $J_j/J_{j-1} \cong W(j) \otimes_k i(W(j))$, where $W(j)$ is the cell module for all j and $W = W(1)$. Now it follows from the canonical isomorphism $\mathrm{DExt}^j_A(X, Y) \cong \mathrm{Tor}^A_j(DY, X)$ that $\mathrm{Tor}^A_j(i(W(s)), W(t)) \cong \mathrm{Ext}^j_A(W(t), Di(W(s))) = 0$ for $j = 1$ and all s, t. Now we apply $i(W) \otimes_A -$ to the exact sequences

$$0 \longrightarrow J_j/J \longrightarrow J_{j+1}/J \longrightarrow W(j+1) \otimes_k i(W(j+1)) \longrightarrow 0,$$

with $j = 2, 3, \ldots, m-1$, and we get that $\mathrm{Tor}_1^A(i(W), B) = 0$. Thus $J \otimes_A J = 0$.

However [**12**, corollary 6·2] says that if J is not zero, then $J \otimes_A J$ is never zero. Hence we must have J to be zero, and the proof is finished.

Combining the results in [**13**], we have the following corollary in which only the last two conditions are new.

Corollary 5·6. *Let A be a cellular algebra with cell modules $W(\lambda)$, $\lambda \in \Lambda$. Then the following are equivalent*:

(1) *A is quasi-hereditary*;
(2) *the Cartan determinant of A is* 1;
(3) *there is a cell chain which is a heredity chain*;
(4) *every cell chain in A is a heredity chain*;
(5) $\mathrm{Ext}^1_A(W(\lambda), W(\mu)^*) = 0$ *for all* $\lambda, \mu \in \Lambda$;
(6) $\mathrm{proj.dim}\, W(\mu) < \infty$ *for all* $\mu \in \Lambda_0$.

In terms of subcategories, the above theorem can be reformulated as follows:

Theorem 5·7. *Let A be a cellular algebra. Then A is quasi-hereditary if and only if* $\mathscr{F}(W(\Lambda)^*) = \mathscr{Y}(W(\Lambda))$.

Proof. If A is quasi-hereditary, then we know by [**13**] that the cell modules are the standard modules and the modules $W(\lambda)^*$ are the costandard modules. Thus we have $\mathscr{F}(W(\Phi)^*) = \mathscr{Y}(W(\Lambda))$. The converse follows from Theorem 5·5 and the definition of $\mathscr{Y}(W(\Lambda))$.

Thus the first cohomology groups of cell modules may play a role in the study of cellular algebras. In this direction, we have the following result on cohomology groups of cell modules.

Proposition 5·8. *Let A be a cellular algebra. Then*:

(1) *if X is an A-module with* $\mathrm{Ext}^1_A(X, W(\Lambda)^*) = 0$, *then* $X \in \mathscr{F}(W(\Lambda))$, *that is,* $\mathscr{W}(W(\Lambda)^*) \subseteq \mathscr{F}(W(\Lambda))$;
(2) $\mathscr{Y}(W(\Lambda)) \subseteq \mathscr{F}(W(\Lambda)^*)$. *In particular, if* $\mathrm{gl.dim}(A) = \infty$, *then* $\mathscr{Y}(W(\Lambda))$ *is properly contained in* $\mathscr{F}(W(\Lambda)^*)$.

Proof. (1) Suppose that X is an A-module with $\mathrm{Ext}^1_A(X, W(\Lambda)^*) = 0$. Let

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_m = A$$

be a cell chain which gives the cell modules $W(\Lambda)$. Assume that $J_j/J_{j-1} \cong W(j) \otimes_k iW(j)$ for all j. Thus $J_j/J_{j-1} \cong W(j)^{m_j}$ as a left module, where $m_j = \dim_k W(j)$. We have the following exact sequence

$$\mathrm{Tor}^A_1(iW(j)^{m_j}, X) \rightarrow J_{j-1} \otimes_A X \rightarrow J_j \otimes_A X \rightarrow W(j) \otimes_k iW(j) \otimes_A X \rightarrow 0.$$

The first term vanishes since $\mathrm{Tor}^A_1(iW(j), X) \cong \mathrm{Ext}^1_A(X, W(j)^*) = 0$. This implies that we have a chain of submodules of X:

$$0 = J_0 \otimes_A X \subset J_1 \otimes_A X \subset J_2 \otimes_A X \subset \cdots \subset J_m \otimes_A X = X$$

with the sections being direct sums of cell modules. Hence X is in $\mathscr{F}(W(\Lambda))$.

(2) follows by applying the dual functor * to the statement (1). This finishes the proof.

Now let us consider the second cohomology groups of cell modules. Comparing with the homological definition of quasi-hereditary algebras, the following question arises naturally.

Question. Let A be a cellular algebra. Are the following statements equivalent?
(1) A is quasi-hereditary;
(2) $\mathrm{Ext}^1_A(W(\Lambda), W(\Lambda)^*) = 0$;
(3) $\mathrm{Ext}^2_A(W(\Lambda), W(\Lambda)^*) = 0$.

Our answer to this question is the following theorem.

THEOREM 5·9. *Let A be a cellular algebra with cell modules $W(\lambda)$. Then A is quasi-hereditary if and only if* $\mathrm{Ext}^2_A(W(\Lambda), W(\Lambda)^*) = 0$.

Proof. We need only to show the '*if*' part. Let

$$0 \subset J_1 \subset J_2 \subset \cdots \subset J_m = A$$

be a cell chain which produces the cell modules $W(\Lambda)$. It follows from $\mathrm{Ext}^2_A(W(\Lambda), W(\Lambda)^*) = 0$ that $\mathrm{Ext}^1_A(J_j, W(\Lambda)^*) = \mathrm{Ext}^2_A(A/J_j, W(\Lambda)^*) = 0$ since A/J_j has a $W(\Lambda)$-filtration. Now we show that J_1 is a heredity ideal. If this is done, then we can use induction to obtain the desired statement.

Since a cell ideal J is either a heredity ideal or $J^2 = 0$, what we have to do is just to exclude the case $J_1^2 = 0$. Now the proof is similar to that of Theorem 5·5. The condition that $\mathrm{Ext}^1_A(J_j, W(\Lambda)^*) = 0$ can be interpreted as $\mathrm{Tor}^A_1(i(W(\Lambda)), J_j) = 0$ by the canonical isomorphism $\mathrm{DExt}^j_A(X, Y) \cong \mathrm{Tor}^A_j(DY, X)$. Since as a right A-module A/J_1 has an $i(W(\Lambda))$-filtration, we know that $\mathrm{Tor}^A_1(A/J_1, J_1) = 0$. Suppose that J_1 is non-zero with $J_1^2 = 0$. Then we have

$$J_1 \otimes_A J_1 \cong \mathrm{Tor}^A_2(A/J_1, A/J_1) \cong \mathrm{Tor}^A_1(A/J_1, J_1) = 0.$$

This implies that J_1 must be zero, a contradiction. Thus J_1 must be a heredity ideal in A. This finishes the proof.

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