MATHEMATICS

Characteristic tilting modules and Ringel duals

XI Changchang (惠昌常)

Department of Mathematics, Beijing Normal University, 100875 Beijing, China (email: xicc@bnu.edu.cn)

Received January 13, 1999

Abstract The characteristic tilting modules of quasi-hereditary algebras which are dual extensions of directed monomial algebras are explicitly constructed; and it is shown that the Ringel dual of the dual extension of an arbitrary hereditary algebra has triangular decomposition and bipartite quiver.

Keywords: **quasi-hereditary algebras, characteristic modules, dual extensions, Ringel duals.**

Quasi-hereditary algebras were introduced by Cline, et al. in order to study highest weight categories in the representation theory of Lie algebras and algebraic groups[1]. Typical examples of quasi-hereditary algebras are Schur algebras[2], the algebras to the blocks of the category $\mathcal{O}$ introduced in ref. [3] and the Temperley-Lieb algebras. Quasi-hereditary algebras seem to become a very interesting class of algebras.

Let A be a quasi-hereditary k-algebra over an algebraically closed field k. Ringel constructed in ref. [4] a new quasi-hereditary algebra $\mathscr{R}(A)$ from A such that $\mathscr{R}(\mathscr{R}(A))$ is Morita equivalent to A, and in fact he used a generalized tilting and cotilting module. This special module is called the characteristic module for the quasi-hereditary algebra A and seems to be of special interest in the representation theory of algebraic groups[5]. The algebra $\mathscr{R}(A)$ is usually called the Ringel dual of A.

Since quasi-hereditary algebras appear always in pair A and $\mathscr{R}(A)$, it is natural to ask the following question: If one of them is known, how does the other look like? To understand the characteristic module and the algebra $\mathscr{R}(A)$, we study in this paper a special class of quasi-hereditary algebras which are dual extensions (for the definition see sec. 1). Our aim is to construct explicitly the characteristic module over the dual extension of a directed monomial algebra. The main result of this paper describes explicitly the quiver of $\mathscr{R}(A)$ for A the dual extension of an arbitrary hereditary algebra. This implies that the quivers of these algebras $\mathscr{R}(A)$ are bipartite, thus generalizing the main result in ref. [6]. We show also that these algebras $\mathscr{R}(A)$ have triangular decompositions and are of global dimension at most 2.

1 Definitions

Let A be a finite-dimensional k-algebra over a field k. By A-mod we denote the category of all finitely generated left A-modules. Maps between A-modules will be written on the right side of the argument; thus the composition of maps $f: M_1 \longrightarrow M_2$ and $g: M_2 \longrightarrow M_3$ will be denoted by fg.

Given a class Θ of A-modules, we denote by $\mathscr{F}(\Theta)$ the class of all A-modules in A-mod which have a finite Θ-filtration; that is, a filtration

$$0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M$$

such that each factor M_{i-1}/M_i is isomorphic to an object in Θ for $1 \leq i \leq t$. For a module $M \in A$-mod, we denote by add (M) the full additive subcategory of A-mod consisting of all finite direct sums of direct summands of M.

Let X be a finite poset in bijective correspondence with the isomorphism classes of simple A-modules. For each $\lambda \in X$, let $E(\lambda)$ be a simple module in the isomorphism class corresponding to λ and $P(\lambda)$ (or $P_A(\lambda)$) a projective cover of $E(\lambda)$ and denote by $\triangle(\lambda)$ the maximal factor module of $P(\lambda)$ with composition factors of the form $E(\mu)$, $\mu \leq \lambda$. Dually, let $Q(\lambda)$ (or $Q_A(\lambda)$) be an injective hull of $E(\lambda)$ and denote by $\nabla(\lambda)$ the maximal submodule of $Q(\lambda)$ with the composition factors of the form $E(\mu)$, $\mu \leq \lambda$. Let $\triangle$ (respectively, ∇) be the full subcategory of all $\triangle(\lambda)$, $\lambda \in X$ (respectively, all $\nabla(\lambda), \lambda \in X$). We call the modules in $\triangle$ the standard modules and the ones in ∇ the costandard modules.

The algebra A is said to be quasi-hereditary with respect to $(X, \leq)$ if for each $\lambda \in X$ we have

(i) $\mathrm{End}_A(\triangle(\lambda))$ is a division ring; and

(ii) $P(\lambda) \in \mathscr{F}(\triangle)$, and moreover, $P(\lambda)$ has a $\triangle$-filtration with quotient $\triangle(\mu)$ for $\mu \geq \lambda$ in which $\triangle(\lambda)$ occurs exactly once.

For a quasi-hereditary algebra A with respect to a poset X we call the elements in X weights and X the weight poset of A. By (A, X) we denote a quasi-hereditary algebra A with the weight poset X.

If a quasi-hereditary algebra has a duality δ on the category A-mod which fixes simple modules, we call it a BGG-algebra (see ref. [7]).

As examples of BGG-algebras, dual extension algebras are constructed in ref. [8]. Let us give the definition more generally in the language of ring theory.

Let C and B be two rings such that there is a common subring S of C and B and there are ideals M in C and N in B with

$$C = S \oplus M, \qquad B = S \oplus N,$$

where $\oplus$ means the direct sum of S-bimodules. We define a multiplication on

$$\mathscr{A}(C, B) := S \oplus M \oplus N \oplus M \otimes_S N$$

by the following law:

$$\begin{aligned}
&(s + m + n + m_1 \otimes n_1)(s' + m' + n' + m_1' \otimes n_1') \\
= \ & ss' + (sm' + ms' + mm') + (sn' + ns' + nn') \\
&+ (m_1' \otimes n_1' + m \otimes n' + mm_1' \otimes n_1' + m_1 \otimes n_1 s' + m_1 \otimes n_1 n')
\end{aligned}$$

for $s, s' \in S$, m, m', m_1, $m_1' \in M$, n, n_1, n', $n_1' \in N$. Then $\mathscr{A}(C, B)$ is an associative ring. We call the algebra $\mathscr{A}(C, B)$ the trivially twisted extension of C and B.

If S is commutative and $B = C^{op}$, then we call $\mathscr{A}(C, B)$ the dual extension of C with respect to the decomposition $C = S \oplus M$. We denote simply by $\mathscr{A}(C)$ the dual extension of C.

We are mainly interested in the case where S is a maximal commutative semisimple subalgebra of C and M is the radical of C. Of particular interest to us is a special case of this construction which arises from the description of an algebra by quivers and relations.

Let C be a finite-dimensional basic algebra over k. As usual, we may assume that C is described by a quiver $Q = (Q_0, Q_1)$ with relations $\{\rho_i \in kQ \mid i \in I_C\}$, where I_C is an index set, (note that we do not specify these relations and we allow multiple arrows). Thus we consider the algebra $kQ^*/\langle\{\rho_i^* \mid i\in I_C\}\rangle$, where Q^* is the opposite quiver of Q and the multiplication $\alpha\beta$ of two arrows α and β means that α comes first and then β follows (for the notation see ref. [9]). For each α from i to j in Q_1, we associate it with an arrow α' from j to i. We denote by Q_1' the set of all such α' with $\alpha\in Q_1$. For a path $\alpha_1\cdots\alpha_m$ we denote by $(\alpha_1\cdots\alpha_m)'$ the path $\alpha'_m\cdots\alpha'_1$ in the quiver (Q_0, Q_1'). With this notation we may define a BGG-algebra.

Let A be the algebra given by the quiver $(Q_0, Q_1\cup Q_1')$ with relations $\{\rho_i \mid i\in I_C\}\cup\{\rho_i' \mid i\in I_C\}\cup\{\alpha\beta' \mid \alpha,\beta\in Q_1\}$. Then it is a finite-dimensional algebra over k. Clearly, A is just the dual extension of C defined above.

Lemma 1. If C has no oriented cycle in its quiver, we may assume that $Q_0=\{1,\dots,n\}$ such that $\mathrm{Hom}_C(P_C(i), P_C(j))=0$ for $i>j$. Then A is a BGG-algebra. Furthermore, the standard A-modules are $\triangle_A(i)=P_C(i)$ for $i\in\{1,\dots,n\}$.

For the proof of this lemma we refer to ref. [8], and for the further properties of the algebra $\mathscr{A}(C)$ one may see refs. [10,11].

2 A construction of certain $\mathscr{A}(C)$-modules

Let k be a field. Let C be a finite-dimensional monomial algebra Thus $C=kQ^*/I^*$, where $Q=(Q_0,Q_1)$ is a finite quiver and I an admissible ideal in kQ generated by monomials.

Given an arrow α in Q_1 with starting vertex $s(\alpha)$ and terminal vertex $t(\alpha)$, we associate with it an arrow α^{-1}, with the same starting and terminal vertex as α. (The notation α^{-1} may not be a good notation here; it really suggests the inverse of α^{-1} in some sense below). Let Q_1^{-1} be the set of all arrows of the form α^{-1}. For a monomial $\alpha_1\alpha_2\cdots a_n$ in Q, we denote by $(\alpha_1\alpha_2\cdots\alpha_n)^{-1}$ the monomial $\alpha_1^{-1}\alpha_2^{-1}\cdots\alpha_n^{-1}$ in the quiver (Q_0, Q_1^{-1}). Let $\bar{Q}=(Q_0, Q_1\cup Q_1^{-1})$ and $\bar{A}$ be the monomial algebra given by the quiver $\bar{Q}$ with relations $\bar{I}$, where $\bar{I}$ is the union of I and I^{-1}. (Here we write I^{-1} for the set $\{w^{-1}\mid w\in I\}$). Note that the algebra A is infinite-dimensional if and only if Q contains an oriented cycle.

In the following we shall construct for each indecomposable projective $\bar{A}$-module a module over $\mathscr{A}(C)$.

Given a vertex $x\in Q_0$, let $T(x)$ have the same vector space as the indecomposable projective $\bar{A}$-module $P_{\bar{A}}(x)$ corresponding to x. Thus $T(x)$ has a k-basis $B(x)$ consisting of monomials in $k\bar{Q}$ starting at the vertex x and not in $\bar{I}$. For any vertex a, let $T(x)_a$ be the k-subspace spanned by all monomials in $B(x)$ having terminal vertex a. Then

$$T(x)=\bigoplus_{a\in Q_0} T(x)_a.$$

For each arrow $\alpha: a\longrightarrow b$ in Q_1, let $T(x)_\alpha: T(x)_a\longrightarrow T(x)_b$ be defined by sending $w\in T(x)_a$ to $w\alpha\in T(x)_b$. For each $\alpha': a\longrightarrow b$ in Q_1', let $T(x)_{\alpha'}: T(x)_a\longrightarrow T(x)_b$ be defined by

$$w\longmapsto\begin{cases} w_1, & \text{if } w=w_1\alpha^{-1},\\ 0, & \text{otherwise}.\end{cases}$$

Note that since we deal with a monomial algebra each monomial w has the unique form $w_1\alpha^{-1}$ if it exists. So the above map is well-defined. By this definition, we see that $T(x)$ is a module over $\mathscr{A}(C)$.

Let $f: P_{\bar{A}}(a)\longrightarrow P_{\bar{A}}(b)$ be an $\bar{A}$-homomorphism. Then it is given by left multiplying an element $w=\sum\lambda_i w_i$ where w_i are words with starting vertex b and terminal vertex a, and $\lambda_i\in$

k. Thus f induces a C-homomorphism from $T(a)$ to $T(b)$. If all w_i end with arrows in Q_1, then f is also an $\mathscr{A}(C)$-homomorphism. Suppose $w_1, \cdots, w_s$ are all paths in the expression of w of the form $w_j = u_j\beta_j^{-1}, \beta_j \in Q_1$. Let $U(a, b, f)$ be the $\mathscr{A}(C)$- submodule of $T(b)$ generated by $u_j, 1 \leq j \leq s$. Then f induces an $\mathscr{A}(C)$-homomorphism from $T(a)$ to $T(b)/U(a, b, f)$.

Conversely, let f be an $\mathscr{A}(C)$-homomorphism from $T(a)$ to $T(b)$. Then f sends a to an element w of $T(b)$ which may be written in the form $w = \sum_{i=1}^{m} \lambda_i w_i$ with $w_i, 1 \leq i \leq m$, paths in $\bar{Q}$ from b to a. Suppose $w \neq 0$. Since f is also a C-homomorphism, it sends each α in Q_1 with $s(\alpha) = a$ to $w\alpha$. Since $\alpha' \cdot \alpha^{-1} = a$ and the action of α' on the image of α^{-1} under f is w, we see that f sends α^{-1} to $w\alpha^{-1}$ by the definition of the action of α'. By induction on the lenght of the words, we can show that f is just the left multiplication by the element w. Since f is an $\mathscr{A}(C)$-homomorphism, each path w_i terminates with either a vertex in Q_0 or an arrow in Q_1. Clearly, for each element w of this form one can get an $\mathscr{A}(C)$-homomorphism from $T(a)$ to $T(b)$ by mapping a to w. Note that there are also $\mathscr{A}(C)$-homomorphisms which send a to zero; they are not of this form.

3 Properties of the module $T(a)$

In the following we denote by C a finite-dimensional monomial k-algebra given by a quiver $Q = (Q_0, Q_1)$ with relations, and by A the dual extension of C. By A-Mod we denote the category of all left A-modules. Given a class Θ of A-modules in A-Mod, we say that a module M has an -filtration in Θ if there is a filtration

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$$

of submodules of M such that $\bigcup_i F_i = M$ and $F_{i+1}/F_i \in \Theta$. Dually, we have the notion of -cofiltration: if there is a filtration

$$\cdots \subseteq F_2 \subseteq F_1 \subseteq F_0 = M$$

of submodules of M with $\bigcap_i F_i = 0$ and $F_i/F_{i+1} \in \Theta$. Let us denote by $\mathscr{F}^+(\Theta)$ (respectively, $\mathscr{F}^-(\Theta)$) the full subcategory of A-Mod consisting of all modules with an -filtration (respectively, an -cofiltration) in Θ.

Lemma 2. Let a be a vertex in Q_0. Then

(1) $T(a)$ contains an A-submodule $\triangle(a)$ which is isomorphic to $P_C(a)$ considered as an A- module.

(2) There is an exact sequence

$$0 \longrightarrow \triangle(a) \xrightarrow{i_a} T(a) \xrightarrow{p_a} X(a) \longrightarrow 0,$$

where $X(a)$ is an A- module with an -filtration in $\{\triangle(b) \mid b \in Q_0\}$.

Dually, we have

(1') $T(a)$ has a factor A-module $\nabla(a)$ which is isomorphic to C^{op}-module $Q_{C^{op}}(a)$ considered as an A-module.

(2') There is an exact sequence

$$0 \longrightarrow Y(a) \longrightarrow T(a) \xrightarrow{\pi_a} \nabla(a) \longrightarrow 0,$$

where $Y(a)$ is an A-module with an -cofiltration in $\{\nabla(b) \mid b \in Q_0\}$.

Proof. (1) Let $\triangle(a)$ be the k-space generated by all paths in $B(a)$ of the form $\alpha_1 \cdots \alpha_m$ of non-negative length with all $\alpha_i \in Q_1$. Then $\triangle(a)$ is an A-submodule of $T(a)$ which is clearly ismorphic to $P_C(a)$ as an A-module.

(2) By the construction of $T(a)$, for any path w in $B(a)$ ending with an arrow α^{-1}, the k-space wC spanned by all paths $wu \in B(a)$ with $u \in \triangle(t(\alpha))$ is a C-module and isomorphic to $P_C(t(\alpha))$ as C-modules. Let M_a be the subset of $B(a)$ consisting of all paths w ending with an arrow α^{-1} and S_i the k-space spanned by all wC, where w runs over all pahts in M_a of length i. Then we have a filtration:

$$F := \triangle(a) \subseteq F_1 \subseteq F_2 \subseteq \cdots$$

of A-submodules of $T(a)$ such that $F_{i+1}/F_i \cong S_{i+1}$ as C-modules for all i. This shows that $T(a)/\triangle(a)$ has an -filtration in $\{\triangle(b) \mid b \in Q_0\}$.

The rest of the lemma can be proved dually.

Lemma 3. Let $\triangle = \{\triangle(a) \mid a \in Q_0\}$ and $\nabla = \{\nabla(a) \mid a \in Q_0$. Then we have $T(\lambda) \in \mathscr{F}^+(\triangle) \cap \mathscr{F}^-(\nabla)$.

Lemma 4. $\mathrm{Tor}_i^A(DT(a), T(b)) = 0$ for $i \geq 1$, where D is the usual dual $\mathrm{Hom}_k(-, k)$.

Proof. Since the direct limit of flat modules is flat and $M \otimes_A \varinjlim F_i \cong \varinjlim M \otimes_A F_i$, we see that $\mathrm{Tor}_j^A(M, \varinjlim F_i) \cong \varinjlim \mathrm{Tor}_j^A(M, F_i)$ for all j. It follows from ref. [12] that $D\mathrm{Ext}_A^j(X, Y) \cong \mathrm{Tor}_j^A(DY, X)$ for any finitely generated module X. Now let F_i be the filtration of $T(b)$ given in the proof of Lemma 2. Then the direct limit of F_i is just the module $T(b)$. Since $T(b)$ as C^{op}-module is a direct sum of modules of the forms $Q_{C^{op}}(x), x \in Q_0$, and the ring C^{op} is a noetherian ring, we know that as C^{op}-module $T(b)$ is injective. Then $\mathrm{Tor}_j^A(DT(a)), T(b)) = \mathrm{Tor}_j^A(DT(a), \varinjlim F_i) = \varinjlim \mathrm{Tor}_j^A(DT(a), F_i) = \varprojlim \mathrm{Ext}_A^j(F_i, T(a))$. Now suppose $j \geq 1$ and we show that $\mathrm{Ext}_A^j(F_i, T(a)) = 0$ for all i. Since each F_i has a finite $\triangle$-filtration, it is enough to show that $\mathrm{Ext}_A^j(\triangle(x), T(a)) = 0$ for all $x \in Q_0$. But this follows from $\mathrm{Ext}_A^j(\triangle(x), T(a)) \cong \mathrm{Ext}_{C^{op}}^j(E(x), {}_{C^{op}}T(a)) = 0$ since $A_{C^{op}}$ is projective and $\triangle(x) \cong A \otimes_{C^{op}} E(x)$. Thus, by induction on i, we can show that $\mathrm{Ext}_A^j(F_i, T(a)) = 0$.

Lemma 5. Suppose Q does not contain any oriented cycle. Then $T(a)$ is indecomposable.

Proof. Note that $T(a)$ is a finite-dimensional module and $\dim_k T(a)_a = 1$. Let M be the indecomposable direct summand of $T(a)$ containing a. We show that M contains $B(a)$ by induction on the length of the paths in $B(a)$. If the lenght is zero then it is true. Suppose all paths in $B(a)$ of lenght i are contained in M. Let $w = w_1\beta$ with $\beta \in Q_1 \cup Q_1^{-1}$ of length $i+1$. Since M is a submodule of $T(a)$ containing w_1, there holds $w \in M$ if $\beta \in Q_1$. Now suppose $\beta = \alpha^{-1}$. If w_1 is of length 0 then $w \in M$ since $a \in M$ and M contains all paths of the form $\alpha_1^{-1} \cdots \alpha_r^{-1}$ with $\alpha_i \in Q_1$. If w_1 ends with an arrow in Q_1 then kw_1 forms a C^{op}-module. Since $T(a)$ is in $\mathscr{F}^-(\nabla)$, it is as C^{op}-module a direct sum of modules of the forms $Q_{C^{op}}(x), x \in Q_0$. Thus the module ${}_{C^{op}}M$ is also a direct sum of indecomposable injective C^{op}-modules. Let $\nabla(b)$ be the direct summand containing kw_1. (Note that such a direct summand exists since kw_1 appears as socle in the module ${}_{C^{op}}M$.) Then $w \in \nabla(b)$ since $\alpha' \cdot w = w_1$. Hence $w \in M$. If w_1 ends with an arrow in Q_1^{-1} we may assume w_1 is of the form $w_2\beta_1^{-1}\cdots\beta_r^{-1}$ such that w_2 ends with an arrow in Q_1 or $w_2 = a$. In this case we may consider the C^{op}-module kw_2 instead of kw_1 in the above discussion. With a similar argument, we can show that $w \in M$. Thus $M = T(a)$ is indecomposable.

Lemma 6. If $T(a)$ is finite-dimensional, then $T(a)$ is isomorphic to $\mathrm{Hom}_k(T(a),k)$, where the A-module structure on $DT(a):=\mathrm{Hom}_k(T(a),k)$ is given by

$$(x\cdot f)(w)=f(x'\cdot w),\ x\in A, w\in T(a)\quad\text{and}\quad f\in \mathrm{Hom}_k(T(a),k).$$

Proof. Let $B(a)^*=\{w^*\mid w\in B(a)\}$ be the dual basis of $B(a)$. We define a bijective k-linear map ϕ as the map sending $w\in B(a)$ to $(w^{-1})^*\in B(a)^*$. Then one can verify that ϕ is an A-homomorphism. Hence ϕ is an isomorphism.

Note that if $T(a)$ is infinite-dimensional then $T(a)\not\cong DT(a)$ as A-modules.

4 Application

As a consequence we apply the previous construction to the case where the monomial algebra C is directed, namely there exitsts no oriented cycle in its quiver Q. First let us recall the following theorem which characterizes the canonical modules over quasi-hereditary algebras. For the proof one may refer to ref. [4]. (Note that a proof of part (3) can also be found in ref. [1]).

Theorem A. Let A be a quasi-hereditary algebra with the weight poset X.

(1) The intersection $\mathscr{F}(\triangle)\cap\mathscr{F}(\nabla)$ contains exactly $|X|$ isomorphism classes of indecomposable modules, where $|X|$ is the cardinality of X. They may be parametrized as $T(\lambda),\lambda\in X$, such that the following holds: There are exact sequences

$$(a)\ 0\longrightarrow\triangle(\lambda)\longrightarrow T(\lambda)\longrightarrow X(\lambda)\longrightarrow 0,$$
$$(b)\ 0\longrightarrow Y(\lambda)\longrightarrow T(\lambda)\longrightarrow \nabla(\lambda)\longrightarrow 0,$$

where $X(\lambda)$ is filtered by $\triangle(\mu)$'s for certain $\mu<\lambda$ and $Y(\lambda)$ by $\nabla(\mu)$'s for certain $\mu<\lambda$. In particular, $T(\lambda)$ has a unique composition factor isomorphic to $E(\lambda)$ and all other composition factors are of the form $E(\mu)$ with $\mu<\lambda$, where $E(\mu)$ denotes the simple A-module corresponding to the weight $\mu\in X$.

(2) Put $T=\bigoplus_{\lambda\in X}T(\lambda)$ and $\mathscr{R}(A)=\mathrm{End}_A(T)$. Then T is a tilting-cotilting module and $\mathscr{R}(A)$ is a quasi-hereditary algebra, with standard modules $\triangle_{\mathscr{R}(A)}(\lambda)=\mathrm{Hom}_A(T,\nabla(\lambda))$, where the weight poset of $\mathscr{R}(A)$ is X^{op}.

(3)

$$\mathrm{Ext}_A^n(\triangle(\lambda),\nabla(\mu))=\begin{cases}k, & \text{if}\quad n=0\quad\text{and}\quad\lambda=\mu,\\ 0, & \text{otherwise}.\end{cases}$$

The modules $T(\lambda)$ are called canonical modules, the tilting-coltilting module T is called the characteristic module for (A,X), and the algebra $\mathscr{R}(A)=\mathrm{End}_A(T)$ is usually called the Ringel dual of A.

Note that if the quiver Q of C has no oriented cycle then there is a natural order on Q_0: We say that $\lambda\leq\mu$ in Q_0 if there is a path from μ to λ in the quiver Q.

Suppose we are given a finite-dimensional monomial algebra C having no oriented cycle in its quiver Q. Then $A=\mathscr{A}(C)$ is a quasi-hereditary algebra. Moreover, the module $T(a)$ constructed in sec. 2 is a finite-dimensional A-module because the algebra $\bar{A}$ is a finite-dimensional k-algebra, and belongs to $\mathscr{F}(\triangle)\cap\mathscr{F}(\nabla)$. Hence $T(a)$ are the canonical modules of the quasi-hereditary algebra A. We restate the results in sec. 3 as follows:

Lemma 7. Let C be a finite-dimensional directed monomial k-algebra and A its dual extension. Then for each $a\in Q_0$,

(1) there is a unique submodule $\triangle(a)$ of $T(a)$ such that $\triangle(a)$ are isomorphic to $P_C(a)$ as an A-module and the factor module $X(a)$ is a module in $\mathscr{F}\{\triangle(b)\mid b<a\}$.

(2) there is a unique factor module $\nabla(a)$ of $T(a)$ such that $\nabla(a)$ are isomorphic to $Q_{C^{op}}(a)$ *as an* A-module and the kernel $Y(a)$ is a module in $\mathscr{F}\{\nabla(b) \mid b < a\}$.

5 Ringel duals of dual extensions of hereditary algebras

From now on we suppose $A = \mathscr{A}(C)$ is the dual extension of a finite-dimensional hereditary algebra C given by a quiver Q. In this case, we know that $\mathscr{F}(\triangle)$ is closed under submodules and $\mathscr{F}(\nabla)$ is closed under factor modules. Hence the modules $X(a)$ and $Y(a)$ in Lemma 7 are direct sums of the canonical modules and isomorphic to each other since A is a BGG-algebra. The following lemma gives an explicit description.

Lemma 8. Let $[a:b]$ denote the multiplicity of the composition factor $E(b)$ in the projective C-module $P_C(a)$.

(1) There is an exact sequence

$$0 \longrightarrow \triangle(a) \xrightarrow{i_a} T(a) \xrightarrow{p_a} \bigoplus_{c<a} T(c)^{[a:c]} \longrightarrow 0,$$

(2) there is an exact sequence

$$0 \longrightarrow \bigoplus_{c<a} T(c)^{[a:c]} \xrightarrow{l_a} T(a) \xrightarrow{\pi_a} \nabla(a) \longrightarrow 0.$$

Proof. Suppose $a \neq b$. Since $[a:b]$ is the number of all paths in Q starting at a and ending at b, we see that this number is equal to the cardinality of the set I_a of all paths $w = w_1\beta^{-1}$ in $B(a)$ such that w_1 is a path in Q, $\beta \in Q_1$ and $t(\beta) = b$. Let $w \in I_a$ with $t(w) = c$. We denote by X_w the space spanned by $wB(c)$. Since $X_w \cap \triangle(a) = 0$ as subspace of $T(a)$, we see that as vector spaces $(X_w + \triangle(a))/\triangle(a)$ are isomorphic to $T(c)$ by sending $w_1 \in B(c)$ to $ww_1 + \triangle(a)$. Note that $X_w + \triangle(a)$ is also an A-module. Thus the foregoing isomorphism is also an isomorphism of A-modules. Now it follows from the construction of k-basis of $T(a)$ that the first exact sequence in the lemma exists. The rest of the lemma can be proved dually.

Theorem 1. Let C be a finite dimensional hereditary algebra given by quiver Q. Then for every $x, y \in Q_0$, there holds

$$\dim_k \mathrm{irr}_{\mathscr{T}}(T(x), T(y)) = \begin{cases} [x:y], & \text{if } x < y \text{ and } y \text{ is maximal, or} \\ & \text{if } x > y \text{ and } x \text{ is maximal,} \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathscr{T} = \mathrm{add}(T)$ and $\mathrm{irr}_{\mathscr{T}}(T(x), T(y)) := \mathrm{rad}_{\mathscr{T}}(T(x), T(y))/\mathrm{rad}^2_{\mathscr{T}}(T(x), T(y))$ is the bimodule of irreducible maps from $T(x)$ to $T(y)$ in $\mathscr{T}$.

Proof. By Lemma 2.2 of ref. [6], it remains to prove that $\mathrm{irr}_{\mathscr{T}}(T(x), T(y)) = 0$ if neither x nor y is maximal. Since the Ringel dual does not contain a loop, we have to examine the following different cases.

(i) x and y are not comparable. Then every map $f \in \mathrm{Hom}_A(T(x), T(y))$ factors through p_x since $T(y)$ has no composition factor $E(x)$ and $f|_{\triangle(x)} = 0$.

(ii) $x < y$. If f is a non-zero homomorphism from $T(x)$ to $T(y)$ and $f|_{\triangle(x)} = 0$ then f factors through $X(x)$ by Lemma 8. Suppose $f|_{\triangle(x)} \neq 0$. By the construction, f maps x to a non-zero element w in $T(y)$ and the map f is just given by the left multiplication of w. Since y is not maximal, there exists an arrow $\alpha: y \leftarrow z \in Q_1$. Let g be the map from $T(x)$ to $T(z)$ defined by $x \mapsto \alpha^{-1}w$. Then g is an A-homomorphism. Let h be the composition of the canonical projection of $T(z)$ onto $X(z)$ and the canonical projection of $X(z)$ onto its direct summand

$X_{\alpha^{-1}}$ (see the proof of Lemma 8). Then $X_{\alpha^{-1}} \cong T(y)$ and $f = gh$ by the construction of modules $T(a)$.

(iii) $x > y$. The dual of (ii). This finishes the proof.

Remark. (1) It follows from the above theorem that the quiver of $\mathscr{R}(A)$ is bipartite, that is, the vertex set is a disjoint union of $Q'_0 = \{a \in Q_0\} \mid a$ is maximal$\}$ and $Q''_0 = Q_0 \setminus Q'_0$, and the arrows are between Q'_0 and Q''_0. This generalizes the main result in ref. [6].

(2) In general the quiver of a Ringel dual may not be of the form described in Theorem 1 since the structure of the characteristic module may be different from the one constructed in sec. 2. For this one may see ref. [13] and the recent work ref. [14].

6 The global dimension of $\mathscr{R}(A)$

Since the Ringel dual $\mathscr{R}(A)$ is the endomorphism ring of a generalized tilting module T, we know from the tilting theory (Proposition 3.4 of ref. [15]) that the global dimension of $\mathscr{R}(A)$ is between the two numbers gl.dim (A) − proj.dim (T) and gl.dim(A) + proj.dim (T). Under the hypothesis in sec. 5 we will see that the global dimension of $\mathscr{R}(A)$ is equal to that of A.

Lemma 9. Let A be the dual extension of a finite-dimensional hereditary algebra C. Then $\mathscr{R}(A)$ has a triangular decomposition. More precisely, there are directed subalgebras E and E^{op} of $\mathscr{R}(A)$ satisfying the following conditions:

(i) $E \cap E^{op}$ is the maximal semisimple subalgebra E_0 of $\mathscr{R}(A)$ generated by the identity maps $id_{T(a)}$, $a \in Q_0$ considered as elements in $\mathscr{R}(A)$.

(ii) The multiplication map

$$E \otimes_{E_0} E^{op} \longrightarrow \mathscr{R}(A)$$

is bijective.

Proof. Let E be the k-space spanned by $id_{T(a)}$, $a \in Q_0$ and all monomials in π_w, where π_w are defined as components of p_a with $a \in Q_0$ in Lemma 8. Then E is a directed subalgebra of $\mathscr{R}(A) := \mathrm{End}_A(\oplus_{b \in Q_0} T(b))$. We can identify $E \cdot id_{T(a)}$ with the space spanned by $id_{T(a)}$ and all monomials $\pi_{w_1}\pi_{w_2}\cdots\pi_{w_m}$ with $t(w_i) = s(w_{i+1})$ for $1 \le i \le m-1$ and $t(w_m) = a$. We shall show that E is a $\triangle$-subalgebra of $\mathscr{R}(A)$, namely the map $E \cdot id_{T(a)} \longrightarrow \mathrm{Hom}_A(T, \nabla(A)) = \mathrm{Hom}_A(T, T(a))/\mathrm{Hom}_A(T, \oplus_{v<a}T(v))$ given by $f \mapsto fp$ is an isomorphism, where $p = \pi_a$ is as in Lemma 8.

(1) This is an injective map. We pick a non-zero element $f \in E \cdot id_{T(a)}$. Then f is a linear combination of monomials $\pi_{w_1}\pi_{w_2}\cdots\pi_{w_m}$. Since the canonical projection π_w maps w to $t(w) \in T(t(w))$, the map f is always surjective, and therefore $fp \neq 0$.

(2) It is surjective. If $0 \neq f \in \mathrm{Hom}_A(T, \nabla(a))$, then $f = \bar{f}p$ with $\bar{f}: T(x) \longrightarrow T(a)$. Note that $a \le x$ since $f \neq 0$. If $x = a$ then $\bar{f}$ is an automorphism of $T(a)$. By the construction of $T(a)$, $\bar{f}$ is a scalar of the identity map $id_{T(a)}$ and hence in $E \cdot id_{T(a)}$. If $a < x$ then by Lemma 8, we can decompose $\bar{f}$ into $\bar{f} = \pi_{w_1}f_1 + \cdots + \pi_{w_n}f_n$ with $f_j: T(t(w_j)) \longrightarrow T(a)$ and $t(w_j) < x$. If $f_j p = 0$ then $f_j \in \mathrm{Hom}_A(T, \oplus_{v<a}T(v))$; if $f_j p \neq 0$ then $a \le t(w_j)$. In this case we may repeat the above discussion. Since the poset Q_0 is finite, after finitely many steps, $\bar{f}$ is of the form

$$\bar{f} = \sum_j \lambda_j \pi_{w_1}\pi_{w_2}\cdots\pi_{w_{m_j}} + g$$

with $t(w_{m_j}) = a$ and $g \in \mathrm{Hom}_A(T, \oplus_{v<a} T(v))$. Then $(\bar{f} - g)p = f$. This shows that our map is surjective.

By the definition of a $\triangle$-subalgebra ref. [16], we see that E is a $\triangle$-subalgebra of $\mathscr{R}(A)$. Since $\mathscr{R}(A)$ is a BGG-algebra, it follows that $\mathscr{R}(A)$ has a triangular decomposition $E \otimes_{E_0} E^{op}$ with E^{op} generated by the maps $id_{T(a)}$ and all monomials in i_w, where i_w are components of the maps l_a, $a \in Q_0$ in Lemma 8.

Theorem 2. Let A be the dual extension of a finite-dimensional hereditary algebra C. Then

$$\mathrm{gl.dim}\mathscr{R}(A) = \begin{cases} 0, & \text{if } C \text{ is semisimple;} \\ 2, & \text{otherwise.} \end{cases}$$

Proof. It follows from Lemma 8 that the projective dimension of $\triangle_{\mathscr{R}(A)}(a) = \mathrm{Hom}_A(T, \nabla(a))$ is at most one. Thus by ref. [17] the category $\mathscr{F}(\nabla_{\mathscr{R}(A)})$ is closed under factor modules and the category $\mathscr{F}(\triangle_{\mathscr{R}(A)})$ is closed under submodules. By Lemma 9, the standard $\mathscr{R}(A)$-modules are just the projective E-modules. Hence E is a hereditary algebra and $\mathrm{gl.dim}\, \mathscr{R}(A) \leq 2 \cdot \mathrm{gl.dim}\,(E)$. If C is semisimple, then $C = A = \mathscr{R}(A)$ and $\mathrm{gl.dim}\, \mathscr{R}(A) = 0$. If C is not semisimple, it is easy to see that $\mathscr{R}(A)$ is not hereditary and hence $\mathrm{gl.dim}\, \mathscr{R}(A) \geq 2$. Thus $\mathrm{gl.dim}\mathscr{R}(A) = 2 = \mathrm{gl.dim}\,(A)$.

Usually, a quasi-hereditary algebra A and its Ringel dual $\mathscr{R}(A)$ may have different global dimensions. The following lemma gives a suffcient condition for having equal global dimension.

Lemma 10. Let A be a quasi-hereditary algebra of global dimension $2m$, and suppose the characteristic module of A is of projective dimension at most one. If $\mathscr{R}(A)$ has a triangular decomposition $E \otimes_{E_0} E^{op}$ such that the standard $\mathscr{R}(A)$-modules are semisimple on restriction to E^{op}, then $\mathrm{gl.dim}\, \mathscr{R}(A) = \mathrm{gl.dim}(A)$.

Proof. We know from Theorem 1.1 of ref. [16] that the global dimension of $\mathscr{R}(A)$ is an even number. Thus this even number is between $2m - 1$ and $2m + 1$ and must equal $2m$.

7 Questions

Before we state our questions, let us recall the definition of algebraically compact modules. Given a left A-module M, let $a = (a_{ij})_{ij}$ be an $m \times n$ matrix, and $b = (b_i)_i$ a vector of length m, with entries a_{ij}, b_i in A. We denote by $U(a, b)$ the set of elements $y \in M$ such that there are elements $x_1, \cdots, x_n \in M$ with

$$\sum a_{ij}x_j = b_i y \quad \text{for} \quad 1 \leq i \leq m.$$

This is a subgroup of M. A subgroup of this form $U(a, b)$ is called a finitely definable subgroup of M. The module M is called algebraically compact (or pure injective) provided for every codirected system of finitely definable subgroups M_i of M, the canonical map $M \longrightarrow \varprojlim M/M_i$ is surjective.

The following questions related to the module $T(a)$ are open.

Let C be an arbitrary finite-dimensional monomial k-algebra and A the dual extension of C.

Question 1. Is ${}_AT(a)$ always algebraically compact?

Question 2. Which properties may the algebra $\mathrm{End}_A(\oplus_{a \in Q_0} T(a))$ have?

Question 3. Suppose C is a directed algebra. Can we describe the relations for the Ringel dual of $\mathscr{A}(C)$?

Acknowlegements. The author would like to thank C. M. Ringel for the hospitality during his stay in Bielefeld. He also wants to thank S. König for useful conversations and suggestions. The author was partially supported by the National Natural Science Foundation of China (Grant No. 19831070).

The author would like to dedicate this paper to Prof. Bo-xun Zhou on his 70th birthday.

References

1. Cline, E., Parshall, B., Scott, L., Finite dimensional algebras and highest weight categories, J. Reine Angew. Math., 1988, 391:85.
2. Green, J. A., Polynomial representations of GL_n, Lecture Notes in Math., Vol. 830, Berlin: Springer-Verlag, 1980.
3. Bernstein, I. N., Gelfand, I. M., Gelfand, S. I., A category of $\mathfrak{g}$-modules, Funct. Anal. Appl., 1976, 10:67.
4. Ringel, C. R., The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences, Math. Zeit., 1991, 208:209.
5. Donkin, S. M., On tilting modules for algebraic groups, Math. Zeit, 1993, 212:39.
6. Deng, B. M., Xi, C. C., Ringel duals of quasi-hereditary algebras, Comm. Algebra, 1996, 24:2825.
7. Irving, R.S., BGG-algebras and the BGG reciprocity principle, J. Algebra, 1990, 135:363.
8. Xi, C. C., Quasi-hereditary algebras with a duality, J. Reine Angew. Math., 1994, 449:201.
9. Ringel, C. M., Tame algebras and integral quadratic forms, Lect. Notes in Math., Vol. 1099, Berlin: Springer-Verlag, 1984.
10. Xi, C. C., Global dimensions of dual extension algebras, Manuscr. Math., 1995, 88:25.
11. Deng, B. M., Xi, C. C., Quasi-hereditary algebras which are dual extensions of algebras, Comm. Algebra, 1994, 22:4717.
12. Eilenberg, S., Cartan, H., Homological algebra, Princeton-New Jersey: Princeton University Press, 1956.
13. Deng, B. M., Xi, C. C., Quasi-hereditary algebras which are twisted double incidence algebras of posets, Contributions to Algebra and Geometry, 1995, 36:37.
14. König, S.; Xi, C. C., Strong symmetry defined by twisting modules, applied to quasi-hereditary algebras with triangular decomposition and vanishing radical cube, Commun. Math. Phys., 1998, 197:427.
15. Happle, D., Triangulated categories in the representation theory of finite groups and finite-dimensional algebras, Cambridge University Press, London Math. Notes Series 119, 1988.
16. König, S., On the global dimensions of quasi-hereditary algebras with triangular decomposition, Proc. Amer. Math. Soc., 1996, 124:1993.
17. Dlab, V.; Ringel, C. M., The module theoretical approach to quasi-hereditary algebras, in Representations of algebras and related topics, LMS Lecture Notes Series 168 (eds. Tachikawa, H., Brenner, S.), London: Cambridge University Press, 1992, 200 - 224.