Homological ring epimorphisms and recollements II: Algebraic K-theory

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Abstract

For a homological ring epimorphism from a ring R to another ring S, we prove that if the module ${}_RS$ has a finite-type resolution, then the algebraic K-theory space of R decomposes as a product of the ones of S and a differential graded algebra. In addition, if the homological ring epimorphism induces a recollement of derived module categories of rings, then the differential graded algebra involved can be replaced by a usual ring. This result is then applied to noncommutative localizations and to homological exact pairs introduced in the first paper of this series. For example, we get a long Mayer-Vietoris sequence of higher algebraic K-groups for homological Milnor squares, including a result of Karoubi.

Contents

1 Introduction

Recall that a ring epimorphism $R \to S$ between rings with identity is said to be *homological* if the derived module category of the ring S can be regarded as a full subcategory of the derived module category of the ring

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R by restriction. For a homological noncommutative localization $\lambda : R \to S$ of rings, Neeman and Ranicki in [16] have discovered a remarkable long exact sequence of algebraic K-groups:

$$\cdots \longrightarrow K_{n+1}(S) \longrightarrow K_n(\mathcal{R}) \longrightarrow K_n(R) \longrightarrow K_n(S) \longrightarrow K_{n-1}(\mathcal{R}) \longrightarrow$$
$$\cdots \longrightarrow K_0(\mathcal{R}) \longrightarrow K_0(R) \longrightarrow K_0(S)$$

for all $n \in \mathbb{N}$, where $\mathcal{R}$ is an exact category determined by λ. This result extends many results in the literature (see [15]). On the one hand, this long sequence, in general, does not have to split into a series of short exact sequences of the corresponding algebraic K-groups, and moreover, the K-theory of the category $\mathcal{R}$ seems not to be easy to handle. On the other hand, there are many homological ring epimorphisms which do not arise from noncommutative localizations, but do give recollements of derived module categories (see the discussion in [6]). As is known, recollements are a generalization of derived equivalences, while derived equivalences preserve algebraic K-theory of rings (see [7]). So, an interesting question for calculation of algebraic K-groups of rings is: When does such a long exact sequence of algebraic K-groups split? Or more generally, can we read off information on algebraic K-theory of rings from recollements of derived module categories? Precisely, we consider the following question:

Question. Let R, S and T be rings with identity. Suppose that there is a recollement among the derived module categories $\mathscr{D}(T)$, $\mathscr{D}(R)$ and $\mathscr{D}(S)$ of the rings T, R and S:

$$\xymatrix{\mathscr{D}(S) \ar[r]^{i_*} & \mathscr{D}(R) \ar@/_1pc/[l] \ar@/^1pc/[l] \ar[r] & \mathscr{D}(T) \ar@/_1pc/[l] \ar@/^1pc/[l]}$$

such that $i_*(S)$ is quasi-isomorphic to a bounded complex of finitely generated projective R-modules. Is the K-theory space $K(R)$ of R homotopy equivalent to the product of the K-theory spaces of S and T? That is, does the following isomorphism hold true:

$$K_n(R) \simeq K_n(S) \oplus K_n(T) \quad \text{for each } \ n \in \mathbb{N}?$$

Here, we denote by $K(\mathscr{E})$ the K-theory space of an exact category $\mathscr{E}$ in the sense of Quillen, by $K(R)$ the K-theory space of the exact category of finitely generated projective R-modules, and by $K_n(R)$ the n-th algebraic K-group of R for each $n \in \mathbb{N}$.

We remark that, without the assumption on $i_*(S)$, the isomorphism $K_n(R) \simeq K_n(S) \oplus K_n(T)$ cannot hold. This was shown by an example in [4, Section 8, Remark (2)] for $n = 0$.

The main purpose of the present paper is to provide an affirmative answer to the above question for homological ring epimorphisms. To attack the question, we will employ ideas from the representation theory of algebras. As a consequence of our methods, we shall establish a long Mayer-Vietoris sequence of higher algebraic K-groups for the so-called homological Milnor squares of rings studied in [3]. This strategy might lead to a bridge between the representation theory of algebras and algebraic K-theory of rings.

Before stating our results precisely, we first recall some definitions.

Let R be a ring with identity. An R-module M has a *finite-type resolution* provided that there is a finite projective resolution by finitely generated projective R-modules, that is, there is an exact sequence $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ for some $n \in \mathbb{N}$ such that all R-modules P_j are projective and finitely generated.

Let X and Y be pointed topological spaces. A map $f : X \to Y$ is called a *homotopy equivalence* if there is a map $g : Y \to X$ such that $fg : X \to X$ and $gf : Y \to Y$ are pointed-homotopic to the identities of X and Y, respectively. Here, by a map between pointed topological spaces we always mean a pointed continuous map. If there is a homotopy equivalence between X and Y, then we say that X and Y are *homotopy equivalent*, and simply write $X \xrightarrow{\sim} Y$.

For a differential graded algebra $\mathbb{A}$, we denote by $K(\mathbb{A})$ the algebraic K-theory space defined in Subsection 3.5, and by $K_n(\mathbb{A})$ the n-th homotopy group of $K(\mathbb{A})$ for $n \in \mathbb{N}$.

Our general result on homological ring epimorphisms reads as follows.

Theorem 1.1. *Let $\lambda : R \to S$ be a homological ring epimorphism.*

(1) *If ${}_RS$ admits a finite-type resolution, then there is a differential graded ring $\mathbb{T}$ determined by λ such that $K(R) \xrightarrow{\sim} K(S) \times K(\mathbb{T})$ as K-theory spaces, and therefore*

$$K_n(R) \simeq K_n(S) \oplus K_n(\mathbb{T}) \quad \textit{for all } \ n \in \mathbb{N}.$$

(2) *Suppose that there exists a ring T and a recollement among the derived module categories $\mathscr{D}(T)$, $\mathscr{D}(R)$ and $\mathscr{D}(S)$ of the rings T, R and S:*

$$\mathscr{D}(S) \xrightarrow{i_*} \mathscr{D}(R) \longrightarrow \mathscr{D}(T)$$

where i_* is the restriction functor induced from λ. If the module ${}_RS$ or S_R has a finite-type resolution, then

$$K(R) \xrightarrow{\sim} K(S) \times K(T)$$

as K-theory spaces, and therefore

$$K_n(R) \simeq K_n(S) \oplus K_n(T) \quad \textit{for all } \ n \in \mathbb{N}.$$

Theorem 1.1 provides a partial answer to the above question and extends both [5, Theorem 1.1 (2)] and some cases in [27]. As a consequence of the proof of Theorem 1.1, we have the following corollary on noncommutative localizations, which, under the finite-type condition, provides a strong result (compare with [16, Theorem 0.5]). Note that the terminology “noncommutative localization” was originally called “universal localization” in the literature.

Corollary 1.2. *Let R be a ring and Σ a set of injective homomorphisms between finitely generated projective R-modules. Suppose that the universal localization $R \to R_\Sigma$ of R at Σ is homological and that the left R-module R_Σ has a finite-type resolution. Then*

$$K(R) \xrightarrow{\sim} K(R_\Sigma) \times K(\mathscr{E})$$

where $\mathscr{E}$ is the small exact category of (R,Σ)-torsion modules which are exactly those finitely presented R-modules M of projective dimension at most 1 such that $R_\Sigma \otimes_R M = 0 = \mathrm{Tor}_1^R(R_\Sigma, M)$.

As an application of our methods developed in this paper, we consider exact pairs introduced in [3] (see also Subsection 5.2 below). In this case, we get a long Mayer-Vietoris sequence of K-groups for homological Milnor squares of rings.

Theorem 1.3. *Let (λ,μ) be an exact pair of ring homomorphisms $\lambda : R \to S$ and $\mu : R \to T$, and let $S \sqcup_R T$, together with the ring homomorphisms $\rho : S \to S \sqcup_R T$ and $\phi : T \to S \sqcup_R T$, be the coproduct of S and T over R. Suppose that λ is a homological ring epimorphism and $\mathrm{Tor}_i^R(T,S) = 0$ for all $i > 0$. Then the following statements hold true:*

(1) *The sequence of K-theory spaces*

$$K(R) \xrightarrow{\left(-K(\lambda),K(\mu)\right)} K(S) \times K(T) \xrightarrow{\left(\begin{smallmatrix} K(\rho) \\ K(\phi) \end{smallmatrix}\right)} K(S \sqcup_R T)$$

is a weak homotopy fibration, where $-K(\lambda)$ denotes the composite of $K(\lambda)$ with $K([1])$. In particular, there is a long exact sequence of K-groups:

$$\cdots \longrightarrow K_{n+1}(S \sqcup_R T) \longrightarrow K_n(R) \xrightarrow{\left(-K_n(\lambda),K_n(\mu)\right)} K_n(S) \oplus K_n(T) \xrightarrow{\left(\begin{smallmatrix} K_n(\rho) \\ K_n(\phi) \end{smallmatrix}\right)} K_n(S \sqcup_R T) \longrightarrow K_{n-1}(R) \longrightarrow$$

$$\cdots \longrightarrow K_0(R) \longrightarrow K_0(S)\oplus K_0(T) \longrightarrow K_0(S\sqcup_R T)$$

for all $n\in\mathbb{N}$.

(2) *If, in addition, the module* ${}_RS$ *or* T_R *has a finite-type resolution, then*

$$K(R)\times K(S\sqcup_R T) \stackrel{\sim}{\longrightarrow} K(S)\times K(T)$$

as K-theory spaces, and therefore

$$K_n(R)\oplus K_n(S\sqcup_R T)\simeq K_n(S)\oplus K_n(T) \quad \textit{for all}\ \ n\in\mathbb{N}.$$

We remark that, by [3, Lemma 3.8 (2)], the coproduct $S\sqcup_R T$ in Theorem 1.3 is actually isomorphic to the endomorphism ring $\mathrm{End}_T(T\otimes_R S)$.

As an immediate consequence of Theorem 1.3, we get a result of Karoubi, namely Corollary 5.3, which provides a long exact sequence of algebraic K-groups for localizations. As another consequence of Theorem 1.3, we have the following result on a class of homological Milnor squares.

Corollary 1.4. (1) *Let* R *be a ring with two ideals* I_1 *and* I_2 *such that* $I_1\cap I_2=0$. *Suppose that the canonical ring homomorphism* $R\to R/I_1$ *is homological. If the left* R*-module* I_1 *or the right* R*-module* I_2 *has a finite-type resolution, then*

$$K_n(R)\oplus K_n(R/(I_1+I_2))\simeq K_n(R/I_1)\oplus K_n(R/I_2)$$

for all $n\in\mathbb{N}$.

(2) *Suppose that* $\lambda: R\to S$ *is a homomorphism of rings and* M *is an* S-S*-bimodule. If* λ *is a homological ring epimorphism, then*

$$K_n(R)\oplus K_n(S\ltimes M)\simeq K_n(S)\oplus K_n(R\ltimes M)$$

for all $n\in\mathbb{N}$, *where* $S\ltimes M$ *stands for the trivial extension of* S *by* M.

This paper is organized as follows: In Section 2, we briefly recall some definitions and basic facts on triangulated categories, homological ring epimorphisms and recollements. In Section 3, we first recall the algebraic K-theories developed by Waldhausen for Waldhausen categories and Schlichting for Frobenius pairs, and then introduce our definition of algebraic K-theory spaces for differential graded algebras, which is a modification of Schlichting's definition in [20]. In Section 4, we prove the main result, Theorem 1.1. But, before starting with our proof, we first consider homotopy-split injections for K-theory spaces as a preparation, and then prove the first part of Theorem 1.1, which shows that, in general, the algebraic K-theory of recollements induced from homological ring epimorphisms involves differential graded algebras. With the help of the first part of Theorem 1.1, we then give proofs of the second part of Theorem 1.1 and its Corollary 1.2. In Section 5, we apply our results in the previous sections to homological exact pairs defined in the first paper [3] of this series, and get a long Mayer-Vietoris sequence of K-groups, which shows Theorem 1.3. As an immediate consequence of Theorem 1.3, we reobtain a Mayer-Vietoris sequence in Corollary 5.3, due originally to Karoubi, for positive K-theory of localizations. At the end of this section, we deduce Corollary 1.4 from Theorem 1.3. In Section 6, we illustrate our results by an example which shows that the differential graded algebra in Theorem 1.1 (1) cannot be substituted by its underlying ring (just forgetting the differential).

2 Preliminaries

In this section, we shall fix notation which will be employed throughout the paper, and provide some basic facts which will be used in our later proofs.

2.1 General terminology and notation on categories

Let $\mathcal{C}$ be an additive category.

We always assume that a full subcategory $\mathcal{B}$ of $\mathcal{C}$ is closed under isomorphisms, that is, if $X \in \mathcal{B}$ and $Y \in \mathcal{C}$ with $Y \simeq X$, then $Y \in \mathcal{B}$.

Given two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, we denote the composite of f and g by fg which is a morphism from X to Z, while given two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ among three categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, we denote the composite of F and G by GF which is a functor from $\mathcal{C}$ to $\mathcal{E}$.

Let $\mathrm{Ker}(F)$ and $\mathrm{Im}(F)$ be the kernel and image of the functor F, respectively. That is, $\mathrm{Ker}(F) := \{X \in \mathcal{C} \mid FX \simeq 0\}$ and $\mathrm{Im}(F) := \{Y \in \mathcal{D} \mid \exists X \in \mathcal{C}, FX \simeq Y\}$. In particular, $\mathrm{Ker}(F)$ and $\mathrm{Im}(F)$ are closed under isomorphisms in $\mathcal{C}$ and $\mathcal{D}$, respectively.

An additive functor $F : \mathcal{A} \to \mathcal{B}$ between two additive categories $\mathcal{A}$ and $\mathcal{B}$ is called an *equivalence up to factors* if F is fully faithful and each object of $\mathcal{B}$ is isomorphic to a direct summand of the image of an object of $\mathcal{A}$ under F.

Let $\mathcal{A}$ be a triangulated category and $\mathcal{X}$ a full triangulated subcategory of $\mathcal{A}$. Then, essentially due to Verdier, there exists a triangulated category $\mathcal{A}/\mathcal{X}$, and a triangle functor $q : \mathcal{A} \to \mathcal{A}/\mathcal{X}$ with $\mathcal{X} \subseteq \mathrm{Ker}(q)$ such that q has the following universal property: If $q' : \mathcal{A} \to \mathcal{T}$ is a triangle functor with $\mathcal{X} \subseteq \mathrm{Ker}(q')$, then q' factorizes uniquely through $\mathcal{A} \xrightarrow{q} \mathcal{A}/\mathcal{X}$ (see [14, Theorem 2.18]). The category $\mathcal{A}/\mathcal{X}$ is called the *Verdier quotient* of $\mathcal{A}$ by $\mathcal{X}$, and the functor q is called the *Verdier localization functor*. In this case, $\mathrm{Ker}(q)$ is the full subcategory of $\mathcal{A}$ consisting of direct summands of all objects in $\mathcal{X}$. We remark that the objects of the category $\mathcal{A}/\mathcal{X}$ are the same as the objects of $\mathcal{A}$ (see [14, Chapter 2] for details).

A sequence $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ of triangle functors F and G between triangulated categories is said to be *exact* if the following four conditions are satisfied:

(i) The functor F is fully faithful.

(ii) The composite $GF : \mathcal{A} \to \mathcal{C}$ of F and G is zero.

(iii) The image $\mathrm{Im}(F)$ of F is equal to the kernel of G.

(iv) The functor G induces an equivalence from the Verdier quotient of $\mathcal{B}$ by $\mathrm{Im}(F)$ to $\mathcal{C}$.

Clearly, if $\mathcal{X}$ is closed under direct summands in $\mathcal{A}$, then we have an exact sequence of triangulated categories:

$$\mathcal{X} \hookrightarrow \mathcal{A} \xrightarrow{q} \mathcal{A}/\mathcal{X} \ .$$

Let $\mathcal{T}$ be a triangulated category with small coproducts (that is, coproducts indexed over sets exist in $\mathcal{T}$).

An object $U \in \mathcal{T}$ is said to be *compact* if $\mathrm{Hom}_{\mathcal{T}}(U, -)$ commutes with small coproducts in $\mathcal{T}$. The full subcategory of $\mathcal{T}$ consisting of all compact objects is denoted by $\mathcal{T}^c$.

For any non-empty class $\mathscr{S}$ of objects in $\mathcal{T}$, we denote by $\mathrm{Tria}(\mathscr{S})$ (respectively, $\mathrm{thick}(\mathscr{S})$) the smallest full triangulated subcategory of $\mathcal{T}$ containing $\mathscr{S}$ and being closed under small coproducts (respectively, direct summands). If $\mathscr{S}$ consists of only one object U, then we simply write $\mathrm{Tria}(U)$ and $\mathrm{thick}(U)$ for $\mathrm{Tria}(\{U\})$ and $\mathrm{thick}(\{U\})$, respectively. The notation $\mathrm{Tria}(\mathscr{S})$ without referring to $\mathcal{T}$ will not cause any confusions because this notation can be clarified from the contexts of our considerations.

The following facts are in the literature (see [14, Proposition 1.6.8] and [3, Section 2.1]).

Lemma 2.1. (1) *If $\mathcal{T}_0$ is a full triangulated subcategory of $\mathcal{T}$ such that $\mathcal{T}_0$ is closed under countable coproducts, then $\mathcal{T}_0$ is closed under direct summands in $\mathcal{T}$.*

(2) *Let $\mathcal{T}'$ be a triangulated category with small coproducts, and let $F : \mathcal{T} \to \mathcal{T}'$ be a triangle functor. If F preserves small coproducts, then $F(\mathrm{Tria}(U)) \subseteq \mathrm{Tria}(F(U))$ for any $U \in \mathcal{T}$.*

Finally, we mention a special case of the result [14, Theorem 4.4.9] for $\beta = \aleph_0$.

Lemma 2.2. *Let $\mathcal{S}$ be a triangulated category with small coproducts. Let $\mathcal{R} \subseteq \mathcal{S}$ be a full triangulated subcategory, closed under the formation of the coproducts in $\mathcal{S}$ of any set of its objects. Let $\mathcal{T} := \mathcal{S}/\mathcal{R}$. Assume further that there exist: (i) A set of objects $S \subseteq \mathcal{S}^c$, so that $\mathcal{S} = \mathrm{Tria}(S)$. (ii) A set of objects $R \subseteq \mathcal{R} \cap \mathcal{S}^c$, so that $\mathcal{R} = \mathrm{Tria}(R)$. Then the following hold true:*

(1) *The inclusion $\mathcal{R} \subseteq \mathcal{S}$ takes compact objects to compact objects, and so does the Verdier localization functor $\mathcal{S} \to \mathcal{T}$. In other words, we have a commutative diagram*

$$\begin{array}{ccccc} \mathcal{R}^c & \longrightarrow & \mathcal{S}^c & \longrightarrow & \mathcal{T}^c \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathcal{R} & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{T}. \end{array}$$

Moreover, we have $\mathrm{Tria}(\mathcal{R})^c = \mathrm{Tria}(\mathcal{R}) \cap \mathcal{S}^c = \mathrm{thick}(\mathcal{R})$.

(2) *The composite $\mathcal{R}^c \to \mathcal{S}^c \to \mathcal{T}^c$ in the above diagram must vanish, since it is just the restriction to $\mathcal{R}^c$ of a vanishing functor on $\mathcal{R}$. We therefore have a factorization of $\mathcal{S}^c \to \mathcal{T}^c$ as*

$$\mathcal{S}^c \longrightarrow \mathcal{S}^c/\mathcal{R}^c \xrightarrow{\ i\ } \mathcal{T}^c.$$

The functor $i : \mathcal{S}^c/\mathcal{R}^c \to \mathcal{T}^c$ is an equivalence up to factors.

2.2 Complexes over module categories

Throughout the paper, by a ring we always mean an associative ring with identity.

Let R be a ring. We denote by R-Mod, R-proj and $\mathscr{P}^{<\infty}(R)$ the categories of left R-modules, finitely generated projective left R-modules and left R-modules having finite-type resolutions, respectively. As usual, the complex, homotopy and derived categories of R-Mod are denoted by $\mathscr{C}(R)$, $\mathscr{K}(R)$ and $\mathscr{D}(R)$, respectively. Clearly, $\mathscr{D}(R) = \mathrm{Tria}(R)$. By usual convention, we write $\mathscr{D}^c(R)$ for $\mathscr{D}(R)^c$.

For each $n \in \mathbb{Z}$, we denote the n-th cohomology functor by $H^n(-) : \mathscr{D}(R) \to R$-Mod.

Now we briefly recall the definitions of Hom-complexes and tensor complexes.

Let $(X^\bullet, d_{X^\bullet})$ and $(Y^\bullet, d_{Y^\bullet})$ be complexes in $\mathscr{C}(R)$. The Hom-complex of $X^\bullet$ and $Y^\bullet$ over R is a complex $\mathrm{Hom}_R^\bullet(X^\bullet, Y^\bullet) := \big(\mathrm{Hom}_R^n(X^\bullet, Y^\bullet), d^n_{X^\bullet, Y^\bullet}\big)_{n\in\mathbb{Z}}$ where

$$\mathrm{Hom}_R^n(X^\bullet, Y^\bullet) := \prod_{p\in\mathbb{Z}} \mathrm{Hom}_R(X^p, Y^{p+n})$$

and the differential $d^n_{X^\bullet, Y^\bullet}$ of degree n is given by

$$(h^p)_{p\in\mathbb{Z}} \mapsto \big(h^p d_{Y^\bullet}^{p+n} - (-1)^n d_{X^\bullet}^p h^{p+1}\big)_{p\in\mathbb{Z}}$$

for $(h^p)_{p\in\mathbb{Z}} \in \mathrm{Hom}_R^n(X^\bullet, Y^\bullet)$.

Let $Z^\bullet$ be another object in $\mathscr{C}(R)$. We define

$$\circ :\ \mathrm{Hom}_R^\bullet(X^\bullet, Y^\bullet) \times \mathrm{Hom}_R^\bullet(Y^\bullet, Z^\bullet) \longrightarrow \mathrm{Hom}_R^\bullet(X^\bullet, Z^\bullet),\ \ (f, g) \mapsto (f^p g^{p+m})_{p\in\mathbb{Z}}$$

for $f := (f^p)_{p\in\mathbb{Z}} \in \mathrm{Hom}_R^m(X^\bullet, Y^\bullet)$ and $g := (g^p)_{p\in\mathbb{Z}} \in \mathrm{Hom}_R^n(Y^\bullet, Z^\bullet)$ with $m, n \in \mathbb{Z}$. Thus the operation $\circ$ is associative and distributive, and therefore $\mathrm{Hom}_R^\bullet(X^\bullet, X^\bullet)$ is a $\mathbb{Z}$-graded ring. For simplicity, the Hom-complex $\mathrm{Hom}_R^\bullet(X^\bullet, X^\bullet)$ is denoted by $\mathrm{End}_R^\bullet(X^\bullet)$. In fact, $\mathrm{End}_R^\bullet(X^\bullet)$ is a differential graded ring (see Section 3.5 for definition) and will be called the *dg endomorphism ring* of $X^\bullet$. Note that $H^n(\mathrm{End}_R^\bullet(X^\bullet)) \simeq \mathrm{Hom}_{\mathscr{K}(R)}(X^\bullet, X^\bullet[n])$ for any $n \in \mathbb{Z}$.

Moreover, the above-defined operation $\circ$ satisfies the following identity:

$$(f\circ g)\,d^{m+n}_{X^\bullet,Z^\bullet} = f\circ (g)d^{n}_{Y^\bullet,Z^\bullet} + (-1)^n (f)d^{m}_{X^\bullet,Y^\bullet}\circ g.$$

Let $W^\bullet$ be a chain complex over R^{op}-Mod. Then the tensor complex of $W^\bullet$ and $X^\bullet$ over R is a complex $W^\bullet\otimes^\bullet_R X^\bullet := \big(W^\bullet\otimes^n_R X^\bullet, \partial^n_{W^\bullet,X^\bullet}\big)_{n\in\mathbb{Z}}$ where

$$W^\bullet\otimes^n_R X^\bullet := \bigoplus_{p\in\mathbb{Z}} W^p\otimes_R X^{n-p}$$

and the differential $\partial_{W^\bullet,X^\bullet}$ of degree n is given by

$$w\otimes x\mapsto (w)d^p_{W^\bullet}\otimes x + (-1)^p w\otimes (x)d^{n-p}_{X^\bullet}$$

for $w\in W^p$ and $x\in X^{n-p}$.

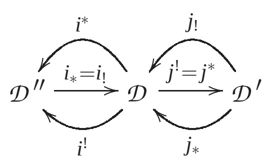
Let S be another ring and $M^\bullet$ a complex of R-S-bimodules. The total left-derived functor of $M^\bullet\otimes^\bullet_S -$ is denoted by $M^\bullet\otimes^{\mathbb{L}}_S - : \mathscr{D}(S)\to\mathscr{D}(R)$, and the total right-derived functor of $\mathrm{Hom}^\bullet_R(M^\bullet,-)$ is denoted by $\mathbb{R}\mathrm{Hom}_R(M^\bullet,-):\mathscr{D}(R)\to\mathscr{D}(S)$. Clearly, $\big(M^\bullet\otimes^{\mathbb{L}}_S -, \mathbb{R}\mathrm{Hom}_R(M^\bullet,-)\big)$ is an adjoint pair of triangle functors.

2.3 Recollements and homological ring epimorphisms

In this subsection, we recall the notion of recollements which were introduced by Beilinson, Bernstein and Deligne (see [1]), and are widely used in algebraic geometry and representation theory. Typical examples of recollements can be constructed from homological ring epimorphisms.

Let $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories with shift functors denoted universally by [1].

We say that $\mathcal{D}$ is a *recollement* of $\mathcal{D}'$ and $\mathcal{D}''$ if there are six triangle functors as in the following diagram



such that

(1) the 4 pairs $(i^*,i_*),(i_!,i^!),(j_!,j^!)$ and (j^*,j_*) are adjoint pairs of functors;

(2) the 3 functors i_*, j_* and $j_!$ are fully faithful;

(3) the composite of two functors in each row is zero, that is, $i^!\,j_*=0$ (and thus also $j^!\,i_!=0$ and $i^*\,j_!=0$); and

(4) there are 2 canonical triangles in $\mathcal{D}$ for each object $X\in\mathcal{D}$:

$$j_!\,j^!(X)\longrightarrow X\longrightarrow i_*i^*(X)\longrightarrow j_!\,j^!(X)[1],$$

$$i_!i^!(X)\longrightarrow X\longrightarrow j_*j^*(X)\longrightarrow i_!i^!(X)[1],$$

where $j_!\,j^!(X)\to X$ and $i_!i^!(X)\to X$ are counit adjunction maps, and where $X\to i_*i^*(X)$ and $X\to j_*j^*(X)$ are unit adjunction maps.

It is known that, up to equivalence of categories, recollements of triangulated categories are the same as torsion torsion-free triples (TTF-triples) of triangulated categories (see, for example, [4, Section 2.3] for details). In the following lemma we mention some facts about recollements for later proofs.

Lemma 2.3. *Suppose that the above recollement is given. Then the following hold:*

(a) *The images of the three fully faithful functors* $j_!$, i_* *and* j_* *are closed under direct summands in* $\mathcal{D}$.

(b) *The Verdier quotients of* $\mathcal{D}$ *by the images of the triangle functors* $j_!$ *and* i_* *are equivalent to* $\mathcal{D}''$ *and* $\mathcal{D}'$*, respectively.*

(c) *Assume that* $\mathcal{D}$, $\mathcal{D}'$ *and* $\mathcal{D}''$ *admit small coproducts. Then both* $j_!$ *and* i^* *preserve compact objects. Suppose further that* $\mathcal{D}$ *is compactly generated, that is, there is a set* S *of compact objects in* $\mathcal{D}$ *such that* $\mathrm{Tria}(S) = \mathcal{D}$*, then* i_* *preserves compact objects if and only if so is* $j^!$*. In this case, we can obtain a "half recollement" of subcategories of compact objects:*

$$\xymatrix{(\mathcal{D}'')^c \ar[r]^{i_*} & \mathcal{D}^c \ar@/_1pc/[l]_{i^*} \ar[r]^{j^!} & (\mathcal{D}')^c \ar@/_1pc/[l]_{j_!}}$$

Note that (a) and (b) follow from [2, Chapter I, Proposition 2.6], while (c) follows from [2, Chapter III, Lemma 1.2(1) and Chapter IV, Proposition 1.11].

A typical example of recollements is provided by homological ring epimorphisms. Recall that a ring epimorphism $\lambda : R \to S$ is said to be *homological* if $\mathrm{Tor}_n^R(S,S) = 0$ for all $n > 0$ (see [8, 16]). This is also equivalent to saying that the restriction functor $D(\lambda_*) : \mathscr{D}(S) \to \mathscr{D}(R)$ is fully faithful.

The following result can be concluded from [17, Section 4].

Lemma 2.4. *Let* $\lambda : R \to S$ *be a homological ring epimorphism. Then there is a recollement of triangulated categories:*

$$\xymatrix{\mathscr{D}(S) \ar[r]^{i_*} & \mathscr{D}(R) \ar@/_1pc/[l]_{i^*} \ar@/^1pc/[l] \ar[r]^{j^!} & \mathrm{Tria}({}_RQ^\bullet) \ar@/_1pc/[l]_{j_!} \ar@/^1pc/[l]}$$

where $Q^\bullet$ *is the two-term complex* $0 \to R \xrightarrow{\lambda} S \to 0$ *with* R *and* S *in degrees* 0 *and* 1*, respectively, and where* $j_!$ *is the canonical embedding and*

$$j^! = Q^\bullet \otimes_R^{\mathbb{L}} -,\ i^* = S \otimes_R^{\mathbb{L}} -,\ i_* = D(\lambda_*).$$

Thus, if we define $\mathscr{Y} := \{Y \in \mathscr{D}(R) \mid \mathrm{Hom}_{\mathscr{D}(R)}(X,Y) = 0 \ \text{ for any } \ X \in \mathrm{Tria}({}_RQ^\bullet)\}$, then it follows from Lemma 2.4 that

$$\mathscr{Y} = \{Y \in \mathscr{D}(R) \mid \mathrm{Hom}_{\mathscr{D}(R)}(Q^\bullet, Y[n]) = 0 \text{ for } \ n \in \mathbb{Z}\} = \{Y \in \mathscr{D}(R) \mid Q^\bullet \otimes_R^{\mathbb{L}} Y = 0\},$$

and that i_* induces an equivalence $\mathscr{D}(S) \xrightarrow{\simeq} \mathscr{Y}$.

In general, for a ring R, the categories $\mathscr{D}(R)$ and $\mathscr{D}(R^{\mathrm{op}})$ are not triangle equivalent. Nevertheless, with the help of Lemma 2.4, we can establish the following result which will be used in the proof of Theorem 1.1.

Lemma 2.5. *Let* $\lambda : R \to S$ *be a homological ring epimorphism. Then the following are equivalent for a ring* T*:*

(1) *There is a recollement of derived categories:*

$$\xymatrix{\mathscr{D}(S) \ar[r]^{D(\lambda_*)} & \mathscr{D}(R) \ar@/_1pc/[l] \ar@/^1pc/[l] \ar[r] & \mathscr{D}(T) \ar@/_1pc/[l] \ar@/^1pc/[l]}$$

(2)*There is a recollement of derived categories:*

$$\xymatrix{\mathscr{D}(S^{\mathrm{op}}) \ar[r]^{D(\lambda_*)} & \mathscr{D}(R^{\mathrm{op}}) \ar@/_1pc/[l] \ar@/^1pc/[l] \ar[r] & \mathscr{D}(T^{\mathrm{op}}) \ar@/_1pc/[l] \ar@/^1pc/[l]}$$

Proof. Observe that if $\lambda : R \to S$ is a homological ring epimorphism, then so is the map $\lambda : R^{\rm op} \to S^{\rm op}$ by [8, Theorem 4.4]. Moreover, it follows from [17, Corollary 3.4] that (1) holds if and only if there is a complex $P^\bullet \in \mathscr{C}^b(R\text{-proj})$ such that $\mathrm{Tria}(P^\bullet) = \mathrm{Tria}({}_RQ^\bullet)$, $\mathrm{End}_{\mathscr{D}(R)}(P^\bullet) \simeq T$ and $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, P^\bullet[n]) = 0$ for any $n \neq 0$, where $Q^\bullet$ is the complex $0 \to R \to S \to 0$. However, for such a complex $P^\bullet$, we always have

$$\mathrm{Hom}_{\mathscr{D}(R^{\rm op})}(P^{\bullet *}, P^{\bullet *}[n]) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, P^\bullet[n]) \ \text{ for all } \ n \in \mathbb{Z},$$

where $P^{\bullet *} := \mathrm{Hom}_R(P^\bullet, R) \in \mathscr{C}^b(R^{\rm op}\text{-proj})$. So, to prove that (1) and (2) are equivalent, it is enough to prove the following statement:

If $P^\bullet \in \mathscr{C}^b(R\text{-proj})$ such that $\mathrm{Tria}(P^\bullet) = \mathrm{Tria}({}_RQ^\bullet)$, then $\mathrm{Tria}(P^{\bullet *}) = \mathrm{Tria}(Q^\bullet_R)$.

In fact, let $P^\bullet$ be such a complex and define

$$\mathscr{Y}' := \{Y \in \mathscr{D}(R^{\rm op}) \mid \mathrm{Hom}_{\mathscr{D}(R^{\rm op})}(X, Y) = 0 \ \text{ for } \ X \in \mathrm{Tria}(P^{\bullet *})\}.$$

Since $P^\bullet \in \mathscr{C}^b(R\text{-proj})$, we have $P^{\bullet *} \in \mathscr{C}^b(R^{\rm op}\text{-proj})$. It follows from [4, Lemma 2.8] that there is a recollement:

$$\mathscr{Y}' \xrightarrow{\ \mu\ } \mathscr{D}(R^{\rm op}) \longrightarrow \mathrm{Tria}(P^{\bullet *})$$

where μ is the inclusion. This implies that

$$(a) \qquad \mathrm{Tria}(P^{\bullet *}) = \{X \in \mathscr{D}(R^{\rm op}) \mid \mathrm{Hom}_{\mathscr{D}(R^{\rm op})}(X, Y) = 0 \ \text{ for } \ Y \in \mathscr{Y}'\}.$$

Furthermore, we remark that

$$\mathscr{Y}' = \{Y \in \mathscr{D}(R^{\rm op}) \mid \mathrm{Hom}_{\mathscr{D}(R^{\rm op})}(P^{\bullet *}, Y[n]) = 0 \text{ for } \ n \in \mathbb{Z}\} = \{Y \in \mathscr{D}(R^{\rm op}) \mid \mathbb{R}\mathrm{Hom}_{R^{\rm op}}(P^{\bullet *}, Y) = 0\},$$

and that

$$\mathbb{R}\mathrm{Hom}_{R^{\rm op}}(P^{\bullet *}, -) \simeq - \otimes^{\mathbb{L}}_R P^\bullet : \ \mathscr{D}(R^{\rm op}) \longrightarrow \mathscr{D}(\mathbb{Z})$$

by [3, Section 2.1]. Thus $\mathscr{Y}' = \{Y \in \mathscr{D}(R^{\rm op}) \mid Y \otimes^{\mathbb{L}}_R P^\bullet = 0\}$. However, by Lemma 2.1 (2), for a given $Y \in \mathscr{D}(R^{\rm op})$, the left-derived tensor functor $Y \otimes^{\mathbb{L}}_R - : \mathscr{D}(R) \to \mathscr{D}(\mathbb{Z})$ sends $\mathrm{Tria}(Q^\bullet)$ (respectively, $\mathrm{Tria}({}_RP^\bullet)$) to zero if and only if $Y \otimes^{\mathbb{L}}_R Q^\bullet = 0$ (respectively, $Y \otimes^{\mathbb{L}}_R P^\bullet = 0$). Since $\mathrm{Tria}(P^\bullet) = \mathrm{Tria}({}_RQ^\bullet)$ by assumption, we certainly obtain $\mathscr{Y}' = \{Y \in \mathscr{D}(R^{\rm op}) \mid Y \otimes^{\mathbb{L}}_R Q^\bullet = 0\}$.

Since $\lambda : R^{\rm op} \to S^{\rm op}$ is also a homological ring epimorphism, we obtain another recollement by Lemma 2.4:

$$\mathscr{D}(S^{\rm op}) \xrightarrow{\ D(\lambda_*)\ } \mathscr{D}(R^{\rm op}) \xrightarrow{\ G\ } \mathrm{Tria}(Q^\bullet_R) \quad (F : \mathrm{Tria}(Q^\bullet_R) \to \mathscr{D}(R^{\rm op}))$$

where F is the inclusion and G is the tensor functor $- \otimes^{\mathbb{L}}_R Q^\bullet$. This implies that $\mathrm{Im}(D(\lambda_*)) = \mathrm{Ker}(G)$ and

$$(b) \qquad \mathrm{Tria}(Q^\bullet_R) = \{X \in \mathscr{D}(R^{\rm op}) \mid \mathrm{Hom}_{\mathscr{D}(R^{\rm op})}(X, Y) = 0 \ \text{ for } \ Y \in \mathrm{Ker}(G)\}.$$

Since $\mathscr{Y}' = \mathrm{Ker}(G)$, we conclude from (a) and (b) that $\mathrm{Tria}(P^{\bullet *}) = \mathrm{Tria}(Q^\bullet_R)$. This finishes the proof of Lemma 2.5. $\square$

3 Algebraic K-theory

In this section, we briefly recall some basics on algebraic K-theory of Waldhausen categories and Frobenius pairs developed in [24] and [20], respectively. And we then discuss algebraic K-theory of differential graded algebras and prove a few lemmas as preparations for proofs of the main results.

3.1 K-theory spaces of small Waldhausen categories

Let us first recall some elementary notion and facts about the K-theory of small Waldhausen categories (see [24, 23, 18]).

Let $\mathcal{C}$ be a small Waldhausen category, that is, a pointed category (equipped with a zero object) with cofibrations and weak equivalences. In [24, Section 1.3], Waldhausen has defined a K-theory space $K(\mathcal{C})$ for $\mathcal{C}$, which is a pointed topological space, and an n-th homotopy group $K_n(\mathcal{C})$ of $K(\mathcal{C})$ for each $n \in \mathbb{N}$, which is called the n-th K-group of $\mathcal{C}$. Clearly, if a Waldhausen category $\mathcal{C}'$ is *essentially small*, that is, the isomorphism classes of objects of $\mathcal{C}'$ form a set, then the definition of Waldhausen K-theory still makes sense for $\mathcal{C}'$ because, in this case, one can choose a small Waldhausen subcategory $\mathcal{C}$ of $\mathcal{C}'$ such that $\mathcal{C}$ is equivalent to $\mathcal{C}'$, and define the K-theory of $\mathcal{C}'$ through that of $\mathcal{C}$.

Note that $K(\mathcal{C})$ is always homotopy equivalent to a CW-complex. In fact, this follows from the following observation: The classifying space of a small category has the structure of a CW-complex and the loop space of a CW-complex is homotopy equivalent to a CW-complex (see [13]), while $K(\mathcal{C})$ is the loop space of a classifying space constructed from $\mathcal{C}$.

The K-theory space defined by Waldhausen is natural in the following sense: Each exact functor $F : \mathcal{C} \to \mathcal{D}$ between Waldhausen categories $\mathcal{C}$ and $\mathcal{D}$ induces a continuous map $K(F) : K(\mathcal{C}) \to K(\mathcal{D})$ of (pointed) topological spaces, and a homomorphism $K_n(F) : K_n(\mathcal{C}) \to K_n(\mathcal{D})$ of abelian groups for each $n \in \mathbb{N}$. If $G : \mathcal{D} \to \mathcal{E}$ is another exact functor between Waldhausen categories, then $K(GF) = K(F)K(G)$ in our notation.

The cartesian product $\mathcal{C} \times \mathcal{C}$ of a Waldhausen category $\mathcal{C}$ is again a Waldhausen category with cofibrations and weak equivalences defined in an obvious way.

Note that finite coproducts always exist in $\mathcal{C}$, and that the coproduct functor

$$\sqcup : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}, \ (M,N) \mapsto M \sqcup N \ \text{ for all } \ M,N \in \mathcal{C},$$

is an exact functor between Waldhausen categories. More important, with the induced map $K(\sqcup) : K(\mathcal{C}) \times K(\mathcal{C}) \to K(\mathcal{C})$, the space $K(\mathcal{C})$ becomes a homotopy-associative pointed H-space, and the homomorphism $K_n(\sqcup) : K_n(\mathcal{C}) \times K_n(\mathcal{C}) \to K_n(\mathcal{C})$ is actually given by $(y,z) \mapsto y+z$ for $y,z \in K_n(\mathcal{C})$.

Recall that a pointed space (X,e) with X a topological space and $e \in X$ is called a *homotopy-associative pointed H-space* (see [22, Chapter 7]) if there is a pointed map $(-,-) : X \times X \to X$ satisfying the following two conditions:

(1) The maps $(e,-)$ and $(-,e)$ are pointed-homotopic to the identity Id_X of X.

(2) The respective composites of the following maps:

$$X \times (X \times X) \xrightarrow{Id_X \times (-,-)} X \times X \xrightarrow{(-,-)} X \quad \text{and} \quad (X \times X) \times X \xrightarrow{(-,-) \times Id_X} X \times X \xrightarrow{(-,-)} X$$

are pointed-homotopic.

Clearly, the associated point $e_{\mathcal{C}}$ of $K(\mathcal{C})$ corresponds to the image of the map $K(\{0\}) \to K(\mathcal{C})$ induced from the inclusion $\{0\} \hookrightarrow \mathcal{C}$, where 0 denotes the zero object of $\mathcal{C}$.

Next, we shall discuss some additivity of exact functors between Waldhausen categories.

Let $\mathcal{C}_i$ be a small Waldhausen category for $i = 1,2$. Denote by $\lambda_i : \mathcal{C}_i \to \mathcal{C}_1 \times \mathcal{C}_2$ and $p_i : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_i$ the canonical injection and projection, respectively. Then $K(\mathcal{C}_1 \times \mathcal{C}_2) = K(\mathcal{C}_1) \times K(\mathcal{C}_2)$ and

$$K(\lambda_1) : K(\mathcal{C}_1) \to K(\mathcal{C}_1 \times \mathcal{C}_2), \ c_1 \mapsto (c_1, e_{\mathcal{C}_2}),$$

$$K(\lambda_2) : K(\mathcal{C}_2) \to K(\mathcal{C}_1 \times \mathcal{C}_2), \ c_2 \mapsto (e_{\mathcal{C}_1}, c_2),$$

$$K(p_i) : K(\mathcal{C}_1 \times \mathcal{C}_2) \to K(\mathcal{C}_i), \ (c_1, c_2) \mapsto c_i$$

for $c_i \in K(\mathcal{C}_i)$ with $i = 1,2$.

On the one hand, if $G : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}$ is an exact functor, then $K(G) : K(\mathcal{C}_1) \times K(\mathcal{C}_2) \to K(\mathcal{C})$ is given by the composite of the following two maps:

$$K(G_1) \times K(G_2) : K(\mathcal{C}_1) \times K(\mathcal{C}_2) \to K(\mathcal{C}) \times K(\mathcal{C}) \text{ and } K(\sqcup) : K(\mathcal{C}) \times K(\mathcal{C}) \to K(\mathcal{C})$$

where $G_i : \mathcal{C}_i \to \mathcal{C}$ is defined to be the composition of λ_i with G. This is due to the following identities:

$$G(C_1, C_2) = G\big((C_1, 0) \sqcup (0, C_2)\big) = G(C_1, 0) \sqcup G(0, C_2) = G_1(C_1) \sqcup G_2(C_2)$$

for $C_i \in \mathcal{C}_i$.

On the other hand, if $H : \mathcal{C} \to \mathcal{C}_1 \times \mathcal{C}_2$ is an exact functor, then

$$K(H) = \big(K(H_1), K(H_2)\big) : K(\mathcal{C}) \to K(\mathcal{C}_1) \times K(\mathcal{C}_2)$$

where $H_i : \mathcal{C} \to \mathcal{C}_i$ is defined to be the composition of H with p_i.

Finally, we recall some definitions and basic facts in homotopy theory for later proofs. For more details, we refer the reader to [26, Chapters III and IV] and [22, Chapter 7]. Those readers who are familiar with homotopy theory may skip the rest of this subsection.

Let $(Y, y_0) \xrightarrow{g} (Z, z_0)$ be a map of pointed topological spaces. The *homotopy fibre* $F(g)$ of g is defined to be the following pointed topological space

$$F(g) := \{(\omega, y) \mid \omega : [0,1] \to Z,\, y \in Y,\, (0)\omega = z_0,\, (1)\omega = (y)g\}$$

with the base-point $\big(c_{z_0}, y_0\big)$, where c_{z_0} is the constant path $t \mapsto z_0$ for $t \in [0,1]$. If we define $h : F(g) \to Y$ by $(\omega, y) \mapsto y$ for any $(\omega, y) \in F(g)$, then there is a long exact sequence of homotopy groups:

$$\cdots \longrightarrow \pi_{n+1}(Z, z_0) \longrightarrow \pi_n\big(F(g), (c_{z_0}, y_0)\big) \xrightarrow{\pi_n(h)} \pi_n(Y, y_0) \xrightarrow{\pi_n(g)} \pi_n(Z, z_0) \longrightarrow \pi_{n-1}\big(F(g), (c_{z_0}, y_0)\big) \longrightarrow$$

$$\cdots \longrightarrow \pi_0\big(F(g), (c_{z_0}, y_0)\big) \longrightarrow \pi_0(Y, y_0) \longrightarrow \pi_0(Z, z_0)$$

where $\pi_n(Z, z_0)$ denotes *the n-th homotopy group* of (Z, z_0) for each $n \in \mathbb{N}$ (see [26, Corollary IV. 8.9]).

A sequence $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$ of pointed topological spaces is called *a homotopy fibration* if the composite of f and g is equal to the constant map which sends every x in X to the base-point of Z, and if the natural map

$$X \longrightarrow F(g), \quad x \mapsto \big(c_{z_0}, (x)f\big) \text{ for } x \in X$$

is a homotopy equivalence.

The sequence $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$ of pointed topological spaces is called *a weak homotopy fibration* if there is a pointed topological space (Z', z_0'), and two pointed maps $g_1 : Y \to Z'$ and $g_2 : Z' \to Z$ with $g = g_1 g_2$ such that

(1) the sequence $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g_1} (Z', z_0')$ is a homotopy fibration, and that

(2) g_2 induces an injection $\pi_0(Z', z_0') \to \pi_0(Z, z_0)$ and a bijection $\pi_n(Z', z_0') \to \pi_n(Z, z_0)$ for $n > 0$.

Assume that $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$ is a weak homotopy fibration. Then there is a long exact sequence of homotopy groups:

$$\cdots \longrightarrow \pi_{n+1}(Z, z_0) \longrightarrow \pi_n(X, x_0) \xrightarrow{\pi_n(f)} \pi_n(Y, y_0) \xrightarrow{\pi_n(g)} \pi_n(Z, z_0) \longrightarrow \pi_{n-1}(X, x_0) \longrightarrow$$

$$\cdots \longrightarrow \pi_0(X, x_0) \longrightarrow \pi_0(Y, y_0) \longrightarrow \pi_0(Z, z_0)$$

for all $n \in \mathbb{N}$, and g_2 induces a weak equivalence from the loop space $\Omega(Z', z_0')$ of (Z', z_0') to the one of (Z, z_0).

3.2 Frobenius pairs

We recall some definitions given in [20].

By a *Frobenius category* we mean an exact category (see [18, 10]) with enough projective and injective objects such that projectives and injectives coincide. A map between two Frobenius categories is an exact functor which preserves projective objects.

Let $\mathcal{C}$ be a Frobenius category.

We denote by $\mathcal{C}$-proj the full subcategory of $\mathcal{C}$ consisting of all projective objects. It is well known that the factor category $\underline{\mathcal{C}}$ of $\mathcal{C}$ modulo $\mathcal{C}$-proj, called the *stable category* of $\mathcal{C}$, is a triangulated category. Moreover, two objects X and Y of $\mathcal{C}$ are isomorphic in $\underline{\mathcal{C}}$ if and only if $X \oplus P \simeq Y \oplus Q$ in $\mathcal{C}$ for some $P, Q \in \mathcal{C}$-proj. In particular, $X \simeq 0$ in $\underline{\mathcal{C}}$ if and only if $X \in \mathcal{C}$-proj.

A subcategory $\mathcal{X}$ of $\mathcal{C}$ is called a *Frobenius subcategory* of $\mathcal{C}$ if $\mathcal{X}$ is a Frobenius category and the inclusion $\mathcal{X} \subseteq \mathcal{C}$ is a fully faithful map of Frobenius categories. In this case, $\mathcal{X}$-proj $\subseteq \mathcal{C}$-proj, and a morphism in $\mathcal{X}$ factorizes through $\mathcal{X}$-proj if and only if it factorizes through $\mathcal{C}$-proj. This implies that the inclusion $\mathcal{X} \subseteq \mathcal{C}$ induces a fully faithful inclusion $\underline{\mathcal{X}} \subseteq \underline{\mathcal{C}}$ of triangulated categories. In general, $\underline{\mathcal{X}}$ does not have to be a triangulated subcategory of $\underline{\mathcal{C}}$ since $\underline{\mathcal{X}}$ is not necessarily closed under isomorphisms in $\underline{\mathcal{C}}$. However, by our convention, the image of the inclusion $\underline{\mathcal{X}} \subseteq \underline{\mathcal{C}}$ is indeed a triangulated subcategory of $\underline{\mathcal{C}}$.

A pair $\mathbf{C} := (\mathcal{C}, \mathcal{C}_0)$ of Frobenius categories is called a *Frobenius pair* if $\mathcal{C}$ is a small category and $\mathcal{C}_0$ is a Frobenius subcategory of $\mathcal{C}$. A map from a Frobenius pair $(\mathcal{C}, \mathcal{C}_0)$ to another Frobenius pair $(\mathcal{C}', \mathcal{C}'_0)$ is a map of Frobenius categories $\mathcal{C} \to \mathcal{C}'$ such that it restricts to a map from $\mathcal{C}_0$ to $\mathcal{C}'_0$ (see [20, Section 4.3]).

Let $\mathbf{C} := (\mathcal{C}, \mathcal{C}_0)$ be a Frobenius pair. Then the image of the inclusion $\underline{\mathcal{C}_0} \subseteq \underline{\mathcal{C}}$ is a triangulated subcategory of $\underline{\mathcal{C}}$. So we can form the Verdier quotient of $\underline{\mathcal{C}}$ by this image, denoted by

$$\mathscr{D}_F(\mathbf{C}) := \underline{\mathcal{C}}/\underline{\mathcal{C}_{\mathbf{0}}}$$

which is called the *derived category* of the Frobenius pair $\mathbf{C}$. Here, we use the same notation $\underline{\mathcal{C}}/\underline{\mathcal{C}_0}$ as in [20] to denote the derived category of $\mathbf{C}$, but the meaning of $\underline{\mathcal{C}}/\underline{\mathcal{C}_0}$ in our paper is slightly different from the one in [20] because we require that the image of an inclusion functor is closed under isomorphisms. Nevertheless, all results in [20] work with this modified definition of derived categories.

Clearly, if $\mathcal{C}_0 = \mathcal{C}$-proj, then $\mathscr{D}_F(\mathbf{C}) = \underline{\mathcal{C}}$. In this case, we shall often write $\mathcal{C}$ for the Frobenius pair $(\mathcal{C}, \mathcal{C}\text{-proj})$.

The category $\mathcal{C}$ of a Frobenius pair $\mathbf{C} := (\mathcal{C}, \mathcal{C}_0)$ can be regarded as a small Waldhausen category (for definition, see [24] or [5]): The inflations in $\mathcal{C}$ form the cofibrations of $\mathcal{C}$, and the morphisms in $\mathcal{C}$ which are isomorphisms in $\mathscr{D}_F(\mathbf{C})$ form the weak equivalences of $\mathcal{C}$. In this note, we shall write $\mathbf{C}$ for the Waldhausen category $\mathcal{C}$ to emphasize the role of $\mathcal{C}_0$. According to our foregoing notation, we denote by $\mathcal{C}$ the Waldhausen category defined by the Frobenius pair $(\mathcal{C}, \mathcal{C}\text{-proj})$. For the Waldhausen category $\mathbf{C}$, we denote the K-theory space of $\mathbf{C}$ in the sense of Waldhausen by $K(\mathbf{C})$ which is a pointed topological space, and the n-th K-group of $K(\mathbf{C})$ by $K_n(\mathbf{C})$ for each $n \in \mathbb{N}$.

It is known that $K_0(\mathbf{C})$ is naturally isomorphic to the Grothendieck group $K_0(\mathscr{D}_F(\mathbf{C}))$ of the small triangulated category $\mathscr{D}_F(\mathbf{C})$ (see [23, Section 1.5.6], [25, Chapter IV, Proposition 8.4] and [21, Proposition 3.2.22]).

Let $G : \mathbf{C} \to \mathbf{C}'$ be a map of Frobenius pairs. On the one hand, G automatically induces a triangle functor $\mathscr{D}_F(G) : \mathscr{D}_F(\mathbf{C}) \to \mathscr{D}_F(\mathbf{C}')$, which sends $X \in \mathcal{C}$ to $G(X) \in \mathcal{C}'$. On the other hand, $G : \mathcal{C} \to \mathcal{C}'$ is an exact functor of associated Waldhausen categories, which induces a continuous map $K(G) : K(\mathbf{C}) \to K(\mathbf{C}')$.

In this paper, we assume that all Waldhausen categories considered arise from Frobenious pairs.

3.3 Examples of Frobenius pairs and their derived categories

Two typical examples of Frobenius pairs are of our interest.

(a) The first typical example of Frobenius pairs is provided by the categories of bounded complexes over exact categories.

Let $\mathscr{E}$ be a small exact category (for definition, see [18] and [10]). We denote by $\mathscr{C}^b(\mathscr{E})$ the category of bounded chain complexes over $\mathscr{E}$. Then $\mathscr{C}^b(\mathscr{E})$ is a small, exact category with degreewise split conflations, that is, a sequence $X^\bullet \to Y^\bullet \to Z^\bullet$ is a conflation in $\mathscr{C}^b(\mathscr{E})$ if $X^i \to Y^i \to Z^i$ is isomorphic to the split conflation $X^i \to X^i \oplus Z^i \to Z^i$ for each $i \in \mathbb{Z}$. Actually, $\mathscr{C}^b(\mathscr{E})$ is even a Frobenius category in which projective objects are exactly bounded contractible chain complexes over $\mathscr{E}$. Recall that a chain complex $X^\bullet$ is called *contractible* when the identity on $X^\bullet$ is null-homotopic. Moreover, the stable category of $\mathscr{C}^b(\mathscr{E})$ is the usual bounded homotopy category $\mathscr{K}^b(\mathscr{E})$, that is, $\mathscr{D}_F(\mathscr{C}^b(\mathscr{E})) = \mathscr{K}^b(\mathscr{E})$.

Recall that a complex $X^\bullet = (X^i, d^i)_{i\in\mathbb{Z}}$ over $\mathscr{E}$ is called *acyclic* if d^i is a composite of a deflation π^i with an inflation λ^i such that (λ^i, π^{i+1}) is a conflation for all *i*. Let $\mathscr{C}^b_{ac}(\mathscr{E}) \subseteq \mathscr{C}^b(\mathscr{E})$ be the full subcategory of objects which are homotopy equivalent to acyclic chain complexes over $\mathscr{E}$. Then $\mathscr{C}^b_{ac}(\mathscr{E})$ contains all projective objects of the Frobenius category $\mathscr{C}^b(\mathscr{E})$, and is closed under extensions, kernels of deflations as well as cokernels of inflations in $\mathscr{C}^b(\mathscr{E})$. Thus $\mathscr{C}^b_{ac}(\mathscr{E})$ inherits a Frobenius structure from $\mathscr{C}^b(\mathscr{E})$ and

$$\mathbf{C} := \left(\mathscr{C}^b(\mathscr{E}), \mathscr{C}^b_{ac}(\mathscr{E})\right)$$

is a Frobenius pair. In particular, the pair $\mathbf{C}$ (or the associated category $\mathscr{C}^b(\mathscr{E})$) can be regarded as a Waldhausen category: A chain map $f^\bullet : X^\bullet \to Y^\bullet$ in $\mathscr{C}^b(\mathscr{E})$ is called a cofibration if $f^i : X^i \to Y^i$ is a split inflation in $\mathscr{E}$ for each $i \in \mathbb{Z}$; a weak equivalence if the mapping cone of $f^\bullet$ belongs to $\mathscr{C}^b_{ac}(\mathscr{E})$. Moreover, $\mathscr{D}_F(\mathbf{C})$ coincides with the bounded derived category $\mathscr{D}^b(\mathscr{E})$ of $\mathscr{C}^b(\mathscr{E})$, which is defined as follows:

Let $\mathscr{E}'$ be an arbitrary exact category. The objects of $\mathscr{D}^b(\mathscr{E}')$ are the objects of $\mathscr{C}^b(\mathscr{E}')$. The morphisms of $\mathscr{D}^b(\mathscr{E}')$ are obtained from the chain maps by formally inverting the maps whose mapping cones are acyclic (as complexes of objects in $\mathscr{E}'$). For example, if $\mathscr{E}'$ is the usual exact category R-Mod with R a ring, then $\mathscr{D}^b(\mathscr{E}')$ is the usual derived category $\mathscr{D}^b(R)$. Further, any exact functor $F : \mathscr{E}_1 \to \mathscr{E}_2$ between exact categories induces a triangle functor $D(F) : \mathscr{D}^b(\mathscr{E}_1) \to \mathscr{D}^b(\mathscr{E}_2)$. For more details, see [10].

Assume that the exact structure of $\mathscr{E}$ is induced from an abelian category $\mathscr{A}$. That is, $\mathscr{E} \subseteq \mathscr{A}$ is a full subcategory such that it is closed under extensions, and that a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ with all terms in $\mathscr{E}$ is a conflation in $\mathscr{E}$ if and only if $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is an exact sequence in $\mathscr{A}$. Furthermore, assume that $\mathscr{E}$ is closed under kernels of epimorphisms in the abelian category. In this case, the chain map $f^\bullet : X^\bullet \to Y^\bullet$ is a weak equivalence in $\mathbf{C}$ if and only if $f^\bullet$ is a quasi-isomorphism in $\mathscr{C}(\mathscr{A})$, that is, $H^i(f^\bullet) : H^i(X^\bullet) \to H^i(Y^\bullet)$ is an isomorphism in $\mathscr{A}$ for each $i \in \mathbb{Z}$.

Note that an exact category $\mathscr{E}$ itself can also be understood as a Waldhausen category with cofibrations being inflations, and weak equivalences being isomorphisms. Up to now, there are at least three algebraic K-theory spaces associated with a small exact category $\mathscr{E}$: The Quillen K-theory space of the exact category $\mathscr{E}$, the Waldhausen K-theory space with respect to the Waldhausen category $\mathscr{E}$, and the Waldhausen K-theory space of the Waldhausen category defined by the Frobenius pair $\left(\mathscr{C}^b(\mathscr{E}), \mathscr{C}^b_{ac}(\mathscr{E})\right)$. However, these spaces are the same up to homotopy equivalence (see [24, Section 1.9]) and [23, Theorem 1.11.7]). So, in this paper, we always identify these spaces.

(b) The next example of Frobenius pairs is constructed from categories of finitely generated projective modules.

Let R be a ring. Then the category R-proj of finitely generated projective R-modules is a small exact category with split, short exact sequences as its conflations. Clearly, this exact structure on R-proj is induced from the usual exact structure of the abelian category R-Mod. Following Quillen [18], the *algebraic K-theory space* $K(R)$ of R is defined to be the space $K(R\text{-proj})$ of R-proj, and the *n-th algebraic K-group* $K_n(R)$ of R to be the n-th homotopy group of $K(R)$.

According to (a), the pairs $\left(\mathscr{C}^b(R\text{-proj}), \mathscr{C}^b_{ac}(R\text{-proj})\right)$ and $\left(\mathscr{C}^b(\mathscr{P}^{<\infty}(R)), \mathscr{C}^b_{ac}(\mathscr{P}^{<\infty}(R))\right)$ are Frobenius pairs. In this way, both $\mathscr{C}^b(R\text{-proj})$ and $\mathscr{C}^b(\mathscr{P}^{<\infty}(R))$ can be regarded as small Waldhausen categories. Note that $\mathscr{P}^{<\infty}(R)$ is a small exact category.

It is easy to see that $\mathscr{C}^b_{ac}(R\text{-proj})$ consists of all bounded contractible chain complexes over R-proj, which are exactly projective objects in the Frobenius category $\mathscr{C}^b(R\text{-proj})$, that is, $\mathscr{C}^b(R\text{-proj})\text{-proj} = \mathscr{C}^b_{ac}(R\text{-proj})$. Thus $\mathscr{D}_F(\mathscr{C}^b(R\text{-proj}))$ is the bounded homotopy category $\mathscr{K}^b(R\text{-proj})$. Since each compact object of $\mathscr{D}(R)$ is quasi-isomorphic to an object of $\mathscr{C}^b(R\text{-proj})$, we know that $\mathscr{K}^b(R\text{-proj})$ is equivalent to $\mathscr{D}^c(R)$ via the Verdier localization functor $\mathscr{K}(R) \to \mathscr{D}(R)$.

Hence, we see that $K(R)$, $K(\mathscr{C}^b(R\text{-proj}))$ and $K(\mathbf{C})$ with $\mathbf{C} := \big(\mathscr{C}^b(R\text{-proj}), \mathscr{C}^b_{ac}(R\text{-proj})\big)$ are homotopy equivalent, and therefore their algebraic K_n-groups are all isomorphic.

Note that $\mathscr{C}^b(\mathscr{P}^{<\infty}(R))\text{-proj} \neq \mathscr{C}^b_{ac}(\mathscr{P}^{<\infty}(R))$ in general.

Let S be another ring and $M^\bullet$ a bounded complex of R-S-bimodules. If ${}_RM^\bullet \in \mathscr{C}^b(R\text{-proj})$, then the tensor functor $M^\bullet \otimes^\bullet_S - : \mathscr{C}^b(S\text{-proj}) \to \mathscr{C}^b(R\text{-proj})$ is a well-defined map of Frobenius pairs.

Finally, we establish a useful result about constructing maps between some special Frobenius pairs.

Lemma 3.1. *Let R and S be rings, and let $Q^\bullet \in \mathscr{C}^b(R \otimes_{\mathbb{Z}} S^{\mathrm{op}})$ such that ${}_RQ^n \in \mathscr{P}^{<\infty}(R)$ for all $n \in \mathbb{Z}$. Consider the following Frobenius pairs:*

$$\mathbf{A} := \big(\mathscr{C}^b(S\text{-proj}), \mathscr{C}^b_{ac}(S\text{-proj})\big) \text{ and } \mathbf{B} := \big(\mathscr{C}^b(\mathscr{P}^{<\infty}(R)), \mathscr{C}^b_{ac}(\mathscr{P}^{<\infty}(R))\big).$$

Then the following statements hold:

(1) *The functor ${}_RQ^\bullet \otimes^\bullet_S - : \mathbf{A} \to \mathbf{B}$ is a well-defined map of Frobenius pairs.*

(2) *The induced functor $\mathscr{D}_F({}_RQ^\bullet \otimes^\bullet_S -) : \mathscr{D}_F(\mathbf{A}) \to \mathscr{D}_F(\mathbf{B})$ of derived categories is given by the composition of the following functors:*

$$\mathscr{D}_F(\mathbf{A}) = \mathscr{D}_F(\mathscr{C}^b(S\text{-proj})) = \mathscr{K}^b(S\text{-proj}) \xrightarrow{{}_RQ^\bullet \otimes^\bullet_S -} \mathscr{K}^b(\mathscr{P}^{<\infty}(R)) \xrightarrow{q} \mathscr{D}^b(\mathscr{P}^{<\infty}(R)) = \mathscr{D}_F(\mathbf{B})$$

where q is the Verdier localization functor.

Proof. Recall that $\mathscr{C}^b(S\text{-proj})$ and $\mathscr{C}^b(\mathscr{P}^{<\infty}(R))$ are Frobenius categories in which the conflations are degreewise split exact sequences of chain complexes, and the projective objects are bounded contractible chain complexes over S-proj and $\mathscr{P}^{<\infty}(R)$, respectively.

Since $Q^\bullet \in \mathscr{C}^b(R \otimes_{\mathbb{Z}} S^{\mathrm{op}})$ with ${}_RQ^n \in \mathscr{P}^{<\infty}(R)$ for all $n \in \mathbb{Z}$, we have ${}_RQ^\bullet \in \mathscr{C}^b(\mathscr{P}^{<\infty}(R))$, and therefore $G := {}_RQ^\bullet \otimes^\bullet_S - : \mathscr{C}^b(S\text{-proj}) \to \mathscr{C}^b(\mathscr{P}^{<\infty}(R))$ is an additive functor. Clearly, G preserves both degreewise split conflations and contractible chain complexes. Thus G is a map of Frobenius categories. In particular, G induces a triangle functor $\mathscr{K}^b(S\text{-proj}) \to \mathscr{K}^b(\mathscr{P}^{<\infty}(R))$ of homotopy categories. To show (1), it remains to check that G can restrict to a functor $\mathscr{C}^b_{ac}(S\text{-proj}) \to \mathscr{C}^b_{ac}(\mathscr{P}^{<\infty}(R))$. However, this follows from the following two observations:

(I) $\mathscr{C}^b_{ac}(S\text{-proj})$ consists of all bounded contractible chain complexes over S-proj, which are exactly projective objects in the Frobenious category $\mathscr{C}^b(S\text{-proj})$.

(II) All bounded contractible chain complexes over $\mathscr{P}^{<\infty}(R)$ belong to $\mathscr{C}^b_{ac}(\mathscr{P}^{<\infty}(R))$.

Thus G is a map of Frobenius pairs. This shows (1).

Recall that $\mathscr{D}_F(G) : \mathscr{D}_F(\mathbf{A}) \to \mathscr{D}_F(\mathbf{B})$ is defined by $X \mapsto G(X)$ for $X \in \mathscr{C}^b(S\text{-proj})$. Clearly, (2) holds. □

3.4 Fundamental theorems in algebraic K-theory of Frobenius pairs

Now, we recall some basic results on algebraic K-theory of Frobenious pairs in terms of derived categories. Our main reference in this section is the paper [20] by Schlichting.

The following localization theorem may trace back to the localization theorem in [18, Section 5, Theorem 5] for exact categories, the fibration theorem in [24, Theorem 1.6.4] for Waldhausen categories, and the localization theorem in [23, Theorem 1.8.2] for complicial biWaldhausen categories. For a proof of the present form, we refer the reader to [20, Propositions 3 and 5, p.126 and p.128]. Also, the approximation and cofinality theorems are taken from [20, Propositions 3 and 4].

Lemma 3.2. (1) Localization Theorem:

Let $\mathbf{A} \xrightarrow{F} \mathbf{B} \xrightarrow{G} \mathbf{C}$ *be a sequence of Frobenius pairs. If the sequence* $\mathscr{D}_F(\mathbf{A}) \xrightarrow{\mathscr{D}_F(F)} \mathscr{D}_F(\mathbf{B}) \xrightarrow{\mathscr{D}_F(G)} \mathscr{D}_F(\mathbf{C})$ *of derived categories is exact, then the induced sequence* $K(\mathbf{A}) \xrightarrow{K(F)} K(\mathbf{B}) \xrightarrow{K(G)} K(\mathbf{C})$ *of* K*-theory spaces is a homotopy fibration, and therefore there is a long exact sequence of* K*-groups*

$$\cdots \longrightarrow K_{n+1}(\mathbf{C}) \longrightarrow K_n(\mathbf{A}) \xrightarrow{K_n(F)} K_n(\mathbf{B}) \xrightarrow{K_n(G)} K_n(\mathbf{C}) \longrightarrow K_{n-1}(\mathbf{A}) \longrightarrow$$

$$\cdots \longrightarrow K_0(\mathbf{A}) \longrightarrow K_0(\mathbf{B}) \longrightarrow K_0(\mathbf{C}) \longrightarrow 0$$

for all $n \in \mathbb{N}$.

(2) Approximation Theorem:

Let $G : \mathbf{B} \to \mathbf{C}$ *be a map of Frobenius pairs. If the associated functor* $\mathscr{D}_F(G) : \mathscr{D}_F(\mathbf{B}) \to \mathscr{D}_F(\mathbf{C})$ *of derived categories is an equivalence, then the induced map* $K(G) : K(\mathbf{B}) \to K(\mathbf{C})$ *of* K*-theory spaces is a homotopy equivalence. In particular,* $K_n(G) : K_n(\mathbf{B}) \xrightarrow{\simeq} K_n(\mathbf{C})$ *for all* $n \in \mathbb{N}$.

(3) Cofinality Theorem:

Let $G : \mathbf{B} \to \mathbf{C}$ *be a map of Frobenius pairs. If the associated functor* $\mathscr{D}_F(G) : \mathscr{D}_F(\mathbf{B}) \to \mathscr{D}_F(\mathbf{C})$ *of derived categories is an equivalence up to factors, then the induced map* $K(G) : K(\mathbf{B}) \to K(\mathbf{C})$ *of* K*-theory spaces gives rise to an injection* $K_0(G) : K_0(\mathbf{B}) \longrightarrow K_0(\mathbf{C})$ *and an isomorphism:* $K_n(G) : K_n(\mathbf{B}) \xrightarrow{\simeq} K_n(\mathbf{C})$ *for all* $n > 0$.

Note that the surjectivity of the last map in the long exact sequence in Lemma 3.2 (1) follows from the fact that $K_0(\mathbf{C})$ is isomorphic to the Grothendieck group $K_0\big(\mathscr{D}_F(\mathbf{C})\big)$ of $\mathscr{D}_F(\mathbf{C})$.

The following result is a slight variation of [20, Section 6.1] which has been mentioned there without proof. For the convenience of the reader, we include here a proof (see also [16, Lemma 2.5] for a special case).

Lemma 3.3. Thickness Theorem:

Let $\mathbf{C} := (\mathcal{C}, \mathcal{C}_0)$ *be a Frobenius pair. Suppose that there is a triangulated category* $\mathscr{C}$ *together with a triangle equivalence* $G : \mathscr{D}_F(\mathbf{C}) \to \mathscr{C}$. *Let* $\mathscr{X}$ *be a full triangulated subcategory of* $\mathscr{C}$. *Define* $\mathcal{X}$ *to be the full subcategory of* $\mathcal{C}$ *consisting of objects* X *such that* $G(X) \in \mathscr{X}$. *Then the following statements are true:*

(1) *The category* $\mathcal{X}$ *contains* $\mathcal{C}_0$ *and is closed under extensions in* $\mathcal{C}$. *Moreover,* $\mathcal{X}$ *naturally inherits a Frobenius structure from* $\mathcal{C}$, *and becomes a Frobenius subcategory of* $\mathcal{C}$ *such that* $\mathcal{X}$-proj = $\mathcal{C}$-proj.

(2) *Both* $\mathbf{X} := (\mathcal{X}, \mathcal{C}_0)$ *and* $\mathbf{C}_{\mathscr{X}} := (\mathcal{C}, \mathcal{X})$ *are Frobenius pairs, and the inclusion functor* $\mathcal{X} \to \mathcal{C}$ *and the identity functor* $\mathcal{C} \to \mathcal{C}$ *induce the following commutative diagram of triangulated categories:*

$$\begin{array}{ccccc}
\mathscr{D}_F(\mathbf{X}) & \hookrightarrow & \mathscr{D}_F(\mathbf{C}) & \longrightarrow & \mathscr{D}_F(\mathbf{C}_{\mathscr{X}}) \\
\downarrow \simeq & & G \downarrow \simeq & & \downarrow \simeq \\
\mathscr{X} & \hookrightarrow & \mathscr{C} & \longrightarrow & \mathscr{C}/\mathscr{X}
\end{array}$$

(3) *If* $\mathscr{X}$ *is closed under direct summands in* $\mathscr{C}$, *then both rows in the diagram of* (2) *are exact sequences of triangulated categories.*

Proof. (1) By definition of $\mathscr{D}_F(\mathbf{C}) := \underline{\mathcal{C}}/\underline{\mathcal{C}_0}$, the objects of $\mathscr{D}_F(\mathbf{C})$ are the same as the objects of $\mathcal{C}$. Thus, if $M \in \mathcal{C}_0$ or $M \in \mathcal{C}$-proj, then $M \simeq 0$ in $\mathscr{D}_F(\mathbf{C})$. This implies that $\mathcal{X}$ contains both $\mathcal{C}_0$ and $\mathcal{C}$-proj. Since G is a triangle functor and $\mathscr{X}$ is a full triangulated subcategory of $\mathscr{C}$, it is easy to see that $\mathcal{X}$ is closed under extensions in $\mathcal{C}$.

Since $\mathcal{X}$ is closed under extensions in $\mathcal{C}$, we can endow $\mathcal{X}$ with an exact structure induced from the one of $\mathcal{C}$, namely, a sequence $X \to Y \to Z$ with all terms in $\mathcal{X}$ is called a conflation in $\mathcal{X}$ if it is a conflation in

$\mathcal{C}$. Then one can check that, with this exact structure, $\mathcal{X}$ becomes an exact category. Now, we claim that $\mathcal{X}$ is even a Frobenius category such that $\mathcal{X}$-proj = $\mathcal{C}$-proj. Indeed, it suffices to show that if $L \to P \to N$ is a conflation in $\mathcal{C}$ with $P \in \mathcal{C}$-proj, then $L \in \mathcal{X}$ if and only if $N \in \mathcal{X}$. Actually, such a conflation can be extended to a distinguished triangle $L \to P \to N \to L[1]$ in $\underline{\mathcal{C}}$, and further, to a distinguished triangle in $\mathscr{D}_F(\mathbf{C})$. Since $P \simeq 0$ in $\mathscr{D}_F(\mathbf{C})$, we have $N \simeq L[1]$ in $\mathscr{D}_F(\mathbf{C})$. As $\mathscr{X}$ is closed under shifts in $\mathscr{C}$ and G is a triangle functor, we know that $G(L) \in \mathscr{X}$ if and only if $G(N) \in \mathscr{X}$. In other words, $L \in \mathcal{X}$ if and only if $N \in \mathcal{X}$. This verifies the claim.

(2) Note that $\mathcal{C}_0 \subseteq \mathcal{X} \subseteq \mathcal{C}$ and $\mathcal{C}_0$-proj $\subseteq \mathcal{X}$-proj $= \mathcal{C}$-proj. Thus $\mathbf{X} := (\mathcal{X}, \mathcal{C}_0)$ and $\mathbf{C}_{\mathscr{X}} := (\mathcal{C}, \mathcal{X})$ are Frobenius pairs.

Recall that $\mathscr{D}_F(\mathbf{X}) := \underline{\mathcal{X}}/\underline{\mathcal{C}_0}$ and $\mathscr{D}_F(\mathbf{C}_{\mathscr{X}}) := \underline{\mathcal{C}}/\underline{\mathcal{X}}$. Clearly, the inclusion functor $\lambda : \mathcal{X} \to \mathcal{C}$ and the identity functor $Id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ are maps from the Frobenius pair $\mathbf{X}$ to the Frobenius pairs $\mathbf{C}$, and from $\mathbf{C}$ to $\mathbf{C}_{\mathscr{X}}$, respectively. So we have two triangle functors $\mathscr{D}_F(\lambda) : \underline{\mathcal{X}}/\underline{\mathcal{C}_0} \to \underline{\mathcal{C}}/\underline{\mathcal{C}_0}$ and $\mathscr{D}_F(Id_{\mathcal{C}}) : \underline{\mathcal{C}}/\underline{\mathcal{C}_0} \to \underline{\mathcal{C}}/\underline{\mathcal{X}}$, which are induced from the inclusion $\underline{\mathcal{X}} \subseteq \underline{\mathcal{C}}$ and the identity functor of $\underline{\mathcal{C}}$, respectively.

Clearly, $\underline{\mathcal{X}}$ contains $\underline{\mathcal{C}_0}$, that is, the objects of $\underline{\mathcal{C}_0}$ is a subclass of the objects of $\underline{\mathcal{X}}$ with the morphism set $\mathrm{Hom}_{\underline{\mathcal{C}_0}}(X,Y) = \mathrm{Hom}_{\underline{\mathcal{X}}}(X,Y)$ for all objects X,Y in $\mathcal{C}_0$. Since the inclusion $\underline{\mathcal{X}} \subseteq \underline{\mathcal{C}}$ is fully faithful, the functor $\mathscr{D}_F(\lambda)$ is also a fully faithful inclusion which gives rise to the following commutative diagram:

$$(*) \qquad \begin{array}{ccc} \underline{\mathcal{X}}/\underline{\mathcal{C}_0} & \xhookrightarrow{\mathscr{D}_F(\lambda)} & \underline{\mathcal{C}}/\underline{\mathcal{C}_0} \\ \downarrow \simeq & & G \downarrow \simeq \\ \mathscr{X} & \hookrightarrow & \mathscr{C}. \end{array}$$

Consequently, G induces a triangle equivalence

$$G_1 : (\underline{\mathcal{C}}/\underline{\mathcal{C}_0})/(\underline{\mathcal{X}}/\underline{\mathcal{C}_0}) \xrightarrow{\simeq} \mathscr{C}/\mathscr{X}.$$

By the universal property of the Verdier localization functor $q_1 : \underline{\mathcal{C}} \to \underline{\mathcal{C}}/\underline{\mathcal{X}}$ (respectively, $q_2 : \underline{\mathcal{C}}/\underline{\mathcal{C}_0} \to (\underline{\mathcal{C}}/\underline{\mathcal{C}_0})/(\underline{\mathcal{X}}/\underline{\mathcal{C}_0})$), there is a triangle functor $\phi : \underline{\mathcal{C}}/\underline{\mathcal{X}} \longrightarrow (\underline{\mathcal{C}}/\underline{\mathcal{C}_0})/(\underline{\mathcal{X}}/\underline{\mathcal{C}_0})$ (respectively, $\psi : (\underline{\mathcal{C}}/\underline{\mathcal{C}_0})/(\underline{\mathcal{X}}/\underline{\mathcal{C}_0}) \to \underline{\mathcal{C}}/\underline{\mathcal{X}}$) such that $q_2 q_0 = \phi q_1$ (respectively, $\mathscr{D}_F(Id_{\mathcal{C}}) = \psi q_2$), where $q_0 : \underline{\mathcal{C}} \to \underline{\mathcal{C}}/\underline{\mathcal{C}_0}$ is the Verdier localization functor. Since $q_1 = \mathscr{D}_F(Id_{\mathcal{C}})q_0$, we have

$$\psi\phi q_1 = \psi q_2 q_0 = \mathscr{D}_F(Id_{\mathcal{C}})q_0 = q_1 \quad \text{and} \quad \phi\mathscr{D}_F(Id_{\mathcal{C}})q_0 = \phi q_1 = q_2 q_0.$$

It follows that $\psi\phi = Id$ and $\phi\mathscr{D}_F(Id_{\mathcal{C}}) = q_2$. As $\phi\psi q_2 = \phi\psi\phi\mathscr{D}_F(Id_{\mathcal{C}}) = \phi\mathscr{D}_F(Id_{\mathcal{C}}) = q_2$, we obtain $\phi\psi = Id$. Thus ϕ is a triangle isomorphism.

Now, we define $\overline{G} := G_1\phi : \underline{\mathcal{C}}/\underline{\mathcal{X}} \to \mathscr{C}/\mathscr{X}$. Then the following diagram of triangulated categories

$$(**) \qquad \begin{array}{ccc} \underline{\mathcal{C}}/\underline{\mathcal{C}_0} & \xrightarrow{\mathscr{D}_F(Id_{\mathcal{C}})} & \underline{\mathcal{C}}/\underline{\mathcal{X}} \\ G \downarrow \simeq & & \overline{G} \downarrow \simeq \\ \mathscr{C} & \xrightarrow{q} & \mathscr{C}/\mathscr{X}. \end{array}$$

is commutative, where q is the Verdier localization functor. Now, (2) follows from $(*)$ and $(**)$.

(3) In this case, $\mathscr{X}$ is the kernel of the localization functor $q : \mathscr{C} \to \mathscr{C}/\mathscr{X}$. Thus (3) follows. $\square$.

3.5 Algebraic K-theory of differential graded algebras

In this subsection, we shall give a definition of K-theory spaces of differential graded algebras, which generalizes the one of K-theory spaces of ordinary rings and modifies slightly the definition in [20]. But, at the level of homotopy groups, the two definitions give the isomorphic algebraic K_n-groups for $n \in \mathbb{N}$.

Throughout this subsection, k stands for an arbitrary but fixed commutative ring (for example, the ring $\mathbb{Z}$ of integers), and all rings considered here are k-algebras. Note that each ordinary ring with identity can be regarded as a $\mathbb{Z}$-algebra.

Let $\mathbb{A}$ be a differential graded (dg) associative and unitary k-algebra, that is, $\mathbb{A} = \oplus_{n\in\mathbb{Z}} A^n$ is a $\mathbb{Z}$-graded k-algebra with a differential $d^n : A^n \to A^{n+1}$ such that $(A^n, d^n)_{n\in\mathbb{Z}}$ is a chain complex of k-modules and

$$(xy)d^{m+n} = x(yd^n) + (-1)^n(xd^m)y$$

for $m \in \mathbb{Z}$, $x \in A^m$ and $y \in A^n$. Thus the map $\mathbb{A} \otimes_k^\bullet \mathbb{A} \to \mathbb{A}$, $a \otimes_k b \mapsto ba$ for $a, b \in \mathbb{A}$, is a chain map.

A left dg $\mathbb{A}$-module $M^\bullet$ is a $\mathbb{Z}$-graded left module $M^\bullet = \oplus_{n\in\mathbb{Z}} M^n$ over the $\mathbb{Z}$-graded k-algebra $\mathbb{A}$, with a differential d such that $(M^n, d)_{n\in\mathbb{Z}}$ is a complex of k-modules, and for any $a \in A^m, x \in M^n$, the following holds:

$$(ax)d^{m+n} = a(xd^n) + (-1)^n(ad^m)x.$$

In particular, each dg $\mathbb{A}$-module is a $\mathbb{Z}$-graded $\mathbb{A}$-module (forgetting the differential).

We should observe that the dg algebra $(\mathbb{A}, d)$ and left dg $\mathbb{A}$-module $M^\bullet$ defined in this paper are actually the dg algebra $(\mathbb{A}^{\mathrm{op}}, d)$ and right dg $\mathbb{A}^{\mathrm{op}}$-module in the sense of [9, Summary], respectively.

For a dg $\mathbb{A}$-module $M^\bullet$, we denote by $M^\bullet[1]$ the shift of $M^\bullet$ by degree 1.

A homomorphism $f^\bullet : M^\bullet \to N^\bullet$ of dg $\mathbb{A}$-modules is a chain map of complexes over k, which commutes with the $\mathbb{A}$-actions on $M^\bullet$ and $N^\bullet$. We say that $f^\bullet$ is a *quasi-isomorphism* if it is a quasi-isomorphism as a chain map of complexes over k, that is, $H^i(f^\bullet) : H^i(M^\bullet) \to H^i(N^\bullet)$ is an isomorphism for every $i \in \mathbb{Z}$. For more details, we refer to [9, Summary].

We denote by $\mathscr{C}(\mathbb{A})$ the category of left dg $\mathbb{A}$-modules. It is known that $\mathscr{C}(\mathbb{A})$ is a Frobenius category (see [9, Section 2]) by declaring a conflation to be a short sequence of dg $\mathbb{A}$-modules such that the underlying sequence of graded $\mathbb{A}$-modules (forgetting differentials) is split exact. The stable category of $\mathscr{C}(\mathbb{A})$ is the dg homotopy category $\mathscr{K}(\mathbb{A})$ in which the objects are the dg $\mathbb{A}$-modules and the morphisms are the homotopy classes of homomorphisms of dg $\mathbb{A}$-modules. By inverting all quasi-isomorphisms of dg $\mathbb{A}$-modules, we obtain the *dg derived category* $\mathscr{D}(\mathbb{A})$ of $\mathbb{A}$. This is a triangulated category and generated by the dg module $\mathbb{A}$, that is, $\mathscr{D}(\mathbb{A}) = \mathrm{Tria}(\mathbb{A})$.

Observe that an ordinary k-algebra A can be regarded as a dg algebra concentrated in degree 0, and that the above-mentioned categories $\mathscr{C}(A)$, $\mathscr{K}(A)$ and $\mathscr{D}(A)$ coincide with the usual complex, homotopy and derived categories of A-modules, respectively. In this case, each dg A-module is exactly a complex of A-modules, and a homomorphism of dg A-modules is a chain map of complexes over A. Moreover, for any $X^\bullet \in \mathscr{C}(A)$, the dg endomorphism algebra $\mathrm{End}_A^\bullet(X^\bullet)$ is a dg algebra with the differential and multiplication $\circ$ given in Subsection 2.2. By the formula on the multiplication $\circ$, if $Y^\bullet \in \mathscr{C}(A)$ is another dg A-module, then the Hom-complex $\mathrm{Hom}_A^\bullet(X^\bullet, Y^\bullet)$ is actually a left dg $\mathrm{End}_A^\bullet(X^\bullet)$- and right dg $\mathrm{End}_A^\bullet(Y^\bullet)$- bimodule.

A dg $\mathbb{A}$-module is said to be *acyclic* if it is acyclic as a complex of k-modules. A dg $\mathbb{A}$-module $M^\bullet$ is said to have the *property* (P) if $\mathrm{Hom}_{\mathscr{K}(\mathbb{A})}(M^\bullet, N^\bullet) = 0$ for any acyclic dg $\mathbb{A}$-module $N^\bullet$. Note that the class of dg $\mathbb{A}$-modules with the property (P) is closed under extensions, shifts, direct summands and direct sums in $\mathscr{C}(\mathbb{A})$. We denote by $\mathscr{K}(\mathbb{A})_p$ the full subcategory of $\mathscr{K}(\mathbb{A})$ consisting of all modules with the property (P). Then $\mathscr{K}(\mathbb{A})_p \subseteq \mathscr{K}(\mathbb{A})$ is a triangulated subcategory containing $\mathbb{A}$ and being closed under direct sums. More important, the Verdier localization functor $q : \mathscr{K}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ restricts to a triangle equivalence $\widetilde{q} : \mathscr{K}(\mathbb{A})_p \xrightarrow{\simeq} \mathscr{D}(\mathbb{A})$ (see [9, Section 3.1]). Particularly, this implies that any quasi-isomorphism between two dg $\mathbb{A}$-modules with the property (P) is an isomorphism in $\mathscr{K}(\mathbb{A})$ and that, for each dg $\mathbb{A}$-module $M^\bullet$, there is a (functorial) quasi-isomorphism ${}_pM^\bullet \to M^\bullet$ of dg $\mathbb{A}$-modules such that ${}_pM^\bullet$ has the property (P).

Let $\mathbb{B}$ be another dg algebra and $U^\bullet$ a dg $\mathbb{B}$-$\mathbb{A}$-bimodule. For a dg $\mathbb{A}$-module $V^\bullet$, we define $U^\bullet \otimes_{\mathbb{A}}^\bullet V^\bullet$ to be the quotient complex of $U^\bullet \otimes_k^\bullet V^\bullet$ modulo the subcomplex $W^\bullet := (W^n)_{n\in\mathbb{Z}}$, where W^n is the k-submodule of $U^\bullet \otimes_k^n V^\bullet$ generated by all elements $ua \otimes v - u \otimes av$ for $u \in U^r$, $a \in \mathbb{A}^s$ and $v \in V^t$ with $r, s, t \in \mathbb{Z}$ and

$n = r+s+t$. Then $U^\bullet \otimes^\bullet_{\mathbb{A}} V^\bullet$ is indeed a dg $\mathbb{B}$-module. This gives rise to the following tensor functor

$$U^\bullet \otimes^\bullet_{\mathbb{A}} - : \mathscr{C}(\mathbb{A}) \longrightarrow \mathscr{C}(\mathbb{B}),\ V^\bullet \mapsto U^\bullet \otimes^\bullet_{\mathbb{A}} V^\bullet.$$

Furthermore, the total left-derived functor $U^\bullet \otimes^{\mathbb{L}}_{\mathbb{A}} - : \mathscr{D}(\mathbb{A}) \to \mathscr{D}(\mathbb{B})$ of this tensor functor is defined by $V^\bullet \mapsto U^\bullet \otimes^\bullet_{\mathbb{A}} ({}_pV^\bullet)$ (see [9, Section 6]). In particular, if $V^\bullet$ has the property (P), then $U^\bullet \otimes^{\mathbb{L}}_{\mathbb{A}} V^\bullet = U^\bullet \otimes^\bullet_{\mathbb{A}} V^\bullet$ in $\mathscr{D}(\mathbb{B})$. Note that if $\mathbb{A}$ and $\mathbb{B}$ are dg algebras concentrated in degree 0, then the above tensor functor and total left-derived functor coincide with the ones defined in Subsection 2.2.

A dg $\mathbb{A}$-module M is called *relatively countable projective* (respectively, *countable projective*) if there is a dg $\mathbb{A}$-module N such that $M \oplus N$ is isomorphic to $\bigoplus_{i\in I} \mathbb{A}[n_i]$ as dg $\mathbb{A}$-modules (respectively, as $\mathbb{Z}$-graded $\mathbb{A}$-modules), where I is a countable set and $n_i \in \mathbb{Z}$. Observe that relatively countable projective modules are countable projective modules and always have the property (P) because $\mathrm{Hom}_{\mathscr{K}(\mathbb{A})}(\mathbb{A}[i], M) \simeq H^{-i}(M)$ for all i.

Let $\mathcal{X}(\mathbb{A})$ be the full subcategory of $\mathscr{C}(\mathbb{A})$ consisting of countable projective $\mathbb{A}$-modules. Then $\mathcal{X}(\mathbb{A})$ is an essentially small category. This is due to the following observation: Let $\mathcal{G}(\mathbb{A})$ be the category of $\mathbb{Z}$-graded $\mathbb{A}$-modules. For every $X := \bigoplus_{i\in\mathbb{Z}} X^i \in \mathcal{G}(\mathbb{A})$, we have the following: (a) The class $\mathcal{U}(X)$ consisting of isomorphism classes of direct summands of X in $\mathcal{G}(\mathbb{A})$ is a set. In fact, there is a surjection from the set of idempotent elements of $\mathrm{End}_{\mathcal{G}(\mathbb{A})}(X)$ to $\mathcal{U}(X)$. (b) The class $\mathcal{V}(X)$ consisting of all dg $\mathbb{A}$-modules with X as the underlying graded $\mathbb{A}$-module is also a set since $\mathcal{V}(X)$ is contained into the set $\{(X, d^i)_{i\in\mathbb{Z}} \mid d^i \in \mathrm{Hom}_k(X^i, X^{i+1})\}$, which is a countable union of sets.

Furthermore, $\mathcal{X}(\mathbb{A})$ is closed under extensions, shifts, direct summands and countable direct sums in $\mathscr{C}(\mathbb{A})$.

Let $\mathscr{C}(\mathbb{A}, \aleph_0)$ be the smallest full subcategory of $\mathcal{X}(\mathbb{A})$ such that it

(1) contains all relatively countable projective $\mathbb{A}$-modules;

(2) is closed under extensions and shifts;

(3) is closed under countable direct sums.

Then $\mathscr{C}(\mathbb{A}, \aleph_0)$ is essentially small, inherits an exact structure from $\mathscr{C}(\mathbb{A})$, and becomes a fully exact subcategory of $\mathscr{C}(\mathbb{A})$. Even more, $\mathscr{C}(\mathbb{A}, \aleph_0)$ is a Frobenius subcategory of $\mathscr{C}(\mathbb{A})$, in which projective-injective objects are the ones of $\mathscr{C}(\mathbb{A})$ belonging to $\mathscr{C}(\mathbb{A}, \aleph_0)$. This can be concluded from the following fact: For each $M \in \mathscr{C}(\mathbb{A})$, there is a canonical conflation $M \to C(M) \to M[1]$ in $\mathscr{C}(\mathbb{A})$ such that $C(M)$ is a projective-injective object of $\mathscr{C}(\mathbb{A})$ (see [9, Section 2.2]). Hence $\mathscr{C}(\mathbb{A}, \aleph_0)$ provides a natural Frobenius pair $(\mathscr{C}(\mathbb{A}, \aleph_0), \mathscr{C}(\mathbb{A}, \aleph_0)\text{-proj})$, and the inclusion $\mathscr{C}(\mathbb{A}, \aleph_0) \subseteq \mathscr{C}(\mathbb{A})$ induces a fully faithful inclusion from the derived category $\mathscr{D}_F(\mathscr{C}(\mathbb{A}, \aleph_0))$ of $\mathscr{C}(\mathbb{A}, \aleph_0)$ to $\mathscr{K}(\mathbb{A})$.

We denote by $\mathscr{K}(\mathbb{A}, \aleph_0)$ the full subcategory of $\mathscr{K}(\mathbb{A})$ consisting of those complexes which are isomorphic in $\mathscr{K}(\mathbb{A})$ to objects of $\mathscr{C}(\mathbb{A}, \aleph_0)$. Then $\mathscr{K}(\mathbb{A}, \aleph_0)$ is a triangulated subcategory of $\mathscr{K}(\mathbb{A})$ by the condition (2), and the inclusion $\mathscr{D}_F(\mathscr{C}(\mathbb{A}, \aleph_0)) \subseteq \mathscr{K}(\mathbb{A}, \aleph_0)$ is a triangle equivalence. Since the full subcategory of $\mathcal{X}(\mathbb{A})$ consisting of all dg $\mathbb{A}$-modules with the property (P) satisfies the above conditions (1)-(3), we deduce that each object of $\mathscr{C}(\mathbb{A}, \aleph_0)$ has the property (P). This implies that $\mathscr{K}(\mathbb{A}, \aleph_0) \subseteq \mathscr{K}(\mathbb{A})_p$. Furthermore, by definition, $\mathscr{C}(\mathbb{A}, \aleph_0)$ is closed under countable direct sums in $\mathscr{C}(\mathbb{A})$, and therefore $\mathscr{K}(\mathbb{A}, \aleph_0)$ is closed under countable direct sums in $\mathscr{K}(\mathbb{A})_p$. It follows from Lemma 2.1 (1) that $\mathscr{K}(\mathbb{A}, \aleph_0)$ is closed under direct summands in $\mathscr{K}(\mathbb{A})_p$.

Now, let $\mathscr{X}(\mathbb{A})$ be the full subcategory of $\mathscr{D}(\mathbb{A})$ consisting of all those objects which are isomorphic in $\mathscr{D}(\mathbb{A})$ to the images of objects of $\mathscr{K}(\mathbb{A}, \aleph_0)$ under the equivalence $\widetilde{q} : \mathscr{K}(\mathbb{A})_p \xrightarrow{\simeq} \mathscr{D}(\mathbb{A})$. Then $\mathscr{X}(\mathbb{A})$ is a triangulated subcategory of $\mathscr{D}(\mathbb{A})$ closed under direct summands, and $\widetilde{q}$ induces a triangle equivalence from $\mathscr{K}(\mathbb{A}, \aleph_0)$ to $\mathscr{X}(\mathbb{A})$. In all, we have

$$\mathscr{D}_F(\mathscr{C}(\mathbb{A}, \aleph_0)) \subseteq \mathscr{K}(\mathbb{A}, \aleph_0) \subseteq \mathscr{K}(\mathbb{A})_p,\ \ \mathscr{X}(\mathbb{A}) \subseteq \mathscr{D}(\mathbb{A})$$

and

$$\mathscr{D}_F(\mathscr{C}(\mathbb{A}, \aleph_0)) \xrightarrow{\simeq} \mathscr{K}(\mathbb{A}, \aleph_0) \xrightarrow{\simeq} \mathscr{X}(\mathbb{A})$$

as triangulated categories.

Recall that a dg $\mathbb{A}$-module M is called a *finite cell module* if there is a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$$

of dg $\mathbb{A}$-modules such that, for each $0 \leq i \leq n-1 \in \mathbb{N}$, the quotient module M_{i+1}/M_i is isomorphic to $\mathbb{A}[n_i]$ for some $n_i \in \mathbb{Z}$ (see [11, Part III]). Clearly, each finite cell $\mathbb{A}$-module belongs to $\mathscr{C}(\mathbb{A}, \aleph_0)$. Moreover, the category of finite cell $\mathbb{A}$-modules is closed under extensions in $\mathscr{C}(\mathbb{A}, \aleph_0)$. Actually, this category is a Frobenius subcategory of $\mathscr{C}(\mathbb{A}, \aleph_0)$, in which projective-injective objects are the ones of $\mathscr{C}(\mathbb{A}, \aleph_0)$ belonging to this subcategory.

An object $M \in \mathscr{D}(\mathbb{A})$ is said to be *compact* if $\mathrm{Hom}_{\mathscr{D}(\mathbb{A})}(M, -)$ commutes with direct sums in $\mathscr{D}(\mathbb{A})$. Let $\mathscr{D}^c(\mathbb{A})$ be the full subcategory of $\mathscr{D}(\mathbb{A})$ consisting of all compact objects. Then $\mathscr{D}^c(\mathbb{A})$ is the smallest full triangulated subcategory of $\mathscr{D}(\mathbb{A})$ containing $\mathbb{A}$ and being closed under direct summands. In fact, each compact object of $\mathscr{D}(\mathbb{A})$ is a direct summand of a finite cell module in $\mathscr{D}(\mathbb{A})$ (see [9, Section 5]). This implies the following chain of full subcategories: $\mathscr{D}^c(\mathbb{A}) \subseteq \mathscr{X}(\mathbb{A}) \subseteq \mathscr{D}(\mathbb{A})$.

Now, we define $\mathcal{W}_{\mathbb{A}}$ to be the full subcategory of $\mathscr{C}(\mathbb{A}, \aleph_0)$ consisting of all those objects in $\mathscr{C}(\mathbb{A}, \aleph_0)$ such that they are isomorphic in $\mathscr{D}(\mathbb{A})$ to compact objects of $\mathscr{D}(\mathbb{A})$. Clearly, $\mathcal{W}_{\mathbb{A}}$ is essentially small. Moreover, by applying Lemma 3.3 to the Frebenius pair $\mathscr{C}(\mathbb{A}, \aleph_0)$ and the equivalence $\mathscr{D}_F(\mathscr{C}(\mathbb{A}, \aleph_0)) \xrightarrow{\simeq} \mathscr{X}(\mathbb{A})$ with the triangulated subcategory $\mathscr{D}^c(\mathbb{A})$ of $\mathscr{X}(\mathbb{A})$, we deduce that $\mathcal{W}_{\mathbb{A}}$ is a Frobenius subcategory of $\mathscr{C}(\mathbb{A}, \aleph_0)$ with the same projective objects, and that the following diagram of triangulated categories commutes:

$$(\star)\qquad \begin{array}{ccccccc} \mathscr{D}_F(\mathcal{W}_{\mathbb{A}}) & \hookrightarrow & \mathscr{D}_F(\mathscr{C}(\mathbb{A}, \aleph_0)) & \hookrightarrow & \mathscr{K}(\mathbb{A})_p & \hookrightarrow & \mathscr{K}(\mathbb{A}) \\ \downarrow \simeq & & \downarrow \simeq & & \tilde{q}\downarrow \simeq & \swarrow q & \\ \mathscr{D}^c(\mathbb{A}) & \hookrightarrow & \mathscr{X}(\mathbb{A}) & \hookrightarrow & \mathscr{D}(\mathbb{A}) & & \end{array}$$

From now on, we regard $\mathcal{W}_{\mathbb{A}}$ as a Waldhausen category in the sense of Subsection 3.2, namely, it arises exactly from the Frobenius pair $(\mathcal{W}_{\mathbb{A}}, \mathcal{W}_{\mathbb{A}}\text{-proj})$.

We define the *algebraic K-theory space of the dg k-algebra* $\mathbb{A}$ to be the space $K(\mathcal{W}_{\mathbb{A}})$, denoted by $K(\mathbb{A})$. For $n \in \mathbb{N}$, the n-th *K-group* of $\mathbb{A}$ is defined to be the n-th homotopy group of $K(\mathbb{A})$, denoted by $K_n(\mathbb{A})$. Note that $K_0(\mathbb{A})$ is isomorphic to $K_0(\mathscr{D}_F(\mathcal{W}_{\mathbb{A}}))$, the Grothendieck group of the (essentially small) triangulated category $\mathscr{D}_F(\mathcal{W}_{\mathbb{A}})$ of the Frobenius pair $(\mathcal{W}_{\mathbb{A}}, \mathcal{W}_{\mathbb{A}}\text{-proj})$(see Subsection 3.2).

Consequently, we have obtained the following result.

Lemma 3.4. *The Verdier localization functor $\mathscr{K}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$ induces a triangle equivalence: $\mathscr{D}_F(\mathcal{W}_{\mathbb{A}}) \xrightarrow{\simeq} \mathscr{D}^c(\mathbb{A})$. In particular, $K_0(\mathcal{W}_{\mathbb{A}})$ is isomorphic to the Grothendieck group $K_0(\mathscr{D}^c(\mathbb{A}))$ of $\mathscr{D}^c(\mathbb{A})$.*

To illustrate our definition of K-theory spaces of dg algebras, we first establish the following result.

Lemma 3.5. *Let $\mathcal{F}_{\mathbb{A}}$ be the full subcategory of $\mathcal{W}_{\mathbb{A}}$ consisting of all finite cell $\mathbb{A}$-modules. Then the inclusion $\mathcal{F}_{\mathbb{A}} \to \mathcal{W}_{\mathbb{A}}$ induces an injection $K_0(\mathcal{F}_{\mathbb{A}}) \to K_0(\mathcal{W}_{\mathbb{A}})$ and an isomorphism $K_n(\mathcal{F}_{\mathbb{A}}) \xrightarrow{\sim} K_n(\mathcal{W}_{\mathbb{A}})$ for each $n > 0$.*

Proof. Note that $\mathcal{F}_{\mathbb{A}}$ is a Frobenius subcategory of $\mathcal{W}_{\mathbb{A}}$ and that the inclusions $\mathcal{F}_{\mathbb{A}} \subseteq \mathcal{W}_{\mathbb{A}} \subseteq \mathscr{C}(\mathbb{A})$ induce fully faithful inclusions $\mathscr{D}_F(\mathcal{F}_{\mathbb{A}}) \subseteq \mathscr{D}_F(\mathcal{W}_{\mathbb{A}}) \subseteq \mathscr{K}(\mathbb{A})_p$ (see Subsection 3.2).

To show that the inclusion $\mathscr{D}_F(\mathcal{F}_{\mathbb{A}}) \to \mathscr{D}_F(\mathcal{W}_{\mathbb{A}})$ is an equivalence up to factors, we shall compare the images of these two categories under the equivalence $\tilde{q} : \mathscr{K}(\mathbb{A})_p \to \mathscr{D}(\mathbb{A})$ in the above diagram ($\star$). In fact, by Lemma 3.4, the restriction of the functor $\tilde{q}$ to $\mathscr{D}_F(\mathcal{W}_{\mathbb{A}})$ gives rise to a triangle equivalence $\mathscr{D}_F(\mathcal{W}_{\mathbb{A}}) \xrightarrow{\simeq} \mathscr{D}^c(\mathbb{A})$. Let $\mathscr{Y}$ be the smallest full triangulated subcategory of $\mathscr{D}^c(\mathbb{A})$ containing $\mathbb{A}$. Since the objects of $\mathscr{D}_F(\mathcal{F}_{\mathbb{A}})$ are the same as the ones of $\mathcal{F}_{\mathbb{A}}$, the image of the restriction of the functor $\tilde{q}$ to $\mathscr{D}_F(\mathcal{F}_{\mathbb{A}})$ is contained in $\mathscr{Y}$, and therefore is equal to $\mathscr{Y}$. Thus $\tilde{q}$ induces a triangle equivalence $\mathscr{D}_F(\mathcal{F}_{\mathbb{A}}) \xrightarrow{\simeq} \mathscr{Y}$. Since

$\mathscr{D}^c(\mathbb{A}) = \text{thick}(\mathbb{A})$ and $\mathbb{A} \in \mathscr{Y} \subseteq \mathscr{D}^c(\mathbb{A})$, we have $\text{thick}(\mathscr{Y}) = \mathscr{D}^c(\mathbb{A})$. So the inclusion $\mathscr{Y} \to \mathscr{D}^c(\mathbb{A})$ is an equivalence up to factors. Consequently, the inclusion $\mathscr{D}_F(\mathcal{F}_{\mathbb{A}}) \to \mathscr{D}_F(\mathcal{W}_{\mathbb{A}})$ induced from $\mathcal{F}_{\mathbb{A}} \subseteq \mathcal{W}_{\mathbb{A}}$ is also an equivalence up to factors. Now, Lemma 3.5 follows from Lemma 3.2 (3). $\square$.

Remark 3.6. In [20, Section 12.3], a K-theory spectrum $\mathbb{K}(\mathcal{F}_{\mathbb{A}})$ is defined for the category $\mathcal{F}_{\mathbb{A}}$. Moreover, it is known in [20, Theorem 8] that, for each $n \in \mathbb{N}$, the n-th homology group of $\mathbb{K}(\mathcal{F}_{\mathbb{A}})$ is given by

$$\pi_n\big(\mathbb{K}(\mathcal{F}_{\mathbb{A}})\big) = \begin{cases} K_n(\mathcal{F}_{\mathbb{A}}) & \text{if } n > 0, \\ K_0(\mathscr{D}^c(\mathbb{A})) & \text{if } n = 0. \end{cases}$$

Thus Lemmas 3.5 and 3.4 show that $\pi_n\big(\mathbb{K}(\mathcal{F}_{\mathbb{A}})\big) \simeq K_n(\mathbb{A})$ for all $n \in \mathbb{N}$, and therefore, at the level of homotopy groups, our definition of K-theory for dg algebras is isomorphic to the one defined by Schlichting in [20].

The following result, together with Lemma 3.5, may explain the advantage of defining K-theory of arbitrary dg algebras by using the category $\mathcal{W}_{\mathbb{A}}$ rather than $\mathcal{F}_{\mathbb{A}}$.

Lemma 3.7. *Let A be an algebra with identity, and let $\mathbb{A}$ be the dg algebra A concentrated in degree 0. Then $K(A) \xrightarrow{\sim} K(\mathbb{A})$ as K-theory spaces.*

Proof. Clearly, $\mathscr{C}(\mathbb{A}) = \mathscr{C}(A)$, $\mathscr{K}(\mathbb{A}) = \mathscr{K}(A)$ and $\mathscr{D}(\mathbb{A}) = \mathscr{D}(A)$. In particular, $\mathscr{D}^c(\mathbb{A}) = \mathscr{D}^c(A)$. By the construction of $\mathcal{W}_{\mathbb{A}}$, we see that $\mathscr{C}^b(A\text{-proj}) \subseteq \mathcal{W}_{\mathbb{A}}$ and $\mathscr{C}^b(A\text{-proj})\text{-proj} = \mathscr{C}^b_{ac}(A\text{-proj}) \subseteq \mathcal{W}_{\mathbb{A}}\text{-proj}$. Thus the inclusion $j : \mathscr{C}^b(A\text{-proj}) \to \mathcal{W}_{\mathbb{A}}$ is a fully faithful map of Frobenius pairs. In other words, $\mathscr{C}^b(A\text{-proj})$ is a Frobenius subcategory of $\mathcal{W}_{\mathbb{A}}$. This implies that the triangle functor $\mathscr{D}_F(j) : \mathscr{D}_F(\mathscr{C}^b(A\text{-proj})) \to \mathscr{D}_F(\mathcal{W}_{\mathbb{A}})$ is fully faithful (see Subsection 3.2). Now we show that $\mathscr{D}_F(j)$ is an equivalence. On the one hand, the localization functor $q : \mathscr{K}(A) \to \mathscr{D}(A)$ induces an equivalence $q_1 : \mathscr{D}_F(\mathcal{W}_{\mathbb{A}}) \to \mathscr{D}^c(A)$ by Lemma 3.4. On the other hand, the composite of the following functors:

$$\mathscr{K}^b(R\text{-proj}) = \mathscr{D}_F(\mathscr{C}^b(A\text{-proj})) \xrightarrow{\mathscr{D}_F(j)} \mathscr{D}_F(\mathcal{W}_{\mathbb{A}}) \xrightarrow{q_1} \mathscr{D}^c(A)$$

is also an equivalence induced by q. Thus $\mathscr{D}_F(j)$ is a triangle equivalence. By Lemma 3.2 (2), we know that $K(A) \xrightarrow{\sim} K(\mathcal{W}_{\mathbb{A}}) =: K(\mathbb{A})$ as K-theory spaces. $\square$

The following result is in the literature [7, Proposition 6.7 and Corollary 3.10] where proofs use knowledge on model categories. For the convenience of the reader, we include here another proof based on the facts mentioned in the present paper.

Lemma 3.8. *Let $\lambda : \mathbb{B} \to \mathbb{A}$ be a homomorphism of dg algebras which is a quasi-isomorphism. Then the functor $\mathbb{A} \otimes^{\bullet}_{\mathbb{B}} - : \mathscr{C}(\mathbb{B}) \to \mathscr{C}(\mathbb{A})$ induces a homotopy equivalence $K(\mathbb{B}) \xrightarrow{\sim} K(\mathbb{A})$ of K-theory spaces. In particular, if $H^i(\mathbb{A}) = 0$ for all $i \neq 0$, then $K(\mathbb{A}) \xrightarrow{\sim} K(H^0(\mathbb{A}))$.*

Proof. Note that the functor $\mathbb{A} \otimes^{\bullet}_{\mathbb{B}} - : \mathcal{W}_{\mathbb{B}} \to \mathcal{W}_{\mathbb{A}}$ is a well-defined map of Frobenius pairs and that objects belonging to $\mathcal{W}_{\mathbb{B}}$ or $\mathcal{W}_{\mathbb{A}}$ always have the property (P). So we can form the following commutative diagram of functors:

$$\begin{array}{ccc} \mathscr{D}_F(\mathcal{W}_{\mathbb{B}}) & \xrightarrow{\mathscr{D}_F(\mathbb{A} \otimes^{\bullet}_{\mathbb{B}} -)} & \mathscr{D}_F(\mathcal{W}_{\mathbb{A}}) \\ \simeq \big\downarrow & & \big\downarrow \simeq \\ \mathscr{D}^c(\mathbb{B}) & \xrightarrow{\mathbb{A} \otimes^{\mathbb{L}}_{\mathbb{B}} -} & \mathscr{D}^c(\mathbb{A}) \end{array}$$

where the equivalences in vertical direction are induced by the localization functors $\mathscr{K}(\mathbb{B}) \to \mathscr{D}(\mathbb{B})$ and $\mathscr{K}(\mathbb{A}) \to \mathscr{D}(\mathbb{A})$, respectively (see Lemma 3.4). Since $\lambda : \mathbb{B} \to \mathbb{A}$ is a quasi-isomorphism, it follows from

[11, Proposition 4.2] that the functor $\mathbb{A}\otimes^{\bullet}_{\mathbb{B}} -$ induces a triangle equivalence $\mathscr{D}(\mathbb{B}) \xrightarrow{\sim} \mathscr{D}(\mathbb{A})$ which restricts to an equivalence $\mathscr{D}^c(\mathbb{B}) \xrightarrow{\simeq} \mathscr{D}^c(\mathbb{A})$ (see also [9, Section 3.1]). Thus the functor

$$\mathscr{D}_F(\mathbb{A}\otimes^{\bullet}_{\mathbb{B}} -) : \mathscr{D}_F(\mathcal{W}_{\mathbb{B}}) \longrightarrow \mathscr{D}_F(\mathcal{W}_{\mathbb{A}})$$

is a triangle equivalence. Now, the first part of Lemma 3.8 follows from Lemma 3.2 (2).

Suppose that $\mathbb{A} := (A^i, d^i)_{i\in\mathbb{Z}}$ with $H^i(\mathbb{A}) = 0$ for all $i \neq 0$. We define $\tau^{\leq 0}(\mathbb{A})$ to be the following dg algebra:

$$\cdots \longrightarrow A^{-3} \xrightarrow{d^{-3}} A^{-2} \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} \mathrm{Ker}(d^0) \longrightarrow 0 \longrightarrow \cdots.$$

Then there exist two canonical quasi-isomorphisms $\tau^{\leq 0}(\mathbb{A}) \to \mathbb{A}$ and $\tau^{\leq 0}(\mathbb{A}) \to H^0(\mathbb{A})$ of dg algebras. It follows from the first part of Lemma 3.8 that

$$K(\tau^{\leq 0}(\mathbb{A})) \xrightarrow{\sim} K(\mathbb{A}) \text{ and } K(\tau^{\leq 0}(\mathbb{A})) \xrightarrow{\sim} K(H^0(\mathbb{A})).$$

Combining these homotopy equivalences with Lemma 3.7, we see that $K(\mathbb{A}) \xrightarrow{\sim} K(H^0(\mathbb{A}))$ as K-theory spaces. $\square$

The following result will be used in proofs of our main results.

Lemma 3.9. *Let A be an algebra and $P^{\bullet} \in \mathscr{C}^b(A\text{-proj})$. Define $\mathbb{S} := \mathrm{End}^{\bullet}_R(P^{\bullet})$ and $\mathcal{P}$ to be the full subcategory of $\mathscr{C}^b(A\text{-proj})$ consisting of all those complexes which, regarded as objects in $\mathscr{D}(A)$, belong to* $\mathrm{Tria}(P^{\bullet})$*. Then $K(\mathbb{S}) \xrightarrow{\sim} K(\mathcal{P})$ as K-theory spaces.*

Proof. We remark that $\mathcal{P}$ is a Frobenius subcategory of $\mathscr{C}^b(A\text{-proj})$ such that its derived category $\mathscr{D}_F(\mathcal{P})$ is equivalent to $\mathscr{X} := \mathrm{Tria}(P^{\bullet}) \cap \mathscr{D}^c(A)$ via the Verdier localization functor $q : \mathscr{K}(A) \to \mathscr{D}(A)$.

In fact, since $\mathscr{X}$ is a full triangulated subcategory of $\mathscr{D}^c(A)$ and $\mathscr{D}_F(\mathscr{C}^b(A\text{-proj})) = \mathscr{K}^b(A\text{-proj}) \xrightarrow{\simeq} \mathscr{D}^c(A)$, we see that $\mathcal{P}$ is exactly the full subcategory of $\mathscr{C}^b(A\text{-proj})$, in which the objects are complexes in $\mathscr{C}^b(A\text{-proj})$ such that they are isomorphic in $\mathscr{D}^c(A)$ to objects of $\mathscr{X}$. Hence, by Lemma 3.3, $\mathcal{P}$ is a Frobenius subcategory of $\mathscr{C}^b(R\text{-proj})$ and the functor q induces an equivalence $q_1 : \mathscr{D}_F(\mathcal{P}) \xrightarrow{\simeq} \mathscr{X}$.

Now we view A as a dg algebra concentrated in degree 0, and let $\mathcal{X}$ be the full subcategory of $\mathcal{W}_A$ consisting of those objects that are isomorphic in $\mathscr{D}^c(A)$ to objects of $\mathscr{X}$. Then, applying Lemma 3.3 to the Frobenius pair $\mathcal{W}_A$ and the equivalence $\mathscr{D}_F(\mathcal{W}_A) \to \mathscr{D}^c(A)$ in Lemma 3.4, we get a Frobenius pair $(\mathcal{X}, \mathcal{W}_A\text{-proj})$ which is included in the Frobenius pair $(\mathcal{W}_A, \mathcal{W}_A\text{-proj})$, and an equivalence $q_2 : \mathscr{D}_F(\mathcal{X}) \xrightarrow{\simeq} \mathscr{X}$ induced from the functor q. Note that $\mathcal{X}\text{-proj} = \mathcal{W}_A\text{-proj}$. Recall that, for a dg algebra $\mathbb{A}$, $\mathcal{W}_{\mathbb{A}}\text{-proj}$ consists of all those objects which are homotopy equivalent to the zero object in $\mathscr{C}(\mathbb{A})$.

In the following, we first show that $K(\mathcal{P}) \xrightarrow{\sim} K(\mathcal{X})$, and then that $K(\mathbb{S}) \xrightarrow{\sim} K(\mathcal{X})$ as K-theory spaces. With these two homotopy equivalences in mind, we will obviously have $K(\mathbb{S}) \xrightarrow{\sim} K(\mathcal{P})$, as desired.

Let us check that $K(\mathcal{P}) \xrightarrow{\sim} K(\mathcal{X})$. Actually, it follows from $\mathscr{C}^b(A\text{-proj}) \subseteq \mathcal{W}_A$ that $\mathcal{P} \subseteq \mathcal{X}$. Since $\mathcal{P}\text{-proj} = \mathscr{C}^b_{ac}(A\text{-proj}) \subseteq \mathcal{W}_A\text{-proj} = \mathcal{X}\text{-proj}$, the inclusion $\mu : \mathcal{P} \to \mathcal{X}$ of Frobenius categories induces a fully faithful functor $\mathscr{D}_F(\mu) : \mathscr{D}_F(\mathcal{P}) \to \mathscr{D}_F(\mathcal{X})$. Since $q_1 = q_2\mathscr{D}_F(\mu)$, we see that $\mathscr{D}_F(\mu)$ is an equivalence. Thus the map $K(\mathcal{P}) \to K(\mathcal{X})$ is a homotopy equivalence by Lemma 3.2 (2).

Next, we prove that there is a homotopy equivalence $K(\mathbb{S}) \xrightarrow{\sim} K(\mathcal{X})$.

To prove this statement, we define $G := P^{\bullet}\otimes^{\bullet}_{\mathbb{S}} - : \mathscr{C}(\mathbb{S}) \to \mathscr{C}(A)$ and claim that $G : \mathcal{W}_{\mathbb{S}} \to \mathcal{X}$ is a map of Frobenius pairs. We first show that G is well-defined, that is, $G(\mathcal{W}_{\mathbb{S}}) \subseteq \mathcal{X}$. In fact, as a graded A-module, $P^{\bullet}$ is equal to $\bigoplus_{i\in\mathbb{Z}} P^i$, which is a finitely generated projective A-module. Let $\mathcal{X}(\mathbb{S})$ and $\mathcal{X}(A)$ be the full subcategories of $\mathscr{C}(\mathbb{S})$ and $\mathscr{C}(A)$ consisting of countable projective modules, respectively. Then, due to $G(\mathbb{S}) = P^{\bullet}\otimes^{\bullet}_{\mathbb{S}}\mathbb{S} \simeq P^{\bullet}$, we see that the functor $G : \mathcal{X}(\mathbb{S}) \to \mathcal{X}(A)$ is well defined. Note that $G(\mathbb{S}) \simeq P^{\bullet} \in \mathscr{C}^b(A\text{-proj}) \subseteq \mathscr{C}(A, \aleph_0)$ and the functor $G : \mathcal{X}(\mathbb{S}) \to \mathcal{X}(A)$ preserves conflations and commutes with both shifts and countable direct sums. This implies that the following full subcategory

$$G^{-1}(\mathscr{C}(A, \aleph_0)) := \{X \in \mathcal{X}(\mathbb{S}) \mid G(X) \in \mathscr{C}(A, \aleph_0)\}$$

of $\mathcal{X}(\mathbb{S})$ contains all relatively countable projective $\mathbb{A}$-modules, and is closed under extensions, shifts and countable direct sums.

Since $\mathscr{C}(\mathbb{S}, \aleph_0)$ is the smallest subcategory of $\mathcal{X}(\mathbb{S})$ which admits these properties, we have $\mathscr{C}(\mathbb{S}, \aleph_0) \subseteq G^{-1}(\mathscr{C}(A, \aleph_0))$. Thus $G(\mathscr{C}(\mathbb{S}, \aleph_0)) \subseteq \mathscr{C}(A, \aleph_0)$ and $G : \mathscr{C}(\mathbb{S}, \aleph_0) \to \mathscr{C}(A, \aleph_0)$ is a well-defined functor.

Furthermore, since each $M \in \mathscr{C}(\mathbb{S}, \aleph_0)$ always has the property (P), we see that $P^\bullet \otimes^{\mathbb{L}}_{\mathbb{S}} M = G(M)$ in $\mathscr{D}(A)$. So, to show that $G(\mathcal{W}_{\mathbb{S}}) \subseteq \mathcal{X}$, it suffices to prove that if $M \in \mathcal{W}_{\mathbb{S}}$, then $P^\bullet \otimes^{\mathbb{L}}_{\mathbb{S}} M \in \mathscr{X}$. Actually, let $M \in \mathcal{W}_{\mathbb{S}}$. Then $M \in \mathscr{D}^c(\mathbb{S})$. As $P^\bullet \otimes^{\mathbb{L}}_{\mathbb{S}} \mathbb{S} \simeq P^\bullet \in \mathscr{D}^c(A)$, the functor $P^\bullet \otimes^{\mathbb{L}}_{\mathbb{S}} - : \mathscr{D}(\mathbb{S}) \to \mathscr{D}(A)$ preserves compact objects. This implies $P^\bullet \otimes^{\mathbb{L}}_{\mathbb{S}} M \in \mathscr{D}^c(A)$. Note that $P^\bullet \otimes^{\mathbb{L}}_{\mathbb{S}} -$ commutes with direct sums and $\mathscr{D}(\mathbb{S}) = \mathrm{Tria}(\mathbb{S})$. By Lemma 2.1 (2), we have $P^\bullet \otimes^{\mathbb{L}}_{\mathbb{S}} M \in \mathrm{Tria}(P^\bullet)$. Thus $P^\bullet \otimes^{\mathbb{L}}_{\mathbb{S}} M \in \mathrm{Tria}(P^\bullet) \cap \mathscr{D}^c(A) = \mathscr{X}$.

As a result, we have $G(\mathcal{W}_{\mathbb{S}}) \subseteq \mathcal{X}$. Since G always preserves conflations and homotopy equivalences, we know that G sends projective objects of $\mathcal{W}_{\mathbb{S}}$ to the ones of $\mathcal{X}$, and therefore $G : \mathcal{W}_{\mathbb{S}} \to \mathcal{X}$ is a map of Frobenius pairs.

Finally, we show that the functor $\mathscr{D}_F(G) : \mathscr{D}_F(\mathcal{W}_{\mathbb{S}}) \to \mathscr{D}_F(\mathcal{X})$ induced from G is a triangle equivalence.

Indeed, by [9, Section 3.1], the functor $G = P^\bullet \otimes^{\bullet}_{\mathbb{S}} - : \mathscr{C}(\mathbb{S}) \to \mathscr{C}(A)$ induces a triangle equivalence:

$$P^\bullet \otimes^{\mathbb{L}}_{\mathbb{S}} - : \mathscr{D}(\mathbb{S}) \xrightarrow{\simeq} \mathrm{Tria}(P^\bullet),$$

which restricts to an equivalence $\mathscr{D}^c(\mathbb{S}) \xrightarrow{\simeq} \mathscr{X}$ since $\mathscr{X}$ coincides with the full subcategory of $\mathrm{Tria}(P^\bullet)$ consisting of all compact objects in $\mathrm{Tria}(P^\bullet)$ by Lemma 2.2 (1). Moreover, according to Lemma 3.4, the localization functor $\mathscr{K}(\mathbb{S}) \to \mathscr{D}(\mathbb{S})$ induces an equivalence $\tilde{q} : \mathscr{D}_F(\mathcal{W}_{\mathbb{S}}) \xrightarrow{\simeq} \mathscr{D}^c(\mathbb{S})$. Consequently, we can form the following commutative diagram of functors:

$$\begin{array}{ccc}
\mathscr{D}_F(\mathcal{W}_{\mathbb{S}}) & \xrightarrow{\mathscr{D}_F(G)} & \mathscr{D}_F(\mathcal{X}) \\
\simeq \downarrow \tilde{q} & & q \downarrow \simeq \\
\mathscr{D}^c(\mathbb{S}) & \underset{\simeq}{\xrightarrow{P^\bullet \otimes^{\mathbb{L}}_{\mathbb{S}} -}} & \mathscr{X}
\end{array}$$

Thus $\mathscr{D}_F(G) : \mathscr{D}_F(\mathcal{W}_{\mathbb{S}}) \xrightarrow{\simeq} \mathscr{D}_F(\mathcal{X})$ is an equivalence. This implies that $K(\mathbb{S}) \xrightarrow{\sim} K(\mathcal{X})$ by Lemma 3.2 (2). As $K(\mathcal{P}) \xrightarrow{\simeq} K(\mathcal{X})$, we see that $K(\mathbb{S}) \xrightarrow{\sim} K(\mathcal{P})$ as K-theory spaces. □

As a further preparation for proofs of our main results, we now recall a useful fact about K-theory spaces of ordinary rings, which is a revisited version of a special case of the classical 'resolution theorem' due to Quillen (see [18, Section 4, Corollary 2]). For more general arrangement of this result for exact categories, we refer the reader to [21, Proposition 3.3.8]. For the convenience of the reader, we include here a proof for this special case.

Lemma 3.10. *Let A be a ring. Then the following are true.*

(1) *The inclusions* A-proj $\hookrightarrow \mathscr{P}^{<\infty}(A) \hookrightarrow A$-Mod *of exact categories induce equivalences:*

$$\mathscr{K}^b(A\text{-proj}) \xrightarrow{\simeq} \mathscr{D}^b(\mathscr{P}^{<\infty}(A)) \xrightarrow{\simeq} \mathscr{D}^c(A).$$

(2) *The inclusion* A-proj $\hookrightarrow \mathscr{P}^{<\infty}(A)$ *induces a homotopy equivalence* $K(A) \xrightarrow{\sim} K(\mathscr{P}^{<\infty}(A))$, *where* $K(\mathscr{P}^{<\infty}(A))$ *is the K-theory space of the exact category* $\mathscr{P}^{<\infty}(A)$.

Proof. (1) Recall that $\mathscr{D}^b(\mathscr{P}^{<\infty}(A))$ denotes the bounded derived category of the exact category $\mathscr{P}^{<\infty}(A)$ defined in Subsection 3.3. By the dual of [10, Theorem 12.1], the inclusion A-proj $\to \mathscr{P}^{<\infty}(A)$ induces a fully faithful functor $\mathscr{K}^b(A\text{-proj}) \to \mathscr{D}^b(\mathscr{P}^{<\infty}(A))$. Actually, this functor is also dense since each module in $\mathscr{P}^{<\infty}(A)$ has a finite resolution by finitely generated projective A-modules. Thus $\mathscr{K}^b(A\text{-proj}) \xrightarrow{\simeq} \mathscr{D}^b(\mathscr{P}^{<\infty}(A))$. Note that the inclusion $\mathscr{P}^{<\infty}(A) \to A$-Mod is an exact functor of exact categories. This directly yields a triangle functor $\mathscr{D}^b(\mathscr{P}^{<\infty}(A)) \to \mathscr{D}(A)$ which factorizes through $\mathscr{D}^c(A)$. Since the composition of the following two functors:

$$\mathscr{K}^b(A\text{-proj}) \xrightarrow{\simeq} \mathscr{D}^b(\mathscr{P}^{<\infty}(A)) \longrightarrow \mathscr{D}^c(A)$$

is an equivalence, we see that the latter is also an equivalence. This shows (1).

(2) The inclusion A-proj $\hookrightarrow \mathscr{P}^{<\infty}(A)$ induces an inclusion functor $\mathscr{C}^b(A\text{-proj}) \hookrightarrow \mathscr{C}^b(\mathscr{P}^{<\infty}(A))$ and a map between the Frobenius pairs

$$\mathbf{A} := \big(\mathscr{C}^b(A\text{-proj}), \mathscr{C}^b_{ac}(A\text{-proj})\big) \text{ and } \mathbf{B} := \big(\mathscr{C}^b(\mathscr{P}^{<\infty}(A)), \mathscr{C}^b_{ac}(\mathscr{P}^{<\infty}(A))\big).$$

Note that $K(\mathbf{A}) \simeq K(A)$ and $K(\mathbf{B}) \simeq K(\mathscr{P}^{<\infty}(A))$. Now, (2) follows from (1) and Lemma 3.2 (2). $\square$

4 Algebraic K-theory of recollements: Proof of Theorem 1.1

The main purpose of this section is to prove Theorem 1.1 and Corollary 1.2. We first make a few preparations.

4.1 Homotopy-split injections on K-theory spaces of Frobenius pairs

As the first step toward the proof of our main result, Theorem 1.1, we will discuss when maps between K-theory spaces, which are induced from maps of Frobenius pairs, are homotopy-split injections.

Let X and Y be two pointed topological spaces. By a map between topological spaces we always mean a pointed and continuous map. Recall that a map $f : X \to Y$ is called a *homotopy-split injection* if there is a map $g : Y \to X$ such that $fg : X \to X$ is pointed-homotopic to the identity map of X. Dually, we can define the homotopy-split surjections.

Homotopy-split injections provide us usually with decompositions of topological spaces. The following result, due to [22, Corollary 7.1.5 and Theorem 7.1.14] and [26, Chapter III, 6.9*], is useful for our considerations.

Lemma 4.1. *Let $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$ be a homotopy fibration. Suppose that X, Y and Z are homotopy equivalent to CW-complexes and that g induces a surjective map $\pi_0(g) : \pi_0(Y, y_0) \to \pi_0(Z, z_0)$ of the 0-th homotopy groups. If the map f is a homotopy-split injection, then Y is homotopy equivalent to the product of X and Z.*

Recall that the K-theory space $K(\mathcal{C})$ of a small Waldhausen category $\mathcal{C}$ is always homotopy equivalent to a CW-complex (see Subsection 3.1). So, in our consideration, we can apply Lemma 4.1 to discuss homotopy-split injections between K-theory spaces of small Waldhausen categories. In fact, by Lemmas 3.2 (1) and 4.1, we have the following consequence for K-theory spaces.

Corollary 4.2. *Let $\mathbf{A} \xrightarrow{F} \mathbf{B} \xrightarrow{G} \mathbf{C}$ be a sequence of Frobenius pairs. Suppose that the sequence $\mathscr{D}_F(\mathbf{A}) \xrightarrow{\mathscr{D}_F(F)} \mathscr{D}_F(\mathbf{B}) \xrightarrow{\mathscr{D}_F(G)} \mathscr{D}_F(\mathbf{C})$ of triangulated categories is exact. If the map $K(F) : K(\mathbf{A}) \to K(\mathbf{B})$ induced by F is a homotopy-split injection, then*

$$K(\mathbf{B}) \xrightarrow{\sim} K(\mathbf{A}) \times K(\mathbf{C}).$$

Next, we establish the following result which generalizes Lemma 3.2 (2).

Lemma 4.3. *Let $H : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{B} \to \mathbf{C}$ be maps of Frobenius pairs such that*

(1) $\mathscr{D}_F(GH) : \mathscr{D}_F(\mathbf{A}) \longrightarrow \mathscr{D}_F(\mathbf{C})$ *is fully faithful, and*

(2) $\mathrm{Im}(\mathscr{D}_F(G)) = \mathrm{Im}(\mathscr{D}_F(GH))$ *in* $\mathscr{D}_F(\mathbf{C})$.

Then the map $K(H) : K(\mathbf{A}) \to K(\mathbf{B})$ induced by H is a homotopy-split injection.

Proof. Let $\mathbf{B} := (\mathcal{B}, \mathcal{B}_0)$, $\mathbf{C} := (\mathcal{C}, \mathcal{C}_0)$ and $\mathscr{X} := \mathrm{Im}(\mathscr{D}_F(G))$. By (1) and (2), we see that $\mathscr{X}$ is a full triangulated subcategory of $\mathscr{D}_F(\mathbf{C})$. Let $\mathcal{X}$ be the full subcategory of $\mathcal{C}$ consisting of all these objects which, viewed as objects of $\mathscr{D}_F(\mathbf{C})$, belong to $\mathscr{X}$, and let $\mathbf{X} := (\mathcal{X}, \mathcal{C}_0)$. Then $\mathbf{X}$ is a Frobenius pair and $\mathscr{D}_F(\mathbf{X}) = \mathscr{X}$ by Lemma 3.3. Clearly, we have $G(\mathcal{B}) \subseteq \mathcal{X}$. This implies that $G : \mathbf{B} \to \mathbf{C}$ induces a map $E : \mathbf{B} \to \mathbf{X}$ of Frobenius pairs, that is, G is a composition of E with the inclusion $\mathbf{X} \hookrightarrow \mathbf{C}$. Now we consider the following commutative diagrams:

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{H} & \mathbf{B} \\ {\scriptstyle EH}\downarrow & \swarrow{\scriptstyle E} & \downarrow{\scriptstyle G} \\ \mathbf{X} & \hookrightarrow & \mathbf{C} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathscr{D}_F(\mathbf{A}) & \xrightarrow{\mathscr{D}_F(H)} & \mathscr{D}_F(\mathbf{B}) \\ {\scriptstyle \mathscr{D}_F(EH)}\downarrow & \swarrow{\scriptstyle \mathscr{D}_F(E)} & \downarrow{\scriptstyle \mathscr{D}_F(G)} \\ \mathscr{D}_F(\mathbf{X}) & \hookrightarrow & \mathscr{D}_F(\mathbf{C}) \end{array}$$

By (1), we see that $\mathscr{D}_F(EH) = \mathscr{D}_F(E)\mathscr{D}_F(H) : \mathscr{D}_F(\mathbf{A}) \to \mathscr{D}_F(\mathbf{X})$ is fully faithful. By (2), we have $\mathscr{X} = \mathrm{Im}(\mathscr{D}_F(GH)) = \mathrm{Im}(\mathscr{D}_F(EH))$. Thus $\mathscr{D}_F(EH) : \mathscr{D}_F(\mathbf{A}) \to \mathscr{D}_F(\mathbf{X})$ is an equivalence between derived categories of Frobenius pairs. Now, it follows from Lemma 3.2 (2) that the map

$$K(EH) = K(H)K(E) : K(\mathbf{A}) \longrightarrow K(\mathbf{X})$$

is a homotopy equivalence. This means that $K(H)$ is a homotopy-split injection. $\square$

As an application of Lemma 4.3, we have the following result which will serve as a preparation for the proof of Theorem 1.1.

Corollary 4.4. *Let R and S be rings, and let $Q^\bullet$ be a complex in $\mathscr{C}^b(R \otimes_{\mathbb{Z}} S^{\mathrm{op}})$ with ${}_RQ^n \in \mathscr{P}^{<\infty}(R)$ for all $n \in \mathbb{Z}$. Let $\mathscr{Y}$ be a full triangulated subcategory of $\mathscr{D}^c(S)$, and define $\mathcal{P} \subseteq \mathscr{C}^b(S\text{-proj})$ to be the full subcategory of objects which belong to $\mathscr{Y}$ as objects in $\mathscr{D}^c(S)$. Then the following hold:*

(1) *The pair $\mathbf{P} := (\mathcal{P}, \mathscr{C}^b_{ac}(S)\text{-proj})$ is a Frobenius pair and the inclusion $\mathbf{P} \subseteq \mathscr{C}^b(S\text{-proj})$ is a map of Frobenius pairs.*

(2) *If the functor $Q^\bullet \otimes^{\mathbb{L}}_S - : \mathscr{D}(S) \to \mathscr{D}(R)$ induces a triangle equivalence $\mathscr{Y} \xrightarrow{\simeq} \mathscr{X} := \mathrm{Tria}({}_RQ^\bullet) \cap \mathscr{D}^c(R)$, then the map $K(\mathbf{P}) \to K(\mathscr{C}^b(S\text{-proj}))$ induced from the inclusion $\mathbf{P} \to \mathscr{C}^b(S\text{-proj})$ is a homotopy-split injection.*

Proof. (1) We consider the Frobenius pair $\mathscr{C}^b(S\text{-proj})$ and the triangle equivalence $\mathscr{D}_F(\mathscr{C}^b(S\text{-proj})) = \mathscr{K}^b(S\text{-proj}) \xrightarrow{\simeq} \mathscr{D}^c(S)$ induced by the canonical localization functor. Then, by Lemma 3.3 (1), we know that $\mathbf{P}$ is a Frobenius subcategory of $\mathscr{C}^b(S\text{-proj})$, and the inclusion $\mathbf{P} \subseteq \mathscr{C}^b(S\text{-proj})$ is a map of Frobenius pairs.

(2) Let $\mathbf{B} := \big(\mathscr{C}^b(\mathscr{P}^{<\infty}(R)), \mathscr{C}^b_{ac}(\mathscr{P}^{<\infty}(R))\big)$. Then $\mathscr{D}_F(\mathbf{B}) = \mathscr{D}^b(\mathscr{P}^{<\infty}(R))$. Since $Q^\bullet \in \mathscr{C}^b(R \otimes_{\mathbb{Z}} S^{\mathrm{op}})$ with ${}_RQ^n \in \mathscr{P}^{<\infty}(R)$ for all $n \in \mathbb{Z}$, it follows from Lemma 3.1 that

$$G := {}_RQ^\bullet \otimes^\bullet_S - : \mathscr{C}^b(S\text{-proj}) \longrightarrow \mathbf{B}$$

is a map of Frobenius pairs and the derived functor $\mathscr{D}_F(G) : \mathscr{D}_F(\mathscr{C}^b(S\text{-proj})) \to \mathscr{D}_F(\mathbf{B})$ is given by the composition of the following functors:

$$\mathscr{D}_F(\mathscr{C}^b(S\text{-proj})) = \!= \mathscr{K}^b(S\text{-proj}) \xrightarrow{{}_RQ^\bullet \otimes^\bullet_S -} \mathscr{K}^b(\mathscr{P}^{<\infty}(R)) \xrightarrow{q} \mathscr{D}^b(\mathscr{P}^{<\infty}(R)) = \!= \mathscr{D}_F(\mathbf{B})$$

where q is the localization functor.

Since $\mathscr{D}(S) = \mathrm{Tria}({}_SS)$ and $Q^\bullet \otimes^{\mathbb{L}}_S S = Q^\bullet$ in $\mathscr{D}(R)$, the image of the functor $Q^\bullet \otimes^{\mathbb{L}}_S - : \mathscr{D}(S) \to \mathscr{D}(R)$ is contained in $\mathrm{Tria}({}_RQ^\bullet)$ by Lemma 2.1 (2). So we write $Q^\bullet \otimes^{\mathbb{L}}_S - : \mathscr{D}(S) \to \mathscr{D}(R)$ as the following composition: $\mathscr{D}(S) \to \mathrm{Tria}({}_RQ^\bullet) \hookrightarrow \mathscr{D}(R)$. Since $Q^\bullet \in \mathscr{C}^b(R \otimes_{\mathbb{Z}} S^{\mathrm{op}})$ with ${}_RQ^n \in \mathscr{P}^{<\infty}(R)$ for all $n \in \mathbb{Z}$, it follows from $\mathscr{P}^{<\infty}(R) \subseteq \mathscr{D}^c(R)$ that ${}_RQ^\bullet \in \mathscr{D}^c(R)$. Note that $\mathscr{K}^b(S\text{-proj}) \xrightarrow{\simeq} \mathscr{D}^c(S)$ and $Q^\bullet \otimes_S^{\mathbb{L}} S = Q^\bullet$ in $\mathscr{D}(R)$. Thus the restriction of $Q^\bullet \otimes_S^{\mathbb{L}} -$ to $\mathscr{D}^c(S)$ is actually a triangle functor from $\mathscr{D}^c(S)$ to $\mathscr{X}$.

Suppose that $Q^\bullet \otimes_S^{\mathbb{L}} - : \mathscr{D}(S) \to \mathscr{D}(R)$ induces a triangle equivalence $\mathscr{Y} \xrightarrow{\simeq} \mathscr{X}$. Let $F : \mathbf{P} \to \mathscr{C}^b(S\text{-proj})$ be the inclusion map of the Frobenius pairs. Now, we show that

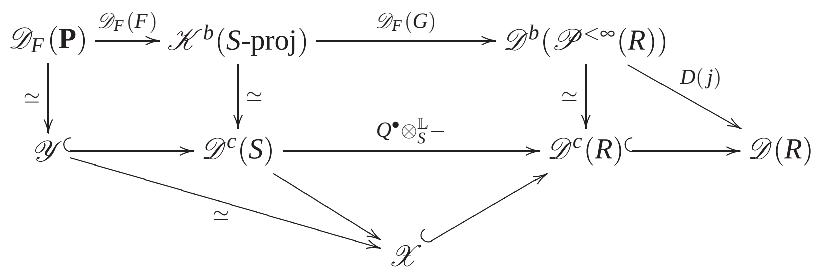
(1) $\mathscr{D}_F(GF) : \mathscr{D}_F(\mathbf{P}) \to \mathscr{D}_F(\mathbf{B}) = \mathscr{D}^b(\mathscr{P}^{<\infty}(R))$ is fully faithful, and

(2) $\mathrm{Im}(\mathscr{D}_F(G)) = \mathrm{Im}(\mathscr{D}_F(GF)) \subseteq \mathscr{D}^b(\mathscr{P}^{<\infty}(R))$.

In fact, by Lemma 3.10, the inclusion $j : \mathscr{P}^{<\infty}(R) \to R\text{-Mod}$ induces a triangle equivalence

$$D(j) : \mathscr{D}^b(\mathscr{P}^{<\infty}(R)) \xrightarrow{\simeq} \mathscr{D}^c(R).$$

According to Lemma 3.3 (2), we can construct the following commutative diagram of triangulated categories (up to natural isomorphism):



This implies that $D(j)\mathscr{D}_F(G)\mathscr{D}_F(F) : \mathscr{D}_F(\mathbf{P}) \to \mathscr{D}^c(R)$ is fully faithful and that

$$\mathrm{Im}\big(D(j)\mathscr{D}_F(G)\big) = \mathrm{Im}\big((Q^\bullet \otimes_S^{\mathbb{L}} -)|_{\mathscr{D}^c(S)}\big) = \mathscr{X} = \mathrm{Im}(D(j)\mathscr{D}_F(G)\mathscr{D}_F(F)).$$

Now, (1) and (2) follow from the equivalence $D(j) : \mathscr{D}^b(\mathscr{P}^{<\infty}(R)) \xrightarrow{\simeq} \mathscr{D}^c(R)$. By Lemma 4.3, we infer that the map $K(F) : K(\mathbf{P}) \to K(\mathscr{C}^b(S\text{-proj}))$ is a homotopy-split injection. $\square$

From the proof of Corollary 4.4, we obtain the following result.

Lemma 4.5. *let R and S be rings, and let $P^\bullet \in \mathscr{C}(R \otimes_{\mathbb{Z}} S^{\mathrm{op}})$ such that ${}_RP^\bullet \in \mathscr{C}^b(R\text{-proj})$, $\mathrm{Hom}_R(P^\bullet, R) \in \mathscr{D}^c(S)$ and the functor $P^\bullet \otimes_S^{\mathbb{L}} - : \mathscr{D}(S) \to \mathscr{D}(R)$ is fully faithful. Assume that there exists a complex $Q^\bullet \in \mathscr{C}^-(S \otimes_{\mathbb{Z}} R^{\mathrm{op}})$ such that ${}_SQ^n \in \mathscr{P}^{<\infty}(S)$ for all $n \in \mathbb{Z}$ and that*

$$Q^\bullet \otimes_R^{\mathbb{L}} - \xrightarrow{\simeq} \mathrm{Hom}_R(P^\bullet, R) \otimes_R^{\mathbb{L}} - : \mathscr{D}(R) \longrightarrow \mathscr{D}(S).$$

Then the map $K(P^\bullet \otimes_S^\bullet -) : K(S) \to K(R)$, induced from the map ${}_RP^\bullet \otimes_S^\bullet - : \mathscr{C}^b(S\text{-proj}) \to \mathscr{C}^b(R\text{-proj})$, is a homotopy-split injection.

Proof. We first point out that $Q^\bullet$ in Lemma 4.5 can be chosen to be a bounded complex, that is, $Q^\bullet \in \mathscr{C}^b(S \otimes_{\mathbb{Z}} R^{\mathrm{op}})$. Indeed, let $Q^\bullet$ be of the from:

$$\cdots \longrightarrow Q^{-2} \xrightarrow{d^{-2}} Q^{-1} \xrightarrow{d^{-1}} Q^0 \xrightarrow{d^0} Q^1 \longrightarrow \cdots \longrightarrow Q^t \longrightarrow 0 \longrightarrow \cdots$$

with some $t \in \mathbb{N}$, and let $P^{\bullet *} := \mathrm{Hom}_R(P^\bullet, R)$. Since $Q^\bullet \otimes_R^{\mathbb{L}} - \xrightarrow{\simeq} P^{\bullet *} \otimes_R^{\mathbb{L}} - : \mathscr{D}(R) \to \mathscr{D}(S)$ by assumption, we have

$$Q^\bullet = Q^\bullet \otimes_R^{\mathbb{L}} R \xrightarrow{\simeq} P^{\bullet *} \otimes_R^{\mathbb{L}} R = P^{\bullet *} \text{ in } \mathscr{D}(S).$$

In particular, $H^n(Q^\bullet) \simeq H^n(P^{\bullet *})$ for all $n \in \mathbb{Z}$. Note that $P^{\bullet *}$ is a bounded complex since ${}_RP^\bullet \in \mathscr{C}^b(R\text{-proj})$. Thus there is an integer $s \leq t$ such that $H^i(Q^\bullet) = 0$ for all $i < s$. Let $W^\bullet$ be the following complex obtained from the canonical truncation in degree s:

$$\cdots \longrightarrow 0 \longrightarrow \mathrm{Coker}(d^{s-1}) \longrightarrow Q^{s+1} \xrightarrow{d^{s+1}} Q^{s+2} \xrightarrow{d^{s+2}} \cdots \longrightarrow Q^t \longrightarrow 0 \longrightarrow \cdots.$$

Then $W^\bullet \in \mathscr{C}^b(S\otimes_{\mathbb{Z}} R^{\mathrm{op}})$ and there is a canonical quasi-isomorphism $f^\bullet : Q^\bullet \to W^\bullet$ in $\mathscr{C}(S\otimes_{\mathbb{Z}} R^{\mathrm{op}})$. In particular, $Q^\bullet \simeq W^\bullet$ in $\mathscr{D}(S)$. Since $P^{\bullet *}\simeq Q^\bullet$ in $\mathscr{D}(S)$ and $P^{\bullet *}\in \mathscr{D}^c(S)$, we see that both $Q^\bullet$ and $W^\bullet$ lie in $\mathscr{D}^c(S)$.

On the one hand, by [9, Lemma 4.2 (d)], the quasi-isomorphism $f^\bullet$ induces a natural isomorphism of derived functors:

$$Q^\bullet \otimes^{\mathbb{L}}_R - \xrightarrow{\simeq} W^\bullet \otimes^{\mathbb{L}}_R - : \mathscr{D}(R) \longrightarrow \mathscr{D}(S),$$

and therefore $W^\bullet \otimes^{\mathbb{L}}_R - \xrightarrow{\simeq} P^{\bullet *} \otimes^{\mathbb{L}}_R - : \mathscr{D}(R)\to \mathscr{D}(S)$. On the other hand, since ${}_SQ^n \in \mathscr{P}^{<\infty}(S)$ for $s+1\leq n\leq t$ and since $\mathscr{P}^{<\infty}(S)\subseteq \mathscr{D}^c(S)$, we see from $W^\bullet \in \mathscr{D}^c(S)$ that the module $\mathrm{Coker}(d^{s-1})$ is in $\mathscr{D}^c(S)$, and therefore it lies in $\mathscr{P}^{<\infty}(S)$. This implies that each term of $W^\bullet$ as an S-module belongs to $\mathscr{P}^{<\infty}(S)$. Thus we can replace $Q^\bullet$ in Lemma 4.5 by the bounded complex $W^\bullet$.

Now, we assume $Q^\bullet \in \mathscr{C}^b(S\otimes_{\mathbb{Z}} R^{\mathrm{op}})$ and define $\mathbf{B} := \big(\mathscr{C}^b(\mathscr{P}^{<\infty}(S)), \mathscr{C}^b_{ac}(\mathscr{P}^{<\infty}(S))\big)$. Then $\mathscr{D}_F(\mathbf{B}) = \mathscr{D}^b(\mathscr{P}^{<\infty}(S))$. Since ${}_SQ^n \in \mathscr{P}^{<\infty}(S)$ for all $n\in\mathbb{Z}$, we have $Q^\bullet \otimes^\bullet_R R \simeq Q^\bullet \in \mathscr{C}^b(\mathscr{P}^{<\infty}(S))$ transparently. By Lemma 3.1, the additive functor

$$G := Q^\bullet \otimes^\bullet_R - : \mathscr{C}^b(R\text{-proj}) \longrightarrow \mathbf{B}$$

is a map of Frobenius pairs. Since ${}_RP^\bullet \in \mathscr{C}^b(R\text{-proj})$, the functor $F := {}_RP^\bullet \otimes^\bullet_S - : \mathscr{C}^b(S\text{-proj}) \to \mathscr{C}^b(R\text{-proj})$ is also a map of Frobenius pairs. Consider the following commutative diagram of triangulated categories:

$$(*)\qquad \begin{array}{ccccc}
\mathscr{K}^b(S\text{-proj}) & \xrightarrow{\mathscr{D}_F(F)} & \mathscr{K}^b(R\text{-proj}) & \xrightarrow{\mathscr{D}_F(G)} & \mathscr{D}^b(\mathscr{P}^{<\infty}(S)) \\
\simeq\downarrow & & \downarrow\simeq & & \simeq\downarrow D(j) \\
\mathscr{D}^c(S) & \xrightarrow{{}_RP^\bullet\otimes^{\mathbb{L}}_S -} & \mathscr{D}^c(R) & \xrightarrow{{}_SQ^\bullet\otimes^{\mathbb{L}}_R -} & \mathscr{D}^c(S)
\end{array}$$

where the equivalence $D(j)$ is induced by the inclusion $j:\ \mathscr{P}^{<\infty}(S)\to S\text{-Mod}$ by Lemma 3.10.

In the following, we claim that the composition of the two functors in the second row of the above diagram is an equivalence.

Indeed, on the one hand, since $(P^\bullet \otimes^{\mathbb{L}}_S -, \mathbb{R}\mathrm{Hom}_R(P^\bullet, -))$ is an adjoint pair and $P^\bullet \otimes^{\mathbb{L}}_S - : \mathscr{D}(S)\to\mathscr{D}(R)$ is fully faithful, the unit adjunction

$$\eta : Id_{\mathscr{D}(S)} \longrightarrow \mathbb{R}\mathrm{Hom}_R(P^\bullet, P^\bullet \otimes^{\mathbb{L}}_S -) : \mathscr{D}(S)\longrightarrow \mathscr{D}(S)$$

is a natural isomorphism. On the other hand, since ${}_RP^\bullet \in \mathscr{C}^b(R\text{-proj})$, we know from [3, Section 2.1] that

$$P^{\bullet *}\otimes^{\mathbb{L}}_R - \xrightarrow{\simeq} \mathbb{R}\mathrm{Hom}_R(P^\bullet, -) : \mathscr{D}(R)\longrightarrow\mathscr{D}(S).$$

Thus

$$Id_{\mathscr{D}(S)} \xrightarrow{\simeq} (P^{\bullet *}\otimes^{\mathbb{L}}_R -)(P^\bullet \otimes^{\mathbb{L}}_S -) : \mathscr{D}(S)\longrightarrow\mathscr{D}(S).$$

Due to the natural equivalence $Q^\bullet \otimes^{\mathbb{L}}_R - \xrightarrow{\simeq} P^{\bullet *}\otimes^{\mathbb{L}}_R - :\ \mathscr{D}(R)\to\mathscr{D}(S)$, we certainly have

$$Id_{\mathscr{D}(S)} \xrightarrow{\simeq} (Q^{\bullet}\otimes^{\mathbb{L}}_R -)(P^\bullet \otimes^{\mathbb{L}}_S -) : \mathscr{D}(S)\longrightarrow\mathscr{D}(S).$$

Consequently, the functor $(Q^\bullet \otimes^{\mathbb{L}}_R -)(P^\bullet\otimes^{\mathbb{L}}_S -) : \mathscr{D}(S)\to\mathscr{D}(S)$ is an equivalence, and therefore so is its restriction to $\mathscr{D}^c(S)$. This finishes the claim.

By the diagram $(*)$, the composite of $\mathscr{D}_F(F)$ with $\mathscr{D}_F(G)$ is an equivalence of derived categories of Frobenius pairs. Now, it follows from Lemma 3.2 (2) that the map $K(GF) = K(F)K(G) : K(S)\to K(S)$ is a homotopy equivalence of K-theory spaces. This shows that $K(F)$ is a homotopy-split injection, completing the proof of Lemma 4.5. $\square$

Remark 4.6. A sufficient condition to guarantee that

$$Q^\bullet \otimes^{\mathbb{L}}_R - \xrightarrow{\simeq} P^{\bullet *} \otimes^{\mathbb{L}}_R - : \mathscr{D}(R) \longrightarrow \mathscr{D}(S)$$

in Lemma 4.5 is that the complexes $Q^\bullet$ and $P^{\bullet *}$ are connected by a series of quasi-isomorphisms among chain complexes over $S \otimes_{\mathbb{Z}} R^{\mathrm{op}}$:

$$Q^\bullet \longleftarrow U_0^\bullet \longrightarrow U_1^\bullet \longleftarrow \cdots \longrightarrow U_{n-1}^\bullet \longleftarrow U_n^\bullet \longrightarrow P^{\bullet *}$$

for some $n \in \mathbb{N}$. For a proof of this fact, we refer the reader to [9, Lemma 4.2 (d)].

4.2 Algebraic K-theory of recollements induced by homological ring epimorphisms

To prove Theorem 1.1, we shall establish the following substantial result, Proposition 4.7, on K-theory spaces of rings which are linked by homological ring epimorphisms. This result involves K-theory spaces of dg algebras, which are introduced in Subsection 3.5, and gives a decomposition of higher algebraic K-groups. The conclusion of our result under the assumption of finite-type resolution is, of course, stronger than the result in [16].

Proof of Theorem 1.1 (1).

For the convenience of references, we restate the first part of Theorem 1.1 more precisely as the following proposition.

Proposition 4.7. *Let $\lambda : R \to S$ be a homological ring epimorphism such that ${}_RS \in \mathscr{P}^{<\infty}(R)$. Then there is a complex $P^\bullet \in \mathscr{C}^b(R\text{-proj})$ such that $\mathrm{Tria}(P^\bullet) = \mathrm{Tria}({}_RQ^\bullet) \subseteq \mathscr{D}(R)$, where $Q^\bullet[1]$ is the mapping cone of λ. Further, if we define $\mathbb{T} := \mathrm{End}^\bullet_R(P^\bullet)$, then $K(R) \xrightarrow{\sim} K(S) \times K(\mathbb{T})$ as K-theory spaces, and therefore*

$$K_n(R) \simeq K_n(S) \oplus K_n(\mathbb{T}) \quad \textit{for all } n \in \mathbb{N}.$$

Moreover, if $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, P^\bullet[i]) = 0$ for all $i \neq 0$, then $K(R) \xrightarrow{\sim} K(S) \times K(T)$ as K-theory spaces, where $T := \mathrm{End}_{\mathscr{D}(R)}(P^\bullet)$. In particular,

$$K_n(R) \simeq K_n(S) \oplus K_n(T) \quad \textit{for all } n \in \mathbb{N}.$$

Proof. Under the assumption ${}_RS \in \mathscr{P}^{<\infty}(R)$, we can choose a complex $S^\bullet$ in $\mathscr{C}^b(R\text{-proj})$ such that ${}_RS$ is isomorphic to $S^\bullet$ in $\mathscr{D}(R)$. So we get a chain map from ${}_RR$ to $S^\bullet$ such that its mapping cone $P^\bullet[1]$ is isomorphic in $\mathscr{D}(R)$ to the mapping cone $Q^\bullet[1]$ of λ. This complex $P^\bullet$ satisfies the property in Proposition 4.7. Now, let us fix such a complex $P^\bullet$ throughout the proof.

By definition, $Q^\bullet$ is the two-term complex $0 \to R \xrightarrow{\lambda} S \to 0$ with R and S in degrees 0 and 1, respectively. Clearly, $Q^\bullet \in \mathscr{C}^b(R \otimes_{\mathbb{Z}} R^{\mathrm{op}})$. Since ${}_RS \in \mathscr{P}^{<\infty}(R)$, we have ${}_RQ^n \in \mathscr{P}^{<\infty}(R)$ for all $n \in \mathbb{Z}$. Therefore, $Q^\bullet$ satisfies the first assumption in Corollary 4.4.

Let

$$\mathscr{P} := \mathrm{Tria}({}_RQ^\bullet) \cap \mathscr{D}^c(R), \quad \mathcal{P} := \{X^\bullet \in \mathscr{C}^b(R\text{-proj}) \mid X^\bullet \in \mathscr{P}\} \quad \text{and} \quad \mathbf{P} := \big(\mathcal{P}, \mathscr{C}^b_{ac}(R\text{-proj})\big).$$

Then it follows from Corollary 4.4 (1) that the inclusion $F : \mathbf{P} \to \mathscr{C}^b(R\text{-proj})$ is a map of Frobenius pairs. Considering the following sequence of Frobenius pairs:

$$\mathscr{C}^b(S\text{-proj}) \xleftarrow{S \otimes_R -} \mathscr{C}^b(R\text{-proj}) \xhookleftarrow{F} \mathbf{P},$$

we then obtain a sequence of triangulated categories:

$$(*) \qquad \mathscr{D}_F(\mathscr{C}^b(S\text{-proj})) \xleftarrow{\mathscr{D}_F(S \otimes_R -)} \mathscr{D}_F(\mathscr{C}^b(R\text{-proj})) \xleftarrow{\mathscr{D}_F(F)} \mathscr{D}_F(\mathbf{P}).$$

Let $\mathscr{P}'$ be the full subcategory of $\mathscr{K}^b(R\text{-proj})$ consisting of all those objects which, regarded as objects of $\mathscr{D}^c(R)$, belong to $\mathscr{P}$. Since $\mathscr{D}_F(\mathscr{C}^b(R\text{-proj})) = \mathscr{K}^b(R\text{-proj}) \xrightarrow{\simeq} \mathscr{D}^c(R)$ and $\mathscr{P}$ is a full triangulated subcategory of $\mathscr{D}^c(R)$, we see from Lemma 3.3 (2) that $\mathscr{D}_F(\mathbf{P}) = \mathscr{P}'$. So the sequence $(*)$ is exactly the following sequence:

$$\mathscr{K}^b(S\text{-proj}) \xleftarrow{\mathscr{D}_F(S\otimes_R -)} \mathscr{K}^b(R\text{-proj}) \xleftarrow{\mathscr{D}_F(F)} \mathscr{P}'$$

Now, we claim that this sequence (of triangulated categories) is exact (see Subsection 2.1 for definition).

Actually, there is a commutative diagram of triangulated categories:

$$\begin{array}{ccccc}
\mathscr{K}^b(S\text{-proj}) & \xleftarrow{\mathscr{D}_F(S\otimes_R -)} & \mathscr{K}^b(R\text{-proj}) & \xleftarrow{\mathscr{D}_F(F)} & \mathscr{P}' \\
\simeq\downarrow & & \downarrow\simeq & & \downarrow\simeq \\
\mathscr{D}^c(S) & \xleftarrow{S\otimes^{\mathbb{L}}_R -} & \mathscr{D}^c(R) & \hookleftarrow & \mathscr{P}
\end{array}$$

in which the square on the right-hand side follows from Lemma 3.3 (2). So it is sufficient to show that the bottom sequence in the above diagram is exact.

Since $\lambda : R \to S$ is a homological ring epimorphism, it follows from Lemma 2.4 that there is a recollement of triangulated categories:

$$(**) \qquad \mathscr{D}(S) \underset{}{\overset{i^*}{\underset{i_*}{\rightleftarrows}}} \mathscr{D}(R) \overset{j_!}{\underset{j^!}{\rightleftarrows}} \mathrm{Tria}({}_RQ^\bullet)$$

where $j_!$ is the canonical inclusion and i_* is the restriction functor $D(\lambda_*)$ induced from λ, and where $j^! := Q^\bullet \otimes^{\mathbb{L}}_R -$ and $i^* := S \otimes^{\mathbb{L}}_R -$.

Since ${}_RS \in \mathscr{P}^{<\infty}(R)$, we have $i_*(S) = {}_RS \in \mathscr{D}^c(R)$. This implies that i_* preserves compact objects. Note that $\mathscr{D}(R)$ is compactly generated by the compact object ${}_RR$. Thus, by Lemma 2.3 (c), we see that $(**)$ gives rise to the following "half recollement" at the level of the subcategories of compact objects:

$$\mathscr{D}^c(S) \overset{i^*}{\underset{i_*}{\rightleftarrows}} \mathscr{D}^c(R) \overset{j_!}{\underset{j^!}{\rightleftarrows}} \mathscr{P} \;= \mathrm{Tria}(Q^\bullet) \cap \mathscr{D}^c(R)$$

which satisfies the following properties:

(a) The inclusion $j_! : \mathscr{P} \longrightarrow \mathscr{D}^c(R)$ is fully faithful. Since both $\mathrm{Tria}({}_RQ^\bullet)$ (see Lemma 2.3 (a)) and $\mathscr{D}^c(R)$ are closed under direct summands in $\mathscr{D}(R)$, we know that $\mathscr{P}$ is also closed under direct summands in $\mathscr{D}^c(R)$.

(b) The composite of $j_!$ with i^* is zero, and the functor i^* induces an equivalence $\mathscr{D}^c(R)/\mathscr{P} \xrightarrow{\simeq} \mathscr{D}^c(S)$ of triangulated categories. In particular, $\mathscr{P}$ coincides with the kernel of the restriction of i^* to $\mathscr{D}^c(R)$.

As a result, the following sequence

$$\mathscr{D}^c(S) \xleftarrow{S\otimes^{\mathbb{L}}_R -} \mathscr{D}^c(R) \xleftarrow{j_!} \mathscr{P}$$

is exact, and therefore so is the sequence $(*)$. This finishes the claim.

By Lemma 3.2, the exactness of $(*)$ implies that the sequence of K-theory spaces:

$$K(S) \xleftarrow{K(S\otimes_R -)} K(R) \xleftarrow{K(F)} K(\mathbf{P})$$

is a homotopy fibration.

Next, we shall apply Corollary 4.4 to show that $K(F)$ is a homotopy-split injection.

By Corollary 4.4, it suffices to check that $j^! : \mathscr{D}(R) \to \mathscr{D}(R)$ induces an auto-equivalence $\mathscr{P} \xrightarrow{\simeq} \mathscr{P}$.

Indeed, by the recollement $(**)$, the composite of $j_!$ with $j^!$ is naturally isomorphic to the identity of $\mathrm{Tria}({}_RQ^\bullet)$. This implies that $j^! : \mathrm{Tria}({}_RQ^\bullet) \longrightarrow \mathrm{Tria}({}_RQ^\bullet)$ is an auto-equivalence. Since $j_!$ always preserves compact objects, we see that $\mathscr{P}$ coincides with the full subcategory of $\mathrm{Tria}({}_RQ^\bullet)$ consisting of all compact objects in $\mathrm{Tria}({}_RQ^\bullet)$ (see also Lemma 2.2 (1)). Thus $j^!$ induces an auto-equivalence $\mathscr{P} \xrightarrow{\simeq} \mathscr{P}$. It follows from Corollary 4.4 that $K(F)$ is a homotopy-split injection. By Corollary 4.2, we see that

$$K(R) \xrightarrow{\sim} K(S) \times K(\mathbf{P})$$

as K-theory spaces.

In the following, we shall apply Lemma 3.9 to prove that $K(\mathbf{P})$ is homotopy equivalent to $K(\mathbb{T})$, where $\mathbb{T} := \mathrm{End}^\bullet_R(P^\bullet)$ is the dg endomorphism ring of $P^\bullet$.

Recall that $P^\bullet \in \mathscr{C}^b(R\text{-proj})$ and $\mathrm{Tria}(P^\bullet) = \mathrm{Tria}({}_RQ^\bullet) \subseteq \mathscr{D}(R)$. In particular, we have

$$\mathscr{P} = \mathrm{Tria}({}_RQ^\bullet) \cap \mathscr{D}^c(R) = \mathrm{Tria}(P^\bullet) \cap \mathscr{D}^c(R) \subseteq \mathscr{D}^c(R).$$

By Lemma 3.9, there is a homotopy equivalence $K(\mathbb{T}) \xrightarrow{\sim} K(\mathbf{P})$. Thus

$$K(R) \xrightarrow{\sim} K(S) \times K(\mathbb{T})$$

as K-theory spaces, and therefore $K_n(R) \simeq K_n(S) \oplus K_n(\mathbb{T})$ for $n \in \mathbb{N}$. This shows the first part of Proposition 4.7.

Note that $H^i(\mathbb{T}) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, P^\bullet[i])$ for each $i \in \mathbb{Z}$. Now, the second part of Proposition 4.7 is a consequence of Lemma 3.8 together with the first part of Proposition 4.7. Thus Theorem 1.1 (1) follows. $\square$

Remark 4.8. By the recollement $(**)$ and Lemma 2.2 (1), we conclude that, under the assumptions of Proposition 4.7, the category $\mathscr{P} := \mathrm{Tria}({}_RQ^\bullet) \cap \mathscr{D}^c(R)$ coincides with any one of the following three categories:

(1) The full subcategory of $\mathrm{Tria}({}_RQ^\bullet)$ consisting of all compact objects in $\mathrm{Tria}({}_RQ^\bullet)$.

(2) The smallest full triangulated subcategory of $\mathscr{D}^c(R)$ which contains $Q^\bullet$ and is closed under direct summands.

(3) The full subcategory of $\mathscr{D}^c(R)$ consisting of all objects $X^\bullet$ such that $S \otimes^{\mathbb{L}}_R X^\bullet = 0$ in $\mathscr{D}(S)$.

Particularly, (3) implies that the category $\mathcal{P}$, defined in the proof of Proposition 4.7, is equal to the full Frobenius subcategory of $\mathscr{C}^b(R\text{-proj})$ consisting of all those complexes $X^\bullet$ such that $S \otimes_R X^\bullet$ is acyclic, that is, $H^i(S \otimes_R X^\bullet) = 0$ for all $i \in \mathbb{Z}$. In the literature, for example, see [16, Theorem 0.5] and [12, Theorem 14.9], it was shown that, for homological ring epimorphisms $\lambda : R \to S$ with certain condidtions, one may get a weak homotopy fibration:

$$(\dagger) \qquad K(\mathbf{P}) \xrightarrow{K(F)} K(R) \xrightarrow{K(S \otimes_R -)} K(S).$$

This implies that, in general, the map $K_0(S \otimes_R -) : K_0(R) \to K_0(S)$ does not have to be surjective. However, Proposition 4.7 shows a stronger conclusion, namely, under the assumption ${}_RS \in \mathscr{P}^{<\infty}(R)$, the sequence $(\dagger)$ splits up to homotopy equivalence:

$$K(R) \xrightarrow{\sim} K(S) \times K(\mathbf{P}).$$

Now we turn to the proof of Theorem 1.1 (2). The main ingredient of the proof is Proposition 4.7.

Proof of Theorem 1.1 (2).

Given a recollement of derived module categories:

$$\mathscr{D}(S) \xrightarrow{i_*} \mathscr{D}(R) \xrightarrow{} \mathscr{D}(T) \quad (\text{with } j_!: \mathscr{D}(T) \to \mathscr{D}(R))$$

we can obtain the following two consequences:

(1) The complex $j_!(T)$ is compact in $\mathscr{D}(R)$ and

$$\mathrm{End}_{\mathscr{D}(R)}(j_!(T)) \simeq T, \ \ \mathrm{Hom}_{\mathscr{D}(R)}(j_!(T), j_!(T)[n]) = 0 \ \text{ for } \ n \neq 0.$$

(2) $\mathrm{Tria}(j_!(T)) = \{X \in \mathscr{D}(R) \mid \mathrm{Hom}_{\mathscr{D}(R)}(X,Y) = 0 \ \text{ for each } Y \in \mathscr{D}(S)\}$.

Moreover, since $\lambda : R \to S$ is a homological ring epimorphism, there is a recollement of triangulated categories by Lemma 2.4:

$$\mathscr{D}(S) \xrightarrow{i_*} \mathscr{D}(R) \xrightarrow{} \mathrm{Tria}({}_RQ^\bullet) \quad (\text{with } F: \mathrm{Tria}({}_RQ^\bullet) \to \mathscr{D}(R))$$

where F is the inclusion and $Q^\bullet[1]$ is the mapping cone of λ. It follows that

$$\mathrm{Tria}({}_RQ^\bullet) = \{X \in \mathscr{D}(R) \mid \mathrm{Hom}_{\mathscr{D}(R)}(X,Y) = 0 \ \text{ for each } Y \in \mathscr{D}(S)\}.$$

By (2), we have $\mathrm{Tria}(j_!(T)) = \mathrm{Tria}({}_RQ^\bullet) \subseteq \mathscr{D}(R)$.

Since $j_!(T)$ is compact in $\mathscr{D}(R)$, we can choose $P^\bullet \in \mathscr{C}^b(R\text{-proj})$ such that $P^\bullet \simeq j_!(T)$ in $\mathscr{D}(R)$. Then $\mathrm{Tria}(P^\bullet) = \mathrm{Tria}({}_RQ^\bullet) \subseteq \mathscr{D}(R)$, and by (1), we have

$$\mathrm{End}_{\mathscr{D}(R)}(P^\bullet) \simeq T \ \text{ and } \ \mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, P^\bullet[n]) = 0 \ \text{ for } \ n \neq 0.$$

Now, suppose that ${}_RS \in \mathscr{P}^{<\infty}(R)$. It follows from Proposition 4.7 that $K(R) \xrightarrow{\sim} K(S) \times K(T)$ as K-theory spaces, and therefore

$$K_n(R) \simeq K_n(S) \oplus K_n(T) \quad \text{for all } \ n \in \mathbb{N}.$$

This finishes the proof of Theorem 1.1 (2) for the case ${}_RS \in \mathscr{P}^{<\infty}(R)$.

Similarly, we can prove Theorem 1.1 (2) for the case $S_R \in \mathscr{P}^{<\infty}(R^{\mathrm{op}})$. In fact, this can be understood from Lemma 2.5 and the following fact: For any ring A, there is a homotopy equivalence $K(A) \xrightarrow{\sim} K(A^{\mathrm{op}})$ (see [18, Sections 1 (3) and 2 (5)]). Thus the proof of Theorem 1.1 has been completed. $\square$.

Proof of Corollary 1.2.

We shall apply Proposition 4.7 and [15, Theorem 0.5] to show Corollary 1.2.

Let $\lambda : R \to S := R_\Sigma$ be the universal localization of R at Σ. By abuse of notation, we identify each homomorphism $P_1 \xrightarrow{f} P_0$ in Σ with the two-term complex $0 \to P_1 \xrightarrow{f} P_0 \to 0$ in $\mathscr{C}^b(R\text{-proj})$, where P_i is in degrees $-i$ for $i = 0, 1$.

First of all, we recall the definition of a small Waldhausen category $\mathcal{R}$. The category $\mathcal{R}$ is the smallest full subcategory of $\mathscr{C}^b(R\text{-proj})$ which

(i) contains all the complexes in Σ,

(ii) contains all acyclic complexes,

(iii) is closed under the formation of mapping cones and shifts,

(iv) contains any direct summands of any of its objects.

We remark that $\mathcal{R}$ was first defined in [16, Definition 0.4] and denoted by **R**. Observe that, in $\mathcal{R}$, the cofibrations are injective chain maps which are degreewise split, and the weak equivalences are homotopy equivalences. Moreover, $\mathcal{R}$ has the following additional properties:

(v) $\mathcal{R}$ is closed under finite direct sums in $\mathscr{C}^b(R\text{-proj})$.

(vi) If $N^\bullet \in \mathcal{R}$ and $M^\bullet \in \mathscr{C}^b(R\text{-proj})$ such that, in $\mathscr{K}^b(R\text{-proj})$, $M^\bullet$ is a direct summand of $N^\bullet$, then $M^\bullet \in \mathcal{R}$. In particular, $\mathcal{R}$ is closed under isomorphisms in $\mathscr{K}^b(R\text{-proj})$.

Actually, these two properties can be deduced from (ii)-(iv) with the help of the following two general facts: Let $X^\bullet, Y^\bullet \in \mathscr{C}^b(R\text{-proj})$. Then

(1) $X^\bullet \oplus Y^\bullet$ is exactly the mapping cone of the zero map from $X^\bullet[-1]$ to $Y^\bullet$.

(2) $X^\bullet \simeq Y^\bullet$ in $\mathscr{K}^b(R\text{-proj})$ if and only if there are two complexes $U^\bullet, V^\bullet \in \mathscr{C}^b_{ac}(R\text{-proj})$ such that $X^\bullet \oplus U^\bullet \simeq Y^\bullet \oplus V^\bullet$ in $\mathscr{C}^b(R\text{-proj})$, because $\mathscr{C}^b(R\text{-proj})$ is a Frobenius category, $\mathscr{C}^b(R\text{-proj})\text{-proj} = \mathscr{C}^b_{ac}(R\text{-proj})$ and $\mathscr{D}_F(\mathscr{C}^b(R\text{-proj})) = \mathscr{K}^b(R\text{-proj})$.

In the following, we shall prove that the Waldhausen category $\mathcal{R}$ coincides with the Waldhausen category $\mathcal{P}$ defined by the Frobenius pair $\mathbf{P} := \big(\mathcal{P}, \mathscr{C}^b_{ac}(R\text{-proj})\big)$, where $\mathcal{P}$ is a full subcategory of $\mathscr{C}^b(R\text{-proj})$ defined by

$$\mathcal{P} := \{X^\bullet \in \mathscr{C}^b(R\text{-proj}) \mid X^\bullet \text{ is isomorphic in } \mathscr{D}(R) \text{ to an object in } \mathrm{Tria}({}_RQ^\bullet)\}$$

and $Q^\bullet$ is the two-term complex $0 \to R \xrightarrow{\lambda} S \to 0$ with R in degree 0.

In fact, by Remark 4.8, we see that $\mathcal{P}$ is the same as the full subcategory of $\mathscr{C}^b(R\text{-proj})$ consisting of those complexes $X^\bullet$ such that $S \otimes_R X^\bullet$ is acyclic (or equivalently, $S \otimes^{\mathbb{L}}_R X^\bullet = 0$ in $\mathscr{D}(S)$). It is easy to see that the latter subcategory satisfies the conditions (i)-(iv). This gives rise to $\mathcal{R} \subseteq \mathcal{P}$.

Next, we show the converse inclusion $\mathcal{P} \subseteq \mathcal{R}$.

Let $\mathscr{R}$ be the full subcategory of $\mathscr{K}^b(R\text{-proj})$ consisting of all objects of $\mathcal{R}$. Then, due to (i)-(vi), we see that $\mathscr{R}$ is a full triangulated subcategory of $\mathscr{K}^b(R\text{-proj})$ containing Σ and being closed under direct summands. Since $\mathscr{D}_F(\mathbf{P}) \subseteq \mathscr{K}^b(R\text{-proj})$, we know from (vi) that $\mathcal{P} \subseteq \mathcal{R}$ if and only if $\mathscr{D}_F(\mathbf{P}) \subseteq \mathscr{R}$. To show $\mathscr{D}_F(\mathbf{P}) \subseteq \mathscr{R}$, it is enough to show that $\mathscr{D}_F(\mathbf{P})$ is exactly the smallest full triangulated subcategory of $\mathscr{K}^b(R\text{-proj})$ which contains Σ and is closed under direct summands.

Indeed, by the proof of Proposition 4.7, $\mathscr{D}_F(\mathbf{P})$ is the full subcategory of $\mathscr{K}^b(R\text{-proj})$ in which the objects, regarded as objects in $\mathscr{D}^c(R)$, belong to $\mathscr{P} := \mathrm{Tria}({}_RQ^\bullet) \cap \mathscr{D}^c(R)$. As $\mathscr{P} = \mathrm{thick}({}_RQ^\bullet)$ by Remark 4.8, $\mathscr{D}_F(\mathbf{P})$ is actually the full subcategory of $\mathscr{K}^b(R\text{-proj})$ in which the objects, regarded as objects in $\mathscr{D}^c(R)$, belong to $\mathrm{thick}({}_RQ^\bullet)$. We claim that $\mathrm{thick}(Q^\bullet) = \mathrm{thick}(\Sigma)$.

Since λ is a homological ring epimorphism, we know from [4, Proposition 3.6] and Lemma 2.4 that $\mathrm{Tria}(\Sigma) = \mathrm{Tria}({}_RQ^\bullet)$ in $\mathscr{D}(R)$. Since both $\mathrm{Tria}(\Sigma)$ and $\mathrm{Tria}({}_RQ^\bullet)$ are closed under small coproducts in $\mathscr{D}(R)$, they have the same subcategories of compact objects, that is, $\mathrm{Tria}(\Sigma)^c = \mathrm{Tria}({}_RQ^\bullet)^c \subseteq \mathscr{D}^c(R)$. Clearly, $\Sigma \subseteq \mathscr{D}^c(R)$ and ${}_RQ^\bullet \in \mathscr{D}^c(R)$ since ${}_RS \in \mathscr{P}^{<\infty}(R) \subseteq \mathscr{D}^c(R)$ by our assumption. By definition, $\mathrm{thick}(\Sigma)$ (respectively, $\mathrm{thick}({}_RQ^\bullet)$) is the smallest full triangulated subcategory of $\mathscr{D}^c(R)$ containing Σ (respectively, $Q^\bullet$) and being closed under direct summands. Then, by Lemma 2.2 (1), $\mathrm{thick}(\Sigma) = \mathrm{Tria}(\Sigma)^c$ and $\mathrm{thick}({}_RQ^\bullet) = \mathrm{Tria}({}_RQ^\bullet)^c$, and therefore $\mathrm{thick}(\Sigma) = \mathrm{thick}({}_RQ^\bullet) \subseteq \mathscr{D}^c(R)$.

Thus $\mathscr{D}_F(\mathbf{P})$ is the full subcategory of $\mathscr{K}^b(R\text{-proj})$ consisting of all those objects which, viewed as objects of $\mathscr{D}^c(R)$, belong to $\mathrm{thick}(\Sigma)$. Since $\mathscr{K}^b(R\text{-proj}) \simeq \mathscr{D}^c(R)$ and $\Sigma \subseteq \mathscr{C}^b(R\text{-proj})$, we conclude that $\mathscr{D}_F(\mathbf{P})$ is equal to the smallest full triangulated subcategory of $\mathscr{K}^b(R\text{-proj})$ containing Σ and being closed under direct summands. Consequently, $\mathcal{P} \subseteq \mathcal{R}$.

Hence $\mathcal{P} = \mathcal{R}$ as full subcategories of $\mathscr{C}^b(R\text{-proj})$. Furthermore, the category $\mathcal{P}$, regarded as a Waldhausen category defined by the Frobenius pair $\mathbf{P}$, has injective chain maps which are degreewise split as cofibrations, and has homotopy equivalences as weak equivalences. This implies that $\mathcal{P} = \mathcal{R}$ as Waldhausen categories.

Suppose that all maps in Σ are injective. Let $\mathscr{E}$ be the exact category of (R,Σ)-torsion modules. Then, it is shown in [15, Theorem 0.5] that $K(\mathcal{R}) \xrightarrow{\sim} K(\mathscr{E})$ as K-theory spaces. Since $\mathcal{P} = \mathcal{R}$ as Waldhausen categories, we obtain $K(\mathbf{P}) := K(\mathcal{P}) \xrightarrow{\sim} K(\mathscr{E})$. As $\lambda: R \to S$ is homological and ${}_RS$ has a finite-type resolution, it follows from the proof of Proposition 4.7 that

$$K(R) \xrightarrow{\sim} K(S) \times K(\mathcal{P}) \xrightarrow{\sim} K(S) \times K(\mathscr{E}).$$

This finishes the proof of Corollary 1.2. $\square$

As a consequence of Proposition 4.7, we have the following result in [27, Lemma 3.1].

Corollary 4.9. *If $\lambda : R \to S$ is an injective ring epimorphism such that ${}_RS$ is projective and finitely generated, then, for each $n \in \mathbb{N}$,*

$$K_n(R) \simeq K_n(S) \oplus K_n\big(\mathrm{End}_R(S/R)\big).$$

Proof. Under our assumption on λ, we see that $\mathrm{Ext}^i_R(S/R, S/R) = 0$ for all $i > 0$. Then the corollary follows from the second part of Proposition 4.7. $\square$

Remark 4.10. In Theorem 1.1 (2), we assume the existence of a recollement of derived module categories of rings. For some necessary and sufficient conditions that vouch for the existence of such a recollement, we refer the interested reader to the preprint [6].

5 Applications to homological exact pairs

In this section, we shall apply our results in the previous sections to homological ring epimorphisms afforded by exact pairs defined in [3].

5.1 A supplement to algebraic K-theory of recollements

Let R be a ring. Recall that $K(R)$ is a homotopy-associative pointed H-space with the multiplication map $K(\sqcup) : K(R) \times K(R) \to K(R)$, which is induced from the coproduct functor $\sqcup : R\text{-proj} \times R\text{-proj} \to R\text{-proj}$ (see Subsection 3.1). For any two maps $f, g : K(R) \to K(R)$, we denote by $f \cdot g : K(R) \to K(R)$ the composite of the following three maps:

$$K(R) \xrightarrow{\Delta} K(R) \times K(R) \xrightarrow{f \times g} K(R) \times K(R) \xrightarrow{K(\sqcup)} K(R),$$

where Δ is the diagonal map $x \mapsto (x, x)$ for $x \in K(R)$.

Observe that the shift functor $[1] : \mathscr{C}^b(R\text{-proj}) \to \mathscr{C}^b(R\text{-proj})$ is also an exact functor of Waldhausen categories, and that the induced map $K([1]) : K(R) \to K(R)$ is a *homotopy inverse* of $K(R)$ in the sense that both $K([1]) \cdot Id_{K(R)}$ and $Id_{K(R)} \cdot K([1])$ are pointed-homotopic to the constant map $K(R) \to K(R)$ defined by $x \mapsto e$, where e is the associated point of $K(R)$. For each $n \in \mathbb{N}$, since $K_n(\Delta) : K_n(R) \to K_n(R) \times K_n(R)$ is still the diagonal map, the homomorphism $K_n([1]) : K_n(R) \to K_n(R)$ is exactly the minus map of the additive groups $K_n(R)$ (see also [23, Corollary 1.7.3]).

Let S be another ring and $N^\bullet$ a bounded complex of S-R-bimodules. If ${}_SN^\bullet \in \mathscr{C}^b(S\text{-proj})$, then the tensor functor $N^\bullet \otimes^\bullet_R - : \mathscr{C}^b(R\text{-proj}) \to \mathscr{C}^b(S\text{-proj})$ is an exact functor of Waldhausen categories. In case $\lambda : R \to S$ is a ring homomorphism, we choose $N^\bullet = {}_SS_R$, where the right R-module structure of S is induced from λ, and denote by $K(\lambda)$ the map $K(S \otimes_R -) : K(R) \to K(S)$.

We first establish the following result on K-theory of recollements.

Lemma 5.1. *Let R_i be a ring for $1 \le i \le 3$, and let $M^\bullet \in \mathscr{C}(R_2 \otimes_{\mathbb{Z}} R_1^{\mathrm{op}})$ and $N^\bullet \in \mathscr{C}(R_3 \otimes_{\mathbb{Z}} R_2^{\mathrm{op}})$ such that ${}_{R_2}M^\bullet \in \mathscr{C}^b(R_2\text{-proj})$ and ${}_{R_3}N^\bullet \in \mathscr{C}^b(R_3\text{-proj})$. Define*

$$F := M^\bullet \otimes^\bullet_{R_1} - : \mathscr{C}^b(R_1\text{-proj}) \longrightarrow \mathscr{C}^b(R_2\text{-proj}) \;\; \textit{and} \;\; G := N^\bullet \otimes^\bullet_{R_2} - : \mathscr{C}^b(R_2\text{-proj}) \longrightarrow \mathscr{C}^b(R_3\text{-proj}).$$

Suppose that there is a recollement of derived module categories:

$$\xymatrix{\mathscr{D}(R_3) \ar[r] & \mathscr{D}(R_2) \ar@/_1pc/[l]_{i^*} \ar@/^1pc/[l] \ar[r] & \mathscr{D}(R_1) \ar@/_1pc/[l]_{j_!} \ar@/^1pc/[l]}$$

such that $j_! = M^\bullet \otimes^{\mathbb{L}}_{R_1} -$ *and* $i^* = N^\bullet \otimes^{\mathbb{L}}_{R_2} -$. *Then the sequence of K-theory spaces*

$$K(R_1) \xrightarrow{K(F)} K(R_2) \xrightarrow{K(G)} K(R_3)$$

is a weak homotopy fibration, and therefore there is a long exact sequence of K-groups:

$$\cdots \longrightarrow K_{n+1}(R_3) \longrightarrow K_n(R_1) \xrightarrow{K_n(F)} K_n(R_2) \xrightarrow{K_n(G)} K_n(R_3) \longrightarrow K_{n-1}(R_1) \longrightarrow$$

$$\cdots \longrightarrow K_0(R_1) \longrightarrow K_0(R_2) \longrightarrow K_0(R_3)$$

for all $n \in \mathbb{N}$.

Proof. Our proof will use some ideas from the proof of [20, Theorem 9].

It follows from the given recollement and Lemma 2.3 that the following sequence

$$\mathscr{D}(R_1) \xrightarrow{j_!} \mathscr{D}(R_2) \xrightarrow{i^*} \mathscr{D}(R_3)$$

of derived module categories is exact (see Subsection 2.1):

(a) The functor $j_!$ is fully faithful and induces an equivalence $\mathscr{D}(R_1) \xrightarrow{\simeq} \mathrm{Tria}({}_{R_2}M^\bullet)$.

(b) The composition of $j_!$ with i^* is zero.

(c) $\mathrm{Ker}(i^*) = \mathrm{Im}(j_!) = \mathrm{Tria}({}_{R_2}M^\bullet)$.

(d) The functor i^* induces an equivalence $\Phi : \mathscr{D}(R_2)/\mathrm{Ker}(i^*) \xrightarrow{\simeq} \mathscr{D}(R_3)$.

Let $\mathscr{X}$ be the full subcategory of $\mathrm{Ker}(i^*)$ consisting of all compact objects. Then, it follows from $M^\bullet \in \mathscr{D}^c(R_2)$ and Lemma 2.2 (1) that $\mathscr{X} \subseteq \mathscr{D}^c(R_2)$. Now, we define $\mathcal{X}$ to be the full subcategory of $\mathscr{C}^b(R_2\text{-proj})$ consisting of the objects which are isomorphic in $\mathscr{D}(R_2)$ to objects of $\mathscr{X}$. Note that $\mathscr{C}^b(R_2\text{-proj})\text{-proj} = \mathscr{C}^b_{ac}(R_2\text{-proj})$ by Example (b) in Subsection 3.3. Applying Lemma 3.3 to the Frobenius pair $\mathscr{C}^b(R_2\text{-proj})$ together with the equivalence $\mathscr{K}^b(R_2\text{-proj}) \xrightarrow{\simeq} \mathscr{D}^c(R_2)$, we see that $\mathcal{X}$ is a Frobenius subcategory of $\mathscr{C}^b(R_2\text{-proj})$ containing all projective objects of $\mathscr{C}^b(R_2\text{-proj})$. Define $\mathbf{X} = \big(\mathcal{X}, \mathscr{C}^b_{ac}(R_2\text{-proj})\big)$ and $\mathbf{Y} = (\mathscr{C}^b(R_2\text{-proj}), \mathcal{X})$. Then, by Lemma 3.3 (2), $\mathbf{X}$ and $\mathbf{Y}$ are Frobenius pairs with $\mathcal{X}\text{-proj} = \mathscr{C}^b(R_2\text{-proj})\text{-proj}$, and we have a sequence of Frobenius pairs:

$$\mathcal{X} \xhookrightarrow{F_2} \mathscr{C}^b(R_2\text{-proj}) \xrightarrow{G_1} \mathbf{Y}$$

where G_1 is the identity functor, such that the following diagram of triangulated categories are commutative:

$$\begin{array}{ccccc}
\mathscr{D}_F(\mathcal{X}) & \xrightarrow{\mathscr{D}_F(F_2)} & \mathscr{D}_F(\mathscr{C}^b(R_2\text{-proj})) & \xrightarrow{\mathscr{D}_F(G_1)} & \mathscr{D}_F(\mathbf{Y}) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\mathscr{X} & \hookrightarrow & \mathscr{D}^c(R_2) & \longrightarrow & \mathscr{D}^c(R_2)/\mathscr{X}
\end{array}$$

Since $\mathscr{X}$ is closed under direct summands in $\mathscr{D}^c(R)$, the bottom sequence in the above diagram is an exact sequence of triangulated categories. By Lemma 3.2 (1) and the definition of K-theory spaces of Frobenius pairs, we deduce that the following sequence of K-theory spaces

$$K(\mathcal{X}) \xrightarrow{K(F_2)} K(R_2) \xrightarrow{K(G_1)} K(\mathcal{Y}).$$

is a homotopy fibration, where $\mathcal{Y}$ is the Waldhausen category defined by the Frobenius pair $\mathbf{Y}$, that is, $\mathcal{Y}$ has the same objects, morphisms and cofibrations as $\mathscr{C}^b(R_2\text{-proj})$, but the weak equivalences in $\mathcal{Y}$ are those chain maps such that their mapping cones lie in $\mathcal{X}$.

Observe that $F(\mathscr{C}^b(R_1\text{-proj})) \subseteq \mathcal{X}$ and that $G(U) \in \mathscr{C}^b_{ac}(R_3\text{-proj})$ for $U \in \mathcal{X}$ since ${}_{R_3}N^\bullet \in \mathscr{C}^b(R_3\text{-proj})$ and $G(U) = i^*(U) = 0$ in $\mathscr{D}(R_3)$. Consequently, F and G induce two canonical maps of Frobenius pairs

$$F_1 : \mathscr{C}^b(R_1\text{-proj}) \longrightarrow \mathcal{X} \ \text{ and } \ G_2 : \mathbf{Y} \longrightarrow \mathscr{C}^b(R_3\text{-proj})$$

such that $F = F_2F_1$ and $G = G_2G_1$, respectively. This can be illustrated by the following diagram of Frobenius pairs:

$$\begin{array}{ccccccc}
\mathscr{C}^b(R_1\text{-proj}) & & \xrightarrow{\quad F \quad} & & \mathscr{C}^b(R_2\text{-proj}) & \xrightarrow{\quad G \quad} & \mathscr{C}^b(R_3\text{-proj}) \\
 & \searrow^{F_1} & & \nearrow^{F_2} & & \searrow^{G_1} \quad \nearrow^{G_2} & \\
 & & \mathcal{X} & & & \mathbf{Y} &
\end{array}$$

Next, we point out that the map $K(F_1) : K(R_1) \to K(\mathcal{X})$ is a homotopy equivalence. Actually, by Lemma 3.2 (2), it suffices to prove that the functor $\mathscr{D}_F(F_1) : \mathscr{D}_F(\mathscr{C}^b(R_1\text{-proj})) \to \mathscr{D}_F(\mathcal{X})$ is a triangle equivalence. This follows from the following commutative diagram:

$$\begin{array}{ccc}
\mathscr{D}_F(\mathscr{C}^b(R_1\text{-proj})) & \xrightarrow{\mathscr{D}_F(F_1)} & \mathscr{D}_F(\mathcal{X}) \\
\simeq \downarrow & & \downarrow \simeq \\
\mathscr{D}^c(R_1) & \underset{\simeq}{\xrightarrow{\quad j_! \quad}} & \mathscr{X}
\end{array}$$

where the equivalence of the second row is due to (a) and (c).

Consequently, the following sequence of K-theory spaces

$$K(R_1) \xrightarrow{K(F)} K(R_2) \xrightarrow{K(G_1)} K(\mathcal{Y}).$$

is a homotopy fibration. So, to prove that the sequence of K-theory spaces

$$K(R_1) \xrightarrow{K(F)} K(R_2) \xrightarrow{K(G)} K(R_3)$$

is a weak homotopy fibration, it is enough to show that the map $K(G_2) : K(\mathcal{Y}) \to K(R_3)$ gives rise to an injection $K_0(G_2) : K_0(\mathcal{Y}) \to K_0(R_3)$ and an isomorphism $K_n(G_2) : K_n(\mathcal{Y}) \xrightarrow{\simeq} K_n(R_3)$ for each $n > 0$.

In fact, by Lemma 3.2 (3), we only need to check that $\mathscr{D}_F(G_2) : \mathscr{D}_F(\mathbf{Y}) \to \mathscr{D}_F(\mathscr{C}^b(R_3\text{-proj}))$ is an equivalence up to factors (see Subsection 2.1 for definition). For this aim, let $\mathscr{Y}$ be the full subcategory of $\mathscr{D}(R_2)/\mathrm{Ker}(i^*)$ consisting of all compact objects. Consider the following canonical exact sequence of triangulated categories:

$$(\dagger) \qquad \mathrm{Ker}(i^*) \hookrightarrow \mathscr{D}(R_2) \longrightarrow \mathscr{D}(R_2)/\mathrm{Ker}(i^*).$$

Since $\mathscr{D}(R_2)$ and $\mathrm{Tria}(M^\bullet)$ are triangulated categories with small coproducts and since ${}_{R_2}M^\bullet \in \mathscr{D}^c(R_2)$, we know from Lemma 2.2 that $(\dagger)$ induces a sequence of the subcategories of compact objects:

$$\mathscr{X} \hookrightarrow \mathscr{D}^c(R_2) \longrightarrow \mathscr{Y}$$

such that $\mathscr{X}$ is closed under direct summands in $\mathscr{D}^c(R_2)$ and that the induced functor $H_1 : \mathscr{D}^c(R_2)/\mathscr{X} \to \mathscr{Y}$ is an equivalence up to factors. Moreover, the equivalence Φ in (d) induces a triangle equivalence $\Phi^c : \mathscr{Y} \xrightarrow{\simeq} \mathscr{D}^c(R_3)$. Define $H_2 : \mathscr{D}^c(R_2)/\mathscr{X} \to \mathscr{D}^c(R_3)$ to be the composite of H_1 with Φ^c. Then H_2 is an equivalence up to factors. Since the following diagram

$$\begin{array}{ccc}
\mathscr{D}_F(\mathbf{Y}) & \xrightarrow{\mathscr{D}_F(G_2)} & \mathscr{D}_F(\mathscr{C}^b(R_3\text{-proj})) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathscr{D}^c(R_2)/\mathscr{X} & \xrightarrow{\quad H_2 \quad} & \mathscr{D}^c(R_3)
\end{array}$$

is commutative, we see that $\mathscr{D}_F(G_2)$ is an equivalence up to factors. This finishes the proof of Lemma 5.1. □

5.2 Long Mayer-Vietoris sequences: Proof of Theorem 1.3

In this section, we shall show Theorem 1.3. Here, we follow the notation introduced in [3].

Throughout this section, we suppose that $\lambda : R \to S$ and $\mu : R \to T$ are ring homomorphisms such that the pair (λ, μ) is exact, that is, the sequence

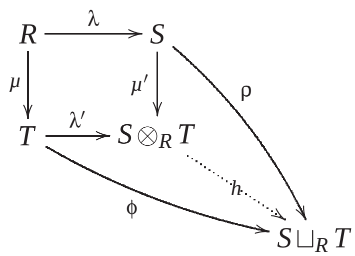
$$0 \longrightarrow R \longrightarrow S \oplus T \xrightarrow{\binom{\mu'}{-\lambda'}} S \otimes_R T \longrightarrow 0$$

of R-R-bimodules is exact, where

$$\lambda' = \lambda \otimes T : t \mapsto 1 \otimes t, \;\; \mu' = S \otimes \mu : s \mapsto s \otimes 1$$

for $t \in T$ and $s \in S$ (see [3] for more information).

Let $S \sqcup_R T$ be the coproduct of the rings S and T over R, and let $\rho : S \to S \sqcup_R T$ and $\phi : T \to S \sqcup_R T$ be the defining ring homomorphisms of the coproduct. Then we have the following commutative diagram:



where h is defined by $s \otimes t \mapsto (s)\rho(t)\phi$ for $t \in T$ and $s \in S$. Note that the square in this diagram is both a push-out and a pull-back. This implies that the mapping cone $Q^\bullet$ of λ is quasi-isomorphic to the mapping cone $Q^\bullet \otimes_R T$ of λ' as complexes.

Given such a diagram, there is a ring homomorphism $\theta : B \to C$ defined in [3]:

$$\theta := \begin{pmatrix} \rho & h \\ 0 & \phi \end{pmatrix} : \quad B = \begin{pmatrix} S & S \otimes_R T \\ 0 & T \end{pmatrix} \longrightarrow C = \begin{pmatrix} S \sqcup_R T & S \sqcup_R T \\ S \sqcup_R T & S \sqcup_R T \end{pmatrix}.$$

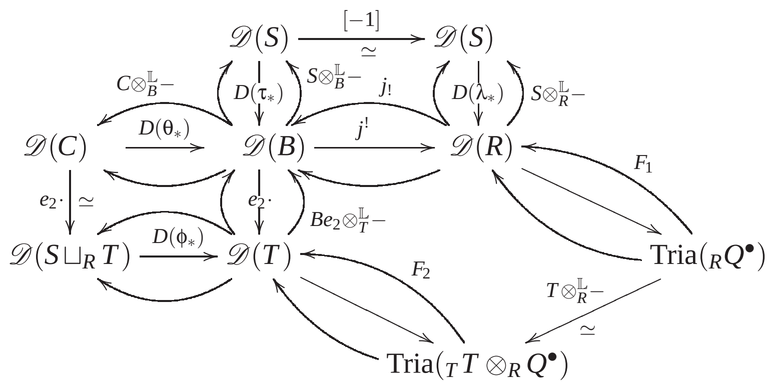
Furthermore, we define $e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in B$ and

$$\varphi : Be_1 \longrightarrow Be_2 : \begin{pmatrix} s \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} s \otimes 1 \\ 0 \end{pmatrix} \text{ for } s \in S.$$

Let $P^\bullet$ be the complex $0 \to Be_1 \xrightarrow{\varphi} Be_2 \to 0$ with Be_1 and Be_2 in degrees -1 and 0, respectively. Then $P^\bullet \in \mathscr{C}^b(B \otimes_{\mathbb{Z}} R^{\mathrm{op}})$ and ${}_BP^\bullet \in \mathscr{C}^b(B\text{-proj})$, where Be_1 and Be_2 are regarded as right R-modules via λ and μ, respectively. Let $P^{\bullet *} := \mathrm{Hom}_B(P^\bullet, B) \in \mathscr{C}^b(R \otimes_{\mathbb{Z}} B^{\mathrm{op}})$, which is isomorphic to the complex $0 \to e_2B \xrightarrow{\varphi_*} e_1B \to 0$ with e_2B and e_1B in degrees 0 and 1, respectively.

In case both λ and ϕ are homological ring epimorphisms, we say that the exact pair (λ, μ) is *homological*, or that the square defined by λ, μ, ϕ and ρ is a *homological Milnor square*. For a homological Milnor square, the following result has been shown in [3, Theorem 1.1 and Corollary 3.11].

Lemma 5.2. *Suppose that λ is a homological ring epimorphism and $\mathrm{Tor}_i^R(T,S)=0$ for all $i>0$. Then there is the following 'pull-back' of recollements of triangulated categories:*

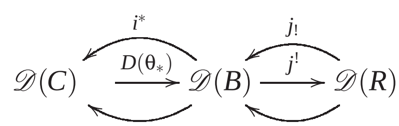


where $\tau: B\to S:=B/(Be_2B)$ is the canonical surjection, F_i is the canonical embedding for $i=1,2$, and

$$j_! = {}_BP^\bullet\otimes_R^{\mathbb{L}} - \ \textit{ and } \ j^! = \mathrm{Hom}_B^\bullet(P^\bullet,-)\simeq P^{\bullet *}\otimes_B^\bullet - .$$

Proof of Theorem 1.3.

(1) By Lemma 5.2, there is a recollement of derived module categories:

(a) 

where $i^*:=C\otimes_B^{\mathbb{L}}-$. Obviously, $P^\bullet\in\mathscr{C}(B\otimes_{\mathbb{Z}}R^{\mathrm{op}})$, ${}_BP^\bullet\in\mathscr{C}^b(B\text{-proj})$, ${}_CC_B\in\mathscr{C}(C\otimes_{\mathbb{Z}}B^{\mathrm{op}})$ and ${}_CC\in\mathscr{C}^b(C\text{-proj})$. By (a) and Lemma 5.1, the following two functors:

$$P^\bullet\otimes_R^\bullet-:\mathscr{C}^b(R\text{-proj})\longrightarrow\mathscr{C}^b(B\text{-proj}) \text{ and } C\otimes_B-:\mathscr{C}^b(B\text{-proj})\longrightarrow\mathscr{C}^b(C\text{-proj})$$

induces a weak homotopy fibration of K-theory spaces:

$$(b)\qquad K(R)\xrightarrow{K(P^\bullet\otimes_R^\bullet-)}K(B)\xrightarrow{K(C\otimes_B-)}K(C).$$

Since B is a triangular matrix ring, we see that the K-theory space $K(B)$ of B is homotopy equivalent to $K(S)\times K(T)$. Thus we get Theorem 1.3 (1) without an explicit description of the maps in the sequence.

In the following, we shall work out the maps in details.

Since C is Morita equivalent to $S\sqcup_R T$, the map $K(e_2C\otimes_C-):K(C)\to K(S\sqcup_R T)$ induced from the exact functor $e_2C\otimes_C:C\text{-proj}\to(S\sqcup_R T)\text{-proj}$ is a homotopy equivalence by Lemma 3.2 (2). Thus we obtain a weak homotopy fibration of K-theory spaces:

$$(c)\qquad K(R)\xrightarrow{K(P^\bullet\otimes_R^\bullet-)}K(B)\xrightarrow{K(e_2C\otimes_B-)}K(S\sqcup_R T).$$

Clearly, $e_2B\simeq T\simeq e_2Be_2$, $Be_2B\simeq Be_2$ and $S\simeq B/(Be_2B)=Be_1$. From the triangular structure of B, we see that the following two maps

$$(d)\qquad K(B)\xrightarrow{\alpha:=\left(K(S\otimes_B-),K(e_2\cdot)\right)}K(S)\times K(T)\xrightarrow{\beta:=\binom{K({}_BS\otimes_S-)}{K(Be_2\otimes_T-)}}K(B)\ .$$

are mutually inverse homotopy equivalences.

Now, by (c) and (d), we obtain a weak homotopy fibration of K-theory spaces:

$$K(R)\xrightarrow{K(P^\bullet\otimes_R^\bullet -)\alpha} K(S)\times K(T)\xrightarrow{\beta K(e_2C\otimes_B -)} K(S\sqcup_R T).$$

It is trivial to check the following natural isomorphisms of functors (see the diagram in Lemma 5.2):

$$(e_2C\otimes_B -)({}_BS\otimes_S -)\xrightarrow{\simeq}(e_2C\otimes_B S)\otimes_S -\xrightarrow{\simeq}(S\sqcup_R T)\otimes_S - : S\text{-proj}\longrightarrow (S\sqcup_R T)\text{-proj},$$

$$(e_2C\otimes_B -)(Be_2\otimes_T -)\xrightarrow{\simeq}(e_2C\otimes_B Be_2)\otimes_T -\xrightarrow{\simeq}(S\sqcup_R T)\otimes_T - : T\text{-proj}\longrightarrow (S\sqcup_R T)\text{-proj},$$

$$(e_2\cdot)(P^\bullet\otimes_R^\bullet -))\xrightarrow{\simeq}(e_2P^\bullet)\otimes_R^\bullet -\xrightarrow{\simeq} T\otimes_R - : R\text{-proj}\longrightarrow T\text{-proj},$$

$$(S\otimes_B -)(P^\bullet\otimes_R^\bullet -))\xrightarrow{\simeq}(S\otimes_B P^\bullet)\otimes_R^\bullet -\xrightarrow{\simeq}\big(P^\bullet/(Be_2B)P^\bullet\big)\otimes_R^\bullet -\xrightarrow{\simeq}(S\otimes_R -)[1] : \mathscr{C}^b(R\text{-proj})\longrightarrow \mathscr{C}^b(S\text{-proj}).$$

Consequently, we have the following homotopic maps

$$K(P^\bullet\otimes_R^\bullet -)\alpha=\big(K(P^\bullet\otimes_R^\bullet -)K(S\otimes_B -),K(P^\bullet\otimes_R^\bullet -)K(e_2\cdot)\big)\xrightarrow{\sim}\big(K((S\otimes_R -)[1]),K(T\otimes_R -)\big)=:\big(-K(\lambda),K(\mu)\big)$$

and

$$\beta K(e_2C\otimes_B -)=\begin{pmatrix} K({}_BS\otimes_S -)K(e_2C\otimes_B -)\\ K(Be_2\otimes_T -)K(e_2C\otimes_B -)\end{pmatrix}\xrightarrow{\sim}\begin{pmatrix} K((S\sqcup_R T)\otimes_S -)\\ K((S\sqcup_R T)\otimes_T -)\end{pmatrix}=:\begin{pmatrix} K(\rho)\\ K(\phi)\end{pmatrix}.$$

Hence, the sequence of K-theory spaces:

$$(e)\qquad K(R)\xrightarrow{\big(-K(\lambda),K(\mu)\big)} K(S)\times K(T)\xrightarrow{\begin{pmatrix} K(\rho)\\ K(\phi)\end{pmatrix}} K(S\sqcup_R T)$$

is a weak homotopy fibration, which yields the long exact sequence of K-groups in Theorem 1.3 (1).

(2) To prove Theorem 1.3 (2), we first show that ${}_BC\in\mathscr{P}^{<\infty}(B)$ if and only if ${}_RS\in\mathscr{P}^{<\infty}(R)$.

In fact, by Lemma 2.3 (c), the functor $D(\lambda_*)$ preserves compact objects if and only if so is $j^!$. That is, ${}_BC\in\mathscr{P}^{<\infty}(B)$ if and only if $j^!(B)\simeq P^{\bullet *}\in\mathscr{D}^c(R)$. Note that $Q^\bullet\otimes_R T\simeq Q^\bullet$ in $\mathscr{D}(R)$ and that ${}_RP^{\bullet *}$ is isomorphic in $\mathscr{C}(R)$ to the direct sum of $S[-1]$ and $(Q^\bullet\otimes_R T)[-1]$. Thus ${}_RP^{\bullet *}\simeq(S\oplus Q^\bullet)[-1]$ in $\mathscr{D}(R)$. Since $Q^\bullet$ is the two-term complex $0\to R\xrightarrow{\lambda} S\to 0$, we infer that $P^{\bullet *}\in\mathscr{D}^c(R)$ if and only if ${}_RS\in\mathscr{D}^c(R)$, and therefore ${}_BC\in\mathscr{P}^{<\infty}(B)$ if and only if ${}_RS\in\mathscr{P}^{<\infty}(R)$.

Next, we show that if ${}_RS\in\mathscr{P}^{<\infty}(R)$, then the sequence (e) splits up to homotopy equivalence.

Let $\mathscr{X}:=\mathrm{Tria}({}_BP^\bullet)\cap\mathscr{D}^c(B)$, and let $\mathcal{X}$ be the full subcategory of $\mathscr{C}^b(B\text{-proj})$ consisting of those objects such that they belong to $\mathscr{X}$ when viewed as objects of $\mathscr{D}(B)$. Then $\mathbf{X}=\big(\mathcal{X},\mathscr{C}^b_{ac}(B\text{-proj})\big)$ is a Frobenius pair, and in this way we consider $\mathcal{X}$ as a Waldhausen category. Since ${}_BP^\bullet\in\mathscr{C}^b(B\text{-proj})$, we see from Lemma 2.2 (1) that $\mathscr{X}$ is equal to the full subcategory of $\mathrm{Ker}(i^*)$ consisting of all compact objects.

Let $L_1:\mathscr{C}^b(R\text{-proj})\to\mathcal{X}$ be the functor induced by $P^\bullet\otimes_R -$ and $L_2:\mathcal{X}\to\mathscr{C}^b(B\text{-proj})$ the inclusion. Then both L_1 and L_2 are maps of Frobenius pairs, and therefore we have a sequence of Frobeniues pairs:

$$\mathscr{C}^b(C\text{-proj})\xleftarrow{C\otimes_B -}\mathscr{C}^b(B\text{-proj})\xleftarrow{L_2}\mathbf{X}$$

Now, by the proof of Lemma 5.1, the sequence (b) can be decomposed into the following form:

$$\begin{array}{ccccc} K(R) & \xrightarrow{K(P^\bullet\otimes_R -)} & K(B) & \xrightarrow{K(C\otimes_B -)} & K(C)\\ & {\scriptstyle K(L_1)}\searrow & \nearrow{\scriptstyle K(L_2)} & & \\ & & K(\mathcal{X}) & & \end{array}$$

such that $K(L_1)$ is a homotopy equivalence.

Since ${}_RS \in \mathscr{P}^{<\infty}(R)$, we get ${}_BC \in \mathscr{P}^{<\infty}(B)$. Further, the recollement (a) implies that $\theta : B \to C$ is a homological ring epimorphism. Thus it follows from the proof of Proposition 4.7 that the following sequence of K-theory spaces:

$$K(\mathcal{X}) \xrightarrow{K(L_2)} K(B) \xrightarrow{K(C\otimes_B -)} K(C)$$

is a homotopy fibration and that $K(L_2)$ is a homotopy-split injection. Therefore the above sequence of K-theory spaces splits up to homotopy equivalence. As $K(L_1)$ is a homotopy equivalence, we further deduce that the map $K(P^\bullet \otimes_R -) : K(R) \to K(B)$ is a homotopy-split injection. Thus the sequence (b) (and also each of the sequences $(c), (d)$ and (e)) splits up to homotopy equivalence. Hence

$$(*) \quad K(R) \times K(S \sqcup_R T) \xrightarrow{\sim} K(S) \times K(T).$$

Finally, we shall show that $(*)$ also holds if $T_R \in \mathscr{P}^{<\infty}(R^{\mathrm{op}})$. By Theorem 1.1, it suffices to prove that $T_R \in \mathscr{P}^{<\infty}(R^{\mathrm{op}})$ if and only if $C_B \in \mathscr{P}^{<\infty}(B^{\mathrm{op}})$.

Indeed, by (a) and Lemma 2.4, we certainly have $\mathrm{Tria}({}_BP^\bullet) = \mathrm{Ker}(i^*) = \mathrm{Im}(j_!) = \mathrm{Tria}({}_BW^\bullet)$, where ${}_BP^\bullet \in \mathscr{C}^b(B\text{-proj})$ and $W^\bullet$ denotes the mapping cone of θ. Then one can follow the proof of Lemma 2.5 to show that the recollement (a) has a dual form:

$$(\widetilde{a}) \qquad \mathscr{D}(C^{\mathrm{op}}) \xrightarrow{D(\theta_*)} \mathscr{D}(B^{\mathrm{op}}) \xrightarrow{\widetilde{j^!}} \mathscr{D}(R^{\mathrm{op}}) \qquad \text{(with } \widetilde{j_!} : \mathscr{D}(R^{\mathrm{op}}) \to \mathscr{D}(B^{\mathrm{op}})\text{)}$$

where

$$\widetilde{j_!} := -\otimes^{\mathbb{L}}_R P^{\bullet *} \ \text{ and } \ \widetilde{j^!} := \mathrm{Hom}^\bullet_{B^{\mathrm{op}}}(P^{\bullet *}, -) \simeq -\otimes^\bullet_B P^\bullet.$$

By Lemma 2.3 (c), we infer that $P^\bullet \in \mathscr{D}^c(R^{\mathrm{op}})$ if and only if $C_B \in \mathscr{P}^{<\infty}(B^{\mathrm{op}})$. Note that $P^\bullet_R$ is isomorphic in $\mathscr{C}(R^{\mathrm{op}})$ to the direct sum of T and the mapping cone $\mathrm{Cone}(\mu') : 0 \to S \xrightarrow{\mu'} S \otimes_R T \to 0$ of μ'. However, since (λ, μ) is an exact pair, it is easy to see that $\mathrm{Cone}(\mu')$ is actually quasi-isomorphic to the mapping cone $\mathrm{Cone}(\mu) : 0 \to R \xrightarrow{\mu} T \to 0$ of the chain map μ. This gives rise to $P^\bullet \simeq T \oplus \mathrm{Cone}(\mu)$ in $\mathscr{D}(R^{\mathrm{op}})$, which implies that $T_R \in \mathscr{P}^{<\infty}(R^{\mathrm{op}})$ if and only if $P^\bullet \in \mathscr{D}^c(R^{\mathrm{op}})$. Thus $T_R \in \mathscr{P}^{<\infty}(R^{\mathrm{op}})$ if and only if $C_B \in \mathscr{P}^{<\infty}(B^{\mathrm{op}})$. $\square$

As a consequence of Theorem 1.3 (1), we reobtain the following result of Karoubi [25, Chapter V, Proposition 7.5 (2)].

Corollary 5.3. *Let A and B be arbitrary rings, and let $f : A \to B$ be a ring homomorphism and Φ a central multiplicatively closed set of nonzerodivisors in A such that the image of Φ under f is a central set of nonzerodivisors in B. Assume that f induces a ring isomorphism $A/sA \xrightarrow{\simeq} B/sB$ for each $s \in \Phi$. Then there is a Mayer-Vietoris sequence*

$$\cdots \longrightarrow K_{n+1}(\Phi^{-1}B) \longrightarrow K_n(A) \longrightarrow K_n(\Phi^{-1}A) \oplus K_n(B) \longrightarrow K_n(\Phi^{-1}B) \longrightarrow K_{n-1}(A) \longrightarrow$$

$$\cdots \longrightarrow K_0(A) \longrightarrow K_0(\Phi^{-1}A) \oplus K_0(B) \longrightarrow K_0(\Phi^{-1}B)$$

for all $n \in \mathbb{N}$, where $\Phi^{-1}A$ stands for the localization of A at Φ.

Proof. Define $R := A$, $S := \Phi^{-1}A$, $T := B$ and $\mu := f$. Let $\lambda : R \to S$ be the canonical map of the localization. By [4, Lemma 6.2], we have $S \sqcup_R T = \Phi^{-1}B$, which is defined by the canonical maps $\rho : \Phi^{-1}A \to \Phi^{-1}B$ and $\phi : B \to \Phi^{-1}B$. Since Φ and $(\Phi)f$ do not contain zerodivisors, both λ and ϕ are injective. As the modules ${}_A\Phi^{-1}A$ and ${}_B\Phi^{-1}B$ are flat, both λ and ϕ are homological ring epimorphisms.

Now, we claim that (λ, μ) is an exact pair. To show this, we first prove that the following well-defined map

$$h : \Phi^{-1}A \otimes_A B \longrightarrow \Phi^{-1}B, \ \ a/s \otimes b \mapsto ((af)b)/(sf)$$

for $a \in A$, $s \in \Phi$ and $b \in B$, is an isomorphism of $\Phi^{-1}A$-B-bimodules. In fact, since $\Phi^{-1}A = \varinjlim_{s\in\Phi} s^{-1}A$, where $s^{-1}A := \{a/s \mid a \in A\} \subseteq \Phi^{-1}A$, we have

$$\Phi^{-1}A \otimes_A B = (\varinjlim_{s\in\Phi} s^{-1}A) \otimes_A B \xrightarrow{\simeq} \varinjlim_{s\in\Phi}(s^{-1}A \otimes_A B) \xrightarrow{\simeq} \varinjlim_{s\in\Phi}(sf)^{-1}B = \Phi^{-1}B.$$

Next, we show that the cokernels of λ and ϕ are isomorphic as A-modules. Actually,

$$\Phi^{-1}A/A = (\varinjlim_{s\in\Phi} s^{-1}A)/A \xrightarrow{\simeq} \varinjlim_{s\in\Phi}(s^{-1}A/A) \xrightarrow{\simeq} \varinjlim_{s\in\Phi}(A/sA).$$

Similarly, $\Phi^{-1}B/B \xrightarrow{\simeq} \varinjlim_{s\in\Phi}(B/sB)$. Since $A/sA \xrightarrow{\simeq} B/sB$ for each $s \in \Phi$, the map f induces an isomorphism of A-modules: $\Phi^{-1}A/A \xrightarrow{\simeq} \Phi^{-1}B/B$, that is, $\mathrm{Coker}(\lambda) \simeq \mathrm{Coker}(\phi)$.

Finally, we point out that the map $\lambda' : B \to \Phi^{-1}A \otimes_A B$, defined by $b \mapsto 1 \otimes b$ for $b \in B$, is injective and that $\mathrm{Coker}(\lambda) \xrightarrow{\simeq} \mathrm{Coker}(\lambda')$. This is due to the equality $\phi = \lambda' h$.

Thus

$$0 \longrightarrow A \xrightarrow{(-\lambda,\mu)} \Phi^{-1}A \oplus B \xrightarrow{\binom{\mu'}{\lambda'}} \Phi^{-1}A \otimes_A B \longrightarrow 0$$

is an exact sequence of A-modules, where $\mu' : \Phi^{-1}A \to \Phi^{-1}A \otimes_A B$ is defined by $x \mapsto x \otimes 1$ for $x \in \Phi^{-1}A$. By definition, the pair (λ, μ) is exact.

Since Φ consists of central, nonzerodivisor elements in A, the A-module ${}_A\Phi^{-1}A$ is flat. Thus $\mathrm{Tor}_i^A(B, \Phi^{-1}A) = 0$ for all $i > 0$. Hence all conditions in Theorem 1.3 are satisfied. Now, Corollary 5.3 follows from Theorem 1.3 (1). $\square$

Proof of Corollary 1.4.

(1) Let $S := R/I_1$, $T := R/I_2$, and let $\lambda : R \to S$ and $\mu : R \to T$ be the canonical surjections. Then, it follows from the proof of [3, Corollary 1.2 (1)] that, under the assumptions of Corollary 1.4 (1), the pair (λ, μ) is exact, the surjective ring homomorphism λ is homological with $\mathrm{Tor}_i^R(T, S) = 0$ for all $i > 0$, and $S \sqcup_R T = R/(I_1 + I_2)$. Now, (1) is an immediate consequence of Theorem 1.3 (2).

(2) Let $T := R \ltimes M$, and let $\mu : R \to T$ be the canonical inclusion from R into T. Assume that λ is a homological ring epimorphism. Then, it follows from the proof of [3, Corollary 1.2 (2)] that (λ, μ) is an exact pair with $\mathrm{Tor}_i^R(T, S) = 0$ for all $i > 0$, and that $S \ltimes M$, together with $\rho : S \hookrightarrow S \ltimes M$ and $\phi : R \ltimes M \to S \ltimes M$ induced from λ, is the coproduct $S \sqcup_R T$ of the rings S and T over R. Now, by Theorem 1.3 (1), we have a long exact sequence of K-groups:

$$\cdots \longrightarrow K_{n+1}(S \ltimes M) \longrightarrow K_n(R) \xrightarrow{\left(-K_n(\lambda), K_n(\mu)\right)} K_n(S) \oplus K_n(R \ltimes M) \xrightarrow{\binom{K_n(\rho)}{K_n(\phi)}} K_n(S \ltimes M) \longrightarrow K_{n-1}(R)$$

$$\cdots \longrightarrow K_0(R) \longrightarrow K_0(S) \oplus K_0(R \ltimes M) \longrightarrow K_0(S \ltimes M)$$

for all $n \in \mathbb{N}$.

Let $\pi : R \ltimes M \to R$ be the canonical surjection. Then $\mu\pi = Id_R$. This implies that the composite of

$$K(\mu) : K(R) \longrightarrow K(R \ltimes M) \ \text{ with } \ K(\pi) : K(R \ltimes M) \longrightarrow K(R)$$

is homotopic to the identity of $K(R)$, and therefore $K_n(\mu)K_n(\pi) = Id_{K_n(R)}$ for all $n \geq 0$. It follows that the map $\left(-K_n(\lambda), K_n(\mu)\right)$ is a split injection for $n \geq 0$. Therefore, for $n > 0$, the map $\binom{K_n(\rho)}{K_n(\phi)}$ is surjective, the sequence

$$0 \longrightarrow K_n(R) \xrightarrow{\left(-K_n(\lambda), K_n(\mu)\right)} K_n(S) \oplus K_n(R \ltimes M) \xrightarrow{\binom{K_n(\rho)}{K_n(\phi)}} K_n(S \ltimes M) \longrightarrow 0$$

is split exact and $K_n(R)\oplus K_n(S\ltimes M)\simeq K_n(S)\oplus K_n(R\ltimes M)$.

To check the isomorphism for the case $n=0$, we use the following known result: If A is a ring and I is a nilpotent ideal of A, then the canonical surjection $A\to A/I$ induces an isomorphism $K_0(A)\to K_0(A/I)$. This implies that

$$K_0(R\ltimes M)\xrightarrow{\simeq}K_0(R)\ \text{ and }\ K_0(S\ltimes M)\xrightarrow{\simeq}K_0(S).$$

Consequently, both $K_0(\mu):K_0(R)\to K_0(R\ltimes M)$ and $K_0(\rho):K_0(S)\to K_0(S\ltimes M)$ are isomorphisms. Hence $K_0(R)\oplus K_0(S\ltimes M)\simeq K_0(S)\oplus K_0(R\ltimes M)$. This completes the proof of Corollary 1.4. □

6 An example

In the following, we give an example to illustrate the key point, Proposition 4.7, in our proof of the main result, Theorem 1.1.

Example 1. Let k be a field, and let R be a k-algebra with the 2×2 matrix ring $M_2(k)$ over k as its vector space, and with the multiplication in R given by

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} a' & b'\\ c' & d'\end{pmatrix}=\begin{pmatrix} aa' & ab'+bd'\\ ca'+dc' & dd'\end{pmatrix}$$

for $a,a',b,b',c,c',d,d'\in k$. Note that R can be depicted as the following quiver algebra with relations

$$(\star)\qquad 1\bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bullet 2\ ,\qquad \alpha\beta=\beta\alpha=0.$$

Let e_i be the idempotent element of R corresponding to the vertex i for $i=1,2$. We consider the universal localization $\lambda:R\to S$ of R at the homomorphism $\varphi:Re_2\to Re_1$ induced by α. This means that, to work out the new algebra S, we need to add a new arrow $\alpha^{-1}:2\to 1$ and two new relations $\alpha\alpha^{-1}=e_1$ and $\alpha^{-1}\alpha=e_2$ to the quiver $(\star)$. Thus we have $\beta=e_2\beta=\alpha^{-1}\alpha\beta=0$ in S since $\alpha\beta=0$. In other words, S can be expressed as the following quiver algebra with relations:

$$1\bullet \underset{\alpha^{-1}}{\overset{\alpha}{\rightleftarrows}} \bullet 2\ ,\qquad \alpha\alpha^{-1}=e_1\ \text{ and }\ \alpha^{-1}\alpha=e_2,$$

which is isomorphic to the usual matrix ring $M_2(k)$ over k. Moreover, the ring homomorphism $\lambda:R\to S$ can be given explicitly by

$$e_1\mapsto e_1,\, e_2\mapsto e_2,\, \alpha\mapsto\alpha,\, \beta\mapsto 0.$$

It is easy to see that $Se_2\simeq Se_1\simeq Re_1$ and $S\simeq Se_1\oplus Se_2\simeq Re_1\oplus Re_1$ as R-modules. In particular, ${}_RS$ is a finitely generated projective R-module and λ is a homological ring epimorphism with ${}_RS\in\mathscr{P}^{<\infty}(R)$.

Now, we define

$$Q^\bullet := 0\longrightarrow Re_2\xrightarrow{\varphi}Re_1\longrightarrow 0\quad\text{ and }\quad P^\bullet := 0\longrightarrow R\xrightarrow{\lambda}S\longrightarrow 0$$

where Re_2 and R are of degree 0. Clearly, $Q^\bullet\in\mathscr{C}^b(R\text{-proj})$ and $P^\bullet[1]$ is the mapping cone of λ. Since $Se_2\simeq Se_1\simeq Re_1$ as R-modules, we infer that $Q^\bullet\simeq P^\bullet$ in $\mathscr{C}(R)$ and $\mathrm{Tria}(Q^\bullet)=\mathrm{Tria}({}_RP^\bullet)\subseteq\mathscr{D}(R)$. Thus all the assumptions of Proposition 4.7 are satisfied. It follows from Proposition 4.7 that

$$K(R)\xrightarrow{\sim}K(S)\times K(\mathbb{T})$$

as K-theory spaces, where $\mathbb{T} := \mathrm{End}^{\bullet}_R(Q^{\bullet})$ is the dg endomorphism algebra of $Q^{\bullet}$ (see Subsection 2.1 for definition).

It is easy to check that the dg algebra $\mathbb{T} := (T^i)_{i\in\mathbb{Z}}$ is given by the following data:

$$T^{-1} = k,\, T^0 = k\oplus k,\, T^1 = k, T^i = 0 \text{ for } i \neq -1,0,1,$$

with the differential:

$$0 \longrightarrow T^{-1} \xrightarrow{0} T^0 \xrightarrow{\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)} T^1 \longrightarrow 0$$

and the multiplication $\circ : \mathbb{T}\times\mathbb{T}\to\mathbb{T}$ (see Subsection 2.1):

$$T^{-1}\circ T^{-1} = T^1\circ T^1 = 0 = T^{-1}\circ T^1 = T^1\circ T^{-1},$$

$$(a,b)\circ(c,d) = (ac,bd),\; f\circ(a,b) = fa,\; (a,b)\circ f = bf,\; g\circ(a,b) = gb,\; (a,b)\circ g = ag,$$

where $(a,b),(c,d)\in T^0$, $f\in T^{-1}$ and $g\in T^1$.

Since $H^1(\mathbb{T}) = 0$, we see that the dg algebra $\mathbb{T}$ is quasi-isomorphic to the following dg algebra $\tau^{\leq 0}(\mathbb{T})$ over k:

$$0 \longrightarrow T^{-1} \xrightarrow{0} \mathrm{Ker}(d^0) \longrightarrow 0$$

where $d^0 = \begin{pmatrix}1\\-1\end{pmatrix} : T^0 \to T^1$. Clearly, the latter algebra is isomorphic to the dg algebra

$$\mathbb{A} := 0 \longrightarrow k \xrightarrow{0} k \longrightarrow 0$$

where the first k is of degree -1 and has a k-k-bimodule structure via multiplication. Thus the algebra structure of $\mathbb{A}$ (by forgetting its differential) is precisely the trivial extension $k\ltimes k$ of k by the bimodule k. Now, by Lemma 3.8, we know that

$$K(\mathbb{T}) \xrightarrow{\sim} K(\tau^{\leq 0}(\mathbb{T})) \xrightarrow{\sim} K(\mathbb{A})$$

as K-theory spaces. Thus $K(R) \xrightarrow{\sim} K(S)\times K(\mathbb{A})$. In particular,

$$K_n(R) \simeq K_n(S)\oplus K_n(\mathbb{A}) \quad \text{for all } n\in\mathbb{N}.$$

It is worth noting that we cannot replace the dg algebra $\mathbb{A}$ in the above isomorphism by the trivial extension $k\ltimes k$ since the algebraic K-theory of dg algebras is different from that of usual rings. In fact, in this example, $K_1(R) = K_1(k)\oplus K_1(k) = k^{\times}\oplus k^{\times}$, $K_1(S) = k^{\times}$ and $K_1(\mathbb{A}) = k^{\times}$, but $K_1(k\ltimes k) = k\oplus k^{\times}$. So $K_1(R)\not\simeq K_1(S)\oplus K_1(k\ltimes k)$. For information on $K_n(k)$ with k a finite field, we refer the reader to [19].

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