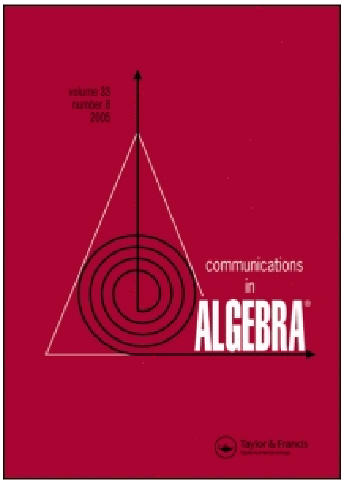

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On wild hereditary algebras with small grwoth numbers

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ON WILD HEREDITARY ALGEBRAS WITH SMALL GROWTH NUMBERS

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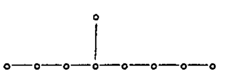
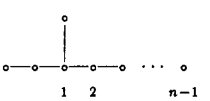
Let A be an artinian algebra. The growth number $\rho(A)$ of the algebra A has recently become important in investigations of the representations of A. For A hereditary it is the maximal real eigenvalue of the Coxeter matrix of A (see **[DR1]** and **[K]**). It measures to some extent the structure of the module category of A (see **[DR1]**,**[K]**). The purpose of this note is to determine all finite–dimensional basic hereditary algebras A over a field k with growth number less than c, where c is the real root of the polynomial $x^3 - x - 1$. In [Z] it was shown that for a regular component $\mathcal{C}$ of the Auslander–Reiten–quiver of A with growth number smaller than c the shape of $\mathcal{C}$ can be determined. We will use the list of all minimal wild hereditary algebras which are finite–dimensional over some field k (compare Appendix) and prove the following

Theorem. Let A be a finite–dimensional connected wild hereditary algebra over a field k. Then $\rho(A) < c$ if and only if the underlying valued graph of A is one of the following:

(1) $\tilde{\tilde{\mathsf{E}}}_7$:



(2) $\mathsf{T}_{2,3,n}$:



, $n \geq 8$

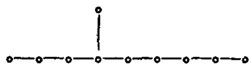
Proof. If the algebra A has the above underlying graph, then we can easily show that the maximal eigenvalue of the corresponding Coxeter–matrix is smaller than c (for the cases $\mathsf{T}_{2,3,n}$ with $n \geq 8$, see **[Z]**).
Now we turn to the converse conclusion. First of all,we need some preparations.

Definition. An algebra A is called minimal wild if A is wild and A/AeA is tame or representation finite for each idempotent $0 \neq e \in A$.

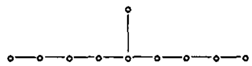
Lemma 1. **(a)** The minimal wild hereditary algebras except $\tilde{\tilde{\mathbf{E}}}_7$ and $\tilde{\tilde{\mathbf{E}}}_8$ have growth number bigger than c.

(b) An algebra with one of the following underlying graphs has growth number greater than c.

(I)

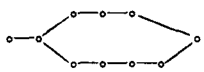


(II)



(c) the algebra with the following underlying graph has the growth number larger than c:

(III)



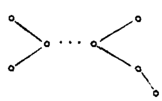
Proof. (a) We check the graphs induced by a minimal wild hereditary algebra in the appendix. According to a well-known theorem in **[BGP]**, for two different orientations of a tree the corresponding Coxeter matrices are similar, so we can choose for each tree a certain orientation and compute the corresponding growth number. For the graphs which contain cycles we should check all possible orientations, but the Bernstein–Gelfand–Ponomarov theorem may reduce the verification to a few cases.

(b) As in (a) we may choose a convenient orientation for both graphs and calculate the largest positive eigenvalue and compare it with c. It turns out that $\rho(I) > c$ and $\rho(II) > c$.

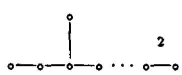
(c) We have to verify the result for five orientations. As in (a) the other orientations can be reduced to one of the five cases.

Lemma 2. The algebra with the following underlying graphs has growth number greater than c:

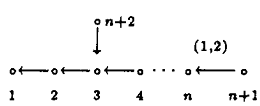
(IV) $\tilde{\tilde{\mathbf{D}}}_n$:



(V)



Proof. If we consider the preprojective components of two algebras A and B with underlying graphs (IV) and (V) respectively, then we can easily see that $\rho(A) = \rho(B)$. Hence it is enough to prove the lemma 2 for (V) in the following case:



We use the notation in **[Z]**, thus

$$P = \begin{bmatrix} 1\,0\,0\,0 \ldots 0\,0\,0 \\ 1\,1\,0\,0 \ldots 0\,0\,0 \\ \vdots\,\vdots\,\vdots \quad \ddots\,\vdots\,\vdots\,\vdots \\ 1\,1\,1\,1 \ldots 1\,0\,0 \\ 1\,1\,1\,1 \ldots 1\,1\,0 \\ 1\,1\,1\,0 \ldots 0\,0\,1 \end{bmatrix},$$

$$I = \begin{bmatrix} 1\,1\,1\,1 \ldots 1\,2\,1 \\ 1\,1\,1\,1 \ldots 1\,2\,1 \\ 0\,0\,1\,1 \ldots 1\,2\,1 \\ 0\,0\,0\,1 \ldots 1\,2\,0 \\ \vdots\,\vdots\,\vdots \quad \ddots\,\vdots\,\vdots\,\vdots \\ 0\,0\,0\,0 \ldots 1\,2\,0 \\ 0\,0\,0\,0 \ldots 0\,1\,0 \\ 0\,0\,0\,0 \ldots 0\,0\,1 \end{bmatrix},$$

and the Coxeter–matrix

$$\Phi_n = -P^{-1}I = \begin{bmatrix} 1 & 0 & 0\,0 \ldots & 0 & 0\,0\,0 \\ -1 & 1 & 0\,0 \ldots & 0 & 0\,0\,0 \\ 0 & -1 & 1\,0 \ldots & 0 & 0\,0\,0 \\ 0 & 0 & -1\,1 \ldots & 0 & 0\,0\,0 \\ \vdots & \vdots & \vdots & \vdots & \vdots\,\vdots\,\vdots \\ 0 & 0 & 0\,0 \ldots & -1 & 1\,0\,0 \\ 0 & 0 & 0\,0 \ldots & 0 & -1\,1\,0 \\ 0 & 0 & -1\,0 \ldots & 0 & 0\,0\,1 \end{bmatrix} \begin{bmatrix} 1\,1\,1\,1 \ldots 1\,1\,2\,1 \\ 0\,1\,1\,1 \ldots 1\,1\,2\,1 \\ 0\,0\,1\,1 \ldots 1\,1\,2\,1 \\ 0\,0\,0\,1 \ldots 1\,1\,2\,0 \\ \vdots\,\vdots\,\vdots \qquad \vdots\,\vdots\,\vdots\,\vdots \\ 0\,0\,0\,0 \ldots 0\,1\,2\,0 \\ 0\,0\,0\,0 \ldots 0\,0\,1\,0 \\ 0\,0\,0\,0 \ldots 0\,0\,0\,1 \end{bmatrix}$$

$$= -\begin{bmatrix} 1 & 1 & 1 & 1 \ldots & 1 & 1 & 2 & 1 \\ -1 & 0 & 0 & 0 \ldots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \ldots & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \ldots & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \ldots & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \ldots & -1 & -1 & -2 & 0 \end{bmatrix}_{(n+2)\times(n+2)}$$

The characteristic polynomial $E_n(\lambda) = det(\Phi_n + \lambda I_n) = \lambda^{n+2} - \lambda^n - \lambda^{n-1} - \lambda^3 - \lambda^2 + 1$. Since $E_n(c) = c^{n+2} - c^n - c^{n-1} - c^3 - c^2 + 1 = c^{n-1}(c^3 - c - 1) - c^3 - c^2 + 1 = -c^3 - c^2 + 1 < 0$, we infer that the maximal real root ρ_n of $E_n(\lambda)$ must be bigger than c. Thus the lemma is proved.

Remark. We can easily show that $\rho_n > \rho_{n+1}$ and $\lim\limits_{n\to\infty} \rho_n = c$.

Lemma 3 [AS]. Suppose a module class $\mathcal{C}$ in A–mod is closed under factor modules and extensions. Let C be an indecomposable in $\mathcal{C}$ which is not Ext–projective and let $0 \to \tau C \to B \to C \to 0$ be an AR–sequence in A–mod. Then there is a commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & C' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & t(\tau C) \arrow[r] \arrow[d] & t(B) \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & A \arrow[r, equal] \arrow[d] & A & & \\
 & 0 & & &
\end{tikzcd}$$

and $0 \to C' \to B' \to C \to 0$ is an AR-sequence in $\mathcal{C}$, where $t(X)$ denotes the maximal submodule of X which is in $\mathcal{C}$.
For the proof of this lemma see **[AS, 3.8]**.

Lemma 4. Let A be a finite-dimensional connected wild hereditary algebra with valued quiver $\vec{Q}$. Let $\vec{Q}'$ be a connected full subquiver of $\vec{Q}$ such that $B := k\vec{Q}'$ is a wild algebra. Then $\rho(B) \leq \rho(A)$.

Proof. Put $\mathcal{C} = B$-mod. Then $\mathcal{C}$ is closed under factor modules and extensions. We take an indecomposable regular B-module X, it is clear that X is not Ext-projective. Using lemma 3 we have $\tau_B X \subseteq \tau_A X$, and therefore $\rho(B) := \lim\limits_{n\to\infty} \sqrt[n]{||\tau_B^n X||} \leq \lim\limits_{n\to\infty} \sqrt[n]{||\tau_A^n X||} = \rho(A)$.

Lemma 5. Every finite-dimensional wild connected hereditary algebra contains a minimal wild hereditary algebra whose ordinary quiver is a full subquiver of that of the given algebra.

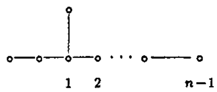
The proof is obvious.

Lemma 6. Let A be a finite-dimensional wild connected algebra with $\rho(A) < c$ and $A = k\vec{Q}$. Suppose there is a full connected subquiver $\vec{Q}'$ of $\vec{Q}$ with trivial values such that the underlying graph Q' is a tree. If $Q_0 = Q' \cup \{x\}$, then there is only one edge α joining x to Q' in Q and the value of this edge is $a(\alpha) \leq 4$.

Proof. Since $Q \neq Q'$, there is a vertex $x \in Q_0 \backslash Q'_0$ such that x is connected to Q' by r edges $\alpha_1, \cdots, \alpha_r$. We will show that $r = 1$.
Suppose $r \geq 2$. Then we can choose a cycle K which is a full subgraph of Q. By lemma 1(a) and $\rho(A) < c$, this cycle K must have trivial valuation. Since A is a wild connected algebra, there is a vertex $y \in Q_0$ which does not lie in K and is such that y is connected to K by n edges. The full subgraph generated by $K \cup \{y\}$ is wild and must contain a minimal wild valued graph. If $n \geq 2$ this must be (5), (6) or (7) in the appendix and these are ruled out by lemma 1(a), and therefore $n = 1$. Then we arrive at a graph which is one of (8) to (13) in the appendix. Again by lemma 1 we obtain $\rho(A) > c$. This contradiction shows that $r = 1$ and finishes the proof of lemma 6.

Proof of the theorem. Let A be a basic finite-dimensional wild connected hereditary algebra with $\rho(A) < c$. By lemma 5 it contains a minimal wild hereditary algebra B whose ordinary quiver is a full subquiver of that of A. From lemma 4 one gets $\rho(B) \leq \rho(A)$. Lemma 1(a) shows that any minimal wild hereditary algebra contained in A must be of type $\tilde{\tilde{\mathsf{E}}}_7$ or $\tilde{\tilde{\mathsf{E}}}_8$. If the algebra B is of the form $\tilde{\tilde{\mathsf{E}}}_7$, then by lemma 6 and lemma 1 the algebra A itself is of type $\tilde{\tilde{E}}_7$. Now we may assume that the algebra B is of type $\tilde{\tilde{E}}_8$. We apply lemma 6 and lemma 2 by induction on the number of vertices to the algebra A. Thus we have that the algebra is of the form

$T_{2,3,n}$:



with $n \geq 8$,

and this has completed the proof of the theorem.

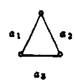
Remark. If we take a finite-dimensional wild hereditary algebra A with $\rho(A) < c$ and a preinjective indecomposable A-module M then the algebra $B = A[M] = \begin{bmatrix} A & M \\ 0 & k \end{bmatrix}$ (one-point extension) is trivially an algebra with $\rho(B) < c$, but not hereditary (see [**R2**]).

Appendix

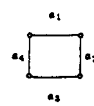
The minimal wild hereditary algebras which are finite-dimensional over a field k

Theorem. *Let k be an arbitrary field. The finite-dimensional minimal wild hereditary algebras are just the algebras with following valued graphs (see* [**DR2**] *for definitions):*

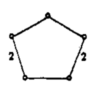
I. Cycle

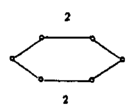
(1) 

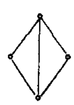
$a_i \leq 4, \sum a_i \geq 4$

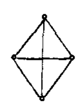
(2) 

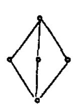
$a_i \leq 3, a_i + a_{i+1} \leq 4, \sum a_i \geq 5$

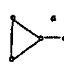
(3) 

(4) 

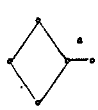
(5) 

(6) 

(7) 

(8) 

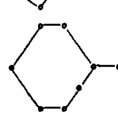
$a \leq 3$

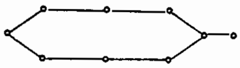
(9) 

$a \leq 2$

(10) 

(11) 

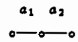
(12) 

(13) 

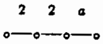
II. Tree

(1) 

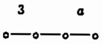
$a \geq 5$

(2) 

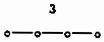
$a_i \leq 4, \sum a_i \geq 5$

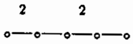
(3a) 

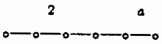
$a \leq 2$

(3b) 

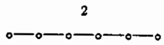
$a \leq 3$

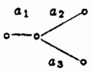
(3c) 

(4) 

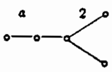
(5a) 

$a \leq 2$

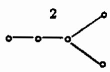
(5b) 

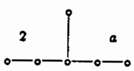
(6) 

$a_i \leq 3, a_i + a_{i+1} \leq 4, \sum a_i \geq 5$

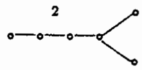
(7a) 

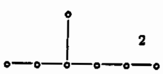
$a \leq 2$

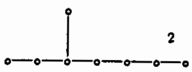
(7b) 

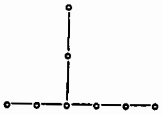
(8a) 

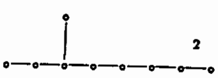
$a \leq 2$

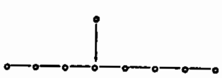
(8b) 

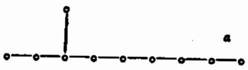
(9) 

(10a) 

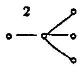
(10b) $\tilde{\tilde{E}}_6$ 

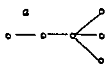
(11a) 

(11b) 

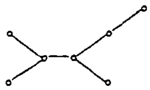
(12) 

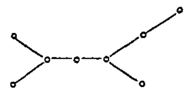
$a \leq 2$

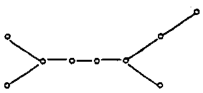
(13) 

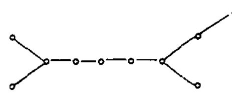
(14) 

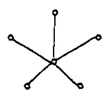
$a \leq 2$

(15) 

(16) 

(17) 

(18) 

(19) 

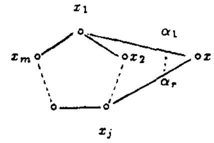
Remark. As usual we do not allow any orientated cycles. To a valued edge $\underset{i}{\circ}\overset{(d_{ij},d'_{ij})}{\text{——}}\underset{j}{\circ}$ we draw simply $\underset{i}{\circ}\overset{a}{\text{——}}\underset{j}{\circ}$ with $a = d_{ij}d'_{ij}$, since we always suppose the valued quiver is symmetrizable.

Sketch of the proof.

Note that if A is a finite dimensional minimal connected wild hereditary algebra then the underlying graph Q of A is either a tree or it contains a cycle. As before the value of an edge α is denoted by $a(\alpha)$. Let $d = max\{a(\alpha) \mid \alpha \in Q_1\}$.
First, suppose the underlying graph Q of A is a tree. If the maximal value $d \geq 5$, then $Q = \circ\overset{d}{\text{——}}\circ$ with $d \geq 5$. So we have to discuss the case $d \leq 4$. For $d \in \{3, 4\}$ we obtain easily the graphs (3b), (3c), some graphs in (2) and a graph in (6) with $a_1 = 3$. In case $d = 1$ we can borrow the list from **[K]** or **[U]**.

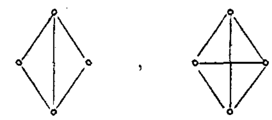
Now assume $d = 2$. We take an edge α with maximal value d and consider the degree of the ends of α case by case. Then we obtain (if necessary, we may repeate the consideration) all the other graphs which are not included in the above known cases by using the results in **[DR2]**.
Second, assume that there is a cycle K with m vertices ($m \geq 3$) in the graph Q. We choose a cycle such that m is minimal.

(i) The cycle K is a trivial valued graph.
Since the algebra is wild, there is at least one vertex x outside of the cycle K, which is connected to K by r edges:

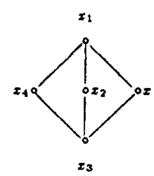


It is clear that $a(\alpha_i) = 1$ for $i = 1, \cdots, r$, because if some $a(\alpha_j) \neq 1$, then we have a cycle K' which has non-trivial valuation, and therefore $Q = K'$. This contradicts the choice of K.

In case $r \geq 2$, we may choose two vertices x_1 and x_j of the cycle K such that x_1 and x_j are connected to x by α_1 and α_j respectively and there is no vertex $y \in \{x_1, \cdots, x_j\} \backslash \{x_1, x_j\}$ which is connected to x. Since our algebra is minimal wild, we must have $2 \leq j \leq 3$. If $j = 2$, then $m = 3$. In this case we obtain the following diagrams:



Now let $j = 3$. Then $m = 4$ and we get the following diagram:

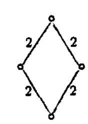


Finally, we consider the case $r = 1$. Thus we get the graphs (8) to (12).

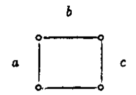
(ii) If the valued graph K has non-trivial valuation, then $Q = K$. So we suppose the valued quiver of the minimal connected wild hereditary algebra is a cycle with non-trivial values and has m vertices. It is easy to see that $m \leq 6$.

We denote the number of edges α with $a(\alpha) > 1$ by s. From the minimality we know that $s \leq 4$. Of course, $s \leq m$ holds.

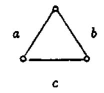
1. $s = 4$. Suppose that $m = 5$ or 6. Thus there is always an edge α with $a(\alpha) = 1$. If we delete one of the points belonging to α (see **[R2]**), then we get a wild algebra, which is a contradiction. Therefore we must have $m = 4$. In this case we have an algebra of the type



2. $s = 3$. Suppose $m = 5$ or 6. If there are two edges α_1 and α_2 with $a(\alpha_i) \neq 1$ such that they have a common vertex, then we obtain a wild algebra by deleting one vertex which does not lie in α_1 or α_2. By the minimality this is impossible. Therefore we have $m \leq 4$.
   In the case $m = 4$ we get easily the following diagram:

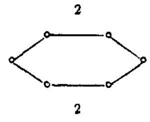


with $1 < a, b, c \leq 3$ and $1 \leq ab, bc \leq 4$. In the case $m = 3$ we obtain the following diagram:

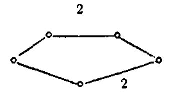


with $1 < a, b, c \leq 4$.

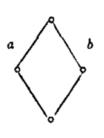
3. $s = 2$. Suppose $m = 5, 6$. Thus we know easily that $a(\alpha_i) \leq 2$ for all α_i. In the case $m = 6$ we infer that the algebra must be of type:



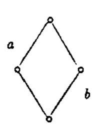
In the case $m = 5$ we have a diagram as follows:



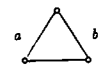
Let $m = 4$. In this case we have $a(\alpha) \leq 3$ for all edges α. If the two non-trivial valued edges have a common vertex, then we arrive at the following diagram:



with $1 < a, b \leq 3$ and $1 < ab \leq 4$. If not, we obtain a diagram as follows:

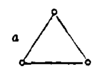


with $1 < a, b \leq 3$.
Now let $m = 3$, thus we get a diagram of the following type:



with $1 < a, b \leq 4$.

4. $s = 1$ We obtain, in this case, the following diagram:



with $2 \leq a \leq 4$.

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