Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 36 (1995), No. 1, 37-71.

Quasi–hereditary Algebras which are Twisted Double Incidence Algebras of Posets

Bangming Deng and **Changchang Xi**[1]
Department of Mathematics, Beijing Normal University
Beijing 100875, P. R. China

Abstract. For each finite poset, we associate it with a family M of matrices and define the corresponding M–twisted double incidence algebra which is a generalization of the construction given by Dyer. In the paper we mainly study the quadratic dual and Ringel dual of the M–twisted double incidence algebra. We prove that if the given poset is a tree then its M-twisted double incidence algebra is a BGG–algebra and the Ringel dual can be determined in detail (with some natural restriction on matrices). Moreover, we show that in the tree case with all matrices non–zero the processes of forming Ringel duals and quadratic duals of M–twisted double incidence algebras, respectively, commute with each other.

Introduction

In his study of Kazhdan–Lusztig–Stanley polynomials with interpretations in the representation theory of finite dimensional quadratic algebras, Dyer introduced in [4] for certain class of posets, including Bruhat intervals or face lattices, families of finite dimensional algebras, and proved that these algebras have many nice properties, and in particular, satisfy the main conjecture mentioned in [4, 1.6]. The algebras constructed from posets satisfying the conditions in [4] are quasi–hereditary algebras with dualities which fix simple modules. Such algebras are called BGG–algebras (see [9], and [3]). It is well–known that each block of the category $\mathcal{O}$ introduced by Bernstein, Gelfand and Gelfand in [1] is equivalent to the module category of a finite dimensional BGG–algebra such that its quadratic dual algebra is also quasi–hereditary. Quasi–hereditary algebras have been introduced by Cline, Parshall and Scott in order to describe the structure of certain finite dimensional algebras which arise naturally in the representation theory of semisimple complex Lie algebras and of reductive algebraic groups, and then are ring–theoretically investigated by Dlab and Ringel in [5]. Recently, Ringel [11] associated with each quasi–hereditary algebra

---

[1] Supported by the Fok Ying Tung Education Foundation

Typeset by $\mathcal{AMS}$-TEX

A another quasi–hereditary algebra $\mathcal{R}(A)$ which we call the Ringel dual of A. Being the endomorphism algebra of a generalized tilting module, the Ringel dual has many applications in the study of symmetric groups and Schur algebras (see [8]).

The main objective of the present paper is the study of Ringel duals and quadratic duals for a class of quasi–hereditary algebras which are defined to be twisted double incidence algebras of posets as well as of the relation between these two dualities.

First we simplify the construction in [4] and construct for any finite poset a family of finite dimensional algebras, here we do not require any restriction on posets and labellings. Since the labellings in our paper are families of matrices, we call them matrix labellings. Of course, if the given poset and the labelling satisfy all conditions in [4, Sect. 3], then our construction yields the same algebra as in [4] whose quasi–heredity is proved by Dyer. (Note that the algebras in [4] are a special class of a much larger class of algebras studied by Dyer in a big program.) In general, the algebras we constructed are not always quasi–hereditary. However, we prove that if the Hasse diagram of the poset is a tree then the constructed algebra is quasi–hereditary. Since in this case all the matrices are symmetric, the defined algebra is even a BGG–algebra.

Let X be a finite poset with a matrix labelling M. We consider the defined algebra $\mathcal{A}_{(X,M)}$, and we call it the M–twisted double incidence algebra of X. To describe the quadratic dual of $\mathcal{A}_{(X,M)}$, we first give a sufficient and necessary condition for $\mathcal{A}_{(X,M)}$ to be a quadratic algebra, and then introduce a new finite dimensional algebra $\mathcal{B}_{(X,M)}$ which is again defined by using the labelling poset X. One of our main results, Theorem 2.9, says that if $\mathcal{A}_{(X,M)}$ is quadratic then the quadratic dual of $\mathcal{A}_{(X,M)}$ is of the form $\mathcal{B}_{(X^{op},-M^t)}$. Usually, the algebra $\mathcal{B}_{(X,M)}$ is not a quasi–hereditary algebra. However, in case X is a tree (namely, the Hasse diagram of X is a tree) or X satisfies the conditions in [4, sect.3], the algebra $\mathcal{B}_{(X,M)}$ becomes quasi–hereditary. For Ringel duals as well as the relation between Ringel duals and quadratic duals we concentrate mainly on the case when X is a tree. The main results are 4.6 and 5.8. Namely, in good cases (if X is a tree and all matrices are invertible), the processes of forming the quadratic dual and the Ringel dual of $\mathcal{A}_{(X,M)}$ respectively, commute with each other.

We organize the paper as follows. In section 0 we recall some basic definitions and facts. In section 1 we generalize the construction in [4]. Section 2 is devoted to the description of quadratic dual of $\mathcal{A}_{(X,M)}$. In order to prove 4.6 and 5.8, we discuss in section 3 the Borel subalgebra and Δ–subalgebra in the sense of König. As a byproduct, we have an upper bound for global dimension of $\mathcal{A}_{(X,M)}$ if it is quasi–hereditary. Using the results in section 3 we prove the main results in the last two sections.

0. Preliminaries

0.1 Let A be a finite dimensional algebra over an algebraically closed field k. We will consider finitely generated left A–modules, maps between A–modules will be written on the right side of the argument, thus the composition of maps $f : M_1 \longrightarrow M_2$ and $g : M_2 \longrightarrow M_3$ will be denoted by fg. The category of all finitely generated modules will be denoted by A–mod. Given a class Θ of A–modules, we denote by $\mathcal{F}(\Theta)$ the class of all A–modules which have a Θ–filtration, that is, a filtration

$$0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M$$

such that each factor M_{i-1}/M_i is isomorphic to an object in Θ for $1 \leq i \leq t$. For a module $M \in A$–mod, we denote by add(M) the full additive subcategory of A–mod consisting of all finite direct sums of direct summands of M.

Let $\wedge$ be a finite poset in bijective correspondence with the isomorphism classes of simple A–modules. For each $\lambda \in \wedge$, let $E(\lambda)$ be a simple module in the isomorphism class corresponding to λ and $P(\lambda)$ (or $P_A(\lambda)$) a projective cover of $E(\lambda)$ and denote by $\Delta(\lambda)$ the maximal factor modules of $P(\lambda)$ with composition factors of the form $E(\mu)$, $\mu \leq \lambda$. Dually, let $Q(\lambda)$ (or $Q_A(\lambda)$) be an injective hull of $E(\lambda)$ and denote by $\nabla(\lambda)$ the maximal submodule of $Q(\lambda)$ with the composition factors of the form $E(\mu)$, $\mu \leq \lambda$. Let Δ (respectively, ∇) be the full subcategory of all $\Delta(\lambda)$, $\lambda \in \wedge$ (respectively, all $\nabla(\lambda)$, $\lambda \in \wedge$). We call the modules in Δ the standard modules and the ones in ∇ the costandard modules.

The algebra A is said to be quasi–hereditary with respect to $(\wedge, \leq)$ if for each $\lambda \in \wedge$ we have

(i) $\mathrm{End}_A(\Delta(\lambda)) \cong k$;

(ii) $P(\lambda) \in \mathcal{F}(\Delta)$, and moreover, $P(\lambda)$ has a Δ–filtration with quotient $\Delta(\mu)$ for $\mu \geq \lambda$ in which $\Delta(\lambda)$ occurs exactly once.

For a quasi–hereditary algebra A with respect to a poset $\wedge$ we call the elements in $\wedge$ weights and $\wedge$ the weight poset of A. By $(A, \wedge)$ we denote a quasi–hereditary algebra A with the weight poset $\wedge$.

If a quasi–hereditary algebra has a duality δ which fixes simple modules, we call it a BGG–algebra (see [9], [3]).

0.2 The usual (and equivalent) definition of quasi–hereditary algebras uses the notion of a heredity ideal. A heredity ideal of an algebra A is an idempotent ideal I, with $I(\mathrm{rad}(A))I = 0$, and such that ${}_AI$ is a projective module (or, equivalently, I_A is a projective right module). A chain of ideals

$$A = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n = 0$$

is called a heredity chain provided I_{t-1}/I_t is a heredity ideal of A/I_t, for $1 \leq t \leq n$. A finite dimensional algebra is quasi–hereditary if and only if it has a heredity chain. Note that a heredity chain may always be refined so that for every t, the indecomposable summands of a fixed module ${}_A(I_{t-1}/I_t)$ are all isomorphic, say isomorphic to some $\Delta(t)$, and then these modules are just the standard modules. Conversely, given an algebra A which satisfies the conditions (i) and (ii), then for every t there exists a maximal left ideal I_t of A which belongs to $\mathcal{F}(\{\Delta(t+1), \cdots, \Delta(s(A))\})$, where $s(A)$ is the number of the isomorphism classes of simple A–modules, it clearly will be a two–sided ideal, and, in this way, we obtain a heredity chain $(I_t)_t$ for A. Note that the A/I_t–modules are just those A–modules which only have composition factors of the form $E(i)$, with $i \leq t$.

0.3 To check whether an ideal in A is a heredity ideal one may use the following result due to Dlab and Ringel.

Lemma [5]. *Let A be a finite dimensional k–algebra and I an ideal generated by the idempotent e in A. If $e(rad(A))e = 0$ then ${}_AI$ is projective if and only if the multiplication map*

$$Ae \underset{eAe}{\otimes} eA \longrightarrow AeA$$

is bijective.

0.4 Now we recall the definition of quadratic algebras and their quadratic duals (see [2]). Consider a k–algebra $A_0 \cong k^n$. Let A_1 be a finite dimensional A_0–bimodule, and form the tensor algebra $T_{A_0}(A_1) := \underset{n\in\mathbb{N}}{\oplus} T^n(A_1)$ over A_0, where $T^n(A_1) := A_1 \otimes_{A_0} \cdots \otimes_{A_0} A_1$ denotes the n–fold tensor product (and $T^0(A_1) = A_0$). An algebra of the form $A = T_{A_0}(A_1)/ < R(A) >$, where $< R(A) >$ is the two–sided ideal of $T_{A_0}(A_1)$ generated by some A_0–subbimodule $R(A)$ of $T^2(A_1)$, is called a (basic) quadratic algebra.

Regard the dual space $A_1^* = \mathrm{Hom}_k(A_1, k)$ as an A_0–bimodule with the action $(v)(afb) = (bva)f$, where $a, b \in A_0$ and $f \in A_1^*$. The quadratic dual of A is defined by $A^! := T_{A_0}(A_1^*)/ < R(A)^\perp >$, where $R(A)^\perp$ is the annihilator $R(A)^\perp := \{f \in A_1^* \otimes_{A_0} A_1^* \mid (R(A))f = 0\}$ of $R(A)$ (we identify $(A_1 \otimes_{A_0} A_1)^*$ with $A_1^* \otimes_{A_0} A_1^*$ by sending $(f \otimes g)^*$ to $g^* \otimes f^*$).

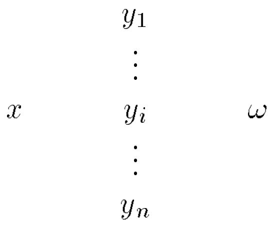
0.5 We fix some notation and terminology used throughout. For a finite poset X we denote by $|X|$ the cardinality of X, and write $x \lessdot y$ (or $y \gtrdot x$) to indicate that $x < y$ and that there is no $z \in X$ satisfying $x < z < y$. For $x, y \in X$ with $x \leq y$ the closed subinterval $[x, y]$ is defined to be the set of all $z \in X$ with $x \leq z \leq y$. A chain from x to y of length n is a sequence $x = x_0 < x_1 < \cdots < x_n = y$ of elements in X. A maximal chain of length n from x to y is a sequence $x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = y$. The minimum (resp. maximum) of the lengths of all chains from x to y is called the minimal (resp. maximal) length of $[x, y]$.

We denote by $k^{n\times m}$ the set of all $n \times m$ matrices over k and by M^t the transpose of a matrix M.

1. Definition of algebras $\mathcal{A}_{(X,M)}$

In this section we define for any finite poset X a family of finite dimensional algebras over an algebraically closed field k. Our construction is a generalization of that given in [4] and includes some special cases in [12]. We prove that $\mathcal{A}_{(X,M)}$ is quasi–hereditary if the Hasse diagram of X is a tree.

1.1 Let k be an algebraically closed field and X a finite poset. We consider each closed subinterval $[x, \omega]$ of X with minimal length 2. Suppose $y_1, \cdots, y_n$ are elements in $[x, \omega]$ such that $x \lessdot y_i \lessdot \omega$, $1 \leq i \leq n$, are all maximal chains from x to ω of length 2, i.e. the Hasse diagram of $[x, \omega]$ has a subdiagram of the following form



and such a diagram is called a mesh diagram of x and ω. With such a mesh we associate a matrix $M_n(x,\omega) \in k^{n\times n}$, say

$$M_n(x,\omega) = (a^{(x,\omega)}_{y_i y_j})_{y_i y_j}.$$

Then we say that X is labelled by matrices, denoted by (X, M), where M is the set of all the labelling matrices, and call M a matrix labelling on X.

1.2 Let X be a finite poset . We first define an associative k–algebra $\mathcal{A}'_X$ with a k–basis consisting of all symbols $x_n \cdots x_1 x_0$, where $n \geq 0$ and $x_i \in X$, $0 \leq i \leq n$, satisfy either $x_{i-1} \lessdot x_i$ or $x_{i-1} \gtrdot x_i$. The multiplication is defined on the basis by setting $(y_m \cdots y_1 y_0)(x_n \cdots x_1 x_0)$ equal to $y_m \cdots y_1 x_n \cdots x_1 x_0$ if $y_0 = x_n$ and 0 otherwise, then extended to $\mathcal{A}'_X$ by linearity. The algebra $\mathcal{A}'_X$ is already introduced by Dyer in [4]. Obviously, there is a k–algebra anti-involution ε of $\mathcal{A}'_X$ defined on the basis vectors by $\varepsilon : x_n \cdots x_1 x_0 \longmapsto x_0 x_1 \cdots x_n$.

1.3 Suppose that we are in the situation of 1.1. For $x \lessdot y$, $x \lessdot z$ in X, we define

$$r_{yxz} = yxz - \sum_{\omega} a_{yz}^{(x,\omega)} y\omega z,$$

where the sum is over all ω satisfying $y \lessdot \omega$ and $z \lessdot \omega$. Note that we allow in the above definition that $y = z$ and that if there is no ω satisfying $y \lessdot \omega$ and $z \lessdot \omega$ then the summation is zero.

1.4 Definition. Let $I(X, M)$ be the ideal of $\mathcal{A}'_X$ generated by elements of the following two types

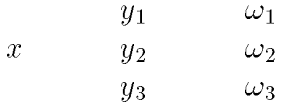
(1.4.1) $\qquad r_{yxz}, \ x, \ y, \ z \in X$ with $x \lessdot y$ and $x \lessdot z$,

(1.4.2) $\qquad u_n \cdots u_1 u_0 - v_m \cdots v_1 v_0, \ \ v_m \cdots v_1 v_0 - u_n \cdots u_1 u_0$, for $u < v$ in X,

where $u = u_n \lessdot \cdots \lessdot u_1 \lessdot u_0 = v$ and $u = v_m \lessdot \cdots \lessdot v_1 \lessdot v_0 = v$ are maximal chains from u to v.

We denote by $\mathcal{A}_{(X,M)}$ the quotient algebra of $\mathcal{A}'_X$ by $I(X, M)$ and call it an M–twisted double incidence algebra of X (This name is suggested by S.König).

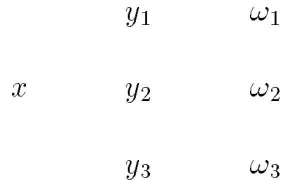
1.5 Remarks and examples. First note that if X satisfies the conditions in [4, 3.1–3.2], then the above definition yields the same algebra as in [4, 3.8]. Certainly, there are a lot of posets which do not satisfy the conditions in [4]. For example, let X be a poset with the following Hasse diagram



Then it is clear that this poset does not fulfill the conditions in [4]. However, we can still get a quadratic algebra $\mathcal{A}_{(X,M)}$ by using 1.4. We take

$$M(x, \omega_1) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad M(x, \omega_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M(x, \omega_3) = \begin{bmatrix} 0 \end{bmatrix}$$

Then $\mathcal{A}_{(X,M)}$ is the path algebra of the quiver



modulo the ideal $I(X,\, M)$.

Second, if we take all the matrices to be zero matrices then we get the algebra $\mathcal{A}(\mathcal{I}(X))$, the dual extension of $\mathcal{I}(X)$, defined in [12] and discussed in [7], where $\mathcal{I}(X)$ is the incidence algebra of X. Recall that the algebra $\mathcal{I}(X)$ by definition is the quotient of $\mathcal{I}'_X$ by the ideal $E(X)$, where $\mathcal{I}'_X$ denotes the subalgebra of $\mathcal{A}'_X$ with basis vectors $x_n \cdots x_1 x_0$ satisfying $x_n \lessdot \cdots \lessdot x_1 \lessdot x_0$, and where $E(X)$ is the ideal of $\mathcal{I}'_X$ generated by the elements of the type $u_n \cdots u_1 u_0 - v_m \cdots v_1 v_0$ for $u = u_n \lessdot \cdots \lessdot u_1 \lessdot u_0 = v$ and $u = v_m \lessdot \cdots \lessdot v_1 \lessdot v_0 = v$ maximal chains from u to v in X.

Finally, if we take all the labelling matrices to be symmetric then the anti–involution ε induces a k–algebra anti–involution of $\mathcal{A}_{(X,M)}$ which fixes all primitive idempotents x with $x \in X$, and we still denote it by ε.

1.6 Proposition. *For a finite poset X with an arbitrary matrix labelling M, the algebra $\mathcal{A}_{(X,M)}$ is finite dimensional.*

Proof. For convenience, we write $A' = \mathcal{A}'_X$, $A = \mathcal{A}_{(X,M)}$ and $I = I(X, M)$, and if there is no confusion in the context, we write sometimes $x_n \cdots x_1 x_0$ for $x_n \cdots x_1 x_0 + I$ in A.

For the proof of this proposition, we show much more, namely the following three properties.

(1) Let x be a maximal element in X. Then Ax can be spanned by the set

$$\{x_n \cdots x_1 x_0 x + I \mid n \in \mathbb{N},\ \ x_n \lessdot \cdots \lessdot x_0 \lessdot x \ \text{ in } \ X\}$$

over k.

To prove (1) it is enough to show the following statement: For $x_n \cdots x_1 x_0 \in A'$, if $x_n \cdots x_1 x_0 x \notin I$ then $x_n \lessdot \cdots \lessdot x_1 \lessdot x_0$. Indeed, if $n = 0$ or 1 then the statement follows directly from the fact that x is a maximal element and $xyz \in I$ for $y \lessdot x$ and $y \lessdot z$. Now suppose that the statement is true for n. Let us take an element $x_{n+1} x_n \cdots x_1 x_0 \in A'$ such that $x_{n+1} x_n \cdots x_1 x_0 x \notin I$. Of course, we have $x_n \cdots x_1 x_0 x \notin I$. By induction hypothesis, we have $x_n \lessdot \cdots \lessdot x_1 \lessdot x_0 \lessdot x$. If $x_{n+1} \lessdot x_n$ then we have what we want. Suppose $x_n \lessdot x_{n+1}$, then

$$x_{n+1} x_n x_{n-1} = \sum_{\omega} a^{(x_n,\omega)}_{x_{n+1} x_{n-1}} x_{n+1} \omega x_{n-1} + r_{x_{n+1} x_n x_{n-1}}$$

in A', where $a^{(x_n,\omega)}_{x_{n+1} x_{n-1}} \in k$ and the sum is over all $\omega \in X$ satisfying $x_{n+1} \lessdot \omega,\ x_{n-1} \lessdot \omega$. Again by induction hypothesis, $\omega x_{n-1} \cdots x_1 x_0 x \in I$ for all ω since $\omega \gtrdot x_{n-1}$. Thus

$$\begin{aligned} x_{n+1} x_n x_{n-1} \cdots x_1 x_0 x = & \sum_{\omega} a^{(x_n,\omega)}_{x_{n+1} x_{n-1}} x_{n+1} \omega x_{n-1} \cdots x_1 x_0 x \\ & + r_{x_{n+1} x_n x_{n-1}} x_{n-2} \cdots x_1 x_0 x \end{aligned}$$

lies in I. This contradiction shows that $x_n \lessdot x_{n+1}$ is impossible, and then finishes the proof of (1).

Dually, we have

($1'$) Let x be a maximal element in X. Then xA can be spanned by the set

$$\{xy_m y_{m-1}\cdots y_0 + I \mid m \in \mathbb{N},\ \ x \gtrdot y_m \gtrdot y_{m-1} \gtrdot \cdots \gtrdot y_0 \ \text{ in } \ X\}$$

over k.

From (1) and ($1'$) we have

(2) $xAx \cong k$ and AxA is finite dimensional.

In fact, that Ax and xA are finite dimensional implies that AxA is also finite dimensional. Moreover, we have by (1) that Ax admits a k–basis $\{x,\, \gamma_1 x,\, \cdots,\, \gamma_m x\}$ with $\gamma_j = x_{j1}x_{j2}\cdots x_{jm_j}$ and $x_{j1} \lessdot x_{j2} \lessdot \cdots \lessdot x_{jm_j} \lessdot x$, where $m_j \geq 1$ is a natural number for $j \leq m$. By ($1'$) xA has as k–basis the set $\{x,\, x\delta_1, \cdots,\, x\delta_m\}$, where $\delta_j = x_{jm_j}\cdots x_{j2}x_{j1}$. Since $\gamma_1 x,\, \cdots,\, \gamma_m x$ are k–linear independent, we know $x_{i1} \neq x_{j1}$ if $i \neq j$. Note that $x\gamma_j x = 0$ for all j according to (1). It is clear that $AxA = \sum_{j=1}^{m} Ax\delta_j$. Since the x_{ji} are pairwise orthogonal idempotents, we have that $AxA = Ax \oplus Ax\delta_1 \oplus \cdots \oplus Ax\delta_m$.

(3) Let x be a maximal element in X. We set $X_1 = X\backslash\{x\}$, equip X_1 with the order relation induced by X and consider the algebra $\mathcal{A}_{(X_1,M_1)}$, where the matrix labelling M_1 of X_1 is the restriction of the given labelling M of X on X_1. Let π_X and π_1 denote the canonical projections from $A' = \mathcal{A}'_X$ to $A = \mathcal{A}_{(X,M)}$ and from A to A/AxA, respectively. Note that the kernel of the composition of π_X and π_1 is $A'xA' + I$. Since X_1 is a full convex subposet of X, the algebra $\mathcal{A}'_{X_1}$ is canonically embedded into A'. This embedding is denoted by μ.

The ideal $I(X_1, M_1)$ of $\mathcal{A}'_{X_1}$ is mapped to zero under the morphism $\mu\pi_X\pi_1$. In fact, if $\rho \in \mathcal{A}'_{X_1}$ is an element of type (1.4.1), then there exists an element $\rho' \in A'$ of the same type such that $\rho' = \rho + a$ with $a \in A'xA'$. In this case we have $(\rho)\mu\pi_X\pi_1 = (\rho' - a)\pi_X\pi_1 = 0 - ((a)\pi_X)\pi_1 = 0$. If $\rho \in A'$ is of type (1.4.2), then $\rho \in I = I(X, M)$, thus $(\rho)\mu\pi_X\pi_1 = 0$. Hence we deduce that $(I(X_1, M_1))\mu\pi_X\pi_1 = 0$. Therefore, $\mu\pi_X\pi_1$ induces an algebra homomorphism $\nu : \mathcal{A}_{(X_1,M_1)} \longrightarrow A/AxA$, that is, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{A}'_{X_1} \arrow[r, "\mu"] \arrow[dd, "\pi_{X_1}"'] & A' \arrow[d, "\pi_X"] \\
 & A \arrow[d, "\pi_1"] \\
\mathcal{A}_{(X_1,M_1)} \arrow[r, "\nu"] & A/AxA
\end{tikzcd}$$

We shall prove that ν is an isomorphism of algebras. It is clear that ν is surjective. It remains to show that ν is injective. In order to do this, we prove first that for any $a \in I$ with $a = r + b$, where $r \in \mathcal{A}'_{X_1}$ and $b \in A'xA'$, there holds $r \in I(X_1, M_1)$. Since I is generated by the elements of types (1.4.1) and (1.4.2), it is enough to prove the statement for such generators. For the generators this becomes obvious.

With the above statement we can show that ν is injective. Taking an $a+I(X_1,\,M_1)\in\mathcal{A}_{(X_1,M_1)}$ with $(a+I(X_1,\,M_1))\nu=0$, we have $0=(a+I(X_1,M_1))\nu=(a)\mu\pi_X\pi_1$. that is, $a\in I+A'xA'$. Thus there are elements $r'\in I$ and $b'\in A'xA'$ such that $a=r'+b'\in\mathcal{A}'_{X_1}$. If we write r' as a sum of $r\in\mathcal{A}'_{X_1}$ and $b\in A'xA'$, then $r\in I(X_1,M_1)$. Hence $a=r+(b+b')$ implies that $a=r\in I(X_1,M_1)$. Therefore, ν is injective.

Using induction on the number of elements of X, one can easily see that $\mathcal{A}_{(X,M)}$ is finite dimensional. □

1.7 In general, the algebra $\mathcal{A}_{(X,M)}$ associated with a matrix labelling poset $(X,\,M)$ may not be quasi–hereditary (see 1.8). However, Dyer proved that $\mathcal{A}_{(X,M)}$ is quasi–hereditary if $(X,\,M)$ satisfies the conditions in [4, Sect. 3]. In the following we prove that if X is a tree, the algebra $\mathcal{A}_{(X,M)}$ is also quasi–hereditary. A further study of quasi–heredity of $\mathcal{A}_{(X,M)}$ will be given in a subsequent paper.

Theorem. *If X is a tree then for any matrix labelling M, the algebra $\mathcal{A}_{(X,M)}$ is quasi–hereditary.*

Proof. From the proof of 1.6 it suffices to prove that $\mathcal{A}_{(X,M)}x\mathcal{A}_{(X,M)}$ is a heredity ideal in $\mathcal{A}_{(X,M)}$ for each maximal element x in X. To this goal we first prove the following fact.

(1) The elements of the form $yxz:=y_0y_1\cdots y_mxz_0z_1\cdots z_n$ with $y=y_0\lessdot y_1\lessdot\cdots\lessdot y_m\lessdot x\gtrdot z_0\gtrdot z_1\gtrdot\cdots\gtrdot z_n=z$ in X form a basis of $\mathcal{A}_{(X,M)}$ over k.

Indeed, to each $x\in X$ we attach a symbol α_x and denote by V_x the vector space over k with the basis set $\{\alpha_y\mid x\leq y, y\in X\}$. For $x\lessdot y$ in X, by $i^y_x:V_y\to V_x$ we denote the inclusion and by $p^x_y:V_x\to V_y$ the k–linear map such that

$$p^x_y(\alpha_z)=\begin{cases}\alpha_y, & \text{if } z=x,\\ \sum\limits_{z\lessdot z'}a_{zz'}\alpha_{z'}, & \text{if } y\leq z,\\ 0, & \text{otherwise.}\end{cases}$$

where $a_{zz'}=a^{(x,y_1)}_{y_0y_0}a^{(y_0,y_2)}_{y_1y_1}\cdots a^{(y_{n-1},z)}_{y_ny_n}a^{(y_n,z')}_{zz}$ and $y=y_0\lessdot y_1\lessdot\cdots\lessdot y_n\lessdot z$ is the unique maximal chain from y to z.

For $x\lessdot y$ in X, one then has

$$i^y_xp^x_y=\sum_{y\lessdot w}a^{(x,w)}_{yy}p^y_wi^w_x\,, \tag{1.7.1}$$

that is, $V=\oplus_{x\in X}V_x$ becomes a module over $\mathcal{A}_{(X,M)}$.

Further, for each $x\in X$, by $j_x:V_x\to V$ and $\pi_x:V\to V_x$ we denote the canonical inclusion and projection respectively. Then by (1.7.1) there is a unique algebra homomorphism $\phi:\mathcal{A}_{(X,M)}\to\operatorname{End}_k(V)$ such that $\phi(x)=\pi_xj_x$ for $x\in X$ and

$$\phi(xy)=\pi_xp^x_yj_y,\qquad \phi(yx)=\pi_yi^y_xj_x$$

if $x\lessdot y$ in X. It is easy to see that for $y\leq x\geq z$ and for $\alpha_y\in V_y$, one has $(\alpha_y)(\phi(yxz))=\alpha_x\in V_z\subset V$. Hence the elements $\phi(yxz)$ with $y\leq x\geq z$ are linearly independent over k, that is, the elements $yxz \in \mathcal{A}_{(X,M)}$ are linearly independent over k. On the other hand, by the definition of $\mathcal{A}_{(X,M)}$ the elements yxz with $y \leq x \geq z$ span $\mathcal{A}_{(X,M)}$ over k. Therefore, the elements yxz form a basis of $\mathcal{A}_{(X,M)}$ over k.

(2) If x is a maximal element in X, the multiplication map

$$\mu : \mathcal{A}_{(X,M)}x \otimes_k x\mathcal{A}_{(X,M)} \longrightarrow \mathcal{A}_{(X,M)}x\mathcal{A}_{(X,M)}$$

is bijective. This is equivalent to that for every $y, z \in X$, the induced map

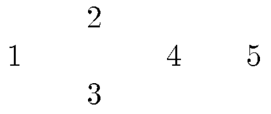
$$\mu_{yz} : y\mathcal{A}_{(X,M)}x \otimes_k x\mathcal{A}_{(X,M)}z \longrightarrow y\mathcal{A}_{(X,M)}x\mathcal{A}_{(X,M)}z$$

is bijective.

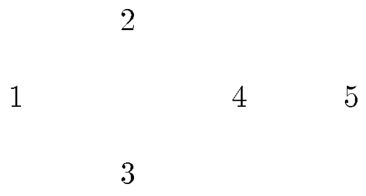
Indeed, the dimensions of $y\mathcal{A}_{(X,M)}x$ and $x\mathcal{A}_{(X,M)}z$ over k are at most 1, and that $y\mathcal{A}_{(X,M)}x \neq 0$ (resp. $x\mathcal{A}_{(X,M)}z \neq 0$) implies $y \leq x$ (resp. $x \geq z$). If $y\mathcal{A}_{(X,M)}x = 0$ or $x\mathcal{A}_{(X,M)}z = 0$, it is obvious that μ_{yz} is bijective. If $y\mathcal{A}_{(X,M)}x \neq 0$ and $x\mathcal{A}_{(X,M)}z \neq 0$, then one has $0 \neq yxz \in \mathcal{A}_{(X,M)}$, and thus μ_{yz} is bijective. Hence $\mathcal{A}_{(X,M)}x\mathcal{A}_{(X,M)}$ is a heredity ideal of $\mathcal{A}_{(X,M)}$.

This finishes the proof. □

1.8 Remark. If the poset X is not a tree, then the statement (1) in the proof of Theorem 1.7 may not be true. Hence in general the defined algebra $\mathcal{A}_{(X,M)}$ is not quasi–hereditary. For example, let X be the poset with the following Hasse diagram:



and with the labelling matrices $M(1,4) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $M(2,5) = [1]$, $M(3,5) = [1]$. Then the algebra $\mathcal{A}_{(X,M)}$ is given by the following quiver



with the relations

$$\alpha\alpha' = \gamma'\gamma, \;\; \alpha\beta' = \beta\alpha' = 0, \;\; \beta\beta' = \delta'\delta, \;\; \gamma\gamma' = \eta'\eta, \;\; \delta\delta' = \eta'\eta,$$
$$\eta\eta' = 0, \;\; \gamma\alpha = \delta\beta, \;\; \alpha'\gamma' = \beta'\delta'.$$

It is easy to check that $\mathcal{A}_{(X,M)}$ is not quasi–hereditary.

2. The quadratic dual of $\mathcal{A}_{(X,M)}$

Let $\mathcal{A}_{(X,M)}$ be the algebra associated with a finite poset X labelled by M, a family of matrices (see section 1). In this section we describe the quadratic dual algebra $\mathcal{A}^!_{(X,M)}$ when $\mathcal{A}_{(X,M)}$ is quadratic. The first question is when $\mathcal{A}_{(X,M)}$ is quadratic, and we have the following

2.1 Lemma. *The algebra $\mathcal{A}_{(X,M)}$ is quadratic if and only if the incidence algebra $\mathcal{I}(X)$ is quadratic.*

This follows directly from the definition of $\mathcal{A}_{(X,M)}$.

2.2 For $x \leq y$ in X, recall that by a maximal chain from x to y of length n we mean a sequence $x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = y$. We say that another chain $x = x'_0 \lessdot x'_1 \lessdot \cdots \lessdot x'_{n'} = y$ is adjacent to the first one if $n = n'$ and there is a unique $1 \leq j \leq n-1$ such that $x_j \neq x'_j$. The poset X is called adjacent if for every $x \leq y$ in X and for every two maximal chains C, D from x to y, there exists a sequence $C = C_0,\ C_1, \cdots, C_m = D$ of maximal chains from x to y such that C_i is adjacent to C_{i+1} for $0 \leq i \leq m-1$.

Lemma. *The incidence algebra $\mathcal{I}(X)$ is quadratic if and only if X is adjacent.*

Proof. It is easy to see that the condition is sufficient. We now prove the necessity of the condition.

Assume that $\mathcal{I}(X)$ is quadratic, then the ideal $E(X)$ is generated by the elements of the form

$$xy_1z - xy_2z \qquad \text{for } x,\ y_1,\ y_2,\ z \in X \text{ with } x \lessdot y_1 \lessdot z,\quad x \lessdot y_2 \lessdot z.$$

Suppose $x \leq y$. Let $C : x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = y$ and $D : x = x'_0 \lessdot x'_1 \lessdot \cdots \lessdot x'_{n'} = y$ be maximal chains from x to y. We shall prove that C and D are pre–adjacent, that is, there is a sequence $C = C_0, C_1, \cdots, C_p = D$ of maximal chains from x to y such that C_i is adjacent to C_{i+1} for $0 \leq i \leq p-1$.

From $x_0x_1\cdots x_n - x'_0x'_1\cdots x'_{n'} \in E(X)$ it follows that

$$(2.2.1)\qquad \begin{aligned} & x_0x_1\cdots x_n - x'_0x'_1\cdots x'_{n'} \\ &= \sum_{i=1}^{m} a_i x_{i0}\cdots x_{is_i}(u_iv_i^-w_i - u_iv_i^+w_i)x_{i,s_i+1}\cdots x_{it_i} \end{aligned}$$

in $\mathcal{I}'_X$, where $a_i \in k$ and $C_i^\sigma : x_{i0} \lessdot \cdots \lessdot x_{is_i} \lessdot u_i \lessdot v_i^\sigma \lessdot w_i \lessdot x_{i,s_i+1} \lessdot \cdots \lessdot x_{it_i}$ are maximal chains in X, $i \in \{1, \cdots m\}$, $\sigma \in \{-, +\}$. It is easy to see that C_i^- is adjacent to C_i^+ for all $1 \leq i \leq m$. By c, d and c_i^σ we then denote the elements $x_0x_1\cdots x_n$, $x'_0x'_1\cdots x'_{n'}$ and $x_{i0}\cdots x_{is_i}u_iv_i^\sigma w_ix_{i,s_i+1}\cdots x_{it_i}$ in $\mathcal{I}'_X$, respectively, where $i = 1, \cdots, m$ and $\sigma = -, +$.

Without loss of generality, we may assume that $x_{i0} = x$, $x_{it_i} = y$ for $1 \leq i \leq m$, i.e. C_i^σ, $i = 1, \cdots, m$, $\sigma = -, +$, are maximal chains from x to y.

Suppose that $C_1^-, C_1^+, \cdots, C_t^-, C_t^+$ are pre–adjacent to C and that $C_{t+1}^-, C_{t+1}^+, \cdots, C_m^-, C_m^+$ are not pre–adjacent to C. We claim that D is pre–adjacent to C. Otherwise, each element in $\{c\} \cup \{c_j^\sigma \mid 1 \le j \le t,\ \sigma = -, +\}$ is not a k–linear combination of the elements in $\{d\} \cup \{c_j^\sigma \mid t+1 \le j \le m,\ \sigma = -, +\}$ in $\mathcal{I}'_X$. According to (2.2.1) we have in $\mathcal{I}'_X$ that

$$c - \sum_{j=1}^{t} a_j(c_j^- - c_j^+) = d + \sum_{j=t+1}^{m} a_j(c_j^- - c_j^+).$$

By linear independence, we get $c - \sum_{j=1}^{t} a_j(c_j^- - c_j^+) = 0$, that is,

$$x_0 x_1 \cdots x_n = c = \sum_{j=1}^{t} a_j(c_j^- - c_j^+) \in E(X).$$

On the other hand, we know from the definition of an incidence algebra that for any maximal chain $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$, there holds $x_0 x_1 \cdots x_n \notin E(X)$. Thus we have a contradiction. This shows D is adjacent to C and finishes the proof. □

2.3 In order to describe $\mathcal{A}^!_{(X,M)}$, we define an algebra $\mathcal{B}_{(X,M)}$ associated with each matrix labelling poset (X, M). The algebra $\mathcal{A}'_X$ is defined as in section 1.

Definition. The algebra $\mathcal{B}_{(X,M)}$ is the quotient of $\mathcal{A}'_X$ by the two–sided ideal $J(X, M)$ generated by the elements of the following two types:

$$(2.3.1) \qquad r_{yxz} = yxz - \sum_{\omega} a_{yz}^{(x\omega)} y\omega z \quad \text{for } x \lessdot y,\ x \lessdot z,$$

where the sum is over all $\omega \in X$ satisfying $y \lessdot \omega$, $z \lessdot \omega$, and

$$(2.3.2) \qquad q_{xy} = \sum_i x x_{i1} \cdots x_{im_i} y, \quad q'_{xy} = \sum_i y x_{im_i} \cdots x_{i1} x, \text{ for } x < y \text{ in } X,$$

where the both sums are over all maximal chains $x \lessdot x_{i1} \lessdot \cdots \lessdot x_{im_i} \lessdot y$ from x to y of length ≥ 2. Note that if there is no maximal chain from x to y of length ≥ 2 in X, then $q_{xy} = 0$.

2.4 Proposition. *For an arbitrary matrix labelling poset (X, M), the algebra $\mathcal{B}_{(X,M)}$ is quadratic.*

Proof. Suppose $x < y$ in X. We denote by $q_{[x,\,y]}$ the sum of all length 2 maximal chains from x to y in X. If there is no such chain then $q_{[x,\,y]} = 0$. It is easy to see that

$$(2.4.1) \qquad q_{xy} = q_{[x,\,y]} + \sum_{x \lessdot u < y} xu \cdot q_{uy}.$$

Further, we denote by $J'(X,\, M)$ the ideal of $\mathcal{B}_{(X,M)}$ generated by the elements of type (2.3.1) and of types $q_{[x,\, y]}$, $q'_{[x,\, y]}$ for $x < y$ in X, where $q'_{[x,\, y]}$ is the image of $q_{[x,\, y]}$ under ε given in 1.2. We shall prove that $J(X,\, M) = J'(X,\, M)$.

(1) To prove $J(X,\, M) \subseteq J'(X,\, M)$ it is enough to show that the elements of type (2.3.2) lie in $J'(X,\, M)$.

Suppose $x < y$ in X. By $l(x,\, y)$ we denote the maximum of lengths of maximal chains from x to y. We use induction on $l(x,\, y)$ to prove $q_{xy} \in J'(X,\, M)$. If $l(x,\, y) = 1$, then $q_{xy} = 0 \in J'(X,\, M)$. Now suppose that for every $x < y$ in X with $1 \leq l(x,\, y) < n$, the element q_{xy} lies in $J'(X,\, M)$. Choose $x < y$ in X with $l(x,\, y) = n$. Then for all $x \lessdot u < y$, there holds $l(u, y) < n = l(x,\, y)$. By induction hypothesis, $q_{uy} \in J'(X,\, M)$ for all u. Then by (2.4.1) one has

$$q_{xy} = q_{[x,\, y]} + \sum_{x \lessdot u < y} xu \cdot q_{uy} \in J'(X,\, M).$$

Similarly, $q'_{xy} \in J'(X,\, M)$. Therefore, $J(X,\, M) \subseteq J'(X,\, M)$.

(2) The inverse inclusion $J'(X,\, M) \subseteq J(X,\, M)$ follows from

$$q_{[x,\, y]} = q_{xy} - \sum_{x \lessdot u < y} xu \cdot q_{uy} \in J(X,\, M).$$

By (1) and (2) we have that $J(X,\, M) = J'(X,\, M)$ is generated by quadratic relations, i.e. $\mathcal{B}_{(X,M)}$ is quadratic. □

2.5 Let $x \in X$ be maximal. We set $X_1 = X \backslash \{x\}$ and equip X_1 with the order relation induced by X. By M_1 we denote the labelling of the restriction of M on X_1. Then we have the following

Proposition. *There is a canonical algebra isomorphism from the algebra* $\mathcal{B}_{(X_1,M_1)}$ *to* $\mathcal{B}_{(X,M)}/\mathcal{B}_{(X,M)}x\mathcal{B}_{(X,M)}$, *where* $\mathcal{B}_{(X,M)}x\mathcal{B}_{(X,M)}$ *denotes the ideal of* $\mathcal{B}_{(X,M)}$ *generated by the idempotent* x.

The proof of this proposition is similar to that of (4) in Proposition 1.6.

2.6 Corollary. *For each* (X, M), *the dimension of* $\mathcal{B}_{(X,M)}$ *over* k *is finite.*

Proof. Let x be a maximal element in X. Using a similar argument in the proof of (1) in Proposition 1.6, one can see that $\mathcal{B}_{(X,M)}x$ is spanned by $\{x_n \cdots x_1 x_0 x + J(X, M) \mid n \in \mathbb{N},\, x_n \lessdot \cdots \lessdot x_1 \lessdot x_0 \lessdot x \text{ in } X\}$ over k and that $x\mathcal{B}_{(X,M)}$ is spanned by $\{x y_m \cdots y_1 y_0 + J(X,\, M) \mid m \in \mathbb{N},\, x \gtrdot y_m \gtrdot \cdots \gtrdot y_1 \gtrdot y_0 \text{ in } X\}$ over k. Thus $\mathcal{B}_{(X,M)}x$ and $x\mathcal{B}_{(X,M)}$ are finite dimensional k–space. Therefore, $\mathcal{B}_{(X,M)}x\mathcal{B}_{(X,M)}$ is a finite dimensional k–space. Using Proposition 2.5 and induction on the number of the elements in X, one can obtain that $\mathcal{B}_{(X,M)}x\mathcal{B}_{(X,M)}$ is finite dimensional. □

2.7 Suppose $\mathcal{A}_{(X,M)}$ is quadratic. Following Manin (see [7]), we write the quadratic algebra in the following form:

$$\mathcal{A}_{(X,M)} \longleftrightarrow \{\mathcal{A}_1,\, R(\mathcal{A}_1) \subseteq \mathcal{A}_1 \otimes_{\mathcal{A}_0} \mathcal{A}_1\},$$

where $\mathcal{A}_0$ is the vector space spanned by $x \in X$ and $\mathcal{A}_1$ is spanned by xy and yx for $x < y$ in X, and where $R(\mathcal{A}_1)$ is the vector space spanned by

$$\begin{aligned} &\{uv_1w - uv_2w \mid u,\, v_1,\, v_2,\, w \in X \ \text{ with } \ u < v_1 < w,\ u < v_2 < w\} \\ &\cup \{r_{yxz} | x,\, y,\, z \in X \ \text{ with } \ x < y,\ x < z\}. \end{aligned}$$

In this case, one can also see that $\mathcal{A}_1 \otimes_{\mathcal{A}_0} \mathcal{A}_1$ is the vector space spanned by

$$\begin{aligned} S = \{xy \otimes yz = xyz \mid x,\, y,\, z \in X \ \text{ with } \ x < y \ \text{ and } \ y < z, \ \text{or} \ x < y \ \text{ and } \ y > z \\ \text{or} \ x > y \ \text{ and } \ y < z, \ \text{or} \ x > y \ \text{ and } \ y > z\}. \end{aligned}$$

By definition, the quadratic dual algebra $\mathcal{A}^!_{(X,M)}$ is given by

$$\mathcal{A}^!_{(X,M)} \longleftrightarrow \{\mathcal{A}^*_1,\ R(\mathcal{A}_1)^\perp \subseteq (\mathcal{A}_1 \otimes_{\mathcal{A}_0} \mathcal{A}_1)^* = \mathcal{A}^*_1 \otimes_{\mathcal{A}_0} \mathcal{A}^*_1\},$$

where $\mathcal{A}^*_1 = \operatorname{Hom}_k(\mathcal{A}_1, k)$, where $R(\mathcal{A}_1)^\perp = \{f \in \mathcal{A}^*_1 \otimes_{\mathcal{A}_0} \mathcal{A}^*_1 \mid (R(\mathcal{A}_1))f = 0\}$ and where the bimodule structure of $\mathcal{A}^*_1$ over $\mathcal{A}_0$ is defined in an obvious way. Here we identify $(\mathcal{A}_1 \otimes_{\mathcal{A}_0} \mathcal{A}_1)^*$ with $\mathcal{A}^*_1 \otimes_{\mathcal{A}_0} \mathcal{A}^*_1$ by sending $(a \otimes b)^*$ to $b^* \otimes a^*$. Then $\mathcal{A}^*_1 \otimes_{\mathcal{A}_0} \mathcal{A}^*_1$ is the vector space spanned by

$$S^* = \{(yz)^* \otimes (xy)^* = (xy \otimes yz)^* = (xyz)^* \mid x,\, y,\, z \in X,\ xyz \in S\}.$$

Lemma. *As a vector space, $R(\mathcal{A})^\perp$ is spanned by the elements of the following two types*

$$(2.7.1) \qquad q^*_{[u,\,w]} = \sum_v (uvw)^*,\ \ q'^*_{[u,\,w]} = \sum_v (wvu)^*, \quad u,\, v,\, w \in X,$$

where the both sums are over all v such that $u < v < w$, and

$$(2.7.2) \qquad r^*_{y\omega z} = (y\omega z)^* + \sum_x a^{(x,\omega)}_{yz} (yxz)^* \ \ \textit{for} \ \ y < \omega,\ z < \omega \ \ \textit{in} \ \ X,$$

where the sum is over all x satisfying $x < y$, $x < z$.

Proof. It is easy to check that the elements of the types (2.7.1) and (2.7.2) lie in $R(\mathcal{A})^\perp$. Conversely, let $\Phi \in R(\mathcal{A})^\perp$, and we write

$$\begin{aligned} \Phi = &\sum_{x < y < z} a_{xyz}(xyz)^* + \sum_{x > y > z} b_{xyz}(xyz)^* + \sum_{\substack{x < y \\ z < y}} c_{xyz}(xyz)^* \\ &+ \sum_{\substack{x > y \\ z > y}} d_{xyz}(xyz)^* =: \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4\,, \end{aligned}$$

where $a_{xyz},\, b_{xyz},\, c_{xyz},\, d_{xyz} \in k$ and Φ_i denotes the i–th term in the sum of the above equality.

Let $[u,\, w]$ be a closed subinterval of minimal length 2 in X with the mesh diagram of u and w of the following form ($s \geq 1$):

$$\begin{array}{ccc} & v_1 & \\ & \vdots & \\ u & v_i & w \\ & \vdots & \\ & v_s & \end{array}$$

then $uv_iw - uv_jw \in R(\mathcal{A})$ for $1 \leq i,\ j \leq s$. Thus $0 = (uv_iw - uv_jw)\Phi = (uv_iw)\Phi - (uv_jw)\Phi = a_{uv_iw} - a_{uv_jw}$, i.e. $a_{uv_iw} = a_{uv_jw}$ for $1 \leq i, j \leq s$. Therefore, $\Phi_1 = \sum\limits_{x \lessdot y \lessdot z} a_{xyz}(xyz)^* = \sum\limits_{u,w} a_{uw}q^*_{[u,\, w]}$, where the sum is over all mesh diagrams of u and w for $u,\, w$ in X, and where $a_{uw} = a_{uv_iw}$ for $1 \leq i \leq s$.

Similarly, $\Phi_2 = \sum\limits_{x \gtrdot y \gtrdot z} b_{xyz}(xyz)^* = \sum\limits_{u,w} b_{uw}{q'}^*_{[u,\, w]}$.

Now consider a full subposet of X with the following Hasse diagram.

$$\begin{array}{cccccccc} & & & e & & & & \\ g_1 & g_2 & \cdots & g_s & h_1 & h_2 & \cdots & h_t \\ & & & f & & & & \end{array} \tag{1}$$

such that $g_1, \cdots, g_s$ (resp. $h_1, \cdots, h_t$) are all the elements v in X satisfying $v \lessdot e$, $v \lessdot f$ (resp. $e \lessdot v$, $f \lessdot v$) with $s + t \geq 1$.

By definition, $r_{eg_if} = eg_if - \sum\limits_{j=1}^{t} a_{ef}^{(g_i,h_j)} eh_jf \in R(\mathcal{A})$, $i = 1, \cdots, s$. Then

$$0 = (r_{eg_if})\Phi = (eg_if)\Phi - \sum_{j=1}^{t} a_{ef}^{(g_i,h_j)}(eh_jf)\Phi = c_{eg_if} - \sum_{j=1}^{t} a_{ef}^{(g_i,h_j)} d_{eh_jf}$$

thus

$$\begin{aligned} \Phi_3 + \Phi_4 &= \sum_{\substack{x \lessdot y \\ z \lessdot y}} c_{xyz}(xyz)^* + \sum_{\substack{x \gtrdot y \\ z \gtrdot y}} d_{xyz}(xyz)^* \\ &= \sum_{e,f} \sum_{j} d_{eh_jf}\big((eh_jf)^* + \sum_{i} a_{ef}^{(g_i,h_j)}(eg_if)^*_i\big) \\ &= \sum_{e,f} \sum_{j} a_{eh_jf} r^*_{(eh_jf)}\,, \end{aligned}$$

where the first sum is over all pairs $(e,\ f)$ such that X contains a full subposet of the form (1).

From the argument above we conclude that $\Phi = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4$ is a k–linear combination of the elements of the types (2.7.1) and (2.7.2). This finishes the proof. $\square$

With the help of the lemma, we now can describe the quadratic dual algebra $\mathcal{A}^!_{(X,M)}$.

2.8 Let X be a finite poset. By X^{op} we denote the opposite poset of X. With each matrix labelling M on X we associate a matrix labelling $-M^t$ on X^{op} as follows: Let $[x,\ z]$ be a closed subinterval of minimal length 2 in X with the following mesh diagram of x and z:

$$\begin{array}{ccc} & y_1 & \\ & \vdots & \\ x & y_i & z \\ & \vdots & \\ & y_n & \end{array}$$

and denote by $M(x,\ z) \in k^{n\times n}$ the labelling matrix to the mesh diagram of x and z. Then $[z,\ x]$ is a closed subinterval of minimal length 2 in X^{op} with the mesh diagram of z and x of the form:

$$\begin{array}{ccc} & y_1 & \\ & \vdots & \\ z & y_i & x \\ & \vdots & \\ & y_n & \end{array}$$

Now we label the mesh diagram of z and x in X^{op} by the matrix $-M(x,\ z)^t$, where $M(x,\ z)^t$ denotes the transpose of $M(x,\ z)$. In such a way, X^{op} becomes a matrix labelling poset, denoted by $(X^{op},\ -M^t)$.

As a consequence of Lemma 2.7, Proposition 2.4 and the construction of the algebra $\mathcal{B}_{(X^{op},\ -M^t)}$, we have the following

Theorem. *For any matrix labelling poset (X, M), if the poset X is adjacent, then*

$$\mathcal{A}^!_{(X,M)} \cong \mathcal{B}_{(X^{op},\ -M^t)}.$$

2.9 Remark. The description of the quadratic dual $\mathcal{A}^!_{(X,M)}$ in Theorem 2.8 is a generalization of that given in [4]. Namely, if the poset and the labelling M satisfy the conditions in [4], then 2.8 is just [4, 3.6(a)].

2.10 One interesting question is when the quadratic dual $\mathcal{A}^!_{(X,M)}$ is quasi–hereditary, i.e. when $\mathcal{B}_{(X^{op},\ -M^t)}$ is quasi–hereditary. The following propositions give a partial answer to this question.

Proposition 1. *If the Hasse diagram of X is a tree then $\mathcal{A}^!_{(X,M)}$ is quasi–hereditary.*

Proof. We shall prove that $\mathcal{B}_{(X,M)}$ is quasi–hereditary for any tree X with arbitrary matrix labelling M. Set $\mathcal{B} = \mathcal{B}_{(X,M)}$. As in the proof of Proposition 1.6 we can show that for each maximal element x in X there holds $\mathrm{End}_{\mathcal{B}}(\mathcal{B}x) \cong k$. Now we prove that the multiplication map $\mathcal{B}x \otimes_k x\mathcal{B} \longrightarrow \mathcal{B}x\mathcal{B}$ is injective. It is equivalent to showing that for each pair y, z in X the induced map

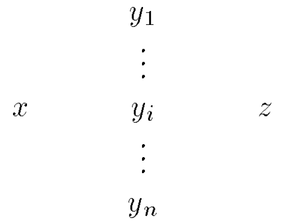
$$\mu : y\mathcal{B}x \otimes_k x\mathcal{B}z \longrightarrow y\mathcal{B}x\mathcal{B}z$$

is injective. If $y = x$ or $y = z$ then μ is clearly bijective. Suppose both $y \neq x$ and $z \neq x$. Since X is a tree, we have that $x_1x_2x_3,\ x_3x_2x_1 \in J(X,\, M)$ for all $x_1 < x_2 < x_3$ in X. Then the space $y\mathcal{B}x$ is at most of dimension 1. Similarly, the space $x\mathcal{B}z$ is at most of dimension 1. If $y\mathcal{B}x = 0$ or $x\mathcal{B}y = 0$, then μ is injective. Suppose that $y\mathcal{B}x \neq 0$ and $x\mathcal{B}z \neq 0$. Then $y\mathcal{B}x = kyx$ with $y < x$ and $x\mathcal{B}z = kxz$ with $z < x$. In this case, $yxz \notin J(X,\, M)$ according to the definition of $\mathcal{B}_{(X,M)}$. Thus μ is injective. Hence if x is maximal in X, then $\mathcal{B}x\mathcal{B}$ is a heredity ideal in $\mathcal{B}_{(X,M)}$.

Now applying induction on the number of elements of X and 2.5, we can see immediately that $\mathcal{B}_{(X,M)}$ is quasi–hereditary for any poset X whose Hasse diagram is a tree. □

Let us remark that if $X = \{1 < 2 < \cdots < n\}$ and if all matrices in the labelling M are [1], then the algebra $\mathcal{B}_{(X,M)}$ is just a finite type block of some q–Schur algebra (see [13]).

Proposition 2. *If X is a poset with the following Hasse diagram ($n \geq 1$):*



Then for an arbitrary labelling matrix $M \in k^{n\times n}$, the algebra $\mathcal{B}_{(X,M)}$ is quasi–hereditary.

Proof. For simplicity, we write $\mathcal{B} = \mathcal{B}_{(X,M)}$ and $J = J(X,\, M)$. By Proposition 2.5, it is enough to prove that the following map

$$\begin{array}{cc} \Psi & \mathcal{B}z \otimes_{z\mathcal{B}z} z\mathcal{B} \longrightarrow \mathcal{B}z\mathcal{B} \\ & a \otimes b \longmapsto ab \end{array}$$

is bijective. It is obvious that Ψ is surjective. To see the injectivity we compare the dimensions of both sides over k. Since $z\mathcal{B}z \cong k$, $\dim_k \mathcal{B}z \otimes_{z\mathcal{B}z} z\mathcal{B} = \dim_k \mathcal{B}z \times \dim_k z\mathcal{B}$. One can easily see, as k–spaces, that $\mathcal{B}z$ has a basis $S_1 = \{z,\, y_1z,\, \cdots,\, y_nz,\, xy_1z,\, \cdots,\, xy_{n-2}z\}$, that $z\mathcal{B}$ has a basis $S_2 = \{z,\, zy_1, \cdots,\, zy_n, zy_1x,\, \cdots,\, zy_{n-2}x\}$ and that $\mathcal{B}z\mathcal{B}$ has a basis $\{ab : a \in S_1,\, b \in S_2\}$, i.e. $\dim_k \mathcal{B}z\mathcal{B} = \dim_k \mathcal{B}z \times \dim_k z\mathcal{B} = 4n^2$. This finishes the proof. □

3. Δ–subalgebras and global dimensions

In the paper [10] S.König defines Δ–subalgebras and Borel subalgebras for quasi–hereditary algebras which might serve as finite–dimensional associative substitutes of Lie theoretic Borel subalgebras. In this section we prove that if the algebra $\mathcal{A}_{(X,M)}$ is quasi–hereditary, it always has a Δ–subalgebra and a Borel subalgebra. The results of this section will be used in the next two sections to calculate the dimensions of Ext–groups.

Let us recall from [10] some definitions.

3.1 Definition. Let $(A, \wedge)$ be a quasi–hereditary algebra with the weight poset $\wedge$ and standard modules $\Delta(\lambda)$, $\lambda \in \wedge$. Let B be a subalgebra of A (both algebras have the same unit element). Then B is called an exact Borel subalgebra of $(A, \wedge)$ if and only if there exists a bijection between the set of weights of B and the set of weights of A such that the following conditions are satisfied:

(D) The algebra $(B, \wedge)$ is directed (i.e. quasi–hereditary with simple standard modules and injective costandard modules) with respect to the partial order induced from the partial order of the set of weights of A;

(T) Tensor induction $A \otimes_B -$ is an exact functor, thus A_B is projective as a right B–module;

(W) For each $\lambda \in \wedge$, the following holds:

$$A \otimes_B E_B(\lambda) \cong \Delta_A(\lambda),$$

where $E_B(\lambda)$ denotes the simple B–module corresponding to the weight λ.

The algebra B is called a strong exact Borel subalgebra of $(A, \wedge)$ if and only if both the following condition (S) and the above conditions (T) and (W) are satisfied.

(S) There is a maximal semisimple subalgebra $S(A)$ of A which is also a maximal semisimple subalgebra of B, thus simple A–modules and simple B-modules coincide.

It is noted by S.König that if a quasi–hereditary algebra has a strong exact Borel subalgebra B then gl.dim.$B \leq$ gl.dim.A (see 3.5 below). There is a dual notion of Δ–subalgebra (see [10]).

3.2 Definition. Let $(A, \wedge)$ be a quasi–herditary algebra and C a subalgebra of A, having the same number of weights. Suppose the weights of C are identified (via certain bijection) with the weights of A, and the set of weights of C in this way is made into a poset. Assume that the algebra $(C^{op}, \wedge)$ is directed (i.e. $(C, \wedge)$ is quasi–hereditary with projective standard modules). Assume that for each weight i the projective A–module $A \otimes_C P_C(i)$ (where $P_C(i)$ is the indecomposable projective C–module corresponding to the weight i) decomposes into a direct sum of exactly one copy $P_A(i)$ (which is the indecomposable projective A–module corresponding to the weight i) and some indecomposable projective A–modules having weights different from i. Fix for each weight i an epimorphism $\kappa(i)$: $A \otimes_C P_C(i) \rightarrow \Delta_A(i)$.

Then C is called a Δ–subalgebra of $(A, \wedge)$ if and only if for each weight i the restriction of $\kappa(i)$ to $P_C(i) \subset A \otimes_C P_C(i)$ is an isomorphism of C–modules:

$$\kappa(i) \mid_{P_C(i)}: P_C(i) \cong \Delta_A(i).$$

If, in addition, C contains a maximal semisimple subalgebra of A, then C is called a strong Δ–subalgebra of $(A, \wedge)$.

3.3 Remark. A special case in the definition is that one takes C to be a subalgebra of A with the same complete set of orthogonal primitive idempotents, that is, $1 = e_1 + \cdots + e_n$ with $e_i \in C$. In this case $A \otimes_C P_C(i) = A \otimes_C Ce_i \cong Ae_i$ and $\kappa(i)$ can be chosen to be the canonical surjective map $Ae_i \to \Delta_A(i)$.

As pointed out in [10] the relation between a Borel subalgebra and a Δ–subalgebra is as follows.

3.4 Theorem [10]. *Let* $(A, \wedge)$ *be a quasi–hereditary algebra and* B *a subalgebra of* A*. Then* B *is an exact Borel subalgebra of* $(A, \wedge)$ *if and only if* B^{op} *is a* Δ*–subalgebra of* $(A^{op}, \wedge)$*.*

3.5 Lemma. *Let* A *be a quasi–hereditary algebra which has a strong exact Borel subalgebra* B*. Then for each natural number* l*, for each* B*–module* M*, and for each* A*–module* N *there is an isomorphism of vector spaces:*

$$Ext^l_A(A \otimes_B M,\, N) \cong Ext^l_B(M,\, N|_B),$$

where $N|_B$ *denotes the restriction of the* A*-module* N *to* B*.*

Proof. The lemma is called Eckmann–Shapiro lemma. For the convenience of readers, we include a proof here. It is enough to show the above isomorphism of Ext–groups. If $l = 0$, this is just the adjunction formula. If M is a projective B–module, there is nothing to be shown. If M is not projective, then we choose an exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P a projective B–module. This yields an exact sequence

$$0 \longrightarrow A \otimes_B K \longrightarrow A \otimes_B P \longrightarrow A \otimes_B M \longrightarrow 0.$$

Thus we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & (A \underset{B}{\otimes} M, {}_AN) \arrow[r] \arrow[d] & (A \underset{B}{\otimes} P, {}_AN) \arrow[r] \arrow[d] & (A \underset{B}{\otimes} K, {}_AN) \arrow[r] \arrow[d] & \mathrm{Ext}^1_A(A \underset{B}{\otimes} M, {}_AN) \arrow[r] & 0 \\
0 \arrow[r] & (M, N|_B) \arrow[r] & (P, N|_B) \arrow[r] & (K, N|_B) \arrow[r] & \mathrm{Ext}^1_B(M, N|_B) \arrow[r] & 0
\end{tikzcd}$$

where all vertical maps are adjunctions and where $(-, -)$ denotes the homomorphism group. This shows that for $l = 1$, the lemma is true. For large l, the isomorphism follows by dimension shift. □

3.6 Now we consider the algebra $\mathcal{A}_{(X,M)}$ constructed in section 1. Suppose that $\mathcal{A}_{(X,M)}$ is quasi–hereditary with respect to X. From the proof of 1.6, the standard $\mathcal{A}_{(X,M)}$–modules are just the indecomposable projective $\mathcal{I}(X)$–modules, where $\mathcal{I}(X)$ is the incidence algebra of the poset X. Hence we have by definition the following

Theorem. *Let X be a finite poset with a matrix labelling M such that the algebra $\mathcal{A}_{(X,M)}$ is quasi–hereditary. Then the incidence algebra $\mathcal{I}(X)$ of X is a strong Δ–subalgebra of $\mathcal{A}_{(X,M)}$.*

If the labelling is symmetric, that is, all matrices are symmetric, then we have an anti–involution ε of the algebra $\mathcal{A}_{(X,M)}$ which fixes the complete set of primitive idempotents. This together with 3.4 yields the following

3.7 Corollary. *Let X be a finite poset with a symmetric matrix labelling such that $\mathcal{A}_{(X,M)}$ is quasi–hereditary. Then $\mathcal{A}_{(X,M)}$ has a strong exact Borel subalgebra.*

Now we shall prove that 3.7 holds more generally. Suppose $\mathcal{A}_{(X,M)}$ is quasi–hereditary.

Let B be the subalgebra of $\mathcal{A}_{(X,M)}$ generated by all elements $x \in X$ and zy with $y < z$ in X. Then B and $\mathcal{A}_{(X,M)}$ have the same maximal semisimple subalgebra $\sum_{x\in X} kx$. If x is a maximal element, then $\mathcal{A}_{(X,M)}x\mathcal{A}_{(X,M)}$ is a heredity ideal by our assumption and $x\mathcal{A}_{(X,M)}$ can be spanned as a vector space by all elements of the form $x_0x_1\cdots x_n$ with $x_n < \cdots < x_1 < x_0 = x$. Hence the costandard module $\nabla_{\mathcal{A}_{(X,M)}}(x)$ is just the injective $\mathcal{A}_{(X,M)}$–module $D(x\mathcal{A}_{(X,M)})$, where $D = \mathrm{Hom}_k(-, k)$. Note that the element $x_0x_1\cdots x_n$ with $x_n < \cdots < x_1 < x_0 = x$ belongs to B. Therefore $xB = x\mathcal{A}_{(X,M)}$. Hence the induction procedure of the proof of 1.6 shows that $\nabla_{\mathcal{A}_{(X,M)}}(x) \cong D(xB)$ as B–modules for every element $x \in X$. It follows then from [10, Theorem A] that B is an exact Borel subalgebra of $\mathcal{A}_{(X,M)}$.

3.8 Theorem. *Let X be a finite poset with a matrix labelling M such that $\mathcal{A}_{(X,M)}$ is quasi–hereditary. Then $\mathcal{A}_{(X,M)}$ has a strong exact Borel subalgebra.*

As a first application of the above results we get an estimate of global dimension of $\mathcal{A}_{(X,M)}$.

Let I^+ be the ideal of $A := \mathcal{A}_{(X,M)}$ generated by all elements xy with $x, y \in X$ and $y < x$. Then $A/I^+ \cong \mathcal{I}(X)$. Similarly, let I^- be the ideal of A generated by all elements xy with $x, y \in X$ and $x < y$. Then $B \cong A/I^-$. Hence every B–module can be regarded as an A–module.

Note that the anti–involution ε of $\mathcal{A}'_X$ induces an anti–isomorphism from $\mathcal{I}(X)$ to B. This implies that $\dim_k\Delta(x) = \dim_k\nabla(x)$ for all $x \in X$.

3.9 Theorem. *Suppose X is a finite poset with a matrix labelling M such that $\mathcal{A}_{(X,M)}$ is quasi–hereditary. If the incidence algebra $\mathcal{I}(X)$ of X has global dimension m, then $m \leq gl.dim.\mathcal{A}_{(X,M)} \leq 2m$.*

Proof. For every simple $\mathcal{I}(X)$–module $E(x)$, we have a miminal projective resolution in $\mathcal{I}(X)$–mod

$$0 \longrightarrow P_{l_x} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow E(x) \longrightarrow 0,$$

where l_x is a natural number and $l_x \leq m$. Since P_i is projective $\mathcal{I}(X)$–module and every indecomposable projective $\mathcal{I}(X)$–module can be considered as a standard $\mathcal{A}_{(X,M)}$–module, it follows from $P_{\mathcal{I}(X)}(x) \cong \Delta_{\mathcal{A}_{(X,M)}}(x) = \mathcal{A}_{(X,M)} \otimes_B E(x)$ and proj.dim.$_B E(x) \leq m$ that the projective dimension of P_i as $\mathcal{A}_{(X,M)}$–module is bounded by m. Using the fact that if $0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0$ is a long exact sequence and proj.dim.$M_i \leq k$ for all i, then proj.dim.$M \leq n+k$, one obtains that gl.dim.$\mathcal{A}_{(X,M)} \leq 2m$. The inequality gl.dim.$\mathcal{A}_{(X,M)} \geq m$ follows from 3.5. □

As a second application we point out the k–dimension of $\mathcal{A}_{(X,M)}$.

3.10 Theorem. *Let $X = \{x_1, \cdots, x_n\}$ be a finite poset with n elements. Suppose M is an arbitrary matrix labelling of X. If $\mathcal{A}_{(X,M)}$ is quasi–hereditary, then*

$$dim_k \mathcal{A}_{(X,M)} = \sum_{i=1}^{n} (dim_k P_{\mathcal{I}(X)}(x_i))^2,$$

where $P_{\mathcal{I}(X)}(x_i)$ is the indecomposable projective $\mathcal{I}(X)$–module corresponding to the weight x_i.

Proof. Let $C = (c_{ij})$ be the Cartan–matrix of $\mathcal{A}_{(X,M)}$. Then $C = (\mathbf{dim}\Delta)^t(\mathbf{dim}\nabla)$ by [6], where $\mathbf{dim}\Delta$ is an $n \times n$ matrix with the i–th row the dimension vector $\underline{\dim}\Delta(x_i)$, and where $\mathbf{dim}\nabla$ is defined in a similar way. Since $\mathcal{A}_{(X,M)}$ is a basic algebra and $\dim_k \Delta(x_i) = \dim_k \nabla(x_i)$, we have

$$\begin{aligned} \dim_k \mathcal{A}_{(X,M)} &= \sum_{1 \leq i,\, j \leq n} c_{ij} = \sum_{i=1}^{n} \dim_k \Delta(x_i) \cdot \dim_k \nabla(x_i) \\ &= \sum_{i=1}^{n} (\dim \Delta(x_i))^2 = \sum_{i=1}^{n} (\dim_k P_{\mathcal{I}(X)}(x_i))^2. \end{aligned}$$

□

To end this section let us mention the following conjecture.

3.11 Conjecture. *Let X be a finite poset with an arbitrary matrix labelling M. Suppose that $\mathcal{A}_{(X,M)}$ is quasi–hereditary, then gl.dim.$\mathcal{A}_{(X,M)} = 2$ gl.dim.$\mathcal{I}(X)$, where $\mathcal{I}(X)$ is the incidence algebra of X.*

4. Ringel duality of $\mathcal{A}_{(X,M)}$

For each quasi–hereditary algebra A, C.M.Ringel has defined in [11] another quasi–hereditary algebra $\mathcal{R}(A)$ which is the endomorphism algebra of the characteristic module for A. His construction $\mathcal{R}$ behaves like a duality on the category of basic quasi–hereditary algebras, namely $\mathcal{R}(\mathcal{R}(A)) \cong A$. Now we call the algebra $\mathcal{R}(A)$ the Ringel dual of A. In this section we study the Ringel dual of $\mathcal{A}_{(X,M)}$ in case the Hasse diagram of X is a tree and M is a symmetric, invertible matrix labelling. In this case we show that $\mathcal{R}(\mathcal{A}_{(X,M)})$ is again of the form $\mathcal{A}_{(Y,\, N)}$ for some matrix labelling poset $(Y,\, N)$.

4.1 Let A be a quasi–hereditary algebra with the weight poset $\wedge$. Then we have the following

(1) The intersection $\mathcal{F}(\Delta_A) \cap \mathcal{F}(\nabla_A)$ contains exactly $s(A)$ indecomposable modules, where $s(A)$ is the cardinality of $\wedge$. They may be parametrized as $T(\lambda)$, $\lambda \in \wedge$ such that the following holds. There are exact sequences

$$\begin{array}{ll} (a) & 0 \longrightarrow \Delta(\lambda) \longrightarrow T(\lambda) \longrightarrow X(\lambda) \longrightarrow 0 \\ (b) & 0 \longrightarrow Y(\lambda) \longrightarrow T(\lambda) \longrightarrow \nabla(\lambda) \longrightarrow 0 \end{array}$$

where $X(\lambda)$ is filtered by $\Delta(\mu)'s$ for certain $\mu < \lambda$ and $Y(\lambda)$ by $\nabla(\mu)$'s for certain $\mu < \lambda$. In particular, $T(\lambda)$ has a unique composition factor isomorphic to $E(\lambda)$ and all other composition factors are of the form $E(\mu)$ with $\mu < \lambda$, where $E(x)$ denotes the simple A–module corresponding to the weight $x \in \wedge$. The modules $T(\lambda)$ are called canonical modules.

(2) The module $T := \bigoplus_{\lambda \in \wedge} T(\lambda)$ which is a tilting–cotilting module is called the characteristic module for $(A, \wedge)$ and $\mathcal{R}(A) = \mathrm{End}_A(T)$ is called the Ringel dual of A, it is also quasi–hereditary, with standard modules $\Delta_{\mathcal{R}(A)}(\lambda) = \mathrm{Hom}_A(T, \nabla(\lambda))$, where the weight poset of $\mathcal{R}(A)$ is $\wedge^{op}$.

(3)

$$\mathrm{Ext}^n_A(\Delta(\lambda), \nabla(\mu)) = \begin{cases} k & \text{if} \quad n = 0 \quad \text{and} \quad \lambda = \mu\,, \\ 0 & \text{otherwise.} \end{cases}$$

For the proofs of the above facts one may see [11].

4.2 Now we consider the algebra $\mathcal{A}_{(X,M)}$. From now on, we suppose that X is a tree. In this case any matrix labelling of X is symmetric. It follows from the definition of the relations of type (1.4.1) that if all the matrices are invertible, that is, they are non–zero elements from k, then we can take all these non–zero elements to be 1, since such two labellings yield two isomorphic algebras. Hence in the following we always assume that all labelling matrices are $[1] \in k^{1\times 1}$ and simply write $\mathcal{A}_X$ for $\mathcal{A}_{(X,M)}$ and $I(X)$ for $I(X, M)$.

Proposition. *Let X be a tree. Then there is an exact sequence*

$$0 \longrightarrow \Delta(\lambda) \longrightarrow T(\lambda) \longrightarrow \bigoplus_{\mu < \lambda} T(\mu) \longrightarrow 0$$

in $\mathcal{A}_X$–mod for all $\lambda \in X$.

Proof. Use induction on the number $|X|$ of elements in X. If $|X| = 1$ there is nothing to prove. Suppose the statement is true for all trees X with $|X| \leq n-1$. Let X be a tree with $|X| = n$ and $\lambda \in X$. If λ is not a maximal element in X then we pick up a maximal element $s \in X$ such that $\lambda < s$. Set $X_1 = X \backslash \{s\}$, then there is an exact sequence in $\mathcal{A}_{X_1}$–mod

$$0 \longrightarrow \Delta_{\mathcal{A}_{X_1}}(\lambda) \longrightarrow T_{\mathcal{A}_{X_1}}(\lambda) \longrightarrow \bigoplus_{\mu < \lambda} T_{\mathcal{A}_{X_1}}(\mu) \longrightarrow 0,$$

where $\mathcal{A}_{X_1}$ denotes the algebra $\mathcal{A}_{(X_1,M_1)}$ with the matrix labelling M_1 of M restricting on $X_1 = X\backslash\{s\}$. Note that $\mathcal{A}_{X_1}$ is isomorphic to the factor algebra of $\mathcal{A}_X$ by the ideal $\mathcal{A}_X s\mathcal{A}_X$.

Since $T(\lambda) \in \mathcal{F}(\{\Delta_{\mathcal{A}_X}(\gamma) | \gamma \neq s\}) = \mathcal{F}(\Delta_{\mathcal{A}_{X_1}})$, the above exact sequence yields a desired one in $\mathcal{A}_X$–mod. Hence we may assume that λ is a maximal element in X. Moreover, with a similar argument as above, we may assume that λ is the largest element in X and that the proposition holds for each $x < \lambda$.

Let $\gamma \in X$ with $\gamma < \lambda$. Then $\dim_k \mathrm{Ext}^1_{\mathcal{A}_X}(\Delta(\gamma), \Delta(\lambda)) = 1$ by Lemma 3.5. Let μ be an element in X with $\mu \lessdot \lambda$. Then we have a non–split exact sequence

$$\eta_{\lambda\mu} \qquad 0 \longrightarrow \Delta(\lambda) \longrightarrow T_{\lambda\mu} \longrightarrow \Delta(\mu) \longrightarrow 0$$

with $T_{\lambda\mu}$ obviously being indecomposable.

For each ν with $\nu < \mu$, using the canonical inclusion map $\Delta(\nu) \longrightarrow \Delta(\mu)$, we can form the following pullback diagram:

$$\begin{tikzcd}
\eta_{\lambda\nu} & 0 \arrow[r] & \Delta(\lambda) \arrow[r] \arrow[d, equal] & T_{\lambda\nu} \arrow[r] \arrow[d] & \Delta(\nu) \arrow[r] \arrow[d] & 0 \\
\eta_{\lambda\mu} & 0 \arrow[r] & \Delta(\lambda) \arrow[r] & T_{\lambda\mu} \arrow[r] & \Delta(\mu) \arrow[r] & 0
\end{tikzcd}$$

We shall prove that the upper exact sequence does not split.

Since λ is the largest element in X and $\mu \lessdot \lambda$, we see from the definition of $\Delta(\mu)$ that $T_{\lambda\mu}$ is isomorphic to $P(\mu)$, the projective $\mathcal{A}_X$–module corresponding to the weight μ, and that $\dim_k \mathrm{Hom}_{\mathcal{A}_X}(P(\nu), P(\mu)) = 2$. In order to show that the exact sequence does not split, it suffices to prove that $\mathrm{Hom}_{\mathcal{A}_X}(\Delta(\nu), P(\mu)) \cong k$.

On one hand, there holds

$$1 \leq \dim_k \mathrm{Hom}_{\mathcal{A}_X}(\Delta(\nu), P(\mu)) \leq \dim_k \mathrm{Hom}_{\mathcal{A}_X}(P(\nu), P(\mu)) = 2.$$

On the other hand, by Lemma 3.5, there holds

$$\dim_k \mathrm{Hom}_{\mathcal{A}_X}(\Delta(\nu), P(\mu)) = \dim_k \mathrm{Hom}_B(E(\nu), P(\mu)|_B),$$

where B denotes the Borel subalgebra of $\mathcal{A}_X$ (see section 3), where $E(\nu)$ is the simple module corresponding to the weight ν, and where $P(\mu)|_B$ denotes the restriction of $P(\mu)$ to B. By the construction of $\mathcal{A}_X$, we have that $\nu P(\mu) = \nu \mathcal{A}_X \mu = k(\nu x_1 x_2 \cdots x_n \mu + I(X)) \oplus k(\nu x_1 \cdots x_n \mu\lambda\mu + I(X))$, where $\nu \lessdot x_1 \lessdot x_2 \lessdot \cdots \lessdot x_n \lessdot \mu$ is the unique maximal chain from ν to μ. For $x_1\nu \in B$, one has

$$x_1\nu \cdot (\nu x_1 x_2 \cdots x_n \mu + I(X)) = x_1 \nu x_1 x_2 \cdots x_n \mu + I(X).$$

Since λ is the largest element in X and all labelling matrices of X are [1], one has by definition that $r_{x_1\nu x_1} = x_1\nu x_1 - x_1 x_2 x_1 \in I(X)$. Then

$$x_1 \nu x_1 x_2 \cdots x_n \mu + I(X) = x_1 x_2 x_1 x_2 x_3 \cdots x_n \mu + I(X).$$

Analogously, one has that $r_{x_2x_1x_2} = x_2x_1x_2 - x_2x_3x_2 \in I(X)$, then

$$x_1x_2x_1x_2x_3\cdots x_n\mu + I(X) = x_1x_2x_3x_2x_3\cdots x_n\mu + I(X).$$

Repeating the above argument, one finally gets

$$\begin{aligned} & x_1\nu\cdot(\nu x_1x_2\cdots x_n\mu + I(X)) = x_1\nu x_1x_2\cdots x_n\mu + I(X) \\ = & x_1x_2x_1x_2x_3\cdots x_n\mu + I(X) = \cdots\cdots \\ = & x_1x_2\cdots x_{n-1}x_n\mu x_n\mu + I(X) = x_1x_2\cdots x_n\mu\lambda\mu + I(X) \neq 0. \end{aligned}$$

Therefore, $\nu x_1x_2\cdots x_n\mu + I(X)$ does not lie in $\mathrm{soc}(P(\mu)|_B)$ since $x_1\nu \in \mathrm{rad}(B)$. From the simplicity of $E(\nu)$ it follows that $\sum_f \mathrm{Im}(f) \subseteq \mathrm{soc}(P(\mu)|_B)$, where the sum is over all $f \in \mathrm{Hom}_B(E(\nu),\, P(\mu)|_B)$. Then

$$\begin{aligned} \mathrm{Hom}_B(E(\nu),\, (P(\mu)|_B)) & \subseteq \mathrm{Hom}_B(E(\nu),\, \mathrm{soc}(P(\mu)|_B)) \cong \nu\cdot\mathrm{soc}(P(\mu)|_B) \\ & \subseteq k(\nu x_1x_2\cdots x_n\mu\lambda\mu + I(X)) \end{aligned}$$

Thus $\dim_k\mathrm{Hom}_B(E(\nu),\, (P(\mu)|_B)) \leq 1$. As a result, $\mathrm{Hom}_{\mathcal{A}_X}(\Delta(\nu),\, P(\mu)) \cong k$. We then conclude that the exact sequence $\eta_{\lambda\mu}$ does not split.

Further, let $\mu_1, \cdots, \mu_s$ be all the elements in X satisfying $\mu_i \lessdot \lambda$, $i = 1, \cdots, s$. Applying $\mathrm{Hom}_{\mathcal{A}_X}(-,\, \Delta(\lambda))$ to the exact sequence

$$0 \longrightarrow \Delta(\mu_i) \longrightarrow T(\mu_i) \longrightarrow \bigoplus_{\gamma\lessdot\mu_i} T(\gamma) \longrightarrow 0,$$

one obtains a long exact sequence

$$\begin{aligned} & 0 \longrightarrow \mathrm{Hom}_{\mathcal{A}_X}(\bigoplus_{\gamma\lessdot\mu_i} T(\gamma), \Delta(\lambda)) \longrightarrow \mathrm{Hom}_{\mathcal{A}_X}(T(\mu_i), \Delta(\lambda)) \longrightarrow \mathrm{Hom}_{\mathcal{A}_X}(\Delta(\mu_i), \Delta(\lambda)) \\ & \longrightarrow \mathrm{Ext}^1_{\mathcal{A}_X}(\bigoplus_{\gamma\lessdot\mu_i} T(\gamma),\, \Delta(\lambda)) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}_X}(T(\mu_i),\, \Delta(\lambda)) \xrightarrow{\phi_i} \mathrm{Ext}^1_{\mathcal{A}_X}(\Delta(\mu_i),\, \Delta(\lambda)) \\ & \longrightarrow \mathrm{Ext}^2_{\mathcal{A}_X}(\bigoplus_{\gamma\lessdot\mu_i} T(\gamma),\, \Delta(\lambda)) \end{aligned}$$

Since $\mathrm{proj.dim.}T(\gamma) \leq 1$, we have that $\mathrm{Ext}^2_{\mathcal{A}_X}(\bigoplus_{\gamma\lessdot\mu_i} T(\gamma),\, \Delta(\lambda)) = 0$, i.e. ϕ_i is surjective. Thus we obtain the following commutative diagram with $\xi_{\lambda\mu_i}$ a non–split exact sequence.

$$\begin{tikzcd}
\eta_{\lambda\mu_i} & 0 \arrow[r] & \Delta(\lambda) \arrow[r] \arrow[d, equal] & T_{\lambda\mu_i} \arrow[r] \arrow[d] & \Delta(\mu_i) \arrow[r] \arrow[d] & 0 \\
\xi_{\lambda\mu_i} & 0 \arrow[r] & \Delta(\lambda) \arrow[r] & T'_{\lambda\mu_i} \arrow[r] & T(\mu_i) \arrow[r] & 0
\end{tikzcd}$$

Now we take the sum of all $\xi_{\lambda\mu_i}$ for all $i = 1, \cdots, s$, and then get the following exact sequence

$$\begin{tikzcd}
\bigoplus_i \xi_{\lambda\mu_i} & 0 \arrow[r] & \bigoplus_i \Delta(\lambda) \arrow[r] \arrow[d, "\sigma"] & \bigoplus_i T'_{\lambda\mu_i} \arrow[r] \arrow[d] & \bigoplus_i T(\mu_i) \arrow[r] \arrow[d, equal] & 0 \\
\xi_\lambda & 0 \arrow[r] & \Delta(\lambda) \arrow[r] & T_\lambda \arrow[r] & \bigoplus_i T(\mu_i) \arrow[r] & 0
\end{tikzcd}$$

Let $\sigma : \underset{i}{\oplus}\Delta(\lambda) \longrightarrow \Delta(\lambda)$ be the map sending $(a_1, \cdots, a_s)$ to $\sum\limits_i a_i$. By constructing the pushout diagram, one obtains an exact sequence ξ_λ (see the above diagram).

We shall prove that $T_\lambda \in \mathcal{F}(\Delta) \cap (\mathcal{F}(\nabla))$. Obviously, $T_\lambda \in \mathcal{F}(\Delta)$. It remains to show $T_\lambda \in \mathcal{F}(\nabla)$, and by [11], it suffices to prove

$$\mathrm{Ext}^1_{\mathcal{A}_X}(\Delta(\nu), T_\lambda) = 0 \quad \text{for all} \quad \nu \leq \lambda$$

since $\text{proj.dim.}_{\mathcal{A}_X}\Delta(\nu) = \text{proj.dim.}_B E(\nu) \leq 1$ for all $\nu \leq \lambda$.

If $\nu = \lambda$, then $\Delta(\nu) = P(\lambda)$ is projective, thus $\mathrm{Ext}^1_{\mathcal{A}_X}(\Delta(\nu), T_\lambda) = 0$. Now we assume that $\nu < \lambda$. Apply $\mathrm{Hom}_{\mathcal{A}_X}(\Delta(\nu), -)$ to ξ_λ, we get the long exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}_X}(\Delta(\nu), \Delta(\lambda)) \longrightarrow \mathrm{Hom}_{\mathcal{A}_X}(\Delta(\nu), T_\lambda) \longrightarrow \mathrm{Hom}_{\mathcal{A}_X}(\Delta(\nu), \underset{i}{\oplus}T(\mu_i))$$

$$\xrightarrow{\delta} \mathrm{Ext}^1_{\mathcal{A}_X}(\Delta(\nu), \Delta(\lambda)) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}_X}(\Delta(\nu), T_\lambda) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}_X}(\Delta(\nu_i), \underset{i}{\oplus}T(\mu_i)).$$

Since $T(\mu_i) \in \mathcal{F}(\nabla)$ for all $1 \leq i \leq s$, one has $\mathrm{Ext}^1_{\mathcal{A}_X}(\Delta(\nu), \underset{i}{\oplus}T(\mu_i)) = 0$ by 4.2 (3). Note that $\dim_k\mathrm{Ext}^1_{\mathcal{A}_X}(\Delta(\nu), \Delta(\lambda)) = 1$. To prove $\mathrm{Ext}^1_{\mathcal{A}_X}(\Delta(\nu), T_\lambda) = 0$, it is enough to prove that the connecting map δ is non–zero. Since X is a tree, there is a unique $1 \leq j \leq s$ such that $\nu \leq \mu_j$. We denote by f_j the composition of the canonical indusion $\Delta(\nu) \longrightarrow T(\mu_j)$ and the canonical embedding $T(\mu_j) \longrightarrow \underset{i}{\oplus}T(\mu_i)$. It is easy to see that $(f_j)\delta = \eta_{\lambda\nu} \neq 0$, i.e. δ is non–zero. Therefore,

$$\mathrm{Ext}^1_{\mathcal{A}_X}(\Delta(\nu), T_\lambda) = 0 \quad \text{for all} \quad \nu \leq \lambda.$$

Thus

$$T_\lambda \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla).$$

It remains to prove $T_\lambda = T(\lambda)$, or equivalently, to prove that T_λ is indecomposable. This follows from the fact $T_\lambda \in \mathcal{F}(\Delta)$ and the fact that all sequences $\xi_{\lambda\mu_i}$ do not split. This finishes the proof. □

4.3 The anti–isomorphism $\varepsilon : \mathcal{A}_X \longrightarrow \mathcal{A}_X$ induces a duality $\mathcal{E} : \mathcal{A}_X - \text{mod} \longrightarrow \mathcal{A}_X - \text{mod}$ which fixes all $E(\lambda)$, $T(\lambda)$, $\lambda \in X$. Dually, by Proposition 4.2, one has an exact sequence

$$0 \longrightarrow \underset{\mu < \lambda}{\oplus} T(\mu) \longrightarrow T(\lambda) \longrightarrow \mathcal{E}(\Delta(\lambda)) = \nabla(\lambda) \longrightarrow 0$$

for each $\lambda \in X$.

Corollary. *For every $x \leq y$ in X, there holds*

$$\dim_k\mathrm{Hom}_{\mathcal{A}_X}(\Delta(x), T(y)) = 1 = \dim_k\mathrm{Hom}_{\mathcal{A}_X}(T(y), \nabla(x)).$$

Proof. Let $x = x_0 < x_1 < \cdots < x_n = y$ be the unique maximal chain from x to y. We shall use induction on n to prove

$$(4.3.1) \qquad \dim_k\mathrm{Hom}_{\mathcal{A}_X}(\Delta(x), T(y)) = 1.$$

If $n = 0$, i.e. $x = y$, this follows from 4.1(1). Suppose that for all $x \leq y$ in X, if the maximal chain from x to y is of length $< n$ then (4.3.1) holds. Now assume that $x < y$ and that $x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = y$ is the maximal chain from x to y. Now apply $\mathrm{Hom}_{\mathcal{A}_X}(\Delta(x), -)$ to the exact sequence

$$0 \longrightarrow \bigoplus_{z \lessdot y} T(z) \longrightarrow T(y) \longrightarrow \nabla(y) \longrightarrow 0,$$

we obtain the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}_X}(\Delta(x), \bigoplus_{z \lessdot y} T(z)) \longrightarrow \mathrm{Hom}_{\mathcal{A}_X}(\Delta(x), T(y)) \longrightarrow \mathrm{Hom}_{\mathcal{A}_X}(\Delta(x), \nabla(y)).$$

Since $x < y$, there holds $\mathrm{Hom}_{\mathcal{A}_X}(\Delta(x), \nabla(y)) = 0$. Thus

$$\dim_k \mathrm{Hom}_{\mathcal{A}_X}(\Delta(x), T(y)) = \dim_k \mathrm{Hom}_{\mathcal{A}_X}(\Delta(x), \bigoplus_{z \lessdot y} T(z)).$$

Because X is a tree, we have $\dim_k \mathrm{Hom}_{\mathcal{A}_X}(\Delta(z), T(x)) = 0$ for $z \lessdot y$, $z \neq x_{n-1}$. Thus

$$\dim_k \mathrm{Hom}_{\mathcal{A}_X}(\Delta(x), T(y)) = \dim_k \mathrm{Hom}_{\mathcal{A}_X}(\Delta(x), T(x_{n-1})).$$

The maximal chain from x to x_{n-1} is then of the length $n - 1 < n$. By induction hypothesis, $\dim_k \mathrm{Hom}_{\mathcal{A}_X}(\Delta(x), T(x_{n-1})) = 1$. Hence $\dim_k \mathrm{Hom}_{\mathcal{A}_X}(\Delta(x), T(y)) = 1$. Dually, $\dim_k \mathrm{Hom}_{\mathcal{A}_X}(T(y), \nabla(x)) = 1$. □

4.4 For all $\mu \lessdot \lambda$, by $i_{\mu\lambda}$ and $\pi_{\lambda\mu}$ we denote the canonical immersion from $T(\mu)$ to $T(\lambda)$ and the canonical projection from $T(\lambda)$ to $T(\mu)$, respectively (see 4.2 and its dual). Further, if $x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = y$ is the unique maximal chain from x to y, then we set $i_{xy} = i_{x_0x_1} i_{x_1x_2} \cdots i_{x_{n-1}x_n}$ and $\pi_{yx} = \pi_{x_nx_{n-1}} \cdots \pi_{x_2x_1} \pi_{x_1x_0}$.

Consider a full additive subcategory $\mathcal{C}$ of A–mod and the objects N_1, $N_2 \in \mathcal{C}$, and let $\mathrm{Irr}_{\mathcal{C}}(N_1, N_2) := \mathrm{rad}_{\mathcal{C}}(N_1, N_2)/\mathrm{rad}^2_{\mathcal{C}} N_1, N_2)$ be the bimodule of irreducible maps from N_1 to N_2 in $\mathcal{C}$.

Proposition. *The two morphisms $i_{\mu\lambda}$ and $\pi_{\lambda\mu}$ for $\mu \lessdot \lambda$ in X are irreducible in $\mathcal{T} := add(\bigoplus_{x \in X} T(x))$. Moreover,*

$$dim_k Irr_{\mathcal{T}}(T(x), T(y)) = \begin{cases} 1 & \textit{if} \quad x \lessdot y \quad \textit{or} \quad y \lessdot x, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof (a) If we have $x < y$ in X, then each $f \in \mathrm{Hom}_{\mathcal{A}_X}(T(x), T(y))$ factors through the canonical inclusion $i_{xy} : T(x) \longrightarrow T(y)$. Indeed, this follows from the fact that $\mathrm{Hom}_{\mathcal{A}_X}(T(u), \nabla(v)) = 0$ for all $u < v$ in X.

(b) For $x \leq y$ in X, if $f \in \mathrm{Hom}_{\mathcal{A}_X}(T(x), T(y))$ is not injective, then f factors through the inclusion $(i_{zy})^t_z : \bigoplus_{z \lessdot x} T(z) \longrightarrow T(y)$, where $(i_{zy})^t_z$ denotes the transpose of $(i_{zy})_z \in$

$k^{1\times n(x)}$ and where $n(x)$ is the number of $z\in X$ satisfying $z\lessdot x$. Indeed, in case $x=y$, we have the following exact sequence

$$0 \longrightarrow \bigoplus_{z\lessdot x} T(z) \xrightarrow{(i_{zx})_z^t} T(x) \xrightarrow{p_x} \nabla(x) \longrightarrow 0.$$

Since $\dim_k \mathrm{Hom}_{\mathcal{A}_X}(T(x), \nabla(x)) = 1$ and f is not injective, we get $fp_x = 0$, i.e. f factors through $(i_{zx})_z^t : \bigoplus_{z\lessdot x} T(z) \longrightarrow T(x)$. Now suppose $x<y$. By (a), there is an $f' : T(x) \longrightarrow T(x)$ such that $f = f' i_{xy}$. Since f is not injective, neither is f'. By the above argument, f' factors through $(i_{zx})_z : \bigoplus_{z\lessdot x} T(z) \longrightarrow T(x)$. Therefore, f factors through $(i_{zx})_z^t\, i_{xy} = (i_{zy})_z^t : \bigoplus_{z\lessdot x} T(z) \longrightarrow T(y)$.

(c) Now suppose that $\mu\lessdot\lambda$ and that $i_{\mu\lambda}$ admits a decomposition $i_{\mu\lambda} = fg$ with $f = (f_{\mu\delta_i})_i : T(\mu) \longrightarrow \bigoplus_{i=1}^{m} T(\delta_i)$ and $g = (g_{\delta_i\lambda})_i^t : \bigoplus_{i=1}^{m} T(\delta_i) \longrightarrow T(\lambda)$ for some $\delta_i \in X$, $1\le i\le m$, that is, $i_{\mu\lambda} = \sum_{i=1}^{m} f_{\mu\delta_i} g_{\delta_i\lambda}$. We shall prove that f is a split monomorphism or that g is a split epimorphism.

From $i_{\mu\lambda} = \sum_{i=1}^{m} f_{\mu\delta_i} g_{\delta_i\lambda}$ and (a), we have

$$T(\lambda) \supset T(\mu) = \mathrm{Im}(i_{\mu\lambda}) \subseteq \sum_{i=1}^{m} \mathrm{Im}(f_{\mu\delta_i} g_{\delta_i\lambda}) \subseteq T(\mu).$$

Thus

$$\sum_{i=1}^{m} \mathrm{Im}(f_{\mu\delta_i} g_{\delta_i\lambda}) = T(\mu) \subset T(\lambda).$$

We claim that there is a δ_j such that $f_{\mu\delta_j} g_{\delta_j\lambda}$ is injective. Otherwise, by (b), $\mathrm{Im}(f_{\mu\delta_i} g_{\delta_i\lambda}) \subseteq \bigoplus_{\gamma\lessdot\mu} T(\gamma)$ for all δ_i. Hence $\sum_{i=1}^{m} \mathrm{Im}(f_{\mu\delta_i} g_{\delta_i\lambda}) \subseteq \bigoplus_{\gamma\lessdot\mu} T(\gamma) \subset T(\mu)$, a contradiction. Thus with $f_{\mu\delta_j} g_{\delta_j\lambda}$ also $f_{\mu\delta_j}$ is injective. This implies $\mu\le\delta_j$ by 4.1(1). If $\mu=\delta_j$, then $f_{\mu\delta_j}$ is invertible, i.e. f is a split monomorphism. In case $\mu<\delta_j$, we shall show that $\delta_j=\lambda$. To this purpose, it is enough to prove that $g_{\delta_j\lambda}$ is injective. Suppose that $g_{\delta_j\lambda}$ is not injective. Using the dual argument of (b) and 4.1(1) we have that $T(\delta_j) \supset \Delta(\delta_j) \subseteq \mathrm{Ker}(g_{\delta_j\lambda})$. Thus $(\Delta(\mu))f_{\mu\delta_j} \subseteq \Delta(\delta_j) \subseteq \mathrm{Ker}(g_{\delta_j\lambda})$ since $\dim_k \mathrm{Hom}_{\mathcal{A}_X}(\Delta(\mu), T(\delta_j)) = 1$ by Corollary 4.3. Hence $\Delta(\mu) \subseteq \mathrm{Ker}(f_{\mu\delta_j} g_{\delta_j\lambda})$ and $f_{\mu\delta_j} g_{\delta_j\lambda}$ is not injective, which is a contradiction. Therefore, $g_{\delta_j\lambda}$ is injective. This implies $\delta_j\le\lambda$. Since $\mu\lessdot\lambda$ and $\mu<\delta_j$, we must have $\delta_j=\lambda$. Then $g_{\delta_j\lambda}$ is invertible, that is, g is a split epimorphism. Hence $i_{\mu\lambda}$ is irreducible. Dually, $\pi_{\lambda\mu}$ is irreducible for $\mu\lessdot\lambda$ in X. The rest of the proposition becomes now obvious. □

4.5 Proposition. (1) *If $\mu\lessdot\lambda$ and $\gamma_1, \cdots, \gamma_s$ are all the elements in X satisfying $\gamma_i\lessdot\mu$, $s\ge 1$, then*

$$i_{\mu\lambda}\pi_{\lambda\mu} = \sum_{i=1}^{s} \pi_{\mu\gamma_i} i_{\gamma_i\mu},$$

(2) *If* $\mu_1 \lessdot \lambda$, $\mu_2 \lessdot \lambda$ *and* $\mu_1 \neq \mu_2$, *then* $i_{\mu_1\lambda}\pi_{\lambda\mu_2} = 0$,
(3) *If* μ *is minimal in* X *and* $\mu \lessdot \lambda$, *then* $i_{\mu\lambda}\pi_{\lambda\mu} = 0$.

Proof. (1) The morphism $\pi_{\lambda\mu}$ is the composition of the following morphisms

$$T(\lambda) \xrightarrow{\pi_1} T(\lambda)/\Delta(\lambda) = \bigoplus_{\delta \lessdot \lambda} T(\delta) \xrightarrow{\pi_2} T(\mu),$$

where π_1 is the canonical projection and π_2 maps the direct summand $T(\mu)$ of $\bigoplus_{\delta \lessdot \lambda} T(\delta)$ identically onto $T(\mu)$ and other summands to zero.

It is easy to see that $\mathrm{Ker}(i_{\mu\lambda}\pi_1) = T(\mu) \cap \Delta(\lambda) = \Delta(\mu)$. Further, $\mathrm{Hom}_{\mathcal{A}_X}(T(\mu), T(\delta)) = 0$ for all $\delta \lessdot \lambda$, $\delta \neq \mu$. Therefore, $i_{\mu\lambda}\pi_{\lambda\mu}$ is the composition of the morphisms

$$T(\mu) \xrightarrow{[\pi_{\mu\gamma_1},\cdots,\pi_{\mu\gamma_s}]} T(\mu)/\Delta(\mu) = \oplus_{i=1}^{s} T(\gamma_i) \xrightarrow{\begin{bmatrix} i_{\gamma_1\mu} \\ \vdots \\ i_{\gamma_s\mu} \end{bmatrix}} ----$$

$--\nabla---$

and

$$\dim_k \mathcal{R}(\mathcal{A}_X) = \sum_{x\in X} (\dim_k \Delta_{\mathcal{R}(\mathcal{A}_X)}(x))^2.$$

On one hand, by the definition of $\mathcal{A}_{X^{op}}$, there holds $\dim_k \Delta_{\mathcal{A}_{X^{op}}}(x) = n_x$, the number of the elements y in X^{op} satisfying $y \leq^{op} x$, or equivalently, $x \leq y$.

On the other hand, there holds $\Delta_{\mathcal{R}(\mathcal{A}_X)}(x) = \mathrm{Hom}_{\mathcal{A}_X}(T, \nabla(x))$ by 4.1(2). Then it follows from 4.1(a) and Corollary 4.3 that

$$\dim_k \Delta_{\mathcal{R}(\mathcal{A}_X)}(x) = \sum_{y\in X} \dim_k \mathrm{Hom}_{\mathcal{A}_X}(T(y), \nabla(x)) = n_x = \dim_k \Delta_{\mathcal{A}_{X^{op}}}(x).$$

Therefore, $\dim_k \mathcal{A}_{X^{op}} = \dim_k \mathcal{R}(\mathcal{A}_X)$, i.e. Ψ is an isomorphism.

In fact, we have proved the following

Theorem. *Let X be a tree labelled by matrices $[1] \in k^{1\times 1}$. Then*

$$\mathcal{R}(\mathcal{A}_X) \cong \mathcal{A}_{X^{op}}.$$

4.7 Remark. If the matrix labelling on a tree X is not invertible, then Theorem 4.6 may be false. Let us consider the poset $X = \{1 < 2 < 3\}$ with $M(1, 3) = [0]$. Then the Ringel dual $\mathcal{R}(\mathcal{A}_{(X,M)})$ is given by the quiver

$$\begin{tikzcd} 1 \arrow[r, "\beta"] & 3 \arrow[r, "\delta"] & 2 \end{tikzcd}$$

with the relation $\beta\alpha = \gamma\delta = \beta\delta\gamma\alpha = 0$. Since $X \cong X^{op}$ as posets, $\mathcal{A}_{(X,M)} \cong \mathcal{A}_{(X^{op}, M)}$. Clearly, $\mathcal{R}(\mathcal{A}_{(X,M)}) \not\cong \mathcal{A}_{(X^{op}, M)}$.

A further discussion of the Ringel dual in case all labellings are zero will appear elsewhere.

5. A relation between the Ringel duality and the quadratic duality

This section is devoted to the study of a relation between the Ringel duality and the quadratic duality of $\mathcal{A}_{(X,M)}$ in case X is a tree with an invertible matrix labelling. In this case we shall show that the processes of forming the dualities commute. To this goal, we first study the Ringel dual of $\mathcal{B}_{(X,M)}$. Note that the algebra $\mathcal{B}_{(X,M)}$ is different from $\mathcal{A}_{(X,M)}$ and, comparing with $\mathcal{A}_{(X,M)}$, may have arbitrary finite global dimension.

The main idea is similar to the one in section 4. For convenience of the reader, we give here proofs but omit some same arguments and refer to section 4.

5.1 From now on, we suppose again that X is a tree with a symmetric, invertible matrix labelling M. The algebra $\mathcal{B}_{(X,M)}$ is defined as in section 2. Using a similar argument in 4.2, we may assume that all labelling matrices in M are $[1] \in k^{1\times 1}$. We have seen in section 2 that $\mathcal{B}_{(X,M)}$ is a quadratic, quasi–hereditary algebra and that the anti–involution ε of $\mathcal{A}'_X$ induces an anti–involution ε' of $\mathcal{B}_{(X,M)}$. Thus $\mathcal{B}_{(X,M)}$ is even a BGG–algebra.

Further, by C' (resp. B') we denote the subalgebra of $\mathcal{B}_{(X,M)}$ generated by $x \in X$ and xy (resp. yx) with $x \lessdot y$ in X. Notice that C' and B' are algebras with radical–square zero. As in section 3 one can show the following

Theorem. *The algebra B' is a strong exact Borel subalgebra of $\mathcal{B}_{(X,M)}$ and the algebra C' a strong Δ–subalgebra of $\mathcal{B}_{(X,M)}$.*

5.2 For simplicity, we write $\mathcal{B}$ for $\mathcal{B}_{(X,M)}$ and J for $J(X, M)$. For each $\lambda \in X$, by $T_{\mathcal{B}}(\lambda)$ we denote the canonical module of $\mathcal{B}$ corresponding to the weight λ and by ${}_{\mathcal{B}}T = \underset{\lambda\in X}{\oplus} T_{\mathcal{B}}(\lambda)$ the characteristic module of $\mathcal{B}$. Then we have

Proposition. *For each $\lambda \in X$, there is an exact sequence*

$$0 \longrightarrow \Delta_{\mathcal{B}}(\lambda) \longrightarrow T_{\mathcal{B}}(\lambda) \longrightarrow \underset{\mu\lessdot\lambda}{\oplus} \Delta_{\mathcal{B}}(\mu) \longrightarrow 0.$$

Dually, there is an exact sequence

$$0 \longrightarrow \underset{\mu\lessdot\lambda}{\oplus} \nabla_{\mathcal{B}}(\mu) \longrightarrow T_{\mathcal{B}}(\lambda) \longrightarrow \nabla_{\mathcal{B}}(\lambda) \longrightarrow 0.$$

Proof. First note that by the construction of $\mathcal{B}$ and the assumption on X, one can see that $\mathrm{rad}^3(\mathcal{B}) = 0$ and that there is an exact sequence

$$0 \longrightarrow \underset{\lambda\lessdot\gamma}{\oplus} \Delta_{\mathcal{B}}(\gamma) \longrightarrow P_{\mathcal{B}}(\lambda) \longrightarrow \Delta_{\mathcal{B}}(\lambda) \longrightarrow 0$$

for each $\lambda \in X$, where $P_{\mathcal{B}}(\lambda)$ is the projective $\mathcal{B}$–module corresponding to the weight λ.

Now we use induction on the number $|X|$ of the elements of X to prove that there is an exact sequence

$$0 \longrightarrow \Delta_{\mathcal{B}}(\lambda) \longrightarrow T_{\mathcal{B}}(\lambda) \longrightarrow \underset{\mu\lessdot\lambda}{\oplus} \Delta_{\mathcal{B}}(\mu) \longrightarrow 0.$$

for each $\lambda \in X$.

If $|X| = 1$, the proposition is clear. Suppose $|X| = n > 1$, and $\lambda \in X$. Using a similar argument in the proof of Proposition 4.2, we may assume that λ is the largest element in X.

For each $\mu \lessdot \lambda$ in X, one has by Lemma 3.5 the equalities

$$\dim_k \mathrm{Hom}_{\mathcal{B}}(\Delta_{\mathcal{B}}(\mu), \Delta_{\mathcal{B}}(\lambda)) = \dim_k \mathrm{Hom}_{B'}(E(\mu), \Delta_{\mathcal{B}}(\lambda)|_{B'}) = 1$$

and

$$\dim_k \mathrm{Ext}^1_{\mathcal{B}}(\Delta_{\mathcal{B}}(\mu), \Delta_{\mathcal{B}}(\lambda)) = \dim_k \mathrm{Ext}^1_{B'}(E(\mu), \Delta_{\mathcal{B}}(\lambda)|_{B'}) = 1.$$

Hence there is a non–split exact sequence

$$\eta_{\lambda\mu} \qquad 0 \longrightarrow \Delta_{\mathcal{B}}(\lambda) \longrightarrow T_{\lambda\mu} \longrightarrow \Delta_{\mathcal{B}}(\mu) \longrightarrow 0.$$

Using a similar argument in the proof of Proposition 4.2, one has that $T_{\lambda\mu} = P_{\mathcal{B}}(\mu)$ and that for each $\nu \lessdot \mu$, we can choose a non–zero morphism $f_{\nu\mu} : \Delta_{\mathcal{B}}(\nu) \to \Delta_{\mathcal{B}}(\mu)$ which gives rise to the following commutative diagram

$$\begin{tikzcd}
\eta_{\lambda\nu} & 0 \arrow[r] & \Delta_{\mathcal{B}}(\lambda) \arrow[r] \arrow[d, equal] & T_{\nu\mu} \arrow[r] \arrow[d] & \Delta_{\mathcal{B}}(\nu) \arrow[r] \arrow[d, "f_{\nu\mu}"] & 0 \\
\eta_{\lambda\mu} & 0 \arrow[r] & \Delta_{\mathcal{B}}(\lambda) \arrow[r] & P_{\mathcal{B}}(\mu) \arrow[r] & \Delta_{\mathcal{B}}(\mu) \arrow[r] & 0
\end{tikzcd}$$

with $\eta_{\lambda\nu}$ a non-split exact sequence (Note that in this situation one should show that $\operatorname{Hom}_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu),\, P_{\mathcal{B}}(\mu)) = 0$. However, this follows from the structure of $P_{\mathcal{B}}(\mu)$ (see the definition of $\mathcal{B}$)).

By forming pushout diagram, we obtain the following commutative diagram:

$$\begin{tikzcd}
\bigoplus\limits_{\mu \lessdot \lambda} \eta_{\lambda\mu} & 0 \arrow[r] & \bigoplus\limits_{\mu \lessdot \lambda} \Delta_{\mathcal{B}}(\lambda) \arrow[r] \arrow[d, "\sigma"] & \bigoplus\limits_{\mu \lessdot \lambda} T_{\lambda\mu} \arrow[r] \arrow[d] & \bigoplus\limits_{\mu \lessdot \lambda} \Delta_{\mathcal{B}}(\mu) \arrow[r] \arrow[d, equal] & 0 \\
\eta_{\lambda} & 0 \arrow[r] & \Delta_{\mathcal{B}}(\lambda) \arrow[r] & T_{\lambda} \arrow[r] & \bigoplus\limits_{\mu \lessdot \lambda} \Delta_{\mathcal{B}}(\mu) \arrow[r] & 0
\end{tikzcd}$$

where σ denotes the map defined by $(a_\mu)_\mu \longmapsto \sum\limits_{\mu} a_\mu$.

We shall prove that $T_\lambda = T_{\mathcal{B}}(\lambda)$. This will follow from the facts (1) and (2) below.

(1) The module T_λ lies in $\mathcal{F}(\Delta_{\mathcal{B}}) \cap \mathcal{F}(\nabla_{\mathcal{B}})$. Indeed, $T_\lambda \in \mathcal{F}(\Delta_{\mathcal{B}})$ follows from the above construction. To prove $T_\lambda \in \mathcal{F}(\nabla_{\mathcal{B}})$ we show that

$$\operatorname{Ext}^i_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu),\, T_\lambda) = 0$$

for all $i \geq 1$ and for all ν in X.

We use induction on i. In the following we prove that $\operatorname{Ext}^1_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu),\, T_\lambda) = 0$ for all $\nu \leq \lambda$ in X.

If $\nu = \lambda$, then $\Delta_{\mathcal{B}}(\nu) = P_{\mathcal{B}}(\lambda)$. Thus $\operatorname{Ext}^1_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu),\, T_\lambda) = 0$. If $\nu < \lambda$, we apply $\operatorname{Hom}_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu),\, -)$ to the exact sequence η_λ and obtain the following long exact sequence

(5.2.1)

$$\begin{aligned}
&0 \longrightarrow \operatorname{Hom}_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), \Delta_{\mathcal{B}}(\lambda)) \longrightarrow \operatorname{Hom}_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), T_\lambda) \longrightarrow \operatorname{Hom}_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu),\, \bigoplus_{\mu \lessdot \lambda} \Delta_{\mathcal{B}}(\mu)) \\
&\xrightarrow{\delta_\nu} \operatorname{Ext}^1_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), \Delta_{\mathcal{B}}(\lambda)) \longrightarrow \operatorname{Ext}^1_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), T_\lambda) \longrightarrow \operatorname{Ext}^1_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu),\, \bigoplus_{\mu \lessdot \lambda} \Delta_{\mathcal{B}}(\mu)).
\end{aligned}$$

Now we consider the following cases:

(i) Case $\nu \lessdot \lambda$. By Lemma 3.5, we have

$$\operatorname{Ext}^1_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu),\, \Delta_{\mathcal{B}}(\lambda)) \cong \operatorname{Ext}^1_{\mathcal{B}'}(E_{\mathcal{B}}(\nu),\, \Delta_{\mathcal{B}}(\lambda)|_{\mathcal{B}'}) \cong k$$

and

$$\operatorname{Ext}^1_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), \underset{\mu<\lambda}{\oplus}\Delta_{\mathcal{B}}(\mu)) \cong \operatorname{Ext}^1_{\mathcal{B}'}(E_{\mathcal{B}}(\nu), \underset{\mu<\lambda}{\oplus}\Delta_{\mathcal{B}}(\mu)|_{\mathcal{B}'}) = 0.$$

where $E_{\mathcal{B}}(\nu)$ denotes the simple $\mathcal{B}$–module corresponding to the weight ν.

On the other hand, by f_ν we denote the canonical embedding from $\Delta_{\mathcal{B}}(\nu)$ to $\underset{\mu<\lambda}{\oplus}\Delta_{\mathcal{B}}(\mu)$. Then $(f_\nu)\delta_\nu = \eta_{\lambda\nu} \neq 0$. Therefore, δ_ν is surjective, i.e. $\operatorname{Ext}^1_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), T_\lambda) = 0$. Moreover, from (5.2.1) we have that

$$\operatorname{Hom}_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), T_\lambda) \cong k$$

for all $\nu < \lambda$.

(ii) Case $\nu < \mu_0 < \lambda$ for some μ_0. Again by Lemma 3.5, we have

$$\operatorname{Hom}_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), \Delta_{\mathcal{B}}(\lambda)) = 0 \text{ and } \operatorname{Hom}_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), \underset{\mu<\lambda}{\oplus}\Delta_{\mathcal{B}}(\lambda)) \cong k.$$

In this case δ_ν is non–zero since $(f_{\nu\mu_0})\delta_\nu = \eta_{\lambda\nu} \neq 0$. This implies that

$$\operatorname{Hom}_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), T_\lambda) = 0.$$

Applying $\operatorname{Hom}_{\mathcal{B}}(-, T_\lambda)$ to the exact sequence

$$0 \longrightarrow \Delta_{\mathcal{B}}(\mu_0) \longrightarrow P_{\mathcal{B}}(\nu) \longrightarrow \Delta_{\mathcal{B}}(\nu) \longrightarrow 0,$$

we get the following exact sequence

$$\begin{aligned} (5.2.2) \quad & 0 \longrightarrow \operatorname{Hom}_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), T_\lambda) \longrightarrow \operatorname{Hom}_{\mathcal{B}}(P_{\mathcal{B}}(\nu), T_\lambda) \longrightarrow \operatorname{Hom}_{\mathcal{B}}(\Delta_{\mathcal{B}}(\mu_0), T_\lambda) \\ & \longrightarrow \operatorname{Ext}^1_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), T_\lambda) \longrightarrow 0. \end{aligned}$$

Obviously, $\operatorname{Hom}_{\mathcal{B}}(P_{\mathcal{B}}(\nu), T_\lambda) \cong k$. By caculating the dimensions of each term in (5.2.2), one can easily see that $\operatorname{Ext}^1_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), T_\lambda) = 0$.

(iii) Case $\nu < \delta < \mu_0 < \lambda$ for some μ_0 and δ in X. Applying $\operatorname{Hom}_{\mathcal{B}}(-, T_\lambda)$ to the exact sequence (here the assumption that λ is the largest element is used)

$$0 \longrightarrow \Delta_{\mathcal{B}}(\delta) \longrightarrow P_{\mathcal{B}}(\nu) \longrightarrow \Delta_{\mathcal{B}}(\nu) \longrightarrow 0,$$

one has the exact sequence

$$\begin{aligned} & 0 \longrightarrow \operatorname{Hom}_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), T_\lambda) \longrightarrow \operatorname{Hom}_{\mathcal{B}}(P_{\mathcal{B}}(\nu), T_\lambda) \longrightarrow \operatorname{Hom}_{\mathcal{B}}(\Delta_{\mathcal{B}}(\delta), T_\lambda) \\ & \longrightarrow \operatorname{Ext}^1_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), T_\lambda) \longrightarrow 0. \end{aligned}$$

In the second case we have seen that $\operatorname{Hom}_{\mathcal{B}}(\Delta_{\mathcal{B}}(\delta), T_\lambda) = 0$. Therefore,

$$\operatorname{Ext}^1_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), T_\lambda) = 0.$$

In all remaining cases, the equality $\operatorname{Ext}^1_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), T_\lambda) = 0$ follows directly from the equality $\operatorname{Ext}^1_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), \Delta_{\mathcal{B}}(\lambda)) = 0 = \operatorname{Ext}^1_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), \underset{\mu<\lambda}{\oplus}\Delta_{\mathcal{B}}(\mu))$, according to Lemma 3.5.

Thus we have proved that $\mathrm{Ext}^i_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), T_\lambda) = 0$ for all ν in X in case $i = 1$. Now suppose that $\mathrm{Ext}^i_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), T_\lambda) = 0$ for all ν in X. Using the exact sequence at the beginning of our proof, we then have

$$\mathrm{Ext}^{i+1}_{\mathcal{B}}(\Delta_{\mathcal{B}}(\nu), T_\lambda) = \mathrm{Ext}^i_{\mathcal{B}}(\bigoplus_{\nu \lessdot \delta} \Delta_{\mathcal{B}}(\delta), T_\lambda) = 0.$$

Therefore, $T_\lambda \in \mathcal{F}(\nabla_{\mathcal{B}})$.

(2) The module T_λ is indecomposable. Indeed, this is an implication of the fact that T_λ lies in $\mathcal{F}(\Delta_{\mathcal{B}})$ and the fact that each sequence $\eta_{\lambda\mu}$ does not split for every $\mu \lessdot \lambda$.

This finishes the proof. □

5.3 Corollary. (1) *For $\mu \lessdot \lambda$ in X, there holds*

$$dim_k Hom_{\mathcal{B}}(\Delta_{\mathcal{B}}(\mu), T_{\mathcal{B}}(\lambda)) = 1 = dim_k Hom_{\mathcal{B}}(T_{\mathcal{B}}(\lambda), \Delta_{\mathcal{B}}(\mu)).$$

Dually,

$$dim_k Hom_{\mathcal{B}}(\nabla_{\mathcal{B}}(\mu), T_{\mathcal{B}}(\lambda)) = 1 = dim_k Hom_{\mathcal{B}}(T_{\mathcal{B}}(\lambda), \nabla_{\mathcal{B}}(\mu)).$$

(2) *For $x \neq y$ in X, there holds $Hom_{\mathcal{B}}(T_{\mathcal{B}}(x), T_{\mathcal{B}}(y)) \neq 0$ if and only if one of the following cases occurs:*

i) $x \lessdot y$,

ii) $y \lessdot x$,

iii) There is an element $z \in X$ such that $z \lessdot x$ and $z \lessdot y$.

Moreover, in all three cases, we have that $dim_k Hom_{\mathcal{B}}(T_{\mathcal{B}}(x), T_{\mathcal{B}}(y)) = 1$.

Proof. This follows directly from Proposition 5.2. □

5.4 For $\lambda \in X$, by i_λ we denote the canonical inclusion from $\Delta_{\mathcal{B}}(\lambda)$ to $T_{\mathcal{B}}(\lambda)$ and by π_λ the canonical projection from $T_{\mathcal{B}}(\lambda)$ to $\nabla_{\mathcal{B}}(\lambda)$. Further, for $\mu \lessdot \lambda$ in X, we denote by $i_{\mu\lambda}$ the canonical inclusion from $\nabla_{\mathcal{B}}(\mu)$ to $T_{\mathcal{B}}(\lambda)$ and by $\pi_{\lambda\mu}$ the canonical projection from $T_{\mathcal{B}}(\lambda)$ to $\Delta_{\mathcal{B}}(\mu)$. Finally, for $\mu \lessdot \lambda$ in X, we set $\phi_{\mu\lambda} = \pi_\mu i_{\mu\lambda}$ and $\psi_{\lambda\mu} = \pi_{\lambda\mu} i_\mu$.

Proposition. *For $\mu \lessdot \lambda$ in X, the morphisms $\phi_{\mu\lambda}$ and $\psi_{\lambda\mu}$ are irreducible in $\mathcal{T}_{\mathcal{B}} := add(\bigoplus_{x \in X} T_{\mathcal{B}}(x))$. Moreover,*

$$dim_k Irr_{\mathcal{T}_{\mathcal{B}}}(T_{\mathcal{B}}(x), T_{\mathcal{B}}(y)) = \begin{cases} 1 & if \quad x \lessdot y \quad or \quad y \lessdot x, \\ 0 & otherwise. \end{cases}$$

Proof. We only prove that $\phi_{\mu\lambda}$ is irreducible. The rest can be proved in a similar way or is obvious. Let $\phi_{\mu\lambda}$ admit a decomposition $\phi_{\mu\lambda} = gh$ with $g = (g_{\mu s_i})_i : T_{\mathcal{B}}(\mu) \longrightarrow \bigoplus_{i=1}^m T_{\mathcal{B}}(s_i)$ and $h = (h_{s_i\lambda})_i^t : \bigoplus_{i=1}^m T_{\mathcal{B}}(s_i) \longrightarrow T_{\mathcal{B}}(\lambda)$ for some s_i in X, $1 \leq i \leq m$, i.e. $\phi_{\mu\lambda} = \sum_{i=1}^m g_{\mu s_i} h_{s_i\lambda}$. By Corollary 5.3, one has that $\mathrm{Hom}_{\mathcal{B}}(T_{\mathcal{B}}(\mu), T_{\mathcal{B}}(\lambda)) = k\phi_{\mu\lambda}$. Therefore, there is an s_j for $1 \leq j \leq m$ such that $g_{\mu s_j} h_{s_j \lambda} = a\phi_{\mu\lambda}$ for an $a \in k\backslash\{0\}$. Then $g_{\mu s_j} \neq 0 \neq h_{s_j \lambda}$. Again by Corollary 5.3, the inequality $g_{\mu s_j} \neq 0$ then implies that $s_j \lessdot \mu$ or $\mu \lessdot s_j$ or that there is an $x \in X$ such that $x \lessdot \mu$ and $x \lessdot s_j$, and the inequality $h_{s_j \lambda} \neq 0$ implies that $s_j \lessdot \lambda$ or $\lambda \lessdot s_j$ or that there is an element $y \in X$ such that $y \lessdot s_j$ and $y \lessdot \lambda$. By checking each case, one can see that the cases $s_j = \mu$ and $s_j = \lambda$ are the only possible cases such that the equality $g_{\mu s_j} h_{s_j \lambda} = a\phi_{\mu\lambda}$ holds. In case $s_j = \mu$, then $h_{s_j \lambda} = h_{\mu\lambda} = b\phi_{\mu\lambda}$ for some $b \in k\backslash\{0\}$. Thus $g_{\mu s_j}$ is invertible since $\mathrm{End}(T_{\mathcal{B}}(\mu))$ is a local ring. Similarly, in case $s_j = \lambda$, one has that $h_{s_j \lambda}$ is invertible. Hence either g is a split monomorphism or h is a split epimorphism, i.e. $\phi_{\mu\lambda}$ is irreducible. □

5.5 Lemma. *For each $\lambda \in X$, there hold $soc(T_{\mathcal{B}}(\lambda)) = soc(\Delta_{\mathcal{B}}(\lambda))$ and $top(T_{\mathcal{B}}(\lambda)) = top(\nabla_{\mathcal{B}}(\lambda))$.*

Proof. If λ is a minimal element in X, then $\Delta_{\mathcal{B}}(\lambda) = E_{\mathcal{B}}(\lambda) = \nabla_{\mathcal{B}}(\lambda)$ and $T_{\mathcal{B}}(\lambda) = E_{\mathcal{B}}(\lambda)$. Now suppose λ is not a minimal element. By Proposition 5.2, we have exact sequences

$$\eta_\lambda \qquad 0 \longrightarrow \Delta_{\mathcal{B}}(\lambda) \xrightarrow{i_\lambda} T_{\mathcal{B}}(\lambda) \xrightarrow{(\pi_{\lambda\mu})_\mu} \bigoplus_{\mu \lessdot \lambda} \Delta_{\mathcal{B}}(\mu) \longrightarrow 0.$$

and

$$\eta'_\lambda \qquad 0 \longrightarrow \bigoplus_{\mu \lessdot \lambda} \nabla_{\mathcal{B}}(\mu) \xrightarrow{(i_{\mu\lambda})^t_\mu} T_{\mathcal{B}}(\lambda) \xrightarrow{\pi_\lambda} \nabla_{\mathcal{B}}(\lambda) \longrightarrow 0.$$

The morphism π_λ induces a morphism $\pi'_\lambda : \mathrm{soc}(T_{\mathcal{B}}(\lambda)) \to \mathrm{soc}(\nabla_{\mathcal{B}}(\lambda)) = E_{\mathcal{B}}(\lambda)$. Since $T_{\mathcal{B}}(\lambda)$ has a unique composition factor isomorphic to $E_{\mathcal{B}}(\lambda)$ and contains a submodule $\Delta_{\mathcal{B}}(\lambda)$, the morphism π'_λ is zero. Thus $\mathrm{soc}(T_{\mathcal{B}}(\lambda))$ is contained in $\bigoplus_{\mu \lessdot \lambda} \nabla_{\mathcal{B}}(\mu)$. Therefore,

$$\mathrm{soc}(T_{\mathcal{B}}(\lambda)) = \mathrm{soc}(\bigoplus_{\mu \lessdot \lambda} \nabla_{\mathcal{B}}(\mu)) = \bigoplus_{\mu \lessdot \lambda} E_{\mathcal{B}}(\mu) = \mathrm{soc}(\Delta_{\mathcal{B}}(\lambda)).$$

Dually, $\mathrm{top}(\nabla_{\mathcal{B}}(\lambda)) = \mathrm{top}(T_{\mathcal{B}}(\lambda))$. □

5.6 Corollary. (1) *For $\mu \lessdot \lambda$, there holds*

$$\phi_{\mu\lambda}\psi_{\lambda\mu} = \sum_{\gamma \lessdot \mu} \psi_{\mu\gamma}\phi_{\gamma\mu}$$

(2) *For $\gamma \lessdot \mu \lessdot \lambda$, there hold $\phi_{\gamma\mu}\phi_{\mu\lambda} = 0$ and $\psi_{\lambda\mu}\psi_{\mu\gamma} = 0$.*

Proof. (1) By definition, $\phi_{\mu\lambda}\psi_{\lambda\mu}$ is the composition of the following morphisms

$$T_{\mathcal{B}}(\mu) \xrightarrow{\pi_\mu} \nabla_{\mathcal{B}}(\mu) \xrightarrow{i_{\mu\lambda}} T_{\mathcal{B}}(\lambda) \xrightarrow{\pi_{\lambda\mu}} \Delta_{\mathcal{B}}(\mu) \xrightarrow{i_\mu} T_{\mathcal{B}}(\mu)$$

and $\sum_{\gamma \lessdot \mu} \psi_{\mu\gamma}\phi_{\gamma\mu}$ is the composition of the morphisms

$$T_{\mathcal{B}}(\mu) \xrightarrow{(\pi_{\mu\gamma})_\gamma} \bigoplus_{\gamma \lessdot \mu} \Delta_{\mathcal{B}}(\gamma) \xrightarrow{\oplus_\gamma i_\gamma} \bigoplus_{\gamma \lessdot \mu} T_{\mathcal{B}}(\gamma) \xrightarrow{\oplus_\gamma \pi_\gamma} \bigoplus_{\gamma \lessdot \mu} \nabla_{\mathcal{B}}(\gamma) \xrightarrow{(i_{\gamma\mu})^t_\gamma} T_{\mathcal{B}}(\mu).$$

On one hand, we consider the composition of the morphisms $j_\lambda := (i_{\nu\lambda})^t_\nu$ and $p_\lambda := (\pi_{\lambda\nu})_\nu$

$$\bigoplus_{\nu \lessdot \lambda} \nabla_{\mathcal{B}}(\nu) \xrightarrow{j_\lambda} T_{\mathcal{B}}(\lambda) \xrightarrow{p_\lambda} \bigoplus_{\nu \lessdot \lambda} \Delta_{\mathcal{B}}(\nu).$$

Since $\mathrm{soc}(\Delta_{\mathcal{B}}(\lambda)) = \bigoplus_{\nu \lessdot \lambda} E_{\mathcal{B}}(\nu) = \mathrm{soc}(\bigoplus_{\nu \lessdot \lambda} \nabla_{\mathcal{B}}(\nu))$, we have that

$$\mathrm{Ker}(j_\lambda p_\lambda) = \Delta_{\mathcal{B}}(\lambda) \cap (\bigoplus_{\nu \lessdot \lambda} \nabla_{\mathcal{B}}(\nu)) = \bigoplus_{\nu \lessdot \lambda} E_{\mathcal{B}}(\nu).$$

Thus $\mathrm{Ker}(i_{\mu\lambda}\pi_{\lambda\mu}) = \nabla_{\mathcal{B}}(\mu) \cap \mathrm{Ker}(j_\lambda p_\lambda) = E_{\mathcal{B}}(\mu)$, i.e. the morphism $i_{\mu\lambda}\pi_{\mu\lambda}$ is the composition of the following canonical projection and inclusion

$$\nabla_{\mathcal{B}}(\mu) \longrightarrow \mathrm{top}(\nabla_{\mathcal{B}}(\mu)) = \bigoplus_{\gamma \lessdot \mu} E_{\mathcal{B}}(\gamma) = \mathrm{soc}(\Delta_{\mathcal{B}}(\mu)) \longrightarrow \Delta_{\mathcal{B}}(\mu).$$

On the other hand, for each $\gamma \lessdot \mu$, the morphism $i_\gamma \pi_\gamma$ is the composition of the following canonical projection and inclusion

$$\Delta_{\mathcal{B}}(\gamma) \longrightarrow \mathrm{top}(\Delta_{\mathcal{B}}(\gamma)) = E_{\mathcal{B}}(\gamma) = \mathrm{soc}(\nabla_{\mathcal{B}}(\gamma)) \longrightarrow \nabla_{\mathcal{B}}(\gamma).$$

As a result, we have that

$$\phi_{\mu\lambda}\psi_{\lambda\mu} = \sum_{\gamma \lessdot \mu} \psi_{\mu\gamma}\phi_{\gamma\mu}$$

(2) This follows from Corollary 5.3. □

5.7 Using a similar discussion in 4.6, we have the following

Theorem. *Let X be a tree labelled by matrices $[1] \in k^{1\times 1}$. Then*

$$\mathcal{R}(\mathcal{B}_X) \cong \mathcal{B}_{X^{op}},$$

where the opposite poset X^{op} of X is also labelled by matrices $[1] \in k^{1\times 1}$.

5.8 As a consequence of 2.9, 4.6 and 5.7 we have the following relation between the quadratic dual and the Ringel dual.

Theorem. *Let X be a tree with an arbitrary invertible matrix labelling. Then*

$$\mathcal{R}(\mathcal{A}^!_{(X,M)}) \cong (\mathcal{R}(\mathcal{A}_{(X,M)}))^!,$$

that is, the Ringel dual and the quadratic dual of $\mathcal{A}_{(X,M)}$ are commutative.

References

[1] I.N.Bernstein, I.M.Gelfand and S.I.Gelfand: *A category of $\mathcal{G}$–modules,* Funct. Anal. Appl. **10** (1976), 67–92.

[2] A.Beilinson, V.Ginsburg and W.Soergel: *Koszul duality patterns in representation theory,* preliminary version of preprint.

[3] E.Cline, B.Parshall and L.Scott: *Finite dimensional algebras and highest weight categories,* J. Reine Angew. Math. **391** (1988), 85–99.

[4] M.J.Dyer: *Kazhdan–Lusztig–Stanley polynomials and quadratic algebras I,* preprint, 1993.

[5] V.Dlab and C.M.Ringel: *Quasi-hereditary algebras,* Ill. J. Math. **33** (1989), 280–291.

[6] V.Dlab and C.M.Ringel: *The module theoretical approach to quasi-hereditary algebras,* LMS Lecture Notes Series 168, "Representations of algebras and related topics", ed. H. Tachikawa and S.Brenner, 1992, 200–224.

[7] B.M.Deng and C.C.Xi: *Quasi-hereditary algebras which are dual extensions of algebras,* Comm. in Algebra **22** (1994), 4717–4735.

[8] K.Erdmann: *Symmetric groups and quasi-hereditary algebras,* preprint.

[9] R.S.Irving: *BGG-algebras and the BGG reciprocity principle,* J. Algebra **135** (1990), 363–380.

[10] S.König: *A guide to exact Borel subalgebras of quasi-hereditary algebras,* Proceeding of ICRA, Carleton–Ottawa Math. LNS 14(1992).

[11] C.M.Ringel: *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences,* Math. Zeit. **208** (1991), 209–223.

[12] C.C.Xi: *Quasi-hereditary algebras with a duality,* J. Reine Angew. Math. **449** (1994), 201–215.

[13] C.C.Xi: *On representation types of q-Schur algebras,* J. Pure Appl. Algebra **84** (1993), 73–84.

Received December 16, 1993; revised version August 25, 1994.