



# Finitistic dimension conjecture and radical-power extensions



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## ABSTRACT

The finitistic dimension conjecture asserts that any finite-dimensional algebra over a field has finite finitistic dimension. Recently, this conjecture is reduced to studying finitistic dimensions for extensions of algebras. In this paper, we investigate those extensions of Artin algebras in which some radical-power of smaller algebras is a nonzero one-sided ideal of bigger algebras. Our result can be formulated for an arbitrary ideal as follows: Let  $B \subseteq A$  be an extension of Artin algebras and  $I$  an ideal of  $B$  such that the full subcategory of  $B/I$ -modules is  $B$ -syzygy-finite. (1) If the extension is right-bounded (for example,  $\text{Gpd}(A_B) < \infty$ ),  $I A \text{rad}(B) \subseteq B$  and  $\text{findim}(A) < \infty$ , then  $\text{findim}(B) < \infty$ . (2) If  $I \text{rad}(B)$  is a left ideal of  $A$  and  $A$  is torsionless-finite, then  $\text{findim}(B) < \infty$ . Particularly, if  $I$  is specified to a power of the radical of  $B$ , then our result not only generalizes some of results in the literature (see Corollary 1.2), but also provides new ways to detect algebras of finite finitistic dimensions.

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## 1. Introduction

Let  $A$  be an Artin algebra. The finitistic dimension of  $A$  is defined to be the supremum of the projective dimensions of finitely generated left  $A$ -modules having finite projective dimension. The famous finitistic dimension conjecture says that any Artin algebra has finite finitistic dimension (see [3, Conjecture (11), p. 410]). This conjecture was initially a question by Rosenberg and Zelinsky, published by Bass in a paper in 1960 (see [4]), and has attracted many mathematicians in the last 5.5 decades. Among them is Maurice Auslander who “is considered to be one of the founders of the modern aspects of the representation theory of

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Artin algebras” (see [14, p. 501]). “One of his main interests in the theory of Artin algebras was the finitistic dimension conjecture and related homological conjectures” (see [14, p. 815]). The conjecture has intimate connections with the solutions of several other open conjectures such as Nakayama conjecture, generalized Nakayama conjecture and Gorenstein symmetry conjecture (see [21] for a survey on these conjectures). Though several special cases for the conjecture to be true have been verified (see, for example, [8,9,11,12,22,23,5]), it is still open in general. Actually, up to the present time, not many practical methods, so far as we know, are available to detect algebras of finite finitistic dimension. It seems necessary to develop some methods for testing finiteness of finitistic dimensions for general algebras or even for some concrete examples.

In the paper [17, Corollary 4.8], the conjecture has been reduced to comparing finitistic dimensions of a pair of algebras instead of focusing only on one single algebra. More precisely, the following two statements are proved to be equivalent for a field  $k$ :

- (1) The finitistic dimension of any finite-dimensional  $k$ -algebra is finite.
- (2) For any extension  $B \subseteq A$  of finite-dimensional  $k$ -algebras such that  $\text{rad}(B)$ , the Jacobson radical of  $B$ , is a left ideal in  $A$ , if  $A$  has finite finitistic dimension, then  $B$  has finite finitistic dimension.

Along this line, the conjecture is further reduced, by a different method, to extensions of algebras with relative global dimension 1, where the ground field is assumed to be perfect (see [20, Proposition 2.9]). Thus it seems quite worthy to consider such kinds of extensions of algebras and to bound the finitistic dimensions of smaller algebras in terms of the ones of bigger algebras which we would like to take as simple as possible.

Let us just mention a few considerations in this direction. In the sequel, we denote by  $\text{gldim}(A)$  and  $\text{findim}(A)$  the global and finitistic dimensions of an algebra  $A$ , respectively; and by  $\text{pd}(A_B)$  the projective dimension of the right  $B$ -module  $A$ .

- (i) Let  $B \subseteq A$  be an extension of Artin algebras such that  $\text{rad}(B)$  is a left ideal in  $A$ . Then:
  - (a) If  $\text{rad}(A) = \text{rad}(B)A$  and  $\text{gldim}(A) \leq 4$ , then  $\text{findim}(B) < \infty$  (see [18, Theorem 3.7]).
  - (b) If  $\text{pd}(A_B) < \infty$  and  $\text{findim}(A) < \infty$ , then  $\text{findim}(B) < \infty$  (see [20, Corollary 1.4]).
- (ii) Let  $C \subseteq B \subseteq A$  be a chain of Artin algebras such that  $\text{rad}(C)$  and  $\text{rad}(B)$  are left ideals in  $B$  and  $A$ , respectively. If  $A$  is representation-finite, then  $\text{findim}(C) < \infty$  (see [17, Theorem 4.5]).

Thus the philosophy of controlling finitistic dimensions by extension algebras works powerfully (see also the example at the end of Section 3).

In this paper, we continue to explore and compare finitistic dimensions of extensions  $B \subseteq A$  of Artin algebras. Since it happens quite often that  $\text{rad}(B)$  itself may not be a left ideal in  $A$  but some of its powers are actually nonzero left or right ideals in  $A$ , our main goal in this paper is to extend results for the case that  $\text{rad}(B)$  is a left ideal in  $A$  to a more general case that  $\text{rad}^s(B)$  is a left (or right) ideal in  $A$  for some positive integer  $s$ . Such kinds of extensions may be called *radical-power* extensions.

We shall carry out our discussion for extensions  $B \subseteq A$  in a broader context by studying an arbitrary ideal of  $B$  rather than a power of the radical of  $B$ . The technical problem we encounter, even in the case of a higher power of the radical, is that the higher syzygies of  $B$ -modules admit no longer  $A$ -module structures. So a crucial ingredient for bounding projective dimensions used in [17,18,20] is missing. To circumvent this problem here, we first use certain submodules of torsionless  $B$ -modules to get  $A$ -module structures, and then establish certain short exact sequences connecting  $B$ -syzygies with the lifted  $A$ -modules. Finally, we employ the Igusa–Todorov function in [12] to estimate upper bounds of projective dimensions.

To state our main result more precisely, let us first introduce some definitions.

Let  $A$  be an Artin algebra and  $M_A$  be a right  $A$ -module. For an  $A$ -module  $X$ , we define  $t_M(X) := \inf\{n \in \mathbb{N} \mid \text{Tor}_j^A(M, X) = 0 \text{ for all } j \geq n + 1\}$ , and  $g_M(A) := \sup\{t_M(X) \mid X \in A\text{-mod with } \text{pd}({}_A X) < \infty\}$ . We call  $g_M(A)$  the *Gorenstein index* of  $A$  relative to  $M$ . In case of an extension  $B \subseteq A$  of Artin algebras, we call  $g_{A_B}(B)$  the Gorenstein index of the extension.

An extension  $B \subseteq A$  of Artin algebras is said to be *right Gorenstein-finite* if the Gorenstein-projective dimension of the right  $B$ -module  $A_B$  is finite. Similarly, left Gorenstein-finite extensions are defined. Further, we introduce a generalization of right Gorenstein-finite extensions: An extension  $B \subseteq A$  of Artin algebras is said to be *right-bounded* if the Gorenstein index of the extension is finite. Note that  $g_M(A) < \infty$  occurs in the following cases:

(a) Following [13], a non-negative integer  $n$  is called a *bound* on the vanishing of  $\text{Tor}^A(M_A, -)$  if, for any  $A$ -module  $Y$ , whenever  $\text{Tor}_p^A(M_A, Y) = 0$  for  $p$  sufficiently large then  $\text{Tor}_p^A(M_A, Y) = 0$  for all  $p \geq n + 1$ . In particular, if  $n$  equals  $\text{pd}(M_A) < \infty$ , or the  $n$ -th syzygy of a projective resolution of  $M_A$  is periodic, then  $n$  is a bound on the vanishing of  $\text{Tor}^A(M_A, -)$ . Thus, if  $n$  is a bound on the vanishing of  $\text{Tor}^A(M_A, -)$ , then  $g_M(A) = n$ .

(b) If the Gorenstein-projective dimension of  $M_A$  is finite, then  $g_M(A)$  is just the Gorenstein-projective dimension of  $M_A$  (see [19, Lemma 4.1]). So, right Gorenstein-finite extensions are right-bounded.

For  $m \geq 0$  and a full subcategory  $\mathcal{C}$  of  $A\text{-mod}$ , the category of all finitely generated left  $A$ -modules, we denote by  $\Omega^m(\mathcal{C})$  the full subcategory of  $A\text{-mod}$  consisting of those  $A$ -modules that are either projective or direct summands of  $m$ -th syzygies of  $A$ -modules in  $\mathcal{C}$ . We say that  $\mathcal{C}$  is  *$m$ -syzygy-finite* if there are only finitely many non-isomorphic indecomposable modules in  $\Omega^m(\mathcal{C})$ , and *syzygy-finite* (or more precisely,  *$A$ -syzygy-finite*) if there is some  $m \geq 0$  such that  $\mathcal{C}$  is  $m$ -syzygy-finite. Particularly, the algebra  $A$  is *representation-finite* (respectively, *torsionless-finite*) if  $A\text{-mod}$  is 0-syzygy-finite (respectively, 1-syzygy-finite). For further information on torsionless-finite algebras, we refer the reader to [15].

Now, our main result can be stated as follows.

**Theorem 1.1.** *Let  $B \subseteq A$  be an extension of Artin algebras and  $I$  an ideal of  $B$  such that the full subcategory of  $B/I$ -modules is  $B$ -syzygy-finite. Then we have the following:*

- (1) *If the extension is right-bounded,  $I A \text{rad}(B) \subseteq B$  and  $\text{findim}(A) < \infty$ , then  $\text{findim}(B) < \infty$ .*
- (2) *If  $I \text{rad}(B)$  is a left ideal of  $A$  and  $A$  is torsionless-finite, then  $\text{findim}(B) < \infty$ .*

If  $I = B$  in Theorem 1.1(1), then we recover [17, Theorem 3.1]. Now, specializing  $I$  to some power of the radical of  $B$ , we then get the following corollary.

**Corollary 1.2.** (1) *Let  $B \subseteq A$  be a right Gorenstein-finite extension of Artin algebras. Suppose that there is an integer  $s \geq 0$  such that  $\text{rad}^s(B)A \text{rad}(B) \subseteq B$  and  $B/\text{rad}^s(B)$  is representation-finite. If  $\text{findim}(A) < \infty$ , then  $\text{findim}(B) < \infty$ .*

(2) *Suppose that  $B \subseteq A$  is an extension of Artin algebras and there is an integer  $s \geq 1$  such that  $\text{rad}^s(B)$  is a left ideal of  $A$  and  $B/\text{rad}^{s-1}(B)$  is representation-finite. If  $A$  is torsionless-finite, then  $\text{findim}(B) < \infty$ .*

Here, we understand  $\text{rad}^0(B) = B$ . In case  $s = 0$  and  $\text{rad}(B)$  is a left ideal in  $A$ , Corollary 1.2(1) coincides with [20, Corollary 1.4] for finiteness of finitistic dimensions. But in the case  $s \geq 1$ , Corollary 1.2(1) seems to be new, compared with the results in [6,13,16–18,20]. Also, remark that Corollary 1.2(2) covers [17, Proposition 4.9] if we take  $s = 1$ . In case  $s = 2$ , the algebra  $B/\text{rad}^{s-1}(B)$  is automatically representation-finite, and therefore Corollary 1.2(2) takes a simple form. Since there are plenty of extensions  $B \subseteq A$  such that  $\text{rad}(B)$  itself is neither a left nor a right ideal in  $A$  but some of its powers is a nonzero left or right ideal in  $A$  (see the examples at the end of Section 3), our corollary generalizes properly some of results in the literature (see [17,18,20]).

Note that the argument in the proof of our main result is different from the earlier ones in [17,18,20], though one idea in common is to use the Igusa–Todorov function. Note also that, compared with the results in [16], our results do not impose any homological conditions (such as finiteness of projective dimension) on ideals or powers of the radical of  $B$  since such conditions seem to be strong for our homological questions. Moreover, our results in this paper recover a result of Green–Zimmermann-Huisgen (see Corollary 3.9) and

provide somewhat handy methods to detect algebras of finite finitistic dimensions, especially those obtained as subalgebras from representation-finite algebras.

The contents of this note are arranged as follows: In Section 2, we first fix some terminology and notation, and then recall some basic facts needed in later proofs. In Section 3, we give proofs of the above-mentioned theorem and corollary. As a byproduct of our results, we re-obtain a main result in [8] (see Corollary 3.9). This section ends with examples to explain how our results can be used. It seems that, for the last example, no previously known results could be applied to detect the finiteness of its finitistic dimension.

## 2. Preliminaries

In this section, we shall briefly recall some basic definitions, fix notation, and collect some known results that are needed in later proofs.

Let  $A$  be an Artin  $R$ -algebra, that is,  $R$  is a commutative Artin ring with identity and  $A$  is a finitely generated unitary  $R$ -module. We denote by  $\text{rad}(A)$  the Jacobson radical of  $A$ . By a module we always mean a finitely generated left module. Given an  $A$ -module  $M$ , we denote by  $\text{rad}_A(M)$  and  $\Omega_A(M)$  the Jacobson radical and the first syzygy of  $M$ , respectively, and by  $\text{pd}({}_A M)$  the projective dimension of  $M$ . Recall that the global dimension of  $A$ , denoted by  $\text{gldim}(A)$ , is defined to be the supremum of projective dimensions of modules in  $A\text{-mod}$ , and the finitistic dimension of  $A$ , denoted by  $\text{findim}(A)$ , is the supremum of projective dimensions of all those  $A$ -modules  $X$  in  $A\text{-mod}$  with  $\text{pd}({}_A X) < \infty$ . So  $\text{findim}(A) \leq \text{gldim}(A)$ .

Recall that an  $A$ -module  $M$  is said to be *Gorenstein-projective* if  $\text{Ext}_A^i(M, {}_A A) = 0 = \text{Ext}_A^i(\text{Tr}(M), {}_A A)$  for all  $i \geq 1$ , where  $\text{Tr}(M)$  is the transpose of  $M$ . We say that  $M$  has *Gorenstein-projective dimension*  $n$  if there is an exact sequence  $0 \rightarrow X_n \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$  such that all  $X_i$  are Gorenstein-projective and that  $n$  is minimal among the lengths of such exact sequences. In this case, we write  $\text{Gpd}({}_A M) = n$ . For further information on Gorenstein homological dimensions, we refer to [2,7,10].

For a homomorphism  $f : X \rightarrow Y$  in  $A\text{-mod}$ , we denote by  $\text{Ker}(f)$  and  $\text{Im}(f)$  the kernel and image of  $f$ , respectively. If  $g : Y \rightarrow Z$  is another homomorphism in  $A\text{-mod}$ , we write  $fg : X \rightarrow Z$  for the composite of  $f$  with  $g$ . In this way,  $\text{Hom}_A(X, Y)$  becomes naturally a left  $\text{End}_A(X)$ - and right  $\text{End}_A(Y)$ -bimodule.

Given an additive full subcategory  $\mathcal{X}$  of  $A\text{-mod}$  and a module  $M$  in  $\mathcal{X}$ , we say that  $M$  is an *additive generator* for  $\mathcal{X}$  if  $\mathcal{X} = \text{add}(M)$ , where  $\text{add}(M)$  is the additive full category of  $A\text{-mod}$  generated by  $M$ .

By an extension  $B \subseteq A$  of algebras we mean a pair of Artin algebras  $A$  and  $B$  such that  $B$  is a subalgebra of  $A$  with the same identity.

To estimate projective dimensions of modules, Igusa–Todorov introduced a useful function  $\Psi$  (see [12]) which is defined as follows: Let  $K(A)$  be the quotient of the free abelian group generated by the isomorphism classes  $[M]$  of modules  $M$  in  $A\text{-mod}$  modulo the relations:

- (i)  $[Y] = [X] + [Z]$  if  $Y \simeq X \oplus Z$ , and
- (ii)  $[P] = 0$  if  $P$  is a projective  $A$ -module.

Thus  $K(A)$  is a free abelian group with a basis consisting of isomorphism classes of non-projective indecomposable  $A$ -modules. By employing the noetherian property of the ring of integers, Igusa and Todorov define a function  $\Psi$  on this abelian group, which takes values of non-negative integers and has the following fundamental properties.

**Lemma 2.1.** [12] *Let  $X$  and  $Y$  be  $A$ -modules.*

- (1) *If  $\text{pd}({}_A X) < \infty$ , then  $\Psi(X) = \text{pd}({}_A X)$ .*
- (2)  *$\Psi(X) \leq \Psi(X \oplus Y)$  and  $\Psi(\bigoplus_{j=1}^n X) = \Psi(X)$  for any natural number  $n \geq 1$ .*
- (3) *If  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  is an exact sequence in  $A\text{-mod}$  with  $\text{pd}({}_A Z) < \infty$ , then  $\Psi(Z) \leq \Psi(X \oplus Y) + 1$ .*

Now, we recall the following well-known homological facts.

**Lemma 2.2.** (1) *If there is an exact sequence  $0 \rightarrow X_s \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow X \rightarrow 0$  in  $A\text{-mod}$ , then  $\text{pd}({}_A X) \leq s + \max\{\text{pd}({}_A X_i) \mid 0 \leq i \leq s\}$ .*

(2) *Each exact sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  of  $A$ -modules induces another two exact sequences*

$$0 \rightarrow \Omega_A(Z) \rightarrow X \oplus P \rightarrow Y \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \Omega_A(Y) \rightarrow \Omega_A(Z) \oplus P' \rightarrow X \rightarrow 0$$

*with  $P$  and  $P'$  projective  $A$ -modules. They are the so-called syzygy shifting.*

Next, we recall a result of Auslander (see [1, Chapt. III, Sec. 2]), which describes the global dimensions of the endomorphism algebras of modules.

**Lemma 2.3.** *Let  $n \geq 2$  be an integer. Suppose that  $A$  is an algebra and  $M$  is an  $A$ -module. Then  $\text{gldim}(\text{End}({}_A M)) \leq n$  if and only if, for each  $A$ -module  $X$ , there is an exact sequence*

$$0 \rightarrow M_{n-2} \rightarrow \dots \rightarrow M_1 \rightarrow M_0 \rightarrow X \rightarrow 0$$

*in  $A\text{-mod}$  such that all  $M_j \in \text{add}(M)$  and  $\text{Hom}_A(M, -)$  preserves the exactness of this sequence.*

We also need the following result from [18].

**Lemma 2.4.** *Let  $B \subseteq A$  be an extension of algebras such that  $\text{rad}(B)$  is a left ideal of  $A$ . Then, for any  $B$ -module  $X$  and integer  $i \geq 2$ , the module  $\Omega_B^i(X)$  has an  $A$ -module structure and there is a projective  $A$ -module  $P$  and an  $A$ -module  $Y$  such that  $\Omega_B^i(X) \simeq \Omega_A(Y) \oplus P$  as  $A$ -modules.*

### 3. Proof of Theorem 1.1

To prove Theorem 1.1, we first have to prove the following lemmas.

**Lemma 3.1.** *Let  $A$  and  $B$  be Artin algebras and  ${}_A M_B$  be an  $A$ - $B$ -bimodule. Suppose that  ${}_A M$  is projective and  $n := g_M(B)$  is finite. Then, for any  $B$ -module  $X$  with  $\text{pd}({}_B X) < \infty$  and for any integer  $m \geq 1$ , we have an isomorphism of  $A$ -modules*

$$\Omega_A(M \otimes_B \Omega_B^{n+m}(X)) \simeq \Omega_A^m(M \otimes_B \Omega_B^{n+1}(X)).$$

**Proof.** We consider the projective cover  $f : P \rightarrow \Omega_B^{n+m-1}(X)$  of  $\Omega_B^{n+m-1}(X)$  and the canonical exact sequence

$$0 \rightarrow \Omega_B^{n+m}(X) \rightarrow P \rightarrow \Omega_B^{n+m-1}(X) \rightarrow 0.$$

Since  $n$  is the Gorenstein index of  $B$  with respect to  $M_B$  and  $n + m > n$ , we have  $\text{Tor}_1^B(M, \Omega_B^{n+m-1}(X)) \simeq \text{Tor}_{n+m}^B(M, X) = 0$ . Thus the following sequence of  $A$ -modules is exact:

$$0 \rightarrow M \otimes_B \Omega_B^{n+m}(X) \rightarrow M \otimes_B P \rightarrow M \otimes_B \Omega_B^{n+m-1}(X) \rightarrow 0 .$$

By assumption, the module  ${}_A M$  is projective. This yields that the  $A$ -module  $M \otimes_B P$  is also projective. Thus  $M \otimes_B \Omega_B^{n+m}(X) \simeq \Omega_A(M \otimes_B \Omega_B^{n+m-1}(X)) \oplus Q$  with  $Q$  a projective  $A$ -module. It follows that  $\Omega_A(M \otimes_B \Omega_B^{n+m}(X)) \simeq \Omega_A^2(M \otimes_B \Omega_B^{n+m-1}(X))$  for all  $m \geq 1$ . Consequently,  $\Omega_A(M \otimes_B \Omega_B^{n+m}(X)) \simeq \Omega_A^m(M \otimes_B \Omega_B^{n+1}(X))$  by repeating the foregoing isomorphism.  $\square$

The next lemma is a consequence of the derived functors of tensor functors.

**Lemma 3.2.** *Let  $B \subseteq A$  be an extension of Artin algebras and  ${}_B X \in B\text{-mod}$ . Then, for any positive integer  $i$ , the following sequence of  $B$ -modules is exact:*

$$0 \longrightarrow \Omega_B^i(X) \longrightarrow A \otimes_B \Omega_B^i(X) \longrightarrow (A/B) \otimes_B \Omega_B^i(X) \longrightarrow 0 .$$

**Proof.** Tensoring the exact sequence  $0 \rightarrow B \rightarrow A \rightarrow A/B \rightarrow 0$  by  $\Omega_B^i(X)$ , we get the following exact sequence of  $B$ -modules:

$$\begin{aligned} 0 \longrightarrow \text{Tor}_1^B(A, \Omega_B^i(X)) \longrightarrow \text{Tor}_1^B(A/B, \Omega_B^i(X)) \longrightarrow \Omega_B^i(X) \longrightarrow A \otimes_B \Omega_B^i(X) \\ \longrightarrow (A/B) \otimes_B \Omega_B^i(X) \longrightarrow 0. \end{aligned}$$

Evidently,  $\text{Tor}_1^B(A/B, \Omega_B^j(X)) \simeq \text{Tor}_{j+1}^B(A/B, X) \simeq \text{Tor}_{j+1}^B(A, X) \simeq \text{Tor}_1^B(A, \Omega_B^j(X))$  for  $j \geq 1$ . Since we are dealing with Artin algebras and their finitely generated modules, the injective map  $\text{Tor}_1^B(A, \Omega_B^i(X)) \rightarrow \text{Tor}_1^B(A/B, \Omega_B^i(X))$  is bijective for  $i \geq 1$ . Consequently, we get the desired exact sequence.  $\square$

The following useful lemma supplies a non-trivial way to lift  $B$ -modules to  $A$ -modules. This observation extends [17, Lemma 0.1] and is probably not previously known in the literature.

**Lemma 3.3.** *Let  $B \subseteq A$  be an extension of Artin algebras and  $I$  be an ideal of  $B$  such that  $I$  is also a left ideal of  $A$ . Then, for any torsionless  $B$ -module  $X$ , its submodule  $IX$  admits an  $A$ -module structure and is actually a torsionless  $A$ -module.*

**Proof.** Let  $X \hookrightarrow P$  be an inclusion with  $P$  a projective  $B$ -module. Then we get an exact sequence  $0 \rightarrow X \hookrightarrow P \rightarrow Y \rightarrow 0$  with  $Y$  the cokernel of the inclusion, and can form the following exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Tor}_1^B(I, Y) & \xrightarrow{\psi} & I \otimes_B X & \xrightarrow{\eta} & I \otimes_B P & \longrightarrow & I \otimes_B Y & \longrightarrow & 0 \\ & & & & & & \downarrow \alpha & & \downarrow \beta & & \\ & & & & & & IP & \longrightarrow & IY & \longrightarrow & 0, \end{array}$$

where  $\alpha$  and  $\beta$  are the multiplication maps. Clearly,  $\alpha$  is an isomorphism since  $P$  is a projective  $B$ -module. It is easy to see  $\text{Im}(\eta) \simeq \text{Im}(\eta\alpha) = IX$ . Thus we have an exact sequence of  $B$ -modules:

$$0 \longrightarrow \text{Tor}_1^B(I, Y) \xrightarrow{\psi} I \otimes_B X \xrightarrow{\eta\alpha} IX \longrightarrow 0.$$

Since  $I$  is a left ideal of  $A$ ,  $\text{Tor}_1^B(I, Y)$  and  $I \otimes_B X$  are  $A$ -modules and the injective homomorphism  $\psi$  is, in fact, a homomorphism of  $A$ -modules. So the  $B$ -module  $IX$ , as the cokernel of  $\psi$ , is endowed with an  $A$ -module structure which is induced from the  $A$ -module  $I \otimes_B X$ . Evidently, the  $A$ -module  $IX$  can be regarded as an  $A$ -submodule of the projective  $A$ -module  ${}_A A \otimes_B P$ .  $\square$

Remark that Lemma 3.3 may be false if  $X$  is not a torsionless module. For a counterexample, we refer the reader to [17, Erratum].

**Proof of Theorem 1.1.** (1) Let  $B \subseteq A$  be a right-bounded extension of Artin algebras and  $I$  be an ideal in  $B$  such that  $IA \text{rad}(B) \subseteq B$  and the full subcategory  $(B/I)\text{-mod}$  of  $B\text{-mod}$  is  $s$ -syzygy-finite (with respect to taking  $B$ -syzygies) for some integer  $s \geq 0$ . Further, we assume that  $n$  is the Gorenstein index of the extension, and define  $m := \text{findim}(A) < \infty$ . Let  $X$  be a  $B$ -module with  $\text{pd}({}_B X) < \infty$ . Then we consider the  $B$ -module  $Y := \Omega_B^{m+n+1}(X)$ , take a projective cover  $\pi : P \rightarrow Y$  of  $Y$ , and form the following exact commutative diagram of  $B$ -modules:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Omega_B(Y) & \longrightarrow & P & \xrightarrow{\pi} & Y \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A \otimes_B \Omega_B(Y) & \longrightarrow & A \otimes_B P & \longrightarrow & A \otimes_B Y \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & (A/B) \otimes_B \Omega_B(Y) & \xrightarrow{\varphi} & (A/B) \otimes_B P & \xrightarrow{\psi} & (A/B) \otimes_B Y \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

where the third column in the diagram is given by Lemma 3.2. The exactness of the second row follows from  $\text{Tor}_1^B(A, Y) = \text{Tor}_1^B(A, \Omega_B^{m+n+1}(X)) \simeq \text{Tor}_{n+m+2}^B(A, X) = 0$  since  $n$  is the Gorenstein index of the extension. Thus there is a projective  $A$ -module  $Q$  such that  $A \otimes_B \Omega_B(Y) \simeq \Omega_A(A \otimes_B Y) \oplus Q$ . So we may rewrite the above diagram as follows:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Omega_B(Y) & \longrightarrow & P & \xrightarrow{\pi} & Y \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Omega_A(A \otimes_B Y) \oplus Q & \longrightarrow & A \otimes_B P & \longrightarrow & A \otimes_B Y \longrightarrow 0 \\
 & & \downarrow f & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{Im}(\varphi) & \longrightarrow & (A/B) \otimes_B P & \longrightarrow & (A/B) \otimes_B Y \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where  $f$  is surjective. Thus we have an exact sequence of  $B$ -modules

$$(*) \quad 0 \longrightarrow \Omega_B(Y) \longrightarrow \Omega_A(A \otimes_B Y) \oplus Q \longrightarrow \text{Im}(\varphi) \longrightarrow 0.$$

By Lemma 3.1, there is the following isomorphism of  $A$ -modules for any  $j \geq 1$ :

$$\Omega_A(A \otimes_B \Omega_B^{n+j}(X)) \simeq \Omega_A^j(A \otimes_B \Omega_B^{n+1}(X)).$$

So, if  $j > \text{pd}({}_B X)$ , then  $\Omega_B^{n+j}(X) = 0$ , and therefore  $\Omega_A^j(A \otimes_B \Omega_B^{n+1}(X)) = 0$  and  $\text{pd}({}_A A \otimes_B \Omega_B^{n+1}(X)) \leq m := \text{findim}(A)$ . In particular, it follows again from Lemma 3.1 that

$$\Omega_A(A \otimes_B \Omega_B^{n+m+1}(X)) \simeq \Omega_A^{m+1}(A \otimes_B \Omega_B^{n+1}(X)) = 0.$$

Hence the sequence (\*) becomes the following exact sequence

$$0 \longrightarrow \Omega_B^{n+m+2}(X) \longrightarrow Q \longrightarrow \text{Im}(\varphi) \longrightarrow 0.$$

If we take  $(s - 1)$ -th syzygy of this sequence, then we get a new exact sequence of  $B$ -modules

$$0 \longrightarrow \Omega_B^{n+m+2+s-1}(X) \longrightarrow \Omega_B^{s-1}(Q) \oplus P' \longrightarrow \Omega_B^{s-1}(\text{Im}(\varphi)) \longrightarrow 0$$

with  $P'$  a projective  $B$ -module. Now, applying the second shifting sequence in Lemma 2.2(2) to this sequence, we obtain the following exact sequence

$$0 \longrightarrow \Omega_B^s(Q) \longrightarrow \Omega_B^s(\text{Im}(\varphi)) \oplus W \longrightarrow \Omega_B^{n+m+s+1}(X) \longrightarrow 0,$$

where  $W$  is a projective  $B$ -module.

Next, we shall prove that  $\text{Im}(\varphi)$  is a  $B/I$ -module, that is,  $I(\text{Im}(\varphi)) = 0$ . In fact,  $\varphi$  is induced by inclusion from  $\Omega_B(Y)$  to  $P$ , and therefore  $\Omega_B(Y) \subseteq \text{rad}({}_B P) = \text{rad}(B)P$ . So an element in  $\text{Im}(\varphi)$  is a finite sum of elements of the form  $(a + B) \otimes_B rp$ , where  $a \in A$ ,  $r \in \text{rad}(B)$  and  $p \in P$ . Hence it is enough to show  $I((a + B) \otimes_B rp) = 0$  in  $(A/B) \otimes_B P$ . If  $r' \in I$ , then it follows from  $I A \text{rad}(B) \subseteq B$  that

$$r'((a + B) \otimes_B rp) = r'((ar + B) \otimes_B p) = (r'ar + B) \otimes_B p = 0 \otimes_B p = 0.$$

This shows that  $\text{Im}(\varphi)$  is a module over  $B/I$ . By assumption, the full subcategory  $(B/I)\text{-mod}$  of  $B\text{-mod}$  is  $s$ -syzygy-finite. So there is an additive generator  ${}_B N$  for  $\Omega_B^s((B/I)\text{-mod})$  such that  $\Omega_B^s(\text{Im}(\varphi)) \in \text{add}(N)$ .

Finally, it follows from Lemma 2.1 that

$$\text{pd}({}_B \Omega_B^{n+m+s+1}(X)) = \Psi({}_B \Omega_B^{n+m+s+1}(X)) \leq 1 + \Psi(\Omega_B^s(Q) \oplus \Omega_B^s(\text{Im}(\varphi)) \oplus W) \leq 1 + \Psi(\Omega_B^s({}_B A) \oplus N).$$

Clearly,  $\Psi(\Omega_B^s({}_B A) \oplus N)$  does not depend upon  $X$ . As a result, we have

$$\text{pd}({}_B X) \leq m + n + s + 1 + \text{pd}({}_B \Omega_B^{n+m+s+1}(X)) \leq m + n + s + 2 + \Psi(\Omega_B^s({}_B A) \oplus N) < \infty.$$

That is,  $\text{findim}(B) < \infty$ . This completes the proof of Theorem 1.1(1).

(2) Let  $X$  be a  $B$ -module with  $\text{pd}({}_B X) < \infty$ . Then we take a projective cover  $P_1 \rightarrow \Omega_B(X)$  of  $\Omega_B(X)$  and get two canonical exact sequences of torsionless  $B$ -modules:

$$\begin{aligned} 0 \longrightarrow \Omega_B^2(X) \longrightarrow P_1 \longrightarrow \Omega_B(X) \longrightarrow 0, \\ 0 \longrightarrow \Omega_B^2(X) \longrightarrow \text{rad}_B(P_1) \longrightarrow \text{rad}_B(\Omega_B(X)) \longrightarrow 0, \end{aligned}$$

which can be used to construct the following exact commutative diagram of  $B$ -modules:



$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & Z & \longrightarrow & I \operatorname{rad}_B(P_1) & \xrightarrow{\varphi} & I \operatorname{rad}_B(\Omega_B(X)) \longrightarrow 0 \\
 & & \downarrow \iota_1 & & \downarrow \iota_2 & & \downarrow \iota_3 \\
 0 & \longrightarrow & \Omega_B^2(X) & \longrightarrow & \operatorname{rad}_B(P_1) & \longrightarrow & \operatorname{rad}_B(\Omega_B(X)) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & T & \longrightarrow & \operatorname{rad}_B(P_1)/I \operatorname{rad}_B(P_1) & \longrightarrow & \operatorname{rad}_B(\Omega_B(X))/I \operatorname{rad}_B(\Omega_B(X)) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where  $Z$  is the kernel of  $\varphi$  and  $T$  is the cokernel of  $\iota_1$ . Thus we have an exact sequence of  $B$ -modules

$$(**) \quad 0 \longrightarrow Z \longrightarrow \Omega_B^2(X) \longrightarrow T \longrightarrow 0$$

with  $T$  a  $B/I$ -module. Note that  $Z$  has an  $A$ -module structure: In fact,  $\varphi$  is the composite of  $I \operatorname{rad}(B) \otimes_B P_1 \rightarrow I \operatorname{rad}(B) \otimes_B \Omega_B(X)$  with the multiplication map  $I \operatorname{rad}(B) \otimes_B \Omega_B(X) \rightarrow I \operatorname{rad}(B)\Omega_B(X) = I \operatorname{rad}_B(\Omega_B(X))$ . Since  $I \operatorname{rad}(B)$  is a left ideal of  $A$  and  $\Omega_B(X)$  is a torsionless  $B$ -module, we see that  $I \operatorname{rad}_B(\Omega_B(X))$  is an  $A$ -module by Lemma 3.3. It turns out that  $\varphi$  becomes a homomorphism of  $A$ -modules, and therefore its kernel  $Z$  is an  $A$ -module.

Since  $P_1$  is a projective  $B$ -module, we have the following inclusions of  $A$ -modules:

$$Z \hookrightarrow I \operatorname{rad}_B(P_1) \simeq I \operatorname{rad}(B) \otimes_B P_1 \hookrightarrow A \otimes_B P_1.$$

Let  $W$  be the cokernel of the inclusion  $Z \hookrightarrow A \otimes_B P_1$ . Then there is a projective  $A$ -module  $Q$  such that  $Z \simeq \Omega_A(W) \oplus Q$ . Thus the exact sequence  $(**)$  can be rewritten as

$$0 \longrightarrow \Omega_A(W) \oplus Q \longrightarrow \Omega_B^2(X) \longrightarrow T \longrightarrow 0.$$

Suppose that the full subcategory  $(B/I)\text{-mod}$  of  $B\text{-mod}$  is  $s$ -syzygy-finite for some  $s \geq 1$ . Then we take the  $(s - 1)$ -th syzygy of the above sequence and get the following exact sequence of  $B$ -modules:

$$0 \longrightarrow \Omega_B^{s-1}(\Omega_A(W) \oplus Q) \longrightarrow \Omega_B^{s+1}(X) \oplus P' \longrightarrow \Omega_B^{s-1}(T) \longrightarrow 0$$

with  $P'$  a projective  $B$ -module. By syzygy shifting, we finally get an exact sequence of the following form:

$$0 \longrightarrow \Omega_B^s(T) \longrightarrow \Omega_B^{s-1}(\Omega_A(W) \oplus Q) \oplus P \longrightarrow \Omega_B^{s+1}(X) \oplus P' \longrightarrow 0,$$

where  $P$  is a projective  $B$ -module. Since  $A$  is torsionless-finite and the full subcategory  $(B/I)\text{-mod}$  of  $B\text{-mod}$  is  $s$ -syzygy-finite, there is an  $A$ -module  $M$  and a  $B$ -module  $N$  such that  $\Omega_A(W) \in \operatorname{add}({}_A M)$  and  $\Omega_B^s(T) \in \operatorname{add}({}_B N)$ . Here  $M$  and  $N$  do not depend upon  $X$ . Thus the Igusa–Todorov function yields the following estimation:

$$\begin{aligned}
 \operatorname{pd}({}_B X) &\leq s + 1 + \operatorname{pd}({}_B \Omega_B^{s+1}(X)) \\
 &\leq s + 1 + 1 + \Psi(\Omega_B^s(T) \oplus \Omega_B^{s-1}(\Omega_A(W) \oplus Q) \oplus P) \\
 &\leq s + 2 + \Psi(N \oplus \Omega_B^{s-1}(M \oplus {}_B A)).
 \end{aligned}$$

Note that  $s$  and  $\Psi(N \oplus \Omega_B^{s-1}(M \oplus_B A))$  are independent of the  $B$ -module  $X$ . Hence  $\text{findim}(B) < \infty$ . This finishes the proof of [Theorem 1.1\(2\)](#).  $\square$

**Remarks.** (1) The above proof shows that, in [Theorem 1.1\(2\)](#), we may replace “ $A$  is torsionless-finite” by “the subcategory  $\Omega_B^{s-1}(\Omega_A(A\text{-mod}))$  is of finite type”.

(2) Given an Artin  $R$ -algebra  $B$  and an ideal  $I$  in  $B$ , if  $B$  is a free  $R$ -module (for instance,  $R$  is a field), then there is a recipe for getting an extension  $B \subseteq A$  of Artin  $R$ -algebras such that  $I$  is a left ideal in  $A$ . In fact, since  $B$  is a free  $R$ -module of finite rank, we can embed  $B$  into a full  $n \times n$  matrix algebra  $\Lambda := M_n(R)$  over  $R$  and define  $A := \{a \in \Lambda \mid aI \subseteq I\}$ . Evidently, the Artin  $R$ -algebra  $A$  contains  $B$  and makes  $I$  into a left ideal.  $\square$

**Proof of Corollary 1.2.** (1) Let  $B \subseteq A$  be a right Gorenstein-finite extension of Artin algebras. Suppose that there is an integer  $s \geq 0$  such that  $\text{rad}^s(B)A\text{rad}(B) \subseteq B$  and that  $B/\text{rad}^s(B)$  is representation-finite. Then we define  $I := \text{rad}^s(B)$ . Evidently,  $I$  satisfies all conditions in [Theorem 1.1\(1\)](#). Hence [Corollary 1.2\(1\)](#) follows directly from [Theorem 1.1\(1\)](#).

(2) Suppose that  $B \subseteq A$  is an extension of Artin algebras such that  $\text{rad}^s(B)$  is a left ideal of  $A$  and that  $B/\text{rad}^{s-1}(B)$  is representation-finite for some integer  $s \geq 1$ . If  $I := \text{rad}^{s-1}(B)$ , then  $I$  fulfills the conditions in [Theorem 1.1\(2\)](#). So [Corollary 1.2\(2\)](#) follows immediately from [Theorem 1.1\(2\)](#).  $\square$

As an immediate consequence of [Corollary 1.2\(1\)](#), we have the following result.

**Corollary 3.4.** *Let  $B \subseteq A$  be a right Gorenstein-finite extension of Artin algebras. Suppose  $A\text{rad}(B) \subseteq \text{rad}(B)A$  and that there is an integer  $s \geq 1$  such that  $\text{rad}^s(B)$  is a right ideal of  $A$  and  $B/\text{rad}^{s-1}(B)$  is representation-finite. If  $\text{findim}(A) < \infty$ , then  $\text{findim}(B) < \infty$ .*

Recall that the representation dimension of an algebra  $A$ , introduced by Auslander in [1], is defined by

$$\text{repdim}(A) := \inf\{\text{gldim}(\text{End}_A(A \oplus D(A) \oplus M)) \mid M \in A\text{-mod}\}.$$

If we strengthen extensions  $B \subseteq A$  as special left Gorenstein-finite extensions, we may relax the assumption on  $B/I$  in [Theorem 1.1\(1\)](#) and get the following result.

**Corollary 3.5.** *Let  $B \subseteq A$  be a right-bounded extension of Artin algebras with  $\text{pd}({}_B A) \leq 1$ , and let  $I$  be an ideal of  $B$  such that  $I A\text{rad}(B) \subseteq B$  and  $\text{repdim}(B/I) \leq 3$ . If  $\text{findim}(A) < \infty$ , then  $\text{findim}(B) < \infty$ .*

**Proof.** As in the proof of [Theorem 1.1\(1\)](#), we get an exact sequence of  $B$ -modules

$$0 \longrightarrow \Omega_B(Q) \longrightarrow \Omega_B(\text{Im}(\varphi)) \oplus W \longrightarrow \Omega_B^{n+m+2}(X) \oplus V \longrightarrow 0,$$

where  $Q$  is a projective  $A$ -module,  $W$  and  $V$  are projective  $B$ -modules. Since  $\text{repdim}(B/I) \leq 3$  and  $\text{Im}(\varphi)$  is a  $B/I$ -module, we can find a  $B/I$ -module  $U$  such that  $\text{gldim}(\text{End}_{B/I}(U)) = \text{repdim}(B/I)$ , and an exact sequence

$$0 \longrightarrow U_1 \longrightarrow U_0 \longrightarrow \text{Im}(\varphi) \longrightarrow 0$$

such that  $U_i \in \text{add}(U)$  by [Lemma 2.3](#). Thus we can form the following exact commutative diagram of  $B$ -modules:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \Omega_B(U_1) & \xlongequal{\quad} & \Omega_B(U_1) & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & K & \longrightarrow & \Omega_B(U_0) \oplus W' & \longrightarrow & \Omega_B^{n+m+1}(X) \oplus V \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \Omega_B(Q) & \longrightarrow & \Omega_B(\text{Im}(\varphi)) \oplus W & \longrightarrow & \Omega_B^{n+m+1}(X) \oplus V \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

where  $W'$  is a projective  $B$ -module. It follows from  $\text{pd}({}_B A) \leq 1$  that  $\text{Ext}_B^1(\Omega_B(Q), \Omega_B(U_1)) = \text{Ext}_B^2(Q, \Omega_B(U_1)) = 0$ , and therefore the first column of the above diagram splits. Thus  $K \simeq \Omega_B(Q \oplus U_1)$ , and consequently, we have the following exact sequence

$$0 \longrightarrow \Omega_B(Q \oplus U_1) \longrightarrow \Omega_B(U_0) \oplus W' \longrightarrow \Omega_B^{n+m+2}(X) \oplus V \longrightarrow 0.$$

Now, the same argument of the proof of [Theorem 1.1\(1\)](#) will lead to [Corollary 3.5](#).  $\square$

The foregoing proof of [Theorem 1.1\(2\)](#) also implies the following conclusion.

**Corollary 3.6.** *Let  $B \subseteq A$  be an extension of Artin algebras such that  $\text{pd}({}_B A) < \infty$  and  $\text{rad}^2(B)$  is a left ideal of  $A$ .*

- (1) *If  $A$  is torsionless-finite, then  $\text{findim}(B) < \infty$ .*
- (2) *If  $\text{gldim}(A) < \infty$ , then  $\text{findim}(B) < \infty$ .*

**Proof.** (1) is clear from [Corollary 1.2\(2\)](#).

(2) We keep the notation in the above proof of [Theorem 1.1\(2\)](#). By [Lemma 2.2\(1\)](#),  $\text{pd}({}_B Z) \leq \text{pd}({}_B A) + \text{gldim}(A)$ . This shows  $\text{pd}({}_B T) < \infty$  for the semisimple module  $T$ . Hence, the argument in the proof of [Theorem 1.1\(2\)](#) yields  $\text{findim}(B) < \infty$ .  $\square$

Now, we consider extensions  $B \subseteq A$  with  $\text{rad}^l(B) = \text{rad}^l(A)$ . For  $l = 1$ , it is known from [[20, Theorem 1.1\(1\)](#)] that  $\text{findim}(B) < \infty$  if  $\text{findim}(A) < \infty$ . For  $l \geq 2$ , we have the following variation of [Corollary 1.2\(2\)](#).

**Corollary 3.7.** *Let  $B \subseteq A$  be an extension of Artin algebras such that  $\text{rad}^l(B) = \text{rad}^l(A)$  for an integer  $l \geq 2$  and that both  $A/\text{rad}^{l-1}(A)$  and  $B/\text{rad}^{l-1}(B)$  are representation-finite. If  $\text{gldim}(A) \leq 2$ , then  $\text{findim}(B) < \infty$ .*

**Proof.** Let  $X$  be a  $B$ -module with  $\text{pd}({}_B X) < \infty$ , and let  $\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$  be a minimal projective resolution of  ${}_B X$ . Then, as in the proof of [Theorem 1.1\(2\)](#), we can construct an exact sequence

$$(**) \quad 0 \longrightarrow Z \longrightarrow \Omega_B^2(X) \longrightarrow T \longrightarrow 0,$$

of  $B$ -modules, which induces another exact sequence of  $A$ -modules

$$0 \longrightarrow Z \longrightarrow \text{rad}_B^l(P_1) \longrightarrow \text{rad}_B^l(\Omega_B(X)) \longrightarrow 0,$$

where these  $A$ -module structures are due to Lemma 3.3 and where  $T$  is a module over  $B/\text{rad}^{l-1}(B)$ . Furthermore, we also have the following exact sequence of  $A$ -modules:

$$0 \longrightarrow \Omega_A^2(A \otimes_B X) \oplus P \longrightarrow A \otimes_B P_1 \longrightarrow A \otimes_B P_0 \longrightarrow A \otimes_B X \longrightarrow 0$$

with  $P$  a projective  $A$ -module. Now, we construct the following exact commutative diagram of  $A$ -modules:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & Z & \longrightarrow & \text{rad}_B^l(P_1) & \longrightarrow & \text{rad}_B^l(\Omega_B(X)) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Omega_A^2(A \otimes_B X) \oplus P & \longrightarrow & A \otimes_B P_1 & \longrightarrow & A \otimes_B P_0 \longrightarrow A \otimes_B X \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & S_1 & \longrightarrow & (A/\text{rad}^l(B)) \otimes_B P_1 & \longrightarrow & S_3 \longrightarrow A \otimes_B X \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Since  $\text{rad}^l(B) = \text{rad}^l(A)$ , we see that  $S_1$  is a module over  $A/\text{rad}^l(A)$  and that  $(A/\text{rad}^l(B)) \otimes_B P_1$ , as an  $A/\text{rad}^l(A)$ -module, is projective. Thus  $S_1$  is a torsionless module over  $A/\text{rad}^l(A)$ . By assumption,  $A/\text{rad}^{l-1}(A)$  is representation-finite, this implies that  $A/\text{rad}^l(A)$  is torsionless-finite (see [1,15]). So there is a module  $M$  over  $A/\text{rad}^l(A)$  such that  $S_1 \in \text{add}(M) = \Omega_{A/\text{rad}^l(A)}^1((A/\text{rad}^l(A))\text{-mod})$ . Since  $\text{gldim}(A) \leq 2$ , the  $A$ -module  $\Omega_A^2(A \otimes_B X)$  is projective and  $Z \simeq \Omega_A(S_1) \oplus Q$  with  $Q$  a projective  $A$ -module. Consequently,  $Z \in \text{add}(\Omega_A(M) \oplus A)$ .

Since  $B/\text{rad}^{l-1}(B)$  is representation-finite, we may find an additive generator  $N$  for  $(B/\text{rad}^{l-1}(B))\text{-mod}$  such that  $\Omega_B(T) \in \text{add}(\Omega_B(N))$ . Now, applying Lemma 2.2 to (\*\*), we have an exact sequence:

$$0 \longrightarrow \Omega_B(T) \longrightarrow Z \oplus P' \longrightarrow \Omega_B^2(X) \longrightarrow 0$$

with  $P'$  a projective  $B$ -module. By Lemma 2.1, we have the following inequalities:

$$\text{pd}(B X) \leq 2 + \text{pd}(B \Omega_B^2(X)) \leq 2 + \Psi(Z \oplus P' \oplus \Omega_B(T)) \leq 2 + \Psi(B \Omega_A(M) \oplus B A \oplus \Omega_B(N)).$$

So  $\text{findim}(B) < \infty$ .  $\square$

In Corollary 3.7, we take  $l = 2$  and get the following corollary.

**Corollary 3.8.** *Let  $B \subseteq A$  be an extension of Artin algebras with  $\text{rad}^2(B) = \text{rad}^2(A)$ . If  $\text{gldim}(A) \leq 2$ , then  $\text{findim}(B) < \infty$ .*

Corollary 3.8 can be used to re-obtain a main result in [8] on algebras with vanishing radical cube.

**Corollary 3.9.** [8] *Suppose that  $B$  is a finite-dimensional  $k$ -algebra of the form  $B = kQ/I$  with vanishing radical cube, where  $Q$  is a quiver and  $I$  is an admissible ideal in the path algebra  $kQ$  of  $Q$  over the field  $k$ . Then  $\text{findim}(B) < \infty$ .*

**Proof.** Since  $\text{rad}^3(B) = 0$ , we may write  $B$  as  $B = kQ_0 \oplus kQ_1 \oplus kQ_2$ , where  $Q_0$  and  $Q_1$  are the sets of vertices and arrows of  $Q$ , respectively, and  $Q_2$  is a set of  $k$ -linearly independent paths of length two in  $B$ .

Then  $B$  is embedded canonically into a triangular matrix algebra  $A := \begin{pmatrix} kQ_0 & & \\ kQ_1 & kQ_0 & \\ kQ_2 & kQ_1 & kQ_0 \end{pmatrix}$  by sending

$b = b_0 + b_1 + b_2$  to  $\begin{pmatrix} b_0 & & \\ b_1 & b_0 & \\ b_2 & b_1 & b_0 \end{pmatrix}$ , where  $b_i \in kQ_i$  for  $0 \leq i \leq 2$ . We can check  $\text{gldim}(A) \leq 2$  and  $\text{rad}^2(B) = \text{rad}^2(A)$ . Thus [Corollary 3.9](#) follows from [Corollary 3.8](#).  $\square$

For a chain of extensions, we have the following result which extends [\[17, Theorem 4.5\]](#) slightly.

**Corollary 3.10.** *Let  $C \subseteq B \subseteq A$  be a chain of extensions of Artin algebras such that  $\text{rad}(C)$  is a left ideal of  $B$  and  $\text{rad}^l(B)$  is a left ideal of  $A$  for some integer  $l \geq 1$ . If  $A$  is torsionless-finite and  $B/\text{rad}^l(B)$  is representation-finite, then  $\text{findim}(C) < \infty$ .*

**Proof.** Let  $X$  be a  $C$ -module with  $\text{pd}(C X) < \infty$ . By [Lemma 2.4](#), there is a  $B$ -module  $Y$  and a projective  $B$ -module  $P$  such that  $\Omega_C^2(X) \simeq \Omega_B(Y) \oplus P$  as  $B$ -modules. Clearly, the  $C$ -modules  $P$  and  $\Omega_B(Y)$  have finite projective dimensions. It follows from [Lemma 2.2\(2\)](#) and the exact sequence

$$0 \longrightarrow \text{rad}_B^l(\Omega_B(Y)) \longrightarrow \Omega_B(Y) \longrightarrow \Omega_B(Y)/\text{rad}_B^l(\Omega_B(Y)) \longrightarrow 0$$

that the following sequence of  $C$ -modules

$$0 \longrightarrow \Omega_C(\Omega_B(Y)/\text{rad}_B^l(\Omega_B(Y))) \longrightarrow \text{rad}_B^l(\Omega_B(Y)) \oplus Q \longrightarrow \Omega_B(Y) \longrightarrow 0$$

is exact, where  $Q$  is a projective  $C$ -module. By [Lemma 3.3](#),  $\text{rad}_B^l(\Omega_B(Y))$  is a torsionless  $A$ -module. Clearly,  $\Omega_B(Y)/\text{rad}_B^l(\Omega_B(Y))$  is a module over  $B/\text{rad}^l(B)$ . Since  $A$  is torsionless-finite and  $B/\text{rad}^l(B)$  is representation-finite, we may assume that  $M$  and  $N$  are additive generators for  $\Omega(A\text{-mod})$  and  $(B/\text{rad}^l(B))\text{-mod}$ , respectively. Thus  $\text{rad}_B^l(\Omega_B(Y)) \in \text{add}(M)$  and  $\Omega_B(Y)/\text{rad}_B^l(\Omega_B(Y)) \in \text{add}(N)$ .

Now, we use [Lemma 2.1](#) to give an upper bound for the projective dimension of  $C X$ :

$$\begin{aligned} \text{pd}(C X) &\leq 2 + \text{pd}(C \Omega_C^2(X)) = 2 + \text{pd}(C \Omega_B(Y) \oplus C P) \\ &\leq 2 + \Psi\left(\Omega_C(\Omega_B(Y)/\text{rad}_B^l(\Omega_B(Y))) \oplus \text{rad}_B^l(\Omega_B(Y)) \oplus Q \oplus P\right) \\ &\leq 2 + \Psi(\Omega_C(N) \oplus M \oplus C B). \end{aligned}$$

So we have  $\text{findim}(C) < \infty$ .  $\square$

Next, we point out that there are lots of right Gorenstein-finite extensions  $B \subseteq A$  satisfying the condition  $\text{rad}(B)A \subseteq B$ , and therefore  $\text{rad}(B)A \text{rad}(B) \subseteq B$ . This is the case of [Corollary 1.2\(1\)](#) for  $s = 1$ . Let us exhibit one such example.

Suppose that  $\Lambda$  is an Artin algebra. We define  $A := M_2(\Lambda)$ , the algebra of  $2 \times 2$  matrices over  $\Lambda$ , and

$$B := \begin{pmatrix} \Lambda & \text{rad}(\Lambda) \\ \Lambda & \Lambda \end{pmatrix}.$$

Then  $A_B$  and  ${}_B A$  are projective and the extension  $B \subseteq A$  is both right- and left-finite. An easy calculation shows that

$$\text{rad}(B) = \begin{pmatrix} \text{rad}(\Lambda) & \text{rad}(\Lambda) \\ \Lambda & \text{rad}(\Lambda) \end{pmatrix}, \quad \text{rad}(A) = \begin{pmatrix} \text{rad}(\Lambda) & \text{rad}(\Lambda) \\ \text{rad}(\Lambda) & \text{rad}(\Lambda) \end{pmatrix} \quad \text{and} \quad \text{rad}(B)A = \begin{pmatrix} \text{rad}(\Lambda) & \text{rad}(\Lambda) \\ \Lambda & \Lambda \end{pmatrix},$$

and that  $\text{rad}(B)$  is neither a left nor a right ideal in  $A$ . But we have  $\text{rad}(A) \subsetneq \text{rad}(B) \subseteq \text{rad}(B)A \subseteq B$ , as desired. If  $\text{findim}(A) < \infty$ , then  $\text{findim}(B) < \infty$  by Corollary 1.2(1) for  $s = 1$ . Moreover, in this case, we can get  $\text{findim}(B) < \infty$  alternatively by Corollary 3.5 since  $\text{rad}^2(B)A\text{rad}(B) \subseteq B$  and since  $B/\text{rad}^2(B)$  always has representation dimension at most 3 by a result of Auslander (see [1, Chapt. III, Sec. 5]).

Finally, we display an example to show how our results developed in this paper can be applied to decide whether certain algebras have finite finitistic dimension. The example shows also that the method of controlling finitistic dimensions by extension algebras seems to be useful.

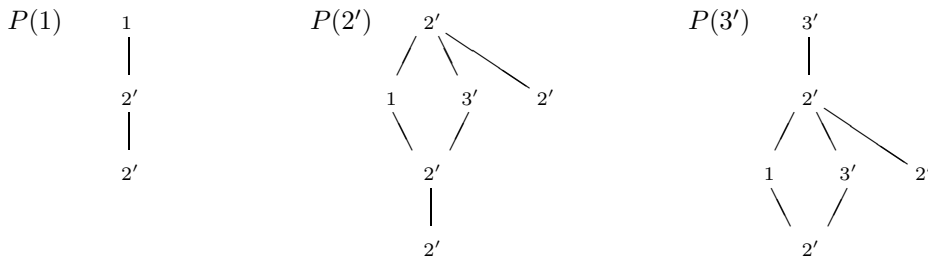
Let  $A$  be an algebra (over a field) given by the following quiver

$$5 \bullet \xleftarrow{\lambda} 2 \bullet \xleftarrow{\epsilon} 3 \bullet \xleftarrow{\xi} 1 \bullet \xleftarrow{\beta} 4 \bullet \xleftarrow{\alpha} 6 \bullet$$

with one relation:  $\alpha\beta\xi\epsilon\lambda = 0$ . Clearly, this algebra is representation-finite. Now, let  $B$  be the subalgebra of  $A$  generated by  $\{e_1, e_{2'} := e_2 + e_4 + e_5, e_{3'} := e_3 + e_6, \lambda, \beta, \alpha + \epsilon, \gamma := \xi\epsilon, \delta := \beta\xi\}$ , where  $e_i$  is the primitive idempotent element of  $A$  corresponding to the vertex  $i$ . Then  $B$  is given by the following quiver

$$1 \bullet \begin{array}{c} \xrightarrow{\gamma} \\ \xleftarrow{\beta} \end{array} \bullet 2' \begin{array}{c} \overset{\curvearrowright}{\lambda} \\ \xrightarrow{\delta} \\ \xleftarrow{\alpha+\epsilon} \end{array} \bullet 3'$$

with relations:  $\beta\gamma = \delta(\alpha + \epsilon)$ ,  $\gamma\delta = \lambda^2 = \lambda\beta = \lambda\delta = (\alpha + \epsilon)\beta\gamma\lambda = 0$ , and the Loewy structures of the indecomposable projective  $B$ -modules are as follows:



It is not difficult to see that  $B$  is representation-infinite and all simple  $B$ -modules have infinite projective dimension. Thus the length  $\ell\ell^\infty(B)$ , defined in [11], is just the Loewy length of  $B$ . Moreover, the algebra  $B$  is neither monomial nor radical-cube vanishing nor standardly stratified nor special biserial. Note that  $B/\text{rad}^3(B)$  is representation-infinite and that all of  $\text{pd}({}_B A)$ ,  $\text{pd}(A_B)$ ,  $\text{pd}({}_B \text{rad}^i(B))$  and  $\text{pd}(\text{rad}^i(B)_B)$  for  $1 \leq i \leq 3$  are infinite. So it is not clear that  $\text{repdim}(B) \leq 3$ . Though  $B$  is embedded into  $A$  of representation dimension 2, the result [18, Theorem 4.2] cannot be applied because  $\text{rad}(B)$  is neither a left nor a right ideal in  $A$ . But we can verify that  $\text{rad}^2(B)$  is an ideal of  $A$ , and therefore  $\text{findim}(B) < \infty$  by Corollary 1.2(2).

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