Canadian Mathematical Society
Conference Proceedings
Volume **24**, 1998

TWISTED DOUBLES OF ALGEBRAS I: DEFORMATIONS OF ALGEBRAS AND THE JONES INDEX

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ABSTRACT. We show that the deformations of certain doubles of a finite–dimensional hereditary algebra are always quasi-hereditary. Moreover, we prove that the 1–deformations of dual extensions of finite dimensional hereditary algebras provide us always multi–matrix algebras. Thus we can get a natural pair of multi–matrix algebras. If the hereditary algebra has radical square zero, then the Jones index of this pair is linked to the spectral radius of the associated Coxeter matrix of the Cartan matrix of the given hereditary algebra.

In the study of quasi–hereditary algebras introduced by Cline, Parshall and Scott in order to study highest weight categories in the representation theory of Lie algebras and algebraic groups (see [CPS]), we constructed in [X1] a class of finite dimensional algebras which is called dual extensions. It turns out that these algebras inherit some nice properties from given algebras. A more general construction is the so–called twisted doubles which were studied in [Dy], [DX1] and [KX]. The twisted doubles are used to construct BGG–algebras introduced in [CPS] and [I], that is, quasi-hereditary algebras with a duality which fixes all simple modules, and moreover, quasi–hereditary algebras which are twisted doubles possess exact Borel subalgebras and Δ-subalgebras, thus having triangular decompositions (see [K] for the definition). Typical examples of twisted doubles are the Schur algebras of finite–representation type and Temperley–Lieb algebras. A lot of quantum groups such as Manin's quantum 2×2 matrices , Dipper and Donkin's quantum groups, and the coordinate rings of quantum symplectic spaces (for a survey see [P]) provides another class of interesting examples of twisted doubles. This means that twisted doubles can be used to construct Hopf-algebras and certain quantum groups, as we will see in [X3].

Naturally, one may think of some kind of perturbations or deformations on the twisted doubles and hope that the resulting algebras could still be quasi-hereditary or have some other nice properties. Motivated by this, we introduce in the present paper the socalled deformations of twisted doubles. Weyl algebras and Woronowicz's C^*–algebras $C(S_\nu U(2))$ (see for example [P]) are examples of this kind of deformations. As the first step we investigate some properties of the deformations of twisted doubles for the class of finite dimensional hereditary algebras. In this case we show that the deformations of twisted doubles of a finite–dimensional hereditary algebra are always quasi-hereditary. Our main result says

1991 Mathematics Subject Classification:16G10,16G20;14D15,16S50.

that for a finite dimensional hereditary algebra C given by a quiver $Q = (Q_0, Q_1)$ the 1-deformation $\mathcal{D}(C)$ of the dual extension $\mathcal{A}(C)$ of C is always a semisimple algebra. Moreover, the decomposition of this 1–deformation into matrix algebras is completely determined by the dimensions of the indecomposable projective C–modules. In this way, we get a natural pair $kQ_0 \subset \mathcal{D}(C)$ of multi–matrix algebras. By tilting theory we may further assume that the algebra C has radical square zero. In this case, the Jones index of this pair is just the spectral radius of the Cartan matrix of $\mathcal{D}(C)$ and can be linked to the spectral radius of the Coxeter matrix of the Cartan matrix of C if C is not representation–finite.

The paper is organized in the following way: We recall the definition of twisted doubles in section 1 and introduce the notion of deformations in section 2. The main result is stated and proved in section 3. The last two sections are devoted to the computation of the Jones index of our natural pair of multi–matrix algebras.

Throughout the paper we denote by k a fixed field.

1. Twisted doubles

The definition of M–twisted double incidence algebras of posets given in [DX1] generalizes the construction in [Dy] where he constructed certain BGG-algebras to approach the ones appearing in the representation theory of Lie algebras and algebraic groups. In the following we present a more general definition from [KX] applicable to any algebra given by a quiver with relations.

Assume that we are given a k–algebra C over a field k defined by a quiver $Q = (Q_0, Q_1)$ and certain relations which we do not have to specify. (The quiver may be an infinite quiver or have multiple arrows or loops. In case the quiver is infinite, we require that it must be locally finite, i.e., for each vertex x, there are only finitely many arrows starting and ending at x). We denote by Q^{op} the opposite quiver of Q, the arrows of which will be denoted by α' with α an arrow i Q. For an arrow α we denote by $s(\alpha)$ and $t(\alpha)$ its starting vertex and the terminal vertex respectively.

First we define twisting labels. Assume we are given an ordered pair of arrows $\alpha : w \leftarrow z$ and $\beta : z \to x$ in Q. Consider all (ordered) pairs (γ_i, δ_i) of arrows $\gamma_i : w \to y_i$, $\delta_i : y_i \leftarrow x$, $1 \leq i \leq m$. With each such pair of arrows we associate an element $l(\alpha, \beta, \gamma_i, \delta_i) \in k$, the **label**. The collection of all labels is called the labelling M of C. This labelling can also be described by all mesh diagrams: For each pair x, w of vertices in Q_0 with a path of length two from w to x there is a unique mesh diagram consisting of all paths $w \to y_i \to x, i = 1, \cdots, m$, of length two from w to x. Then we associate this mesh diagram with a $m \times m$ matrix over the field k, and the labelling M is just the collection of all entries of the associated matrices. Hence we call the labelling a **matrix labelling**.

1.1 Definition. Let C be a k–algebra with a matrix labelling M. We define a new algebra $\mathcal{A}(C, M)$ which is given by the quiver obtained from Q and the opposite quiver Q^{op} by forming the union and identifyiny the vertices, and imposing the following three types of relations:

(1) the relations of the algebra C;

(2) the relations of the algebra C^{op}; and

(3) the **twisting relations**: for each pair α, β of arrows in Q with $s(\alpha) = s(\beta)$, put the relation

$$\alpha'\beta = \sum_i l(\alpha, \beta, \gamma, \delta)\gamma\delta'$$

where the summation runs over all ordered pairs (γ, δ) of arrows in Q such that $s(\gamma) = t(\alpha), s(\delta) = t(\beta)$ and $t(\gamma) = t(\delta)$. Note that we allow $\alpha'\beta = 0$ if such a pair (γ, δ) does not exist.

It is clear that $\mathcal{A}(C, M)$ is an associative k–algebra. If C has the identity then $\mathcal{A}(C, M)$ has also the identity. We call the algebra $\mathcal{A}(C, M)$ the **twisted double** of C with the labelling M.

To illustrate this definition, let us see one example.

1.2 Example. Suppose the algebra C is defined by the quiver

$$\begin{array}{ccccc} & & 2 & & \\ & \alpha \nearrow & & \gamma_1 \searrow\searrow \gamma_2 & \\ 1 & & & & 4 \\ & \beta \searrow & & \nearrow \delta & \\ & & 3 & & \end{array}$$

with relation $\alpha\gamma_1 = \beta\delta$. The labellings for the pair (α, α) are given by 4 elements in the field k: $a_{11} = l(\alpha, \alpha, \gamma_1, \gamma_1), a_{12} = l(\alpha, \alpha, \gamma_1, \gamma_2), a_{21} = l(\alpha, \alpha, \gamma_2, \gamma_1)$ and $a_{22} = l(\alpha, \alpha, \gamma_2, \gamma_2)$, the labellings for the pair (α, β) are given by 2 elemnts, $a_{13} = l(\alpha, \beta, \gamma_1, \delta)$ and $a_{23} = l(\alpha.\beta, \gamma_2, \delta)$. For the pair (β, α), the labellings are $a_{31} = l(\beta, \alpha, \delta, \gamma_1)$ and $a_{32} = l(\beta, \alpha, \delta, \gamma_2)$, finally, the labelling for the pair (β, β) is an elments $a_{33} = l(\beta, \beta, \delta, \delta)$ in the field k. Thus the matrix we have associated is a 3×3 matrix (a_{ij}) and the twisting relations for the twisted double $\mathcal{A}(C, M)$ read as follows:

$$\begin{aligned}
&\alpha'\alpha = a_{11}\gamma_1\gamma_1' + a_{12}\gamma_1\gamma_2' + a_{21}\gamma_2\gamma_1' + a_{22}\gamma_2\gamma_2' \\
&\alpha'\beta = a_{13}\gamma_1\delta' + a_{23}\gamma_2\delta' \\
&\beta'\alpha = a_{31}\delta\gamma_1' + a_{32}\delta\gamma_2' \\
&\beta'\beta = a_{33}\delta\delta' \\
&\gamma_1'\gamma_1 = 0, \quad \gamma_1'\gamma_2 = 0 \\
&\gamma_2'\gamma_1 = 0, \quad \gamma_2'\gamma_2 = 0 \\
&\delta'\delta = 0.
\end{aligned}$$

A special case of the above definition is that we take all labels to be zero. In this case we call the twisted double of C simply the **dual extension** of C (see [X1,1.6]). We shall see in section 3 that the deformation of dual extensions may provide us a very nice semisimple k-algebra. For the discussion of these algebras $\mathcal{A}(C, M)$ one may see [Dy],[DX1],[DX2],[KX] and [X2].

2. Deformations of twisted doubles

In this section we introduce the definition of a deformation of the twisted double of a given algebra C with a matrix labelling M.

Let C be as in the definiton 1.1. For each pair α, β of arrows in Q_1 with the same starting and terminal vertex we attach an element $x(\alpha, \beta) \in k$. Let X be the collection of all these $x(\alpha, \beta)$, we call X an **evaluation**.

2.1 Definition. Let C be a k-algebra with a matrix labelling M and an evaluation X. We define a new algebra $\mathcal{D}(C, M, X)$ which has the same quiver as the algebra $\mathcal{A}(C, M)$ does, and the following relations:

(1) the relations of the algebra C;

(2) the relations of the algebra C^{op}; and

(3) the **deformed twisting relations**: for each pair α, β of arrows in Q with $s(\alpha) = s(\beta)$, put the relation

$$\alpha'\beta = \delta_{t(\alpha),t(\beta)} x(\alpha, \beta) e_{t(\alpha)} + \sum_i l(\alpha, \beta, y_i)\gamma\delta'$$

where $\delta_{u,v}$ is the Kronecker symbol, $e_{t(\alpha)}$ is the idempotent corresponding to the vertex $t(\alpha)$, and the summation runs over all ordered pairs (γ, δ) of arrows in Q such that $s(\gamma) = t(\alpha), s(\delta) = t(\beta)$ and $t(\gamma) = t(\delta)$.

We call the algebra $\mathcal{D}(C, M, X)$ the **deformation** of $\mathcal{A}(C, M)$ at the evaluation X. If X is an evaluation such that $\delta_{t(\alpha),t(\beta)} x(\alpha, \beta) = \delta_{\alpha,\beta}$ for all $\alpha, \beta \in Q_1$, we call $\mathcal{D}(C, M, X)$ the **1–deformation** of $\mathcal{A}(C, M)$. Note that this condition on X is satisfied in the following two cases:

(i) If Q has no multiple arrows and $x(\alpha, \beta) = 1$; or

(ii) if Q has multiple arrows and $x(\alpha, \beta) = \delta_{\alpha,\beta}$ for all $\alpha, \beta \in Q_1$ with the same starting vertex.

Many important algebras are deformations in the above sense. Let us just mention some of them.

2.2 Examples. (1) The polynomial ring $k[T]$ in one variable T is the path algebra of a loop α. The deformations of the twisted doubles of $k[T]$ are the algebras $k < \alpha, \alpha' > / < \alpha'\alpha - \mu\alpha\alpha' - \lambda >$ with $\lambda, \mu \in k$. In case $\lambda = 1 = \mu$, the corresponding algebra is the Weyl algebra and has been studied by many authors.

(2) If we take the algebra C to be the factor algebra of $k < x, y >$ modulo the ideal generated by $xy - \nu yx$, then one can easily verify that Woronowicz's algebra $C(S_\nu U(2))$ is a 1–deformation of a twisted double of C as mentioned in the introduction.

(3) If C is the algebra of all 2×2 upper triangular matrices over k , then the 1–deformation of the dual extension of C is isomorphic to $M_2(k) \oplus M_1(k)$, where $M_n(k)$ stands for the full $n \times n$ matrix algebra over k.

(4) If C is the factor algebra of the algebra of all 3×3 upper triangular matrices modulo the square of the radical, then the 1–deformation of the dual extension of C is isomorphic to $M_2(k) \oplus M_1(k)$.

(5) Let Q be the quiver with two vertices $\{1,2\}$ and two arrows a and b from 1 to 2. We take all $x(\alpha,\beta)=1$ for all arrows. Then the corresponding deformation A of $\mathcal{A}(kQ)$ is a 10–dimensional algebra. Since $A(a-b)=k(a-b)$ and $(a-b)^2=0$, the algebra A is not semisimple (cf.(3)). Note that in this case the algebra is not an 1–deformation since $x(a,b)\neq\delta_{a,b}$.

3. Main result

The main result of this paper is the following theorem.

Theorem. *Let C be a finite dimensional hereditary k-algebra given by a quiver $Q=(Q_0,Q_1)$ and A the 1-deformation of the dual extension of C. Then A is a semisimple k-algebra isomorphic to*

$$\bigoplus_{i\in Q_0} M_{d_i}(k),$$

where $d_i=dim_k Ce_i$ and $M_n(k)$ denotes the full $n\times n$ matrix ring over k.

The proof of the theorem is divided in several lemmas. In the following we denote by C the path algebra kQ defined by the quiver Q and A the 1- deformation of the dual extension of C. For each $\lambda\in Q_0$, choose a k-basis B_λ for the indecomposable kQ-module kQe_λ consisting of monomials in kQ ending at the vertex λ. We begin with the following more general lemma which tells us that the algebra $\mathcal{D}(kQ,M,X)$ has a triangular decomposation and thus is a quasi-hereditary algebra by a result in [K].

3.1 Lemma. *For any matrix labelling M and evaluation X on the algebra kQ, there is a vector space isomorphism*

$$\mu: kQ\otimes_{kQ_0} kQ^{op}\longrightarrow \mathcal{D}(kQ,M,X)$$

given by the multiplication.

Proof. We know that the relations of $\mathcal{D}(kQ,M,X)$ are of the form

$$\alpha'\beta=\delta_{t(\alpha),t(\beta)}x(\alpha,\beta)e_{t(\beta)}+\sum l(\alpha,\beta,\gamma,\delta)\gamma\delta'$$

where $l(\alpha,\beta,\gamma,\delta)$ is a label in M, $\delta_{u,v}$ is the Kronecker symbol, and the summation runs over all pairs (γ,δ) of arrows in Q such that $s(\gamma)=t(\alpha), t(\gamma)=t(\delta)$ and $s(\delta)=t(\beta)$. Thus each element of $\mathcal{D}(kQ,M,X)$ can be written as a linear combination of the monomials of the form $\alpha_1\cdots\alpha_m\beta_1'\cdots\beta_n'$ with n,m positive integers and $\alpha_i,\beta_j\in Q_0\cup Q_1$. By Bergman's diamond lemma [B], These monomials form a k-basis for the algebra $\mathcal{D}(kQ,M,X)$. Thus the surjective map μ is injective and an isomorphism of vector spaces. □

3.2 Lemma. *Let λ be a sink vertex in Q. Then*

$$c'b=\delta_{c,b}e_{t(b)},$$

for all $c, b \in B_\lambda$.

Proof. If c or b is the idempotent element e_λ, then the statement is obviously true since the quiver has no oriented cycle and λ is a sink vertex. So we assume that $c = \alpha_1 \cdots \alpha_m$ and $b = \beta_1 \cdots \beta_n$ with α_i and β_j in Q_1. If $b = c$, then the statement follows, according to the relations in A. If $b \neq c$, then we have to consider the two cases: $m \leq n$ and $m > n$. Since $b \neq c$, there is a natural number j such that $\alpha_i = \beta_i$ for $1 \leq i \leq j$ and $\alpha_{j+1} \neq \beta_{j+1}$. In the case $m \leq n$, we see that $j \leq m$ and $c'b = 0$ if $j < m$. If $j = m$ we must have $\beta_{m+1} \cdots \beta_n = e_\lambda$ since Q has no oriented cycle, but this shows that $c = b$, a contradiction. Hence $c'b = 0$ in case $m \leq n$. Similarly, we can show that the statement is true for $m > n$. □

3.3 Lemma. *Let λ be a sink vertex and $\xi_\lambda := \sum_{b \in B_\lambda} bb'$. Then*

(1) ξ_λ is an idempotent element in A, and the summands bb' are orthogonal idempotents.

(2) ξ_λ is the identity of the ring $Ae_\lambda A$.

(3) Each summand bb' is a primitive idempotent of $Ae_\lambda A$.

Proof. Since $b'b = e_{t(b)}$, there holds $bb'bb' = bb'$, this means that bb' is an idempotent. For distinct monomials b and c in B_λ with positive length, we have $c'b = 0$ in A. Thus (1) follows.

To show (2) and (3), we note that the elements of the forms bc' with $b, c \in B_\lambda$ form a k–basis of $Ae_\lambda A$ by Lemma 3.1. Thus it is easy to check that (2) holds by using the relation $c'b = \delta_{b,c} e_{t(b)}$ for all $b, c \in B_\lambda$. Since an idempotent e is primitive in a ring R if and only if the ring eRe is local, the statement (3) follows also straightforward (here we need that λ is a sink vertex).□

3.4 Lemma. *Suppose λ is a sink vertex. Then $Ae_\lambda A$ is isomorphic to the matrix algebra $M_{d_\lambda}(k)$, with $d_\lambda = dim_k Ce_\lambda$.*

Proof. Let $b_1, \cdots, b_{d_\lambda}$ be a list of the basis B_λ. We define $f_{ij} = b_i b_j'$ for all i, j. Then in A we have

$$f_{ij} f_{lm} = b_i b_j' b_l b_k' = \delta_{j,l} b_i b_k' = \delta_{j,l} f_{ik}.$$

Combinating with Lemma 3.3 we see that $Ae_\lambda A$ is isomorphic to the matrix algebra $M_{d_\lambda}(k)$. □

3.5 Proof of the Theorem. We use induction on the number of the vertices to show the theorem. For $|Q_0| = 1$, the theorem is true. Suppose the theorem is valid for all quivers Q having no oriented cycle and with $|Q_0| < n$. Now let kQ be a finite dimensional algebra with n non-isomorphic simples and λ a sink vertex in Q. We consider the ideal $Ae_\lambda A$ in A and the factor ring $A/Ae_\lambda A$. According to the relations in A, we know that the factor algebra is isomorphic to the 1–deformation A_0 of the path algebra $k\Delta$, where $\Delta = (\Delta_0, \Delta_1)$ is obtained from Q by delating the vertex λ and the arrows ending at λ. By induction, the algebra A_0 is isomorphic to $\oplus_{i \in \Delta_0} M_{d_i}(k)$, where $d_i = \dim_k k\Delta e_i = \dim_k kQe_i$. Thus A is semisimple.

Let ξ_λ be as in Lemma 3.3. Then $\xi_\lambda \in Ae_\lambda A$ is the identity for $Ae_\lambda A$. Since $A\xi_\lambda A \subseteq Ae_\lambda A = Ae_\lambda A\xi_\lambda \subseteq A\xi_\lambda \subseteq A\xi_\lambda A$, we have $A\xi_\lambda A = Ae_\lambda A$. Thus $A\xi_\lambda A = A\xi_\lambda = \xi_\lambda A$. Put $f = 1 - \xi_\lambda$. Then

$$\begin{aligned} A &= A\xi_\lambda \oplus Af \\ &= A\xi_\lambda \oplus \xi_\lambda Af \oplus fAf \\ &= A\xi_\lambda \oplus A\xi_\lambda f \oplus fAf \\ &= A\xi_\lambda \oplus fAf \\ &= A\xi_\lambda A \oplus fAf \end{aligned}$$

This shows that $fAf \cong A/A\xi_\lambda A = A/Ae_\lambda A$ is isomorphic to A_0. By Lemma 3.4, the algebra A is isomorphic to the direct sum of the matrix algebras $M_{d_i}(k), i \in Q_0$.□

The above proof of the theorem shows also the following fact.

3.6 Corollary. *Let M be any finite dimensional kQ^{op}-module with dimension vector $(m_i)_{i\in Q_0}$. Then the induced A-module $A \otimes_{kQ^{op}} M$ is completely reducible and is isomorphic to $\bigoplus_{i\in Q_0} m_i S(i)$, where $S(i)$ denotes the unique (up to isomorphism) simple A-modules corresponding to the block $M_{d_i}(k)$.*□

3.7 Corollary. *If Q be the quiver with $Q_0 = \mathbb{N}$ and $Q_1 = \{i \longrightarrow i+1 \mid i \in \mathbb{N}\}$. Then the 1-deformation of the dual extension of kQ is isomorphic to $\bigoplus_{i=1}^{\infty} M_i(k)$.*

Proof. Let $Q^{(n)}$ be the full subquiver of Q with the vertex set $\{1, 2, \cdots, n\}$ Then the 1-deformation $\mathcal{A}_n$ of the dual extension of $kQ^{(n)}$ is isomorphic to $\bigoplus_{i=1}^{n} M_i(k)$ by the above theorem. Hence, as a direct limit of $\mathcal{A}_n$, the 1-deformation of $\mathcal{A}(kQ)$ is isomorphic to $\bigoplus_i^{\infty} M_i(k)$.□

3.8 Remark. (1) In general the conanical inclusion $kQ^{op} \hookrightarrow A$ is not the universal localization of kQ^{op} using the relative irreducible maps α between indecomposable projective kQ^{op}-modules (see [S]) since $A \otimes_{kQ^{op}} \alpha$ is not an isomorphism of A-modules.

(2) The example 2.2 (4) shows that if the algebra C is not hereditary, then the Theorem may be false.

(3) The following question is still open: Is the 1-deformation of the dual extension of any finite-dimensional algebra semisimple ?

4. Tower theory for the pair $kQ_0 \subset A$

4.1 For each pair $1 \in N \subset M$ of algebras there is a tower

$$1 \in M_0 = N \subset M_1 = M \subset M_2 \subset \cdots$$

by the fundamental construction $M_2 = \mathrm{End}_N({}_N M) \supset M_1$. The rank $\mathrm{rk}(M_k|M_0)$ over M_0 is defined to be the smallest possible number of generators of M_k viewed as a left M_0-module, and the index of N in M is the rate

$$[M : N] := \limsup_{k\to\infty} (rk(M_k|M_0))^{1/k}.$$

For pairs of semisimple k–algebras this index can be equivalently defined as

$$[M : N] = \limsup_{m\to\infty}(\dim_k \underbrace{M \otimes_N \cdots \otimes_N M}_{m \text{ times}})^{1/m}.$$

To compute the index one defines for each pair $1 \in N \subset M$ of multi–matrix algebras the index matrix Λ_N^M. As consequence of the main result, the index matrix of this pair is C^T. For the details of this theory one may refer to [GHJ].

4.2 We are interested in the following pair of multi–matrix algebras. Let kQ be a finite dimensional hereditary k–algebra. We have seen that the 1–deformation A of the dual extension of kQ is a multi–matrix algebra containing kQ as a subalgebra. So we have a pair $1 \in S = kQ_0 \subset A$ of multi–matrix k–algebras. Since the index matrix is C^T, we have by [GHJ,§2] the following theorem.

Theorem. *Let $C = (c_{ij})$ be the Cartan matrix of kQ^{op}. Then for the pair $S \subset A$ we have*
$[A : S] = \rho(C^TC)$, where $\rho(C)$ denotes the spectral radius of C, and the upper index T stands for transpose.

4.3 Suppose that we work with the complex field $\mathbb{C}$ and that Q is connected. It follows from 3.1 that the Cartan matrix of A is C^TC which is nonnegative, symmetric and positive definite. By using the graphic criterion of the irreducibility of a matrix in [LT] one can easily know that C^TC is an irreducible matrix. This implies that there is a positive vector $y = (y_1, \cdots, y_{|Q_0|})$ such that $yC^TC = \rho(C^TC)y$. Since C^T is a nonnegative matrix with diagonal $(1, 1, \cdots, 1)$, we deduce that yC^T has all components bigger than 0. Thus by [GHJ,Th.2.1.6] we have

Proposition. *Let $S \subset A$ be the above pair of semisimple $\mathbb{C}$– algebras and*

$$1 \in M_0 = N \subset M_1 = M \subset M_2 \subset \cdots$$

the tower generated by $S \subset A$ using the fundamental construction. Then
(1) the algebra M_i is generated by A and the idempotents $E_1, \cdots, E_{i-1}$,
(2) the idempotents $E_1, \cdots, E_{i-1}$ satisfy

$$\rho(A)E_jE_kE_j = E_j \quad \text{if } |j-k| = 1, \text{ and}$$
$$E_jE_k = E_kE_j \quad \text{if } |i-j| \geq 2,$$

where $\rho(A)$ denotes the spectral radius of the Cartan matrix of A.

4.4 We say $\Delta = (\Delta_0, \Delta_1)$ is a full subquiver of $Q = (Q_0, Q_1)$ if $\Delta_0 = Q_0$ and Δ_1 is a subset of Q_1. Given a full subquiver Δ of Q, we define the complement of Δ in Q is the full subquiver $\bar{\Delta} = (Q_0, Q_1 \setminus \Delta_1)$. In this case we can consider the 1–deformations of the dual extensions of $k\Delta, k\bar{\Delta}$ and kQ. Let D, B and A be the 1–deformations of the dual extensions of $k\bar{\Delta}, k\Delta$ and kQ respectively. Then we have multi-matrix pairs $1 \in D \subset A$ and $1 \in B \subset A$. For these pairs we have

Lemma. $[A : B] \geq max\{[A : S]/[B : S], [A : S]/[D : S]\}$.

Proof. It follows from [GHJ,2.3.2] that $\Lambda_S^A = \Lambda_B^A \Lambda_S^B$. This yields together with 4.2 the Lemma.

5. Jones index and the eigenvalues of the Coxeter matrix

It is well–known that the Jones index $[A : B]$ of a semisimple pair can be computed by using the spectral radius of the Coxeter matrix of a hereditary algebra $\mathcal{A}(B, A)$ constructed in [DR1]. In this section we shall link the Jones index of the pair of multi–matrix algebras given in the previous section to an eigenvalue of a Coxeter matrix in a direct way. Thus we can compute the Jones index by using the Coxeter matrix of the index matrix.

Throughout this section we assume that kQ is a finite dimensional connected hereditary algebra with radical square zero. Let C be the Cartan matrix of kQ^{op}. Then C is invertible and can be written as $C = I + M$ for some matrix M with $M^2 = 0$. We denote by A the 1–deformation of the dual extension of kQ. For a matrix C we denote by $\rho(C)$ the spectral radius of C.

5.1 Lemma. *Let $\Phi := -C^{-T}C$ be the Coxeter matrix of C and λ an eigenvalue of C^TC. Then*

$$\lambda = \left(\frac{\rho^2 + 1 + \sqrt{\rho^4 + 6\rho^2 + 1}}{2\rho}\right)^2,$$

where ρ^2 is an eigenvalue of the Coxeter matrix Φ.

Proof. We apply the same argument as in [A] to the matrix C^TC.

$$\begin{aligned}
\det(x^2 - C^TC) &= \det(x^2 - C^TC)\det(C^{-1}) \\
&= \det(x^2C^{-1} - C^T) \\
&= \det(x^2(I - M) - (I + M^T)) \\
&= \det((x^2 - 1)I - (x^2M + M^T)) \\
&= x^n \det((x - x^{-1}))I - (xM + x^{-1}M^T)) \\
&= x^n \det((x - x^{-1})I - (M^T + M)).
\end{aligned}$$

Hence x^2 is an eigenvalue of C^TC if and only if $x - x^{-1}$ is an eigenvalue of $M + M^T$. Similarly, for a complex number $\rho \neq 0$, we have that ρ^2 is an eigenvalue of Φ if and only if $\rho + \rho^{-1}$ is an eigenvalue of $M + M^T$. Suppose x is a positive real number such that $\lambda = x^2$. Then there is some $\rho \neq 0 \in \mathbb{C}$ such that $x - x^{-1} = \rho + \rho^{-1}$ and ρ^2 is an eigenvalue of Φ. This means that $x^2 - (\rho + \rho^{-1})x - 1 = 0$. Thus

$$x = \frac{\rho + \rho^{-1} + \sqrt{(\rho + \rho^{-1})^2 + 4}}{2}$$

since x is positive. The lemma follows. □

Since Q is connected, the nonnegative matrix $M + M^T$ is irreducible by [LT]. If we assume that the underlying graph of Q is not Dynkin diagram, then by [PT, 1.5] there is a real number $\lambda \geq 1$ such that $\rho(\Phi) = \lambda^2$ and $\lambda + \lambda^{-1} = \rho(M + M^T)$. Thus we have the following result:

5.2 Proposition. *If the underlying graph of Q is not a Dynkin diagram, then*

$$[A:S] = \left(\frac{\lambda^2+1+\sqrt{\lambda^4+6\lambda^2+1}}{2\lambda}\right)^2,$$

where λ is a real number such that $\lambda^2 = \rho(\Phi)$.

5.3 Corollary. *If kQ is a connected tame hereditary algebra, then*

$$[A:S] = (1+\sqrt{2})^2 = 3+2\sqrt{2}.$$

Proof. This follows from the fact that in our case the spectral radius $\rho(\Phi)$ is equal to 1 (see [DR2]).□

Acknowlegement. The author would like to thank C.M.Ringel and D.Happel for the hospitality during his stay in Bielefeld and Chemnitz, and also thank S.A.Ovsienko for some conversations. The author was partially supported by the Chinese Education Committee for Young Teachers.

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