Recollements of derived categories III: finitistic dimensions

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Abstract

We study homological dimensions of algebras linked by recollements of derived module categories, and establish a series of new upper bounds and relationships among their finitistic, big finitistic and global dimensions. This is closely related to a long-standing conjecture, the finitistic dimension conjecture, in representation theory and homological algebra. Further, we apply our results to a series of situations of particular interest: exact contexts, ring extensions, trivial extensions, pullbacks of rings, and algebras induced from Auslander–Reiten sequences. In particular, we not only extend and amplify Happel's reduction techniques for finitistic dimension conjecture to more general contexts, but also generalize some recent results in the literature.

Contents

1. *Introduction*

Recollements of triangulated categories have been introduced by Beilinson, Bernstein and Deligne in order to decompose derived categories of sheaves into two parts, an open and a closed one (see [**4**]), and thus providing a natural habitat for Grothendieck's six functors. Similarly, recollements of derived module categories can be seen as short exact sequences, describing a derived module category in terms of a subcategory and of a quotient, both of which may be derived module categories themselves, related by six functors that in general are not known. It turns out that recollements provide a very useful framework for understanding connections among three algebraic or geometric objects in which one is interested.

In a series of papers on recollements of derived module categories, we have been addressing basic questions about recollements and rings involved. Our starting point has been infinite-dimensional tilting theory (see [**5**]). While Happel's theorem establishes a derived equivalence between a given ring and the endomorphism ring of a finitely generated tilting module (see [**9**, **14**]), Bazzoni has shown that for a large tilting module one gets instead a recollement relating three triangulated categories, with two of them being the derived categories of the given ring and the endomorphism ring of the large tilting module. In [**5**] we have addressed the question of determining the third category in this recollement as a derived category of a ring and have explained this ring in terms of universal localizations in the sense of Cohn (see [**11**, **17**] for definition). Among the applications has been a counterexample to

the Jordan–Hölder problem for derived module categories. In [6] we have dealt with the problem of constructing recollements in order to relate rings. Our main construction, of exact contexts, can be seen as a far-reaching generalization of pullbacks of rings. In [7] we have used this construction to relate algebraic K-theory of different rings. It turned out that under mild assumptions, the K-theory of an algebra can be fully decomposed under a sequence of recollements.
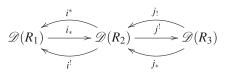
For cohomology and for homological invariants of algebras, such a complete decomposition is not possible. Nevertheless, results by Happel in [15] for the case of bounded derived categories (when fewer recollements exist than in the unbounded case) show that finiteness of finitistic dimension of an algebra can be reduced along a recollement; if such an invariant is finite for the two outer terms, then it is finite for the middle term, too. Note that the particular values of these invariants depend on the ring and are not invariants of the derived category. The present paper aims at extending Happel's reduction techniques for homological conjectures. As in Happel's paper [15] we will focus on finitistic dimensions which include finite global dimensions as a special case.

Recall that the finitistic dimension (respectively, the big finitistic dimension) of a ring R, denoted by fin.dim(R) (respectively, Fin.dim(R)), is by definition the supremum of projective dimensions of those left R-modules having a finite projective resolution by finitely generated (respectively, arbitrary) projective R-modules. Clearly, fin.dim(R) $\leqslant$ Fin.dim(R). Usually, they are quite different (see [23]). The well-known finitistic dimension conjecture states that any Artin algebra has finite finitistic dimension (see, for instance, [2, conjecture (11), p. 410]). This conjecture is a long-standing question [3] and has still not been settled. It is closely related to at least seven other major conjectures in the homological representation theory of algebras (see [2, p. 409–410]).

There are two main directions in this article. First, we provide reduction techniques for homological invariants of unbounded derived module categories, that is, for the most general possible setup (which also has been covered in the preceding articles in this series). The first main result, Theorem 1.1, establishes several inequalities for finitistic and big finitistic dimensions of different rings involved in a recollement of derived module categories. Consequently, we have a criterion for the finiteness of finitistic dimension for each of the three rings in terms of the ones of the other two (see Corollary 3.11). This criterion aims to be applicable by putting conditions on particular objects, not on the whole category. Second, we give a series of applications of our reduction techniques. In particular, the second main result, Theorem 1.2, applies the first main result to the general contexts of the so-called exact contexts introduced in [6], and in addition provides upper and lower bounds for the finitistic dimensions of the three rings involved. A series of corollaries then applies the general results to classes of examples of particular interest, such as ring extensions, trivial extensions, quotient rings and the endomorphism rings of modules related by an almost split sequence.

To describe explicit bounds for finitistic dimensions, we introduce homological widths (or cowidth) of complexes that are quasi-isomorphic to bounded complexes of projective (or injective) modules (see Section 3.1 for details). Broadly speaking, the homological width (respectively, cowidth) defines a map from homotopy equivalence classes of bounded complexes of projective (respectively, injective) modules to the natural numbers. It measures, up to homotopy equivalence, how large the minimal interval of such a complex is in which its non-zero terms are distributed. Particularly, if a module has finite projective (or injective) dimension, then its homological width (or cowidth) is exactly the projective (or injective) dimension. For a complex $X^\bullet$, we denote the homological width and cowidth of $X^\bullet$ by $w(X^\bullet)$ and $cw(X^\bullet)$, respectively.

THEOREM 1.1. *Let R_1, R_2 and R_3 be rings with identity. Suppose that there exists a recollement among the derived module categories $\mathscr{D}(R_3)$, $\mathscr{D}(R_2)$ and $\mathscr{D}(R_1)$ of R_3, R_2 and R_1:*

$$\mathscr{D}(R_1) \xrightarrow[\,\,i^!\,\,]{\substack{i^* \\ \xrightarrow{\,\,i_*\,\,} }} \mathscr{D}(R_2) \xrightarrow[\,\,j_*\,\,]{\substack{j_! \\ \xrightarrow{\,\,j^!\,\,} }} \mathscr{D}(R_3)$$

Then the following hold true.

(1) *Suppose that $j_!$ restricts to a functor $\mathscr{D}^b(R_3) \to \mathscr{D}^b(R_2)$ of bounded derived module categories. Then* $\mathrm{fin.dim}(R_3) \leqslant \mathrm{fin.dim}(R_2) + cw(j^!(\mathrm{Hom}_{\mathbb{Z}}(R_2, \mathbb{Q}/\mathbb{Z})))$ *(respectively,* $\mathrm{Fin.dim}(R_3) \leqslant \mathrm{Fin.dim}(R_2) + cw(j^!(\mathrm{Hom}_{\mathbb{Z}}(R_2, \mathbb{Q}/\mathbb{Z})))$*).*

(2) *Suppose that $i_*(R_1)$ is isomorphic in $\mathscr{D}(R_2)$ to a bounded complex of finitely generated (respectively, arbitrary) projective R_2-modules. Then*
  (a) $\mathrm{fin.dim}(R_1) \leqslant \mathrm{fin.dim}(R_2) + w(i^*(R_2))$ *(respectively,* $\mathrm{Fin.dim}(R_1) \leqslant \mathrm{Fin.dim}(R_2) + w(i^*(R_2))$*),*
  (b) $\mathrm{fin.dim}(R_2) \leqslant \mathrm{fin.dim}(R_1) + \mathrm{fin.dim}(R_3) + w(i_*(R_1)) + w(j_!(R_3)) + 1$ *(respectively,* $\mathrm{Fin.dim}(R_2) \leqslant \mathrm{Fin.dim}(R_1) + \mathrm{Fin.dim}(R_3) + w(i_*(R_1)) + w(j_!(R_3)) + 1$*).*

Note that the assumption of Theorem 1.1 on unbounded derived module categories is weaker than the one on bounded derived module categories, because the existence of recollements of bounded derived module categories implies the one of unbounded derived module categories. This is shown by a recent investigation on recollements at different levels in [**1**, **16**]. So, Theorem 1.1 (see also Corollary 3.13) generalizes the main result in [**15**] since for a recollement of $\mathscr{D}^b(R_j\text{-mod})$ with R_j a finite-dimensional algebra over a field for $1 \leqslant j \leqslant 3$, one can always deduce that $i_*(R_1)$ is compact in $\mathscr{D}(R_2)$. Moreover, Theorem 1.1 extends and amplifies a result in [**22**] because we deal with arbitrary rings instead of Artin algebras, and also yields a generalization of a result in [**19**] for left coherent rings to the one for arbitrary rings (see Corollary 3.9 below).

Note that, in [**15**], one of the key arguments in the proof is that a finite-dimensional algebra has only finitely many non-isomorphic simple modules, while in our general context we do not have this fact and therefore must avoid this kind of arguments. So, the idea of proving Theorem 1.1 will be completely different from the one in both [**15**] and [**22**]. Moreover, our methods also lead to results on upper bounds for global dimensions. For details, we refer the reader to Theorem 3.17.

Now, we apply Theorem 1.1 to recollements constructed in [**6**] and establish relationships among finitistic dimensions of noncommutative tensor products and related rings. First of all, we recall some notions from [**6**].

Let R, S and T be rings with identity, and let $\lambda : R \to S$ and $\mu : R \to T$ be ring homomorphisms. Suppose that M is an S-T-bimodule together with an element $m \in M$. We say that the quadruple (λ, μ, M, m) is an *exact context* if the following sequence

$$0 \longrightarrow R \xrightarrow{(\lambda, \mu)} S \oplus T \xrightarrow{\left(\begin{smallmatrix} \cdot m \\ -m \cdot \end{smallmatrix}\right)} M \longrightarrow 0$$

is an exact sequence of abelian groups, where $\cdot m$ and $m\cdot$ denote the right and left multiplication by m maps, respectively. There is a list of examples in [**6**] that guarantees the ubiquity of exact contexts.

Given an exact context (λ, μ, M, m), there is defined a ring with identity in [**6**], called the *noncommutative tensor product* of (λ, μ, M, m) and denoted by $T \boxtimes_R S$ if the meaning of the exact context is clear. This notion not only generalizes usual tensor products over commutative

rings and captures coproducts of rings, but also plays a key role in describing the left parts of recollements induced from homological exact contexts (see [**6**, Theorem 1.1]).

For an R-module $_RX$, we denote by $\mathrm{flat.dim}(_RX)$ and $\mathrm{proj.dim}(_RX)$ the flat and projective dimensions of X, respectively.

THEOREM 1.2. *Let (λ, μ, M, m) be an exact context with the noncommutative tensor product $T \boxtimes_R S$. Then*

(1) $\mathrm{fin.dim}(R) \leqslant \mathrm{fin.dim}(S) + \mathrm{fin.dim}(T) + \max\{1, \mathrm{flat.dim}(T_R)\} + 1$;

(2) *suppose $\mathrm{Tor}_i^R(T, S) = 0$ for all $i \geqslant 1$. If the left R-module $_RS$ has a finite projective resolution by finitely generated projective modules, then*

    (a) $\mathrm{fin.dim}(T \boxtimes_R S) \leqslant \mathrm{fin.dim}(S) + \mathrm{fin.dim}(T) + 1$,

    (b) $\mathrm{fin.dim}(S) \leqslant \mathrm{fin.dim}\left(\begin{smallmatrix} S & M \\ 0 & T \end{smallmatrix}\right) \leqslant \mathrm{fin.dim}(R) + \mathrm{fin.dim}(T \boxtimes_R S) +$
       $\max\{1, \mathrm{proj.dim}(_RS)\} + 3$.

Note that for the triangular matrix algebra $B := \left(\begin{smallmatrix} S & M \\ 0 & T \end{smallmatrix}\right)$, it is known that $\mathrm{fin.dim}(B) \leqslant \mathrm{fin.dim}(S) + \mathrm{fin.dim}(T) + 1$. But Theorem 1.2(2)(b) provides us with a new upper bound for the finitistic dimension of B in terms of $\mathrm{fin.dim}(T \boxtimes_R S)$ and $\mathrm{fin.dim}(R)$, involving the starting ring R, but without information on $\mathrm{fin.dim}(S)$ and $\mathrm{fin.dim}(T)$. This is non-trivial and somewhat surprising. Moreover, in Theorem 1.2(2), if $\lambda : R \to S$ is a homological ring epimorphism, then we even obtain better estimations: $\mathrm{fin.dim}(S) \leqslant \mathrm{fin.dim}(R)$ and $\mathrm{fin.dim}(T \boxtimes_R S) \leqslant \mathrm{fin.dim}(T)$. In this case, $T \boxtimes_R S$ can be interpreted as the coproduct $S \sqcup_R T$ of the R-rings of S and T.

Theorem 1.2 can be applied to many cases. First, we apply it to ring extensions. This is of particular interest because the finitistic dimension conjecture can be reformulated over perfect fields in terms of ring extensions (see [**21**]). In this case, we get Corollary 3.18 where we do not impose any conditions on the radicals of rings, compared with results in [**20**, **21**].

Next, we apply Theorem 1.2 to trivial extensions and pullback squares of rings. In these cases, we get Corollaries 3.19 and 3.20. Here, the strategy of the proofs is to show first that special exact contexts can be constructed, and then Theorem 1.2 is employed by verifying the Tor-vanishing condition. At last, noncommutative tensor products for these cases have to be described more substantially.

As another application of Theorem 1.2, we consider the finitistic dimensions of algebras arising from idempotent ideals and almost split sequences, see Corollary 3.16 for details.

The paper is sketched as follows: In Section 2, we first recall some necessary definitions and then prove two results on coproducts of rings. In Section 3, we provide proofs of all results in this paper. Especially, we introduce homological widths of complexes and deduce consequences of Theorems 1.1 and 1.2. Moreover, the methods developed in this section can also be used to obtain similar upper bounds for global dimension (see Theorem 3.17).
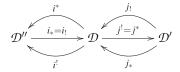
2. *Definitions and conventions*

In this section, we fix notation and briefly recall some definitions. For unexplained ones, we refer the reader to [**6**, **7**].

Throughout the paper, all notations and terminologies are standard. For example, by a ring we mean an associative ring with identity. For a ring R, we denote by R-Mod the category of all left R-modules, and by $\mathscr{C}(R)$, $\mathscr{K}(R)$ and $\mathscr{D}(R)$ the unbounded complex, homotopy and derived categories of R-Mod, respectively. As usual, by adding a superscript $* \in \{-, +, b\}$, we denote their corresponding $*$-bounded categories, for instance, $\mathscr{D}^b(R)$ is the bounded derived category of R-Mod. The full subcategory of compact objects in $\mathscr{D}(R)$ is denoted by $\mathscr{D}^c(R)$. This category is also called the *perfect derived module category* of R. It is known that the localization functor $\mathscr{K}(R) \to \mathscr{D}(R)$ induces a triangle equivalence from the homotopy category of bounded complexes of finitely generated projective R-modules to $\mathscr{D}^c(R)$.

As usual, we write a complex in $\mathscr{C}(R)$ as $X^\bullet = (X^i, d^i_{X^\bullet})_{i \in \mathbb{Z}}$, where $d^i_{X^\bullet} : X^i \to X^{i+1}$ is called the ith differential of $X^\bullet$. Sometimes, for simplicity, we also write $(X^i)_{i \in \mathbb{Z}}$ for $X^\bullet$ without mentioning $d^i_{X^\bullet}$. Given a chain map $f^\bullet : X^\bullet \to Y^\bullet$ in $\mathscr{C}(R)$, its mapping cone is denoted by $\mathrm{Con}(f^\bullet)$. For an integer n, the nth cohomology of $X^\bullet$ is denoted by $H^n(X^\bullet)$. Let $\sup(X^\bullet)$ and $\inf(X^\bullet)$ be the supremum and minimum of indices $i \in \mathbb{Z}$ such that $H^i(X^\bullet) \neq 0$, respectively. If $X^\bullet$ is acyclic, that is, $H^i(X^\bullet) = 0$ for all $i \in \mathbb{Z}$, then we understand that $\sup(X^\bullet) = -\infty$ and $\inf(X^\bullet) = +\infty$. If $X^\bullet$ is not acyclic, then $\inf(X^\bullet) \leqslant \sup(X^\bullet)$, and $H^n(X^\bullet) = 0$ if $\sup(X^\bullet)$ is an integer and $n > \sup(X^\bullet)$ or if $\inf(X^\bullet)$ is an integer and $n < \inf(X^\bullet)$. For a complex $X^\bullet$ in $\mathscr{C}(R)$, if $H^n(X^\bullet) = 0$ for almost all n, then $X^\bullet$ is isomorphic in $\mathscr{D}(R)$ to a bounded complex. So, $\mathscr{D}^b(R)$ is equivalent to the full subcategory of $\mathscr{D}(R)$ consisting of all complexes with finitely many nonzero cohomologies.

As a convention, we write the composite of two homomorphisms $f : X \to Y$ and $g : Y \to Z$ in R-Mod as fg. Thus the image of an element $x \in X$ under f will be written on the opposite of the scalars as $(x)f$ instead of $f(x)$. This convention makes $\mathrm{Hom}_R(X, Y)$ naturally into an $\mathrm{End}_R(X)$-$\mathrm{End}_R(Y)$-bimodule. But, for two functors $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{Y} \to \mathcal{Z}$ of categories, we write $GF : \mathcal{X} \to \mathcal{Z}$ for their composition.

Now we recall the notion of recollements of triangulated categories, which was defined by Beilinson, Bernstein and Deligne in [4] to study derived categories of perverse sheaves over singular spaces. It may be thought as a kind of categorifications of exact sequences in abelian categories.

DEFINITION 2.1. Let $\mathcal{D}, \mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories. We say that $\mathcal{D}$ is a *recollement* of $\mathcal{D}'$ and $\mathcal{D}''$ if there are six triangle functors among the three categories:

such that

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs,
(2) i_*, j_* and $j_!$ are fully faithful functors,
(3) $j^! i_! = 0$ (and thus also $i^! j_* = 0$ and $i^* j_! = 0$), and
(4) for each object $X \in \mathcal{D}$, there are two triangles in $\mathcal{D}$ induced by counit and unit adjunctions:

$$i_! i^!(X) \longrightarrow X \longrightarrow j_* j^*(X) \longrightarrow i_! i^!(X)[1],$$
$$j_! j^!(X) \longrightarrow X \longrightarrow i_* i^*(X) \longrightarrow j_! j^!(X)[1],$$

where the shift functor of triangulated categories is denoted by [1].

By a *half recollement* of triangulated categories among $\mathcal{D}', \mathcal{D}$ and $\mathcal{D}''$, we mean that a quadruple $(i^*, i_*, j_!, j^!)$ of functors satisfying the parts of properties (1)–(4) above which concern them.

Recall that if $\lambda_1 : R_0 \to R_1$ and $\lambda_2 : R_0 \to R_2$ are ring homomorphisms (thus making R_1 and R_2 into R_0-rings), then the *coproduct* of the corresponding R_0-rings, denoted $R_1 \sqcup_{R_0} R_2$, is just the pushout of λ_1 and λ_2 in the category of rings. For the general definition and existence of coproducts of R_0-rings, we refer to [10].

In the following we describe coproducts of rings for two cases in terms of some known constructions. The first one is for trivial extensions, and the second one is for quotient rings.

Given a ring R and an R-R-bimodule M, the *trivial extension* of R by M is a ring, denoted by $R \ltimes M$, with the underlying abelian group $R \oplus M$ and the multiplication:

$(r, m)(r', m') = (rr', rm' + mr')$ for $r, r' \in R$ and $m, m' \in M$. For further information on $R \ltimes M$, we refer the reader to [**12**].

LEMMA 2.2. *Suppose that $\lambda : R \to S$ is a ring epimorphism and M is an S-S-bimodule. Let $\widetilde{\lambda} : R \ltimes M \to S \ltimes M$ be the ring homomorphism induced by λ and let $\mu : R \to R \ltimes M$ be the canonical inclusion. Then the coproduct $S \sqcup_R (R \ltimes M)$ of λ and μ is isomorphic to $S \ltimes M$.*

Proof. Let $\rho : S \to S \ltimes M$ be the canonical inclusion. Note that S and $R \ltimes M$ are R-rings via λ and μ, respectively, and that $\lambda\rho = \mu\,\widetilde{\lambda} : R \to S \ltimes M$. To prove that $S \ltimes M$, together with ρ and $\widetilde{\lambda}$, is the coproduct of S and $R \ltimes M$ over R, we suppose that Λ is an arbitrary ring and that $f : R \ltimes M \to \Lambda$ and $g : S \to \Lambda$ are arbitrary ring homomorphisms such that $\lambda g = \mu f = f|_R$. Then f is determined by the ring homomorphism $f_1 = \lambda g : R \to \Lambda$ and a homomorphism $f_2 : M \to \Lambda$ of R-R-bimodules such that $((M)f_2)^2 = 0$ in Λ. The fact that λ is a ring epimorphism implies that f_2 is also a homomorphism of S-S-bimodules, so that $h : S \ltimes M \to \Lambda$, given by $(s, m)h = (s)g + (m)f_2$ for $s \in S$ and $m \in M$, is a well-defined ring homomorphism, which is the unique one such that $h|_S = g$ and $\widetilde{\lambda}h = f$. □

LEMMA 2.3. *Let R_0 be a ring, and let R_i be an R_0-ring with ring homomorphism $\lambda_i : R_0 \to R_i$ for $i = 1, 2$.*

(1) *If $\lambda_1 : R_0 \to R_1$ is a ring epimorphism, then so is the canonical homomorphism $\rho_2 : R_2 \to R_1 \sqcup_{R_0} R_2$.*

(2) *Let I be an ideal of R_0, and let J be the ideal of R_2 generated by the image $(I)\lambda_2$ of I under the map λ_2. If $R_1 = R_0/I$ and $\lambda_1 : R_0 \to R_1$ is the canonical surjective map, then $R_1 \sqcup_{R_0} R_2 = R_2/J$.*

Proof. (1) Follows from the fact that in any category the opposite of an epimorphism in a pushout diagram is also an epimorphism.

(2) Let $\rho_2 : R_2 \to R_2/J$ be the canonical surjection, and let $\rho_1 : R_1 \to R_2/J$ be the ring homomorphism induced by λ_2 since $J = R_2\,(I)\lambda_2\,R_2 \supseteq (I)\lambda_2$. Now, we claim that R_2/J together with ρ_1 and ρ_2 is the coproduct of R_1 and R_2 over R_0. Clearly, $\lambda_1\,\rho_1 = \lambda_2\rho_2 : R_0 \to R_2/J$. Further, assume that $\tau_1 : R_1 \to S$ and $\tau_2 : R_2 \to S$ are two ring homomorphisms such that $\lambda_2\tau_2 = \lambda_1\tau_1$. Then $(I)\lambda_2\tau_2 = (I)\lambda_1\tau_1 = 0$, and therefore $(J)\tau_2 = 0$. This means that there is a unique ring homomorphism $\delta : R_2/J \to S$ such that $\tau_2 = \rho_2\delta$. It follows that $\lambda_1\tau_1 = \lambda_2\tau_2 = \lambda_2\rho_2\delta = \lambda_1\rho_1\delta$. Since λ_1 is surjective, $\tau_1 = \rho_1\delta$. This shows $R_1 \sqcup_{R_0} R_2 = R_2/J$. □

3. *Proofs of the main results*

This section is devoted to proofs of all results mentioned in the introduction. We start with introducing the so-called homological widths for complexes, and then prove Theorem 1.1 and deduce its corollaries. As a major consequence of Theorem 1.1, we get a proof of Theorem 1.2 and then apply Theorem 1.2 to trivial extensions, pullback rings, ring extensions, idempotent ideals and almost split sequences. Also, we present a result on global dimension (see Theorem 3.17).

3.1. *Homological widths and cowidths of complexes*

As a generalization of finite projective or injective dimensions of modules, we define, in this subsection, homological widths and cowidths for bounded complexes of projective and injective modules, respectively.

Let R be a ring. For an R-module M, we denote by $\mathrm{proj.dim}(M)$, $\mathrm{inj.dim}(M)$ and $\mathrm{flat.dim}(M)$ the projective, injective and flat dimension of M, respectively. As usual, R-Proj is the category of all projective left R-modules, and R-proj is the full subcategory of R-Proj consisting of all finitely generated projective left R-modules. By $\mathscr{P}^{<\infty}(R)$ we denote the full subcategory of R-mod consisting of those R-modules admitting a finite projective resolution with finitely generated terms.

Let $P^{\bullet} := (P^n, d_{P^{\bullet}}^n)_{n \in \mathbb{Z}} \in \mathscr{C}^b(R\text{-Proj})$. We define the homological *width* of $P^{\bullet}$ in the following way:

$$w(P^{\bullet}) := \begin{cases} 0 & \text{if } P^{\bullet} \text{ is acyclic,} \\ \sup(P^{\bullet}) - \inf(P^{\bullet}) + \mathrm{proj.dim}\left(\mathrm{Cok}(d_{P^{\bullet}}^{\inf(P^{\bullet})-1})\right) & \text{otherwise.} \end{cases}$$

Clearly, $0 \leqslant w(P^{\bullet}) < \infty$. Moreover, $P^{\bullet}$ is isomorphic in $\mathscr{K}^b(R\text{-Proj})$ to a complex

$$Q^{\bullet} : 0 \longrightarrow Q^{t-p} \xrightarrow{d^{t-p}} Q^{t-p+1} \xrightarrow{d^{t-p+1}} \cdots \longrightarrow Q^{t-1} \xrightarrow{d^{t-1}} Q^t \longrightarrow P^{t+1} \xrightarrow{d_{P^{\bullet}}^{t+1}} \cdots \longrightarrow P^{s-1} \xrightarrow{d_{P^{\bullet}}^{s-1}} \mathrm{Ker}(d_{P^{\bullet}}^s) \longrightarrow 0$$

with $s := \sup(P^{\bullet})$, $t := \inf(P^{\bullet})$, $p := \mathrm{proj.dim}(\mathrm{Cok}(d_{P^{\bullet}}^{t-1}))$ and each term being projective. Clearly, the sequence

$$0 \longrightarrow Q^{t-p} \xrightarrow{d^{t-p}} Q^{t-p+1} \xrightarrow{d^{t-p+1}} \cdots \longrightarrow Q^{t-1} \xrightarrow{d^{t-1}} Q^t \longrightarrow \mathrm{Cok}(d_{P^{\bullet}}^{t-1}) \longrightarrow 0$$

is a projective resolution of the R-module $\mathrm{Cok}(d_{P^{\bullet}}^{t-1})$. Note that if $P^{\bullet} \in \mathscr{C}^b(R\text{-proj})$, we can choose $Q^{\bullet} \in \mathscr{C}^b(R\text{-proj})$.

Remark that the homological widths of complexes defined here are different from cohomological widths in [**13**].

The following result says that homological widths of bounded complexes of projective modules are preserved under homotopy equivalences.

LEMMA 3.1. *Let $M^{\bullet}$ and $N^{\bullet}$ be in $\mathscr{C}^b(R\text{-Proj})$. If $M^{\bullet} \simeq N^{\bullet}$ in $\mathscr{K}^b(R\text{-Proj})$, then $w(M^{\bullet}) = w(N^{\bullet})$.*

Proof. Recall that $\mathscr{K}^b(R\text{-Proj})$ is the stable category of the Frobenius category $\mathscr{C}^b(R\text{-Proj})$ with projective objects being acyclic complexes. Assume $M^{\bullet} \simeq N^{\bullet}$ in $\mathscr{K}^b(R\text{-Proj})$. Then there exist two acyclic complexes $P^{\bullet}$ and $Q^{\bullet}$ in $\mathscr{C}^b(R\text{-Proj})$ such that $M^{\bullet} \oplus P^{\bullet} \simeq N^{\bullet} \oplus Q^{\bullet}$ in $\mathscr{C}^b(R\text{-Proj})$. This implies that $H^i(M^{\bullet}) \simeq H^i(N^{\bullet})$ and $\mathrm{Cok}(d_{M^{\bullet}}^i) \oplus \mathrm{Cok}(d_{P^{\bullet}}^i) \simeq \mathrm{Cok}(d_{N^{\bullet}}^i) \oplus \mathrm{Cok}(d_{Q^{\bullet}}^i)$ for all $i \in \mathbb{Z}$. Thus $\sup(M^{\bullet}) = \sup(N^{\bullet})$ and $\inf(M^{\bullet}) = \inf(N^{\bullet})$. Moreover, since $\mathrm{Cok}(d_{P^{\bullet}}^i)$ and $\mathrm{Cok}(d_{Q^{\bullet}}^i)$ belong to R-Proj, $\mathrm{proj.dim}(\mathrm{Cok}(d_{M^{\bullet}}^i)) = \mathrm{proj.dim}(\mathrm{Cok}(d_{N^{\bullet}}^i))$. It follows that $w(M^{\bullet}) = w(N^{\bullet})$.   □

Thanks to Lemma 3.1, the definition of homological widths for complexes can be extended slightly to derived categories in the following sense: Given a complex $X^{\bullet} \in \mathscr{D}(R)$, if there is a complex $P^{\bullet} \in \mathscr{C}^b(R\text{-Proj})$ such that $X^{\bullet} \simeq P^{\bullet}$ in $\mathscr{D}(R)$, then we define $w(X^{\bullet}) := w(P^{\bullet})$. This is well defined: If there exists another complex $Q^{\bullet} \in \mathscr{C}^b(R\text{-Proj})$ such that $X^{\bullet} \simeq Q^{\bullet}$ in $\mathscr{D}(R)$, then $P^{\bullet} \simeq Q^{\bullet}$ in $\mathscr{K}^b(R\text{-Proj})$ and $w(P^{\bullet}) = w(Q^{\bullet})$ by Lemma 3.1. So, for such a complex $X^{\bullet}$, its homological width $w(X^{\bullet})$ can be characterized as follows:

$$w(X^{\bullet}) = \min\left\{ \alpha_{P^{\bullet}} - \beta_{P^{\bullet}} \geqslant 0 \,\middle|\, \begin{array}{l} P^{\bullet} \simeq X^{\bullet} \text{ in } \mathscr{D}(R) \text{ for } P^{\bullet} \in \mathscr{C}^b(R\text{-Proj}) \\ \text{with } P^i = 0 \text{ for } i < \beta_{P^{\bullet}} \text{ and } i > \alpha_{P^{\bullet}} \end{array} \right\}.$$

Clearly, if $X \in R$-Mod has finite projective dimension, then $w(X) = \mathrm{proj.dim}(X)$.

Dually, we can define homological cowidths for bounded complexes of injective R-modules.

Let R-Inj denote the category of injective R-modules. Given a complex $I^\bullet := (I^n, d_{I^\bullet}^n)_{n \in \mathbb{Z}} \in \mathscr{C}^b(R\text{-Inj})$, we define the homological *cowidth* of $I^\bullet$ as follows:

$$cw(I^\bullet) := \begin{cases} 0 & \text{if } I^\bullet \text{ is acyclic,} \\ \sup(I^\bullet) - \inf(I^\bullet) + \text{inj.dim}\left(\text{Ker}(d_{I^\bullet}^{\sup(I^\bullet)})\right) & \text{otherwise.} \end{cases}$$

Similarly, if a complex $Y^\bullet$ is isomorphic in $\mathscr{D}(R)$ to a bounded complex $I^\bullet \in \mathscr{C}^b(R\text{-Inj})$, then we define $cw(Y^\bullet) := cw(I^\bullet)$. In particular, if $Y \in R$-Mod has finite injective dimension, then $cw(Y) = \text{inj.dim}(Y)$. Also, we have the following characterization of $cw(Y^\bullet)$:

$$cw(Y^\bullet) = \min \left\{ \alpha_{I^\bullet} - \beta_{I^\bullet} \geqslant 0 \left| \begin{array}{l} I^\bullet \simeq Y^\bullet \text{ in } \mathscr{D}(R) \text{ for } I^\bullet \in \mathscr{C}^b(R\text{-Inj}) \\ \text{with } I^i = 0 \text{ for } i < \beta_{I^\bullet} \text{ and } i > \alpha_{I^\bullet} \end{array} \right. \right\}.$$

Homological widths and cowidths will be used to bound homological dimensions in the next section.

3.2. Proofs and applications of Theorem 1.1

Recall that the *finitistic dimension* of a ring R, denoted by $\text{fin.dim}(R)$, is defined by

$$\text{fin.dim}(R) := \sup\{\text{proj.dim}(_RX) \mid X \in \mathscr{P}^{<\infty}(R)\}.$$

For each $n \in \mathbb{Z}$, we define

$$\mathscr{D}_{\geqslant n}^c(R) := \{X^\bullet \in \mathscr{D}^c(R) \mid X^\bullet \simeq P^\bullet \text{ in } \mathscr{D}^c(R) \text{ with } P^\bullet \in \mathscr{C}^b(R\text{-proj})$$

$$\text{such that } P^i = 0 \text{ for all } i < n\}.$$

From this definition, we have $\mathscr{D}_{\geqslant n}^c(R) \subseteq \mathscr{D}_{\geqslant n'}^c(R)$ whenever $n \geqslant n'$. Since the localization functor $\mathscr{K}(R) \to \mathscr{D}(R)$ induces a triangle equivalence $\mathscr{K}^b(R\text{-proj}) \xrightarrow{\simeq} \mathscr{D}^c(R)$, we have

$$\mathscr{D}^c(R) = \bigcup_{n \in \mathbb{Z}} \mathscr{D}_{\geqslant n}^c(R).$$

Clearly, if $\text{fin.dim}(R) = m < \infty$, then $\mathscr{P}^{<\infty}(R) \subseteq \mathscr{D}_{\geqslant -m}^c(R)$. For the convenience of the later discussions, we also formally set $\mathscr{D}_{\geqslant -\infty}^c(R) := \mathscr{D}^c(R)$ and $\mathscr{D}_{\geqslant +\infty}^c(R) := \{0\}$.

LEMMA 3.2. *Let* $m, n \in \mathbb{N}$. *Then the following statements are true.*

(1) *The full subcategory* $\mathscr{D}_{\geqslant n}^c(R)$ *of* $\mathscr{D}^c(R)$ *is closed under direct summands in* $\mathscr{D}^c(R)$.

(2) *Let* $X^\bullet \in \mathscr{D}_{\geqslant n}^c(R)$, $Z^\bullet \in \mathscr{D}_{\geqslant m}^c(R)$ *and* $s = \min\{n, m\}$. *Then, for any distinguished triangle* $X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1]$ *in* $\mathscr{D}^c(R)$, *we have* $Y^\bullet \in \mathscr{D}_{\geqslant s}^c(R)$.

Proof. (1) Let $M^\bullet \in \mathscr{K}^b(R\text{-proj})$, and let $N^\bullet := (N^i)_{i \in \mathbb{Z}} \in \mathscr{K}^b(R\text{-proj})$ such that $N^i = 0$ for all $i < n$. Suppose that $M^\bullet$ is a direct summand of $N^\bullet$ in $\mathscr{K}^b(R\text{-proj})$, or equivalently, there is a complex $L^\bullet \in \mathscr{C}^b(R\text{-proj})$ such that $M^\bullet \oplus L^\bullet \simeq N^\bullet$ in $\mathscr{K}^b(R\text{-proj})$. Hence $H^i(M^\bullet) = 0$ for all $i < n$. Note that $\mathscr{K}^b(R\text{-proj})$ is the stable category of the Frobenius category $\mathscr{C}^b(R\text{-proj})$ with projective objects being acyclic complexes. So we can find two acyclic complexes $U^\bullet$ and $V^\bullet$ in $\mathscr{C}^b(R\text{-proj})$ such that $M^\bullet \oplus L^\bullet \oplus U^\bullet \simeq N^\bullet \oplus V^\bullet$ in $\mathscr{C}^b(R\text{-proj})$. This implies

$$\text{Cok}(d_{M^\bullet}^{n-1}) \oplus \text{Cok}(d_{L^\bullet}^{n-1}) \oplus \text{Cok}(d_{U^\bullet}^{n-1}) \simeq \text{Cok}(d_{N^\bullet}^{n-1}) \oplus \text{Cok}(d_{V^\bullet}^{n-1}) = N^n \oplus \text{Cok}(d_{V^\bullet}^{n-1}).$$

Since $N^n \oplus \text{Cok}(d_{V^\bullet}^{n-1}) \in R$-proj, we have $\text{Cok}(d_{M^\bullet}^{n-1}) \in R$-proj. It follows that $M^\bullet$ is isomorphic in $\mathscr{K}^b(R\text{-proj})$ to the following truncated complex

$$0 \longrightarrow \text{Cok}(d_{M^\bullet}^{n-1}) \longrightarrow M^{n+1} \longrightarrow M^{n+2} \longrightarrow \cdots \longrightarrow 0.$$

Recall that the localization functor $\mathscr{K}(R) \to \mathscr{D}(R)$ induces a triangle equivalence $\mathscr{K}^b(R\text{-proj}) \xrightarrow{\simeq} \mathscr{D}^c(R)$. Thus (1) follows.

(2) Since $X^\bullet \in \mathscr{D}^c_{\geqslant n}(R)$, there exists a complex $P^\bullet \in \mathscr{C}^b(R\text{-proj})$ with $P^i = 0$ for $i < n$ such that $X^\bullet \simeq P^\bullet$ in $\mathscr{D}^c(R)$. Similarly, there exists another complex $Q^\bullet \in \mathscr{C}^b(R\text{-proj})$ with $Q^i = 0$ for $i < m$ such that $Z^\bullet \simeq Q^\bullet$ in $\mathscr{D}^c(R)$. It follows from the triangle equivalence $\mathscr{K}^b(R\text{-proj}) \xrightarrow{\simeq} \mathscr{D}^c(R)$ that

$$\operatorname{Hom}_{\mathscr{K}^b(R\text{-proj})}(Q^\bullet[-1], P^\bullet) \simeq \operatorname{Hom}_{\mathscr{D}^c(R)}(Q^\bullet[-1], P^\bullet) \simeq \operatorname{Hom}_{\mathscr{D}^c(R)}(Z^\bullet[-1], X^\bullet).$$

Thus the given triangle yields a distinguished triangle in $\mathscr{D}^c(R)$:

$$Q^\bullet[-1] \xrightarrow{f^\bullet} P^\bullet \longrightarrow Y^\bullet \longrightarrow Q^\bullet$$

with $f^\bullet$ a chain map in $\mathscr{C}(R)$. Then $Y^\bullet \simeq \operatorname{Con}(f^\bullet)$ in $\mathscr{D}^c(R)$. Since $\operatorname{Con}(f^\bullet)^i = Q^i \oplus P^i$ for any $i \in \mathbb{Z}$, $\operatorname{Con}(f^\bullet) \in \mathscr{C}^b(R\text{-proj})$ and $\operatorname{Con}(f^\bullet)^i = 0$ for $i < s$. This implies $Y^\bullet \in \mathscr{D}^c_{\geqslant s}(R)$. $\qquad\square$

To investigate relationships among finitistic dimensions of rings in recollements, it may be convenient to introduce the notion of *finitistic dimensions of functors*.

Let R_1 and R_2 be arbitrary rings. Suppose that $\mathscr{X}_1$ and $\mathscr{X}_2$ are full subcategories of $\mathscr{D}(R_1)$ and $\mathscr{D}(R_2)$, respectively. For a given additive functor $F : \mathscr{X}_1 \to \mathscr{X}_2$, we define

$$\inf(F) := \inf\{n \in \mathbb{Z} \mid H^n(F(X)) \neq 0 \text{ for some } X \in R_1\text{-Mod}\} \quad \text{if } R_1\text{-Mod} \subseteq \mathscr{X}_1,$$

$$\operatorname{fin.dim}(F) := \inf\{n \in \mathbb{Z} \mid H^n(F(X)) \neq 0 \text{ for some } X \in \mathscr{P}^{<\infty}(R_1)\} \quad \text{if } \mathscr{P}^{<\infty}(R_1) \subseteq \mathscr{X}_1.$$

Note that $\inf(F) = +\infty$ if and only if $F(X) = 0$ in $\mathscr{D}(R_2)$ for all $X \in R_1\text{-Mod}$. In fact, if there exits some $X \in R_1\text{-Mod}$ such that $H^n(F(X)) \neq 0$ for some integer n, then $\inf(F) \leqslant n$. Moreover, by definition, $\inf(F) \leqslant \operatorname{fin.dim}(F)$, and $\operatorname{fin.dim}(F) \in \mathbb{Z} \cup \{-\infty, +\infty\}$.

LEMMA 3.3. *Let $F : \mathscr{D}(R_1) \to \mathscr{D}(R_2)$ be a triangle functor. Then the following statements are true.*

(1) *If F has a left adjoint $L : \mathscr{D}(R_2) \to \mathscr{D}(R_1)$ with $L(R_2) \in \mathscr{D}^-(R_1)$, then $\inf(F) \geqslant -\sup(L(R_2))$.*

(2) *If F has a right adjoint $G : \mathscr{D}(R_2) \to \mathscr{D}(R_1)$, then F can be restricted to a functor $\mathscr{D}^b(R_1) \to \mathscr{D}^b(R_2)$ if and only if $G(\operatorname{Hom}_{\mathbb{Z}}(R_2, \mathbb{Q}/\mathbb{Z}))$ is isomorphic in $\mathscr{D}(R_1)$ to a bounded complex $I^\bullet$ of injective R_1-modules. In this case, $\inf(F) \geqslant -(m + \operatorname{inj.dim}(\operatorname{Ker}(d^m_{I^\bullet})))$, where $m := \sup(I^\bullet)$ and $d^m_{I^\bullet} : I^m \to I^{m+1}$ is the mth differential of $I^\bullet$.*

Proof. (1) For each $n \in \mathbb{Z}$ and $M \in R_1\text{-Mod}$,

$$H^n(F(M)) \simeq \operatorname{Hom}_{\mathscr{D}(R_2)}(R_2, F(M)[n]) \simeq \operatorname{Hom}_{\mathscr{D}(R_1)}(L(R_2), M[n]).$$

Since $L(R_2) \in \mathscr{D}^-(R_1)$, we have $s := \sup(L(R_2)) < +\infty$. Recall that the localization functor $\mathscr{K}(R_1) \to \mathscr{D}(R_1)$ induces a triangle equivalence $\mathscr{K}^-(R_1\text{-Proj}) \xrightarrow{\simeq} \mathscr{D}^-(R_1)$. So there is a complex $P^\bullet := (P^j)_{j \in \mathbb{Z}} \in \mathscr{C}^-(R_1\text{-Proj})$ with $P^j = 0$ for all $j > s$ such that $P^\bullet \simeq L(R_2)$ in $\mathscr{D}(R_1)$. It follows that

$$H^n(F(M)) \simeq \operatorname{Hom}_{\mathscr{D}(R_1)}(L(R_2), M[n]) \simeq \operatorname{Hom}_{\mathscr{D}(R_1)}(P^\bullet, M[n]) \simeq \operatorname{Hom}_{\mathscr{K}(R_1)}(P^\bullet, M[n]) = 0$$

for all $n < -s$. Thus $\inf(F) \geqslant -s$.

(2) To calculate cohomologies of complexes, we consider the exact functor

$$(-)^\vee := \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) : \ \mathbb{Z}\text{-Mod} \longrightarrow \mathbb{Z}\text{-Mod}.$$

A $\mathbb{Z}$-module U is zero if and only if so is $U^{\vee}$ because $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator for $\mathbb{Z}$-Mod.
Let $X^{\bullet} \in \mathscr{D}(R_2)$. Then

$$H^0(X^{\bullet})^{\vee} = \mathrm{Hom}_{\mathbb{Z}}(H^0(X^{\bullet}), \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hom}_{\mathscr{K}(\mathbb{Z})}(X^{\bullet}, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hom}_{\mathscr{K}(\mathbb{Z})}(R_2 \otimes_{R_2} X^{\bullet}, \mathbb{Q}/\mathbb{Z})$$

$$\simeq \mathrm{Hom}_{\mathscr{K}(R_2)}(X^{\bullet}, R_2{}^{\vee}).$$

Since $R_2{}^{\vee}$ is an injective R_2-module, $\mathrm{Hom}_{\mathscr{K}(R_2)}(X^{\bullet}, R_2{}^{\vee}) \simeq \mathrm{Hom}_{\mathscr{D}(R_2)}(X^{\bullet}, R_2{}^{\vee})$. Thus

$$H^0(X^{\bullet})^{\vee} \simeq \mathrm{Hom}_{\mathscr{D}(R_2)}(X^{\bullet}, R_2{}^{\vee}).$$

Now, let $M \in R_1$-Mod and $n \in \mathbb{Z}$. Then $H^n(F(M))^{\vee} \simeq \mathrm{Hom}_{\mathscr{D}(R_2)}\big(F(M)[n], R_2{}^{\vee}\big)$. Since (F, G)
is an adjoint pair,

$$\mathrm{Hom}_{\mathscr{D}(R_2)}\big(F(M)[n], R_2{}^{\vee}\big) \simeq \mathrm{Hom}_{\mathscr{D}(R_1)}\big(M[n], G(R_2{}^{\vee})\big).$$

This implies that $H^n(F(M)) = 0$ if and only if $\mathrm{Hom}_{\mathscr{D}(R_1)}(M[n], G(R_2{}^{\vee})) = 0$.

Let $W^{\bullet} = G(R_2{}^{\vee})$. To check the sufficiency of (2), it is enough to show
$\mathrm{Hom}_{\mathscr{D}(R_1)}(M[n], W^{\bullet}) = 0$ for almost all n. In fact, if $W^{\bullet}$ is isomorphic in $\mathscr{D}(R_1)$ to a bounded
complex $I^{\bullet}$ of injective R_1-modules, then

$$\mathrm{Hom}_{\mathscr{D}(R_1)}(M[n], W^{\bullet}) \simeq \mathrm{Hom}_{\mathscr{D}(R_1)}(M[n], I^{\bullet}) \simeq \mathrm{Hom}_{\mathscr{K}(R_1)}(M[n], I^{\bullet}) = 0$$

for almost all n.

In the following, we will show the necessity of (2). Suppose that F can be restricted to a
functor $\mathscr{D}^b(R_1) \to \mathscr{D}^b(R_2)$. We first claim $H^n(W^{\bullet}) = 0$ for almost all n, that is, $W^{\bullet} \in \mathscr{D}^b(R_1)$.

Actually, we have the following isomorphisms of abelian groups:

$$H^n(W^{\bullet}) \simeq \mathrm{Hom}_{\mathscr{D}(R_1)}(R_1, G(R_2^{\vee})[n]) \simeq \mathrm{Hom}_{\mathscr{D}(R_2)}(F(R_1), R_2^{\vee}[n]) \simeq H^{-n}(F(R_1)).$$

Since $F(R_1) \in \mathscr{D}^b(R_2)$, $H^n(F(R_1)) = 0$ for almost all n. Thus $H^n(W^{\bullet}) = 0$ for almost all
n, that is, $W^{\bullet}$ is isomorphic in $\mathscr{D}^b(R_1)$ to a bounded complex. Consequently, there exists a
lower-bounded complex $I^{\bullet}$ of injective R_1-modules such that $I^{\bullet} \simeq W^{\bullet}$ in $\mathscr{D}(R_1)$. In particular,
$H^n(I^{\bullet}) \simeq H^n(W^{\bullet})$ for all n. To complete the proof of the necessity of (2), it remains to show
that $I^{\bullet}$ can be chosen to be a bounded complex.

Note that

$$\mathrm{Hom}_{\mathscr{D}(R_2)}(F(M), R_2^{\vee}[n]) \simeq \mathrm{Hom}_{\mathscr{D}(R_1)}(M, W^{\bullet}[n]) \simeq \mathrm{Hom}_{\mathscr{D}(R_1)}(M, I^{\bullet}[n])$$

$$\simeq \mathrm{Hom}_{\mathscr{K}(R_1)}(M, I^{\bullet}[n]).$$

As $F : \mathscr{D}(R_1) \to \mathscr{D}(R_2)$ can be restricted to a functor $\mathscr{D}^b(R_1) \to \mathscr{D}^b(R_2)$ by assumption, we
get $F(M) \in \mathscr{D}^b(R_2)$. Up to isomorphism in $\mathscr{D}(R_2)$, we may assume $F(M) \in \mathscr{C}^b(R_2)$. Since $R_2^{\vee}$
is an injective R_2-module, $\mathrm{Hom}_{\mathscr{D}(R_2)}(F(M), R_2^{\vee}[n]) \simeq \mathrm{Hom}_{\mathscr{K}(R_2)}(F(M), R_2^{\vee}[n]) = 0$ for almost
all n. Thus $\mathrm{Hom}_{\mathscr{K}(R_1)}(M, I^{\bullet}[n]) = 0$ for almost all n. Particularly, there is a natural number
δ_M (depending on M) such that $\mathrm{Hom}_{\mathscr{K}(R_1)}(M, I^{\bullet}[n]) = 0$ for all $n > \delta_M$. We may suppose
that the complex $I^{\bullet}$ is of the following form:

$$0 \longrightarrow I^s \xrightarrow{d^s} I^{s+1} \xrightarrow{d^{s+1}} \cdots \longrightarrow I^m \xrightarrow{d^m} I^{m+1} \xrightarrow{d^{m+1}} \cdots \longrightarrow I^i \xrightarrow{d^i} I^{i+1} \longrightarrow \cdots$$

where all terms I^i are injective and where $s \leqslant m := \sup(I^{\bullet})$ and $H^i(I^{\bullet}) = 0$ for any $i > m$.
Let $V := \bigoplus_{i \geqslant m} \mathrm{Im}(d^i)$. Then

$$\mathrm{Hom}_{\mathscr{K}(R_1)}(V, I^{\bullet}[n]) = 0 \ \text{ for all } \ n > \delta_V.$$

Now we define $t := \max\{m, \delta_V\}$. Then $\mathrm{Hom}_{\mathscr{K}(R_1)}(\mathrm{Im}(d^t), I^{\bullet}[t+1]) = 0$. This implies that the
chain map $\mathrm{Im}(d^t) \to I^{\bullet}[t+1]$, induced from the inclusion $\mathrm{Im}(d^t) \hookrightarrow I^{t+1}$, is homotopic to the
zero map. Therefore, the canonical surjection $I^t \twoheadrightarrow \mathrm{Im}(d^t)$ must split. With I^t then also $\mathrm{Im}(d^t)$

is an injective module. Since $H^i(I^\bullet) = 0$ for any $i > m$, $I^\bullet$ is isomorphic in $\mathscr{D}(R_1)$ to the following bounded complex:

$$0 \longrightarrow I^s \xrightarrow{d^s} I^{s+1} \xrightarrow{d^{s+1}} \cdots \longrightarrow I^m \xrightarrow{d^m} I^{m+1} \xrightarrow{d^{m+1}} \cdots \longrightarrow I^t \xrightarrow{d^t} \operatorname{Im}(d^t) \longrightarrow 0$$

with all of its terms being injective. Thus, up to isomorphism in $\mathscr{D}(R_1)$, we can choose $I^\bullet$ to be a bounded complex of injective modules. This completes the proof of the necessity of (2).

To show the last statement of (2), we note that the R_1-module $\operatorname{Ker}(d^m)$ has a finite injective resolution since $H^i(I^\bullet) = 0$ for all $i > m$. Hence, up to isomorphism in $\mathscr{D}(R_1)$, we can replace $I^\bullet$ by the following bounded complex of injective R_1-modules:

$$0 \longrightarrow I^s \xrightarrow{d^s} I^{s+1} \xrightarrow{d^{s+1}} \cdots \longrightarrow I^{m-1} \longrightarrow \widetilde{I}^m \xrightarrow{\widetilde{d}^m} \widetilde{I}^{m+1} \xrightarrow{\widetilde{d}^{m+1}} \cdots \longrightarrow \widetilde{I}^{m+p-1} \xrightarrow{\widetilde{d}^{m+p-1}} \widetilde{I}^{m+p} \longrightarrow 0$$

where $\operatorname{Ker}(\widetilde{d}^m) = \operatorname{Ker}(d^m)$ and $p := \operatorname{inj.dim}(\operatorname{Ker}(d^m)) \leqslant t$. This implies $\operatorname{Hom}_{\mathscr{K}(R_1)}(M[n], I^\bullet) = 0$ for all $n < -(m+p)$. Since

$$H^n(F(M))^\vee \simeq \operatorname{Hom}_{\mathscr{D}(R_1)}(M[n], W^\bullet) \simeq \operatorname{Hom}_{\mathscr{D}(R_1)}(M[n], I^\bullet) \simeq \operatorname{Hom}_{\mathscr{K}(R_1)}(M[n], I^\bullet),$$

we have $H^n(F(M)) = 0$ for all $n < -(m+p)$. Thus $\inf(F) \geqslant -(m+p)$. $\qquad\square$

We remark that, in Lemma 3.3(2), the R_2-module $I := \operatorname{Hom}_{\mathbb{Z}}(R_2, \mathbb{Q}/\mathbb{Z})$ can be replaced by any injective cogenerator of R_2-Mod. This is due to the fact that G always commutes with direct products. Recall that an R_2-module M is called a *cogenerator* of R_2-Mod if any R_2-module can be embedded into a direct product of copies of M. Clearly, I is an injective cogenerator of R_2-Mod. In case that R_2 is an Artin algebra, there is another injective cogenerator, namely $D(R_2)$ where D is the usual duality of an Artin algebra.

LEMMA 3.4. *Let* $F : \mathscr{D}^c(R_1) \to \mathscr{D}^c(R_2)$ *be a triangle functor. Suppose* $\operatorname{fin.dim}(F) = s > -\infty$ *and* $\operatorname{fin.dim}(R_2) = t < \infty$. *Then*

(1) $F(\mathscr{P}^{<\infty}(R_1)) \subseteq \mathscr{D}^c_{\geqslant s-t}(R_2)$;
(2) *let* $m \in \mathbb{Z}$. *Then, for any* $X \in \mathscr{P}^{<\infty}(R_1)$ *and for any* $Y^\bullet \in \mathscr{D}(R_2)$ *with* $\sup(Y^\bullet) \leqslant m$, *we have* $\operatorname{Hom}_{\mathscr{D}(R_2)}(F(X), Y^\bullet[i]) = 0$ *for all* $i > t - s + m$.

Proof. Note that $s = +\infty$ if and only if $F(X) = 0$ for any $X \in \mathscr{P}^{<\infty}(R_1)$. In this case, both (1) and (2) are true. Now, we assume $s < +\infty$. Thus s is an integer.

(1) Since $F(X) \in \mathscr{D}^c(R_2)$, there exists a complex $Q^\bullet = (Q^j, d^j)_{j \in \mathbb{Z}} \in \mathscr{C}^b(R_2\text{-proj})$ such that $F(X) \simeq Q^\bullet$ in $\mathscr{D}^c(R_2)$. In particular, $H^i(F(X)) \simeq H^i(Q^\bullet)$ for all $i \in \mathbb{Z}$. Since $\operatorname{fin.dim}(F) = s < \infty$, we have $H^i(F(X)) = 0$ for all $i < s$. Thus $H^i(Q^\bullet) = 0$ for all $i < s$. It follows that $Y := \operatorname{Cok}(d^{s-1}) \in \mathscr{P}^{<\infty}(R_2)$, and therefore $Q^\bullet$ is isomorphic in $\mathscr{D}(R_2)$ to the following canonical truncated complex:

$$0 \longrightarrow Y \longrightarrow Q^{s+1} \xrightarrow{d^{s+1}} Q^{s+2} \longrightarrow \cdots \longrightarrow 0.$$

Since $\operatorname{fin.dim}(R_2) = t < \infty$, we have $\operatorname{proj.dim}(_{R_2}Y) \leqslant t$. So the R_2-module Y has a finite projective resolution:

$$0 \longrightarrow P^{s-t} \longrightarrow \cdots \longrightarrow P^{s-1} \longrightarrow P^s \longrightarrow Y \longrightarrow 0$$

with $P^j \in R_2\text{-proj}$ for $s - t \leqslant j \leqslant s$. Consequently, $F(X)$ is isomorphic in $\mathscr{D}(R_2)$ to the following complex

$$P^\bullet : \quad 0 \longrightarrow P^{s-t} \longrightarrow \cdots \longrightarrow P^{s-1} \longrightarrow P^s \longrightarrow Q^{s+1} \xrightarrow{d^{s+1}} Q^{s+2} \longrightarrow \cdots \longrightarrow 0.$$

Clearly, $P^\bullet \in \mathscr{C}^b(R_2\text{-proj})$ and $P^i = 0$ for $i < s - t$. This implies $F(X) \in \mathscr{D}^c_{\geqslant s-t}(R_2)$. Hence $F(\mathscr{P}^{<\infty}(R_1)) \subseteq \mathscr{D}^c_{\geqslant s-t}(R_2)$.

(2) Let $X \in \mathscr{P}^{<\infty}(R_1)$ and $Y^\bullet \in \mathscr{D}(R_2)$ with $\sup(Y^\bullet) \leqslant m < \infty$. Then $H^j(Y^\bullet) = 0$ for $j > m$, and therefore there exists a complex $Z^\bullet \in \mathscr{C}^-(R_2)$ with $Z^r = 0$ for $r > m$, such that $Z^\bullet \simeq Y^\bullet$ in $\mathscr{D}(R_2)$. Moreover, by the proof of (1), there exists another complex $P^\bullet \in \mathscr{C}^b(R_2\text{-proj})$ with $P^i = 0$ for all $i < s - t$, such that $P^\bullet \simeq F(X)$ in $\mathscr{D}(R_2)$. It follows that

$$\operatorname{Hom}_{\mathscr{D}(R_2)}(F(X), Y^\bullet[i]) \simeq \operatorname{Hom}_{\mathscr{D}(R_2)}(P^\bullet, Z^\bullet[i]) \simeq \operatorname{Hom}_{\mathscr{K}(R_2)}(P^\bullet, Z^\bullet[i]) = 0$$

for all $i > t - s + m$. This shows (2). □

LEMMA 3.5. *Let* $F : \mathscr{D}^c(R_1) \to \mathscr{D}^c(R_2)$ *be a fully faithful triangle functor. If* $\operatorname{fin.dim}(F) = s$ *is an integer, then* $\operatorname{fin.dim}(R_1) \leqslant \operatorname{fin.dim}(R_2) - s + \sup(F(R_1))$.

Proof. If $\operatorname{fin.dim}(R_2)$ is infinity, then the right-hand side of the inequality is infinity and the corollary is true. So we assume $\operatorname{fin.dim}(R_2) = t < \infty$. Further, we may assume $R_1 \neq 0$. Since F is fully faithful, we have $0 \neq F(R_1) \in \mathscr{D}^c(R_2)$. This implies $\sup(F(R_1)) < \infty$. Moreover, it is known that, for any $X \in \mathscr{P}^{<\infty}(R_1)$, if there is a natural number n such that $\operatorname{Ext}^i_{R_1}(X, R_1) = 0$ for all $i > n$, then $\operatorname{proj.dim}(_{R_1}X) \leqslant n$. So, to show $\operatorname{fin.dim}(R_1) \leqslant n := t - s + \sup(F(R_1)) < \infty$, it is enough to prove $\operatorname{Ext}^i_{R_1}(X, R_1) = 0$ for all $X \in \mathscr{P}^{<\infty}(R_1)$ and all $i > n$. In fact, since F is fully faithful,

$$\operatorname{Ext}^i_{R_1}(X, R_1) \simeq \operatorname{Hom}_{\mathscr{D}(R_1)}(X, R_1[i]) \simeq \operatorname{Hom}_{\mathscr{D}(R_2)}(F(X), F(R_1)[i]).$$

Due to Lemma 3.4(2), we have $\operatorname{Hom}_{\mathscr{D}(R_2)}(F(X), F(R_1)[i]) = 0$ for all $i > n$. Thus $\operatorname{Ext}^i_{R_1}(X, R_1) = 0$ for all $X \in \mathscr{P}^{<\infty}(R_1)$ and all $i > n$. □

Summarizing Lemmas 3.3 and 3.5 together, we obtain the following useful result.

COROLLARY 3.6. *Let* $F : \mathscr{D}(R_1) \to \mathscr{D}(R_2)$ *be a fully faithful triangle functor such that* $F(R_1) \in \mathscr{D}^c(R_2)$. *Then the following statements hold true.*

(1) *If* F *has a left adjoint* $L : \mathscr{D}(R_2) \to \mathscr{D}(R_1)$ *with* $L(R_2) \in \mathscr{D}^-(R_1)$, *then* $\operatorname{fin.dim}(R_1) \leqslant \operatorname{fin.dim}(R_2) + \sup(L(R_2)) + \sup(F(R_1))$. *If moreover* $L(R_2) \in \mathscr{D}^c(R_1)$, *then* $\operatorname{fin.dim}(R_1) \leqslant \operatorname{fin.dim}(R_2) + w(L(R_2))$.
(2) *If* F *has a right adjoint* $G : \mathscr{D}(R_2) \to \mathscr{D}(R_1)$ *and can be restricted to a functor* $\mathscr{D}^b(R_1) \to \mathscr{D}^b(R_2)$, *then* $\operatorname{fin.dim}(R_1) \leqslant \operatorname{fin.dim}(R_2) + cw(G(\operatorname{Hom}_{\mathbb{Z}}(R_2, \mathbb{Q}/\mathbb{Z})))$.

Proof. If $\operatorname{fin.dim}(R_2)$ is infinity, then the two statements (1) and (2) are trivially true. So, we assume $\operatorname{fin.dim}(R_2) = t < \infty$ and $R_i \neq 0$ for $i = 1, 2$. By assumption, $F(R_1) \in \mathscr{D}^c(R_2)$, and therefore F restricts to a functor $\mathscr{D}^c(R_1) \to \mathscr{D}^c(R_2)$. Since F is fully faithful and $R_1 \neq 0$, we have $F(R_1) \neq 0$. This leads to $\operatorname{fin.dim}(F) \neq +\infty$. Thus $\operatorname{fin.dim}(F) \in \mathbb{Z} \cup \{-\infty\}$.

(1) Since (L, F) is an adjoint pair, $H^n(F(R_1)) \simeq \operatorname{Hom}_{\mathscr{D}(R_2)}(R_2, F(R_1)[n]) \simeq \operatorname{Hom}_{\mathscr{D}(R_1)}(L(R_2), R_1[n])$. It follows from $0 \neq F(R_1) \in \mathscr{D}(R_2)$ that $L(R_2) \neq 0$ in $\mathscr{D}(R_1)$. Since $L(R_2) \in \mathscr{D}^-(R_1)$, we know that $\sup(L(R_2))$ is an integer. By Lemma 3.3(1), $\inf(F) \geqslant -\sup(L(R_2)) > -\infty$, and therefore $\operatorname{fin.dim}(F) \geqslant \inf(F) > -\infty$. Combining this with Lemma 3.5, we have

$$\operatorname{fin.dim}(R_1) \leqslant t - \operatorname{fin.dim}(F) + \sup(F(R_1)) \leqslant t + \sup(L(R_2)) + \sup(F(R_1)).$$

This shows the first part of (1). For the second part of (1), we only need to check $w(L(R_2)) = \sup(L(R_2)) + \sup(F(R_1))$.

In fact, it follows from $L(R_2) \in \mathscr{D}^c(R_1)$ that the homological width of $L(R_2)$ is well defined and there exists a complex

$$P^\bullet : 0 \longrightarrow P^r \xrightarrow{d^r} P^{r-1} \longrightarrow \cdots \longrightarrow P^{s-1} \longrightarrow P^s \longrightarrow 0$$

in $\mathscr{C}^b(R_1\text{-proj})$ with $s = \sup(L(R_2))$ and $s - r = w(L(R_2))$ such that $L(R_2) \simeq P^\bullet$ in $\mathscr{D}(R_1)$ (see Section 3.1). In this case, d^r is not a split injection. Since (L, F) is an adjoint pair,

$$\operatorname{Hom}_{\mathscr{D}(R_1)}(P^\bullet, R_1[n]) \simeq \operatorname{Hom}_{\mathscr{D}(R_1)}(L(R_2), R_1[n]) \simeq \operatorname{Hom}_{\mathscr{D}(R_2)}(R_2, F(R_1)[n]) \simeq H^n(F(R_1))$$

for all $n \in \mathbb{Z}$. This implies $H^n(F(R_1)) = 0$ for all $n > -r$. Moreover, since the map d^r is not a split injection, $\operatorname{Hom}_{\mathscr{D}(R_1)}(P^\bullet, P^r[-r]) \neq 0$. Thus $H^{-r}(F(R_1)) \simeq \operatorname{Hom}_{\mathscr{D}(R_1)}(P^\bullet, R_1[-r]) \neq 0$. This shows $\sup(F(R_1)) = -r$. It follows that $w(L(R_2)) = s - r = \sup(L(R_2)) + \sup(F(R_1))$.

(2) Under the assumption of (2), we see from Lemma 3.3(2) that $\inf(F) \geqslant -(m + \operatorname{inj.dim}(\operatorname{Ker}(d_{I^\bullet}^m)))$, where $I^\bullet \in \mathscr{C}^b(R_1\text{-Inj})$ is defined in Lemma 3.3(2) and $m := \sup(I^\bullet)$. Thus $\operatorname{fin.dim}(F) \geqslant \inf(F) > -\infty$ and

$$\operatorname{fin.dim}(R_1) \leqslant t + m + \operatorname{inj.dim}(\operatorname{Ker}(d_{I^\bullet}^m)) + \sup(F(R_1)) < \infty$$

by Lemma 3.5. Define $W^\bullet := G(\operatorname{Hom}_{\mathbb{Z}}(R_2, \mathbb{Q}/\mathbb{Z}))$. By the proof of Lemma 3.3(2), $W^\bullet \simeq I^\bullet$ in $\mathscr{D}(R_1)$ and $H^n(W^\bullet) \simeq H^{-n}(F(R_1))$ for all $n \in \mathbb{Z}$. This implies $\sup(F(R_1)) = -\inf(W^\bullet) = -\inf(I^\bullet)$. Thus

$$cw(W^\bullet) = \sup(I^\bullet) - \inf(I^\bullet) + \operatorname{inj.dim}(\operatorname{Ker}(d_{I^\bullet}^m)) = m + \sup(F(R_1)) + \operatorname{inj.dim}(\operatorname{Ker}(d_{I^\bullet}^m)).$$

So $\operatorname{fin.dim}(R_1) \leqslant t + cw(W^\bullet)$.　　　　　　　　　　　　　　□

As a consequence of Corollary 3.6, we have the following useful fact.

COROLLARY 3.7. Let $P^\bullet \in \mathscr{C}(R_2 \otimes_{\mathbb{Z}} R_1^{\mathrm{op}})$ such that $_{R_2}P^\bullet \in \mathscr{D}^c(R_2)$. Assume that the following two conditions hold:

(1) $R_1 \simeq \operatorname{End}_{\mathscr{D}(R_2)}(P^\bullet)$ as rings (via multiplication), and $\operatorname{Hom}_{\mathscr{D}(R_2)}(P^\bullet, P^\bullet[n]) = 0$ for all $n \neq 0$;
(2) $P_{R_1}^\bullet$ is isomorphic in $\mathscr{D}(R_1^{\mathrm{op}})$ to a bounded complex

$$F^\bullet : 0 \longrightarrow F^r \longrightarrow F^{r-1} \longrightarrow \cdots \longrightarrow F^{s-1} \longrightarrow F^s \longrightarrow 0$$

of flat R_1^{op}-modules, where $r, s \in \mathbb{Z}$ and $r \leqslant s$.

Then $\operatorname{fin.dim}(R_1) \leqslant \operatorname{fin.dim}(R_2) + s - r$.

Proof. Let $F := P^\bullet \otimes_{R_1}^{\mathbb{L}} - : \mathscr{D}(R_1) \to \mathscr{D}(R_2)$. Then $F(R_1) \simeq {}_{R_2}P^\bullet \in \mathscr{D}^c(R_2)$ and F has a right adjoint $G := \mathbb{R}\operatorname{Hom}_{R_2}(P^\bullet, -) : \mathscr{D}(R_2) \to \mathscr{D}(R_1)$. Since $_{R_2}P^\bullet \in \mathscr{D}^c(R_2)$, the functor F restricts to a functor $F' : \mathscr{D}^c(R_1) \to \mathscr{D}^c(R_2)$. Note that the condition (1) implies that F' is fully faithful. Further, since F commutes with direct sums and $\mathscr{D}(R_1)$ is compactly generated by R_1, F itself is also fully faithful.

Now, we claim that F can be restricted to a functor $\mathscr{D}^b(R_1) \to \mathscr{D}^b(R_2)$. In fact, by Lemma 3.3(2), this is equivalent to claiming that the complex $G(\operatorname{Hom}_{\mathbb{Z}}(R_2, \mathbb{Q}/\mathbb{Z}))$ is isomorphic in $\mathscr{D}(R_1)$ to a bounded complex of injective R_1-modules.

To check the latter, we use the functor $(-)^\vee := \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ and apply G to the injective R_2-module $R_2^\vee$. Then there are the following isomorphisms in $\mathscr{D}(R_1)$:

$$G(R_2^\vee) = \mathbb{R}\operatorname{Hom}_{R_2}(P^\bullet, R_2^\vee) = \operatorname{Hom}_{R_2}^\bullet(P^\bullet, R_2^\vee) \simeq \operatorname{Hom}_{\mathbb{Z}}^\bullet(R_2 \otimes_{R_2} P^\bullet, \mathbb{Q}/\mathbb{Z}) \simeq (P^\bullet)^\vee.$$

Note that $(-)^\vee : R_1^{\mathrm{op}}\text{-Mod} \to R_1\text{-Mod}$ is an exact functor, sending flat R_1^{op}-modules to injective R_1-modules. Thus the condition (2) implies that $(P^\bullet)^\vee$ is isomorphic in $\mathscr{D}(R_1)$ to the following bounded complex of injective R-modules:

$$(F^\bullet)^\vee := 0 \longrightarrow (F^s)^\vee \longrightarrow (F^{s-1})^\vee \longrightarrow \cdots \longrightarrow (F^{r-1})^\vee \longrightarrow (F^r)^\vee \longrightarrow 0,$$

where $(F^s)^\vee$ and $(F^r)^\vee$ are of degrees $-s$ and $-r$, respectively. Consequently, $cw(G(R_2{}^\vee)) = cw((P^\bullet)^\vee) = cw((F^\bullet)^\vee) \leqslant s - r$. Now, it follows from Corollary 3.6(2) that

$$\mathrm{fin.dim}(R_1) \leqslant \mathrm{fin.dim}(R_2) + cw(G(R_2{}^\vee)) \leqslant \mathrm{fin.dim}(R_2) + s - r.$$

This completes the proof. $\square$

Recall that a ring epimorphism $\lambda : R \to S$ is *homological* if and only if the restriction functor $D(\lambda_*) : \mathscr{D}(S) \to \mathscr{D}(R)$ is fully faithful. Note that $D(\lambda_*)$ always has a left adjoint functor $S \otimes_R^{\mathbb{L}} - : \mathscr{D}(R) \to \mathscr{D}(S)$. For further information and advances on homological ring epimorphisms phrased in terms of recollements of derived categories, we refer the reader to [**6**–**8**]. Applying Corollary 3.6(1) to homological ring epimorphisms, we have the following result.

COROLLARY 3.8.   *Let* $\lambda : R \to S$ *be a homological ring epimorphism such that* ${}_R S \in \mathscr{P}^{<\infty}(R)$. *Then* $\mathrm{fin.dim}(S) \leqslant \mathrm{fin.dim}(R)$.

Proof.   If we take $F := D(\lambda_*)$ and $L := S \otimes_R^{\mathbb{L}} -$ in Corollary 3.6(1), then $\mathrm{fin.dim}(S) \leqslant \mathrm{fin.dim}(R) + w(L(S))$. Since $w(L(S)) = \mathrm{proj.dim}({}_S S) = 0$, $\mathrm{fin.dim}(S) \leqslant \mathrm{fin.dim}(R)$. $\square$

The following result extends [**19**, Theorem 1.1] on finitistic dimensions for derived equivalences of left coherent rings to those of arbitrary rings.

COROLLARY 3.9.   *Suppose that* $F : \mathscr{D}(R_1) \to \mathscr{D}(R_2)$ *is a triangle equivalence. Then*

$$|\mathrm{fin.dim}(R_1) - \mathrm{fin.dim}(R_2)| \leqslant w(F(R_1)).$$

Proof.   Suppose that $G : \mathscr{D}(R_2) \to \mathscr{D}(R_1)$ is a quasi-inverse of F. Then (G, F) and (F, G) are adjoint pairs. Since both F and G preserve compact objects, they can be restricted to triangle equivalences of perfect derived categories: $F : \mathscr{D}^c(R_1) \xrightarrow{\simeq} \mathscr{D}^c(R_2)$ and $G : \mathscr{D}^c(R_2) \xrightarrow{\simeq} \mathscr{D}^c(R_1)$. By Corollary 3.6(1), $\mathrm{fin.dim}(R_1) \leqslant \mathrm{fin.dim}(R_2) + w(G(R_2))$ and $\mathrm{fin.dim}(R_2) \leqslant \mathrm{fin.dim}(R_1) + w(F(R_1))$. Thus, to complete the proof, it is enough to show $w(G(R_2)) = w(F(R_1))$.

In fact, up to isomorphism in derived categories, we may assume $F(R_1) \in \mathscr{C}^b(R_2\text{-proj})$ and $G(R_2) \in \mathscr{C}^b(R_1\text{-proj})$.

Without loss of generality, we suppose that $F(R_1)$ is a complex in $\mathscr{C}^b(R_2\text{-proj})$ of the form

$$0 \longrightarrow P^{-r} \xrightarrow{d^{-r}} P^{-r+1} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow 0$$

such that $r = w(F(R_1)) \geqslant 0$. This implies that $H^0(F(R_1)) \neq 0$ and d^{-r} is not a split injection. Since (F, G) is an adjoint pair,

$$\mathrm{Hom}_{\mathscr{K}(R_2)}(F(R_1), R_2[n]) \simeq \mathrm{Hom}_{\mathscr{D}(R_2)}(F(R_1), R_2[n]) \simeq \mathrm{Hom}_{\mathscr{D}(R_1)}(R_1, G(R_2)[n])$$

$$\simeq H^n(G(R_2))$$

for all $n \in \mathbb{Z}$. It follows that $H^i(G(R_2)) = 0$ for $i < 0$ or $i > r$. Further, we claim $H^r(G(R_2)) \neq 0$, and therefore $\sup(G(R_2)) = r$. Actually, since d^{-r} is not a split injection, $\mathrm{Hom}_{\mathscr{K}(R_2)}(F(R_1), P^{-r}[r]) \neq 0$. Thus $0 \neq \mathrm{Hom}_{\mathscr{K}(R_2)}(F(R_1), R_2[r]) \simeq H^r(G(R_2))$. So, up to isomorphism in $\mathscr{K}(R_1)$, the complex $G(R_2)$ has the following form in $\mathscr{C}^b(R_1\text{-proj})$:

$$0 \longrightarrow Q^s \xrightarrow{\varphi^s} Q^{s+1} \longrightarrow Q^{s+2} \longrightarrow \cdots \longrightarrow Q^{r-1} \longrightarrow Q^r \longrightarrow 0$$

such that $0 \leqslant r - s = w(G(R_2))$. In particular, this implies that φ^s is not a split injection. So, to show $w(F(R_1) = w(G(R_2))$, we only need to show $s = 0$.

Indeed, since (G, F) is an adjoint pair,

$$\mathrm{Hom}_{\mathscr{K}(R_1)}(G(R_2), R_1[n]) \simeq \mathrm{Hom}_{\mathscr{D}(R_1)}(G(R_2), R_1[n]) \simeq \mathrm{Hom}_{\mathscr{D}(R_2)}(R_2, F(R_1)[n])$$

$$\simeq H^n(F(R_1))$$

for all $n \in \mathbb{Z}$. On the one hand, if $s < 0$, then $\mathrm{Hom}_{\mathscr{K}(R_1)}(G(R_2), R_1[-s]) \simeq H^{-s}(F(R_1)) = 0$, and therefore $\mathrm{Hom}_{\mathscr{K}(R_1)}(G(R_2), Q^s[-s]) = 0$. This means that φ^s is a split injection, a contradiction. On the other hand, if $s > 0$, then $0 = \mathrm{Hom}_{\mathscr{K}(R_1)}(G(R_2), R_1) \simeq H^0(F(R_1))$. This is also a contradiction. Thus $s = 0$ and $w(F(R_1)) = w(G(R_2))$, as desired. $\qquad\square$

Corollary 3.9 describes a relationship for finitistic dimensions of derived equivalent rings. If we weaken derived equivalences into half recollements of perfect derived module categories, we will obtain the following general result which provides a bound for the finitistic dimension of the middle ring by those of the outer two rings.

PROPOSITION 3.10. *Suppose that there is a half recollement of perfect derived module categories of the rings R_3, R_2 and R_1*

$$\mathscr{D}^c(R_1) \xrightarrow{\ i_* \ } \mathscr{D}^c(R_2) \xrightarrow{\ j^! \ } \mathscr{D}^c(R_3) \ .$$

Then

$$\mathrm{fin.dim}(R_2) \leqslant \mathrm{fin.dim}(R_1) + \mathrm{fin.dim}(R_3) + w\left(i_*(R_1)\right) + w\left(j_!(R_3)\right) + 1.$$

Proof. The proof will be done in several steps. We may suppose $\mathrm{fin.dim}(R_1) < \infty$ and $\mathrm{fin.dim}(R_3) < \infty$. Clearly, if one of R_1 and R_3 is zero, then Proposition 3.10 follows from Corollary 3.9. From now on, we assume $R_1 \neq 0 \neq R_3$.

Step 1. We claim $j_! j^!(\mathscr{P}^{<\infty}(R_2)) \subseteq \mathscr{D}^c_{\geqslant -u}(R_2)$, where $u := \mathrm{fin.dim}(R_3) + w(j_!(R_3)) \geqslant 0$.

Actually, since $j_! : \mathscr{D}^c(R_3) \to \mathscr{D}^c(R_2)$ is fully faithful, $0 \neq j_!(R_3) \in \mathscr{D}^c(R_2)$. This implies $\sup(j_!(R_3)) < \infty$. As $(j_!, j^!)$ is an adjoint pair, one can follow the proof of Lemma 3.3(1) to show $-\sup(j_!(R_3)) \leqslant \inf(j^!)$. Note that $\inf(j^!) \leqslant \mathrm{fin.dim}(j^!)$. Thus $-\infty < -\sup(j_!(R_3)) \leqslant \mathrm{fin.dim}(j^!) \leqslant +\infty$.

Define $u_1 := -\sup(j_!(R_3)) - \mathrm{fin.dim}(R_3)$. Then $u_1 \leqslant \mathrm{fin.dim}(j^!) - \mathrm{fin.dim}(R_3)$. It follows from Lemma 3.4(1) that

$$j^!(\mathscr{P}^{<\infty}(R_2)) \subseteq \mathscr{D}^c_{\geqslant u_1}(R_3).$$

In other words, for any $Y \in \mathscr{P}^{<\infty}(R_2)$, there exists a complex $P_Y^\bullet := (P_Y^n)_{n \in \mathbb{Z}} \in \mathscr{C}^b(R_3\text{-proj})$ with $P_Y^n = 0$ for $n < u_1$ such that $j^!(Y) \simeq P_Y^\bullet$ in $\mathscr{D}^c(R_3)$. Clearly, the complex $P_Y^\bullet$ is of the following form:

$$0 \longrightarrow P_Y^{u_1} \longrightarrow P_Y^{u_1+1} \longrightarrow P_Y^{u_1+2} \longrightarrow \cdots \longrightarrow P_Y^{s(Y)} \longrightarrow 0,$$

where $s(Y)$ depends on Y and $u_1 \leqslant s(Y)$. Since $j_!(R_3) \in \mathscr{D}^c(R_2)$ by the half recollement, $j_!(R_3)$ is isomorphic in $\mathscr{D}^c(R_2)$ to a bounded complex $L^\bullet$ of the form

$$0 \longrightarrow L^{u_2} \longrightarrow L^{u_2+1} \longrightarrow L^{u_2+2} \longrightarrow \cdots \longrightarrow 0$$

such that $u_2 = \sup(j_!(R_3)) - w(j_!(R_3))$ and $L^i \in R_2\text{-proj}$ for all $i \geqslant u_2$ (see Section 3.1). This implies $j_!(R_3) \in \mathscr{D}^c_{\geqslant u_2}(R_2)$. Since $\mathscr{D}^c_{\geqslant u_2}(R_2))$ is closed under direct summands in $\mathscr{D}^c(R_2)$ by Lemma 3.2(1), $j_!(R_3\text{-proj}) \subseteq \mathscr{D}^c_{\geqslant u_2}(R_2)$.

Note that $u = \mathrm{fin.dim}(R_3) + w(j_!(R_3)) = \mathrm{fin.dim}(R_3) + \sup(j_!(R_3)) - u_2 = -(u_1 + u_2)$. Now, we claim $j_! j^!(\mathscr{P}^{<\infty}(R_2)) \subseteq \mathscr{D}^c_{\geqslant -u}(R_2) = \mathscr{D}^c_{\geqslant(u_1+u_2)}(R_2)$.

Actually, for the complex $P_Y^\bullet \in \mathscr{C}^b(R_3\text{-proj})$, there is a canonical distinguished triangle in $\mathscr{D}^c(R_3)$:

$$P_Y^{s(Y)}[-s(Y)] \longrightarrow P_Y^\bullet \longrightarrow P_Y^{\bullet \leqslant s(Y)-1} \longrightarrow P_Y^{s(Y)}[1-s(Y)],$$

where $P_Y^{\bullet \leqslant s(Y)-1}$ is truncated from $P_Y^\bullet$ by replacing $P_Y^{s(Y)}$ with 0, that is,

$$P_Y^{\bullet \leqslant u_1-1}: \quad 0 \longrightarrow P_Y^{u_1} \longrightarrow P_Y^{u_1+1} \longrightarrow \cdots \longrightarrow P_Y^{s(Y)-1} \longrightarrow 0 \longrightarrow 0.$$

This induces a distinguished triangle in $\mathscr{D}^c(R_2)$:

$$j_!\left(P_Y^{s(Y)}\right)[-s(Y)] \longrightarrow j_!(P_Y^\bullet) \longrightarrow j_!\left(P_Y^{\bullet \leqslant s(Y)-1}\right) \longrightarrow j_!\left(P_Y^{s(Y)}\right)[1-s(Y)].$$

Note that $j_!(P_Y^{s(Y)})[-s(Y)] \in \mathscr{D}_{\geqslant s(Y)+u_2}^c(R_2) \subseteq \mathscr{D}_{\geqslant (u_1+u_2)}^c(R_2)$ due to $u_1 \leqslant s(Y)$. Since $P_Y^i \in R_3$-proj for all $u_1 \leqslant i \leqslant s(Y)$, one can apply Lemma 3.2(2) to show $j_!(P_Y^\bullet) \in \mathscr{D}_{\geqslant (u_1+u_2)}^c(R_2)$ by induction on the number of non-zero terms of a complex. It follows from $j^!(Y) \simeq P_Y^\bullet$ that $j_! j^!(Y) \simeq j_!(P_Y^\bullet) \in \mathscr{D}_{\geqslant (u_1+u_2)}^c(R_2)$. This implies $j_! j^!(\mathscr{P}^{<\infty}(R_2)) \subseteq \mathscr{D}_{\geqslant (u_1+u_2)}^c(R_2)$.

Step 2. We show $i_* i^*(\mathscr{P}^{<\infty}(R_2)) \subseteq \mathscr{D}_{\geqslant v}^c(R_2)$, where $v := \mathrm{fin.dim}(R_1) + w(i_*(R_1)) + u + 1$.

First of all, we claim that there is an integer m such that $m \leqslant \mathrm{fin.dim}(i^*) \leqslant +\infty$. Indeed, the given half recollement yields the following canonical triangle

$$(\dagger) \quad j_! j^!(Y) \xrightarrow{\eta_Y} Y \xrightarrow{\varepsilon_Y} i_* i^*(Y) \longrightarrow j_! j^!(Y)[1]$$

in $\mathscr{D}(R_2)$, where η_Y and ε_Y stand for the counit and unit morphisms, respectively. Since $j_! j^!(Y) \in \mathscr{D}_{\geqslant -u}^c(R_2) \subseteq \mathscr{D}^c(R_2)$, we can find a complex $U^\bullet := (U^n)_{n\in\mathbb{Z}} \in \mathscr{C}^b(R_2$-proj) with $U^n = 0$ for all $n < -u \leqslant 0$ such that $j_! j^!(Y) \simeq U^\bullet$ in $\mathscr{D}(R_2)$. It follows that

$$\mathrm{Hom}_{\mathscr{D}(R_2)}(j_! j^!(Y), Y) \simeq \mathrm{Hom}_{\mathscr{D}(R_2)}(U^\bullet, Y) \simeq \mathrm{Hom}_{\mathscr{K}(R_2)}(U^\bullet, Y).$$

So there exists a chain map $f^\bullet: U^\bullet \to Y$ such that its mapping cone $V^\bullet$ is isomorphic to $i_* i^*(Y)$ in $\mathscr{D}(R_2)$. Clearly, $V^0 = U^1 \oplus Y$ and $V^j = U^{j+1}$ for any $j \neq 0$. In particular, $V^j = 0$ for all $j < -u - 1$. Since $i_* : \mathscr{D}^c(R_1) \to \mathscr{D}^c(R_2)$ is fully faithful,

$$H^n(i^*(Y)) \simeq \mathrm{Hom}_{\mathscr{D}(R_1)}(R_1, i^*(Y)[n]) \simeq \mathrm{Hom}_{\mathscr{D}(R_2)}(i_*(R_1), i_* i^*(Y)[n]) \simeq \mathrm{Hom}_{\mathscr{D}(R_2)}(i_*(R_1), V^\bullet[n]).$$

By assumption, $i_*(R_1) \in \mathscr{D}^c(R_2)$ and therefore is isomorphic in $\mathscr{D}^c(R_2)$ to a complex

$$Q^\bullet: \quad 0 \longrightarrow Q^{v_2} \longrightarrow Q^{v_2+1} \longrightarrow \cdots \longrightarrow Q^b \longrightarrow 0$$

in $\mathscr{C}^b(R_2$-proj), where $b := \sup(i_*(R_1))$ and $b - v_2 = w(i_*(R_1))$ (see Section 3.1). Let $m := -u - 1 - b$. Then

$$H^n(i^*(Y)) \simeq \mathrm{Hom}_{\mathscr{D}(R_2)}(i_*(R_1), V^\bullet[n]) \simeq \mathrm{Hom}_{\mathscr{D}(R_2)}(Q^\bullet, V^\bullet[n]) \simeq \mathrm{Hom}_{\mathscr{K}(R_2)}(Q^\bullet, V^\bullet[n]) = 0$$

for all $n < m$. This implies $m \leqslant \mathrm{fin.dim}(i^*) \leqslant +\infty$, as claimed.

Let $v_1 := m - \mathrm{fin.dim}(R_1)$. It follows from Lemma 3.4(1) that $i^*(\mathscr{P}^{<\infty}(R_2)) \subseteq \mathscr{D}_{\geqslant v_1}^c(R_1)$. Now, replacing the pair $(j_!, j^!)$ in the proof of Step 1 with (i^*, i_*), one can similarly show

$$i_* i^*(\mathscr{P}^{<\infty}(R_2)) \subseteq \mathscr{D}_{\geqslant v_1+v_2}^c(R_2).$$

Note that $-(v_1 + v_2) = \mathrm{fin.dim}(R_1) + w(i_*(R_1)) + u + 1 = v \geqslant u + 1 \geqslant 1$.

Step 3. We show $\mathrm{fin.dim}(R_2) \leqslant v = \mathrm{fin.dim}(R_1) + \mathrm{fin.dim}(R_3) + w(i_*(R_1)) + w(j_!(R_3)) + 1$.

Since $j_! j^!(Y) \subseteq \mathscr{D}_{\geqslant -u}^c(R_2)$ and $i_* i^*(Y) \in \mathscr{D}_{\geqslant -v}^c(R_2)$ for $Y \in \mathscr{P}^{<\infty}(R_2)$ with $u < v$, it follows from the triangle $(\dagger)$ and Lemma 3.2(2) that $Y \in \mathscr{D}_{\geqslant -v}^c(R_2)$. Now, let $P^\bullet := (P^n, d^n)_{n\in\mathbb{Z}} \in \mathscr{C}^b(R_2$-proj) such that $P^n = 0$ for all $n < -v$ and $Y \simeq P^\bullet$ in $\mathscr{D}^c(R_2)$. Since Y is an R_2-module, $H^n(P^\bullet) = 0$ for $n \neq 0$ and $H^0(P^\bullet) \simeq Y$. Consequently, $\mathrm{Ker}(d^0) \in R_2$-proj and the following complex

$$0 \longrightarrow P^{-v} \xrightarrow{d^{-v}} P^{-v+1} \xrightarrow{d^{-v+1}} \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} \mathrm{Ker}(d^0) \longrightarrow Y \longrightarrow 0$$

is exact. Thus $\mathrm{proj.dim}(_{R_2}Y) \leqslant v$ and therefore $\mathrm{fin.dim}(R_2) \leqslant v < \infty$. $\qquad\square$

Proof of Theorem 1.1.   We only prove Theorem 1.1 for the little finitistic dimensions, while the proof of the statements for the big finitistic dimension is an easy adaptation of that of the little one and left to the reader.

Note that the triangle functors $j_!$ and i^* in a recollement always take compact objects to compact objects and that $i_*(R_1)$ is compact if and only if $j^!(R_2)$ is compact (for a reference of this fact, one may see, for example, [**7**, Lemma 2.2]). Thus we have a sequence of functors:

$$\mathscr{D}(R_1) \xleftarrow{\ i^*\ } \mathscr{D}(R_2) \xleftarrow{\ j^!\ } \mathscr{D}^c(R_3),$$

where the functor $j_!$ is fully faithful.

Applying Corollary 3.6(2) to the adjoint pair $(j_!, j^!)$, we then obtain (1).
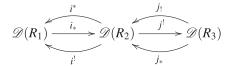
Suppose $i_*(R_1) \in \mathscr{D}^c(R_2)$. Then $j^!(R_2) \in \mathscr{D}^c(R_3)$ and the given recollement in Theorem 1.1 induces a half recollment of perfect derived module categories:

$$\mathscr{D}^c(R_1) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathscr{D}^c(R_2) \underset{j^!}{\overset{j_!}{\rightleftarrows}} \mathscr{D}^c(R_3) \ .$$

Now, the statements (*a*) and (*b*) in (2) follow from Corollary 3.6(1) and Proposition 3.10, respectively. This completes the proof of Theorem 1.1. $\qquad\square$

As a consequence of Theorem 1.1, we have the following corollary.

COROLLARY 3.11.   *Let R_1, R_2 and R_3 be rings. Suppose that there exists a recollement among the derived module categories $\mathscr{D}(R_3)$, $\mathscr{D}(R_2)$ and $\mathscr{D}(R_1)$ of R_3, R_2 and R_1:*

$$\mathscr{D}(R_1) \underset{i^!}{\overset{i^*}{\underset{\longrightarrow}{\rightleftarrows}}} \mathscr{D}(R_2) \underset{j_*}{\overset{j_!}{\underset{\longrightarrow}{\rightleftarrows}}} \mathscr{D}(R_3)$$

Then the following hold true.

(1) *Suppose that $j_!$ restricts to a functor $\mathscr{D}^b(R_3) \to \mathscr{D}^b(R_2)$ of bounded derived module categories. If* fin.dim$(R_2) < \infty$, *then* fin.dim$(R_3) < \infty$.
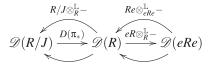
(2) *Suppose that $i_*(R_1)$ is a compact object in $\mathscr{D}(R_2)$. Then we have the following:*

    (a) *if* fin.dim$(R_2) < \infty$, *then* fin.dim$(R_1) < \infty$;

    (b) *if* fin.dim$(R_1) < \infty$ *and* fin.dim$(R_3) < \infty$, *then* fin.dim$(R_2) < \infty$.

As another consequence of Theorem 1.1, we obtain the following corollary which extends the main result [**22**, Theorem] on finitistic dimensions of Artin algebras to the one of arbitrary rings.

COROLLARY 3.12.   *Let R be a ring and e an idempotent element of R. Suppose that the canonical surjection $R \to R/ReR$ is homological with ${}_R ReR \in \mathscr{P}^{<\infty}(R)$. Then* fin.dim$(R/ReR) \leqslant$ fin.dim$(R) \leqslant$ fin.dim$(eRe) +$ fin.dim$(R/ReR) +$ proj.dim$({}_R R/ReR) + 1$.

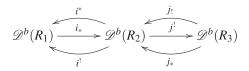*Proof.*   Let $J := ReR$. Since the canonical surjection $R \to R/J$ is homological, there exists a recollement of derived module categories:

$$\mathscr{D}(R/J) \underset{D(\pi_*)}{\overset{R/J \otimes_R^{\mathbb{L}} -}{\rightleftarrows}} \mathscr{D}(R) \underset{eR \otimes_R^{\mathbb{L}} -}{\overset{Re \otimes_{eRe}^{\mathbb{L}} -}{\rightleftarrows}} \mathscr{D}(eRe)$$

Due to $_RJ \in \mathscr{P}^{<\infty}(R)$, we have $D(\pi_*)(R/J) = R/J \in \mathscr{D}^c(R)$ and $w(R/J) = \text{proj.dim}(_RR/J)$. Moreover, $Re \otimes^{\mathbb{L}}_{eRe} eRe = Re$ and $w(_RRe) = 0$. Now, Corollary 3.12 follows from Theorem 1.1(2)(b) and Corollary 3.8. $\qquad\square$

Since a recollement at $\mathscr{D}^b$-level induces a recollement at $\mathscr{D}$-level, the following result is also a straightforward consequence of Theorem 1.1.

COROLLARY 3.13. *Let R_1, R_2 and R_3 be rings. Suppose that there exists a recollement among the derived module categories $\mathscr{D}^b(R_3)$, $\mathscr{D}^b(R_2)$ and $\mathscr{D}^b(R_1)$:*

$$\mathscr{D}^b(R_1) \xrightarrow{\; i_* \;} \mathscr{D}^b(R_2) \xrightarrow{\; j^! \;} \mathscr{D}^b(R_3)$$

such that $i_*(R_1) \in \mathscr{D}^c(R_2)$. Then

$$\text{fin.dim}(R_2) < \infty \text{ if and only if } \max\{\text{fin.dim}(R_1), \text{fin.dim}(R_3)\} < \infty.$$

Remark that Corollary 3.13 generalizes [15, Theorem 2]. In fact, if R_1, R_2 and R_3 are finite-dimensional k-algebras over a field k, satisfying the assumptions in [15, Theorem 2], then there is a recollement among $\mathscr{D}(R_3)$, $\mathscr{D}(R_2)$ and $\mathscr{D}(R_1)$, which can be restricted to a recollement among $\mathscr{D}^b(R_3)$, $\mathscr{D}^b(R_2)$ and $\mathscr{D}^b(R_1)$ such that $i_*(R_1) \in \mathscr{D}^c(R_2)$ by [1, Proposition 4.1, Corollary 4.9 and Theorem 4.6]. Thus [15, Theorem 2] follows from Corollary 3.13.

The existence of a recollement at $\mathscr{D}^b$-level occurs in the following special case (see [16, 18]): Let R be a ring and $J = ReR$ be an ideal generated by an idempotent element e in R such that $_RJ$ is projective and finitely generated and that J_R has finite projective dimension. Then there exists a recollement among $\mathscr{D}^b(eRe)$, $\mathscr{D}^b(R)$ and $\mathscr{D}^b(R/J)$. Remark that, without $\text{proj.dim}(J_R) < \infty$, we may not get a recollement at $\mathscr{D}^b$-level because the left-derived functor $Re \otimes^{\mathbb{L}}_{eRe} - : \mathscr{D}(eRe) \to \mathscr{D}(R)$ may not be restricted to a functor on bounded derived categories. One can construct a desired counterexample from triangular matrix rings.

Applying Corollary 3.12 to triangular matrix rings, we re-obtain the following well-known result (for example, see [12, Corollary 4.21]).

COROLLARY 3.14. *Let R and S be rings, and let M be an S-T-bimodule. Set $B := \left(\begin{smallmatrix} S & M \\ 0 & T \end{smallmatrix}\right)$. Then $\text{fin.dim}(S) \leqslant \text{fin.dim}(B) \leqslant \text{fin.dim}(S) + \text{fin.dim}(T) + 1$.*

Recall from [8] that a morphism $\lambda : Y \to X$ of objects in an additive category $\mathcal{C}$ is said to be *covariant* if the induced map $\text{Hom}_{\mathcal{C}}(X, \lambda) : \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{C}}(X, X)$ is injective, and the induced map $\text{Hom}_{\mathcal{C}}(Y, \lambda) : \text{Hom}_{\mathcal{C}}(Y, Y) \to \text{Hom}_{\mathcal{C}}(Y, X)$ is a split epimorphism of $\text{End}_{\mathcal{C}}(Y)$-modules. Covariant morphisms capture traces of modules, which guarantee the ubiquity of covariant morphisms (see [8]).

For covariant morphisms, we have the following result which follows from Corollary 3.12 and [8, Lemma 3.2].

COROLLARY 3.15. *Let $f : Y \to X$ be a covariant morphism in an additive category $\mathcal{C}$. Then*

$$\text{fin.dim}\left(\text{End}_{\mathcal{C},Y}(X)\right) \leqslant \text{fin.dim}\left(\text{End}_{\mathcal{C}}(Y \oplus X)\right) \leqslant \text{fin.dim}\left(\text{End}_{\mathcal{C}}(Y)\right) + \text{fin.dim}\left(\text{End}_{\mathcal{C},Y}(X)\right) + 2,$$

where $\text{End}_{\mathcal{C},Y}(X)$ is the quotient ring of the endomorphism ring $\text{End}_{\mathcal{C}}(X)$ of X modulo the ideal generated by all those endomorphisms of X which factorize through the object Y.

Consequently, we have the following result.

COROLLARY 3.16. (1) *Let I be an idempotent ideal in a ring R. Then*

$$\mathrm{fin.dim}(R/I) \leqslant \mathrm{fin.dim}\left(\mathrm{End}_R(R \oplus I)\right) \leqslant \mathrm{fin.dim}\left(\mathrm{End}_R({}_RI)\right) + \mathrm{fin.dim}(R/I) + 2.$$

In particular, if ${}_RI$ is, in addition, projective and finitely generated, then

$$\mathrm{fin.dim}(R/I) \leqslant \mathrm{fin.dim}(R) \leqslant \mathrm{fin.dim}\left(\mathrm{End}_R({}_RI)\right) + \mathrm{fin.dim}(R/I) + 2.$$

(2) *Let $0 \to Z \to Y \xrightarrow{f} X \to 0$ be an almost split sequence in R-mod with R an Artin algebra (see [2] for definition), such that $\mathrm{Hom}_R(Y, Z) = 0$. Then*

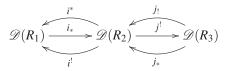$$\mathrm{fin.dim}\left(\mathrm{End}_R(Y \oplus X)\right) \leqslant \mathrm{fin.dim}\left(\mathrm{End}_R(Y)\right) + 2.$$

Proof. (1) Since the inclusion $I \hookrightarrow R$ is a covariant homomorphism in R-Mod and $\mathrm{End}_{R,I}(R) \simeq R/I$, the first statement in (1) follows from Corollary 3.15 immediately. The last statement is a consequence of the fact that R is Morita equivalent to $\mathrm{End}_R(R \oplus I)$.

(2) Under the assumption, we know that f is a covariant map in R-mod, the category of finitely generated R-modules. So, by Corollary 3.15, it is sufficient to show $\mathrm{fin.dim}(\mathrm{End}_{R,Y}(X)) = 0$. In fact, since $\mathrm{End}_R(X)$ is a local algebra and since the ideal of $\mathrm{End}_R(X)$ generated by all homomorphisms which factorize through Y belong to the radical of $\mathrm{End}_R(X)$, the algebra $\mathrm{End}_{R,Y}(X)$ is local. Note that a local Artin algebra has finitistic dimension 0. Therefore $\mathrm{fin.dim}(\mathrm{End}_{R,Y}(X)) = 0$. Now, (2) follows from Corollary 3.15. □

We point out that the methods developed in this paper for finitistic dimensions also work for global dimensions. Recall that, for an arbitrary ring R, we denote by $\mathrm{gl.dim}(R)$ the *global dimensions* of R. By definition, $\mathrm{gl.dim}(R)$ is the supremum of projective dimensions of all left R-modules. Clearly, $\mathrm{fin.dim}(R) \leqslant \mathrm{Fin.dim}(R) \leqslant \mathrm{gl.dim}(R)$; and if $\mathrm{gl.dim}(R) < \infty$, then $\mathrm{Fin.dim}(R) = \mathrm{gl.dim}(R)$. However, the equality $\mathrm{fin.dim}(R) = \mathrm{Fin.dim}(R)$ does not have to hold in general (see [23]).

Concerning global dimensions, we can describe explicitly upper bounds for the global dimension of a ring in terms of the ones of the outer two rings involved in a recollement. These upper bounds imply the finiteness of global dimensions mentioned in [1, Proposition 2.14].

THEOREM 3.17. *Let R_1, R_2 and R_3 be rings. Suppose that there exists a recollement among the derived module categories $\mathscr{D}(R_3)$, $\mathscr{D}(R_2)$ and $\mathscr{D}(R_1)$ of R_3, R_2 and R_1:*

$$\mathscr{D}(R_1) \underset{i^!}{\overset{i^*}{\rightleftarrows}} \xrightarrow{i_*} \mathscr{D}(R_2) \underset{j_*}{\overset{j_!}{\rightleftarrows}} \xrightarrow{j^!} \mathscr{D}(R_3)$$

Then we have the following:

(1) $\mathrm{gl.dim}(R_3) \leqslant \mathrm{gl.dim}(R_2) + cw(j^!(\mathrm{Hom}_{\mathbb{Z}}(R_2, \mathbb{Q}/\mathbb{Z})))$ *and* $\mathrm{gl.dim}(R_1) \leqslant \mathrm{gl.dim}(R_2) + w(i^*(R_2))$;

(2) $\mathrm{gl.dim}(R_2) \leqslant \mathrm{gl.dim}(R_1) + \mathrm{gl.dim}(R_3) + w(i_*(R_1)) + w(j_!(R_3)) + 1$.

Sketch of the Proof. From [1, Proposition 2.14] and its proof, we observe the following two facts.

(i) If $\mathrm{gl.dim}(R_2) < \infty$ or $\mathrm{gl.dim}(R_3) < \infty$, then $i_*(R_1)$ is isomorphic in $\mathscr{D}(R_2)$ to a bounded complex of projective R_2-modules.

(ii) $\mathrm{gl.dim}(R_2) < \infty$ if and only if both $\mathrm{gl.dim}(R_1) < \infty$ and $\mathrm{gl.dim}(R_3) < \infty$. In this case, the recollement among unbounded derived categories can be restricted to a recollement of bounded derived categories.

Moreover, for a ring R, if $\mathrm{gl.dim}(R) < \infty$, then $\mathrm{gl.dim}(R) = \mathrm{Fin.dim}(R)$. Now, Theorem 3.17 becomes a consequence of Theorem 1.1. $\qquad\square$

3.3. *Proofs and applications of Theorem* 1.2

Now we turn to proofs of Theorem 1.2 and its consequences arising from different exact contexts.

Proof of Theorem 1.2. Given an exact context (λ, μ, M, m), we have defined its noncommutative tensor product $T \boxtimes_R S$ and the following two ring homomorphisms:

$$\rho : S \to T \boxtimes_R S, \ s \mapsto 1 \otimes s \ \text{ for } \ s \in S, \ \text{ and } \ \phi : T \to T \boxtimes_R S, \ t \mapsto t \otimes 1 \ \text{ for } \ t \in T.$$

Note that $T \boxtimes_R S$ has $T \otimes_R S$ as its underlying abelian group, while its multiplication is different from the usual tensor product (see [6] for details). Let $B := \left(\begin{smallmatrix} S & M \\ 0 & T \end{smallmatrix}\right)$, $C := M_2(T \boxtimes_R S)$ and

$$\theta := \begin{pmatrix} \rho & \beta \\ 0 & \phi \end{pmatrix} : \ B \longrightarrow C,$$

where $\beta : M \to T \otimes_R S$ is the unique R-R-bimodule homomorphism such that $\phi = (m\cdot)\beta$ and $\rho = (\cdot m)\beta$.

Let

$$\varphi : \begin{pmatrix} S \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} M \\ T \end{pmatrix}, \ \begin{pmatrix} s \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} sm \\ 0 \end{pmatrix} \ \text{ for } \ s \in S.$$

Then φ is a homomorphism of B-R-bimodules. Denote by $P^\bullet$ the mapping cone of φ. Then $P^\bullet \in \mathscr{C}^b(B \otimes_{\mathbb{Z}} R^{\mathrm{op}})$ and $_B P^\bullet \in \mathscr{C}^b(B\text{-proj})$. In particular, $P^\bullet \in \mathscr{D}^c(B)$.

By [6, Theorem 1.1], if $\mathrm{Tor}_i^R(T, S) = 0$ for all $i \geqslant 1$, then there is a recollement of derived categories:

$$(\star) \quad \mathscr{D}(C) \xrightarrow[\overset{\overset{\displaystyle C \otimes_B^{\mathbb{L}} -}{\frown}}{D(\theta_*)}] {} \mathscr{D}(B) \xrightarrow[\overset{\overset{\displaystyle j_!}{\frown}}{j^!}]{} \mathscr{D}(R)$$

where $j_! := {}_B P^\bullet \otimes_R^{\mathbb{L}} -$, $j^! := \mathrm{Hom}_B^\bullet(P^\bullet, -)$ and $D(\theta_*)$ is the restriction functor induced from the ring homomorphism $\theta : B \to C$. First of all, we have the following two easy observations.

(i) $\mathrm{fin.dim}(C) = \mathrm{fin.dim}(T \boxtimes_R S)$ since $C := M_2(T \boxtimes_R S)$ is Morita equivalent to $T \boxtimes_R S$.
(ii) $\mathrm{fin.dim}(B) \leqslant \mathrm{fin.dim}(S) + \mathrm{fin.dim}(T) + 1$. This is well known, see also Corollary 3.14.

We first apply Corollary 3.7 to show Theorem 1.2(1). In fact, by [6, Lemma 5.4], $R \simeq \mathrm{End}_{\mathscr{D}(B)}(P^\bullet)$ as rings (via multiplication) and $\mathrm{Hom}_{\mathscr{D}(B)}(P^\bullet, P^\bullet[n]) = 0$ for any $n \neq 0$. It remains to show that $P_R^\bullet$ is isomorphic in $\mathscr{D}(R^{\mathrm{op}})$ to a bounded complex of flat R^{op}-modules.

Since the exact sequence $0 \to R \xrightarrow{(\lambda, \mu)} S \oplus T \xrightarrow{\left(\begin{smallmatrix} \cdot m \\ -m\cdot \end{smallmatrix}\right)} M \to 0$ is the mapping cone in $\mathscr{C}(R^{\mathrm{op}})$ of the chain map $(\lambda, m\cdot)$ from the complex $\mathrm{Con}(\mu)$ to the complex $\mathrm{Con}(\cdot m)$, we have $\mathrm{Con}(\mu) \simeq \mathrm{Con}(\cdot m)$ in $\mathscr{D}(R^{\mathrm{op}})$, where $\mathrm{Con}(\mu)$ is the complex $0 \to R \xrightarrow{\mu} T \to 0$ with T in degree 0. This implies $P_R^\bullet \simeq T \oplus \mathrm{Con}(\cdot m) \simeq T \oplus \mathrm{Con}(\mu)$ in $\mathscr{D}(R^{\mathrm{op}})$. If $\mathrm{flat.dim}(T_R) = \infty$, then Theorem 1.2(1) is trivially true. So we may suppose $\mathrm{flat.dim}(T_R) < \infty$. Let $t := \max\{1, \mathrm{flat.dim}(T_R)\}$. Then $P^\bullet$ is isomorphic in $\mathscr{D}(R^{\mathrm{op}})$ to a bounded complex

$$F^\bullet := 0 \longrightarrow F^{-t} \longrightarrow F^{-t+1} \longrightarrow \cdots \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow 0$$

such that F^i is a flat R^{op}-module for $-t \leqslant i \leqslant 0$. It follows from Corollary 3.7 that $\mathrm{fin.dim}(R) \leqslant \mathrm{fin.dim}(B) + t \leqslant \mathrm{fin.dim}(S) + \mathrm{fin.dim}(T) + t + 1$. This shows Theorem 1.2(1).

Next, we shall apply Theorem 1.1 to the above recollement $(\star)$ and prove Theorem 1.2(2).

By the proof of [**7**, Theorem 1.3(2)], $D(\theta_*)(C) = {}_BC \in \mathscr{P}^{<\infty}(B)$ if and only if ${}_RS \in \mathscr{P}^{<\infty}(R)$. Suppose ${}_RS \in \mathscr{P}^{<\infty}(R)$. It follows from [**6**, Corollary 5.9(1)] that

$$\mathrm{proj.dim}({}_BC) \leqslant \max\{2, \mathrm{proj.dim}({}_RS) + 1\}.$$

Since $C \otimes_B^{\mathbb{L}} B \simeq C$ in $\mathscr{D}(C)$, it follows from Theorem 1.1(2)(a) that $\mathrm{fin.dim}(C) \leqslant \mathrm{fin.dim}(B)$. Note that $\mathrm{fin.dim}(T \boxtimes_R S) = \mathrm{fin.dim}(C)$ and $\mathrm{fin.dim}(B) \leqslant \mathrm{fin.dim}(S) + \mathrm{fin.dim}(T) + 1$. Thus (a) holds.

As $D(\theta_*)(C) = {}_BC$ and $j_!(R) \simeq {}_BP^\bullet$ in $\mathscr{D}(B)$, we know $w(P^\bullet) = 1$ and

$$w\left(D(\theta_*)(C)\right) = w({}_BC) = \mathrm{proj.dim}({}_BC) \leqslant \max\{2, \mathrm{proj.dim}({}_RS) + 1\}.$$

Now, it follows from Theorem 1.1(2)(b) that

$$\mathrm{fin.dim}(B) \leqslant \mathrm{fin.dim}(R) + \mathrm{fin.dim}(T \boxtimes_R S) + \max\{2, \mathrm{proj.dim}({}_RS) + 1\} + 1 + 1.$$

Clearly, $\mathrm{fin.dim}(S) \leqslant \mathrm{fin.dim}(B)$. Thus (b) holds. $\qquad\square$

We point out the following fact related to Theorem 1.2(2): Suppose that (λ, μ, M, m) is an exact context with $\mathrm{Tor}_i^R(T, S) = 0$ for all $i \geqslant 1$. If $\lambda : R \to S$ is a homological ring epimorphism such that ${}_RS \in \mathscr{P}^{<\infty}(R)$, then $\mathrm{fin.dim}(S) \leqslant \mathrm{fin.dim}(R)$ and $\mathrm{fin.dim}(T \boxtimes_R S) \leqslant \mathrm{fin.dim}(T)$.

In fact, in this case, the Tor-vanishing condition, that is, $\mathrm{Tor}_i^R(T, S) = 0$ for all $i > 0$, is equivalent to that $\phi : T \to T \boxtimes_R S$ is a homological ring epimorphism (see [**6**, Proposition 5.7] for details). Moreover, $T \boxtimes_R S \simeq T \otimes_R S$ as T-S-bimodules. It follows that if ${}_RS \in \mathscr{P}^{<\infty}(R)$, then ${}_TT \boxtimes_R S \in \mathscr{P}^{<\infty}(T)$ by the Tor-vanishing condition. Therefore the above-mentioned fact is a consequence of Corollary 3.8.

COROLLARY 3.18. *Suppose that $S \subseteq R$ is an extension of rings, that is, S is a subring of R with the same identity. Let R' be the endomorphism ring of the S-module R/S, and let $R' \boxtimes_S R$ be the noncommutative tensor product of the exact context determined the extension. Then*

(1) $\mathrm{fin.dim}(S) \leqslant \mathrm{fin.dim}(R) + \mathrm{fin.dim}(R') + \max\big\{1, \mathrm{flat.dim}((R/S)_S),$
    $\mathrm{flat.dim}(\mathrm{Hom}_S(R, R/S)_S)\big\} + 1$;
(2) *suppose that the left S-module R is projective and finitely generated. Then*
    (a) $\mathrm{fin.dim}(R' \boxtimes_S R) \leqslant \mathrm{fin.dim}(R) + \mathrm{fin.dim}(R') + 1$;
    (b) $\mathrm{fin.dim}(R) \leqslant \mathrm{fin.dim}(S) + \mathrm{fin.dim}(R' \boxtimes_S R) + 4$.

Proof. Let $\tau : S \subseteq R$ be the inclusion from S into R, and let $\pi : R \to R/S$ be the canonical surjection. We define

$$\sigma :\, S \longrightarrow R' = \mathrm{End}_S(R/S), \quad s \mapsto (r \mapsto rs) \text{ for } s \in S \text{ and } r \in R/S$$

to be the right multiplication map. Then the quadruple $(\tau, \sigma, \mathrm{Hom}_S(R, R/S), \pi)$ determined by the extension is an exact context (see the examples in [**6**, Section 3]) and its noncommutative tensor product $R' \boxtimes_S R$ is well defined. If ${}_SR$ is flat, then $\mathrm{Tor}_i^S(R', R) = 0$ for all $i \geqslant 1$. Particularly, under the assumption on ${}_SR$ in Corollary 3.18(2), the quadruple fulfils the Tor-vanishing condition in Theorem 1.2(2).

Now, we apply Theorem 1.2 to the exact context $(\tau, \sigma, \mathrm{Hom}_S(R, R/S), \pi)$, and see that the statements (a) and (b) in Corollary 3.18 follow from the statements (a) and (b) in Theorem 1.2, respectively. To show Corollary 3.18(1), we shall apply Theorem 1.2(1). For this aim, we shall prove

$$\mathrm{flat.dim}(R'_S) \leqslant \max\left\{\mathrm{flat.dim}\left(\mathrm{Hom}_S(R, R/S)_S\right), \mathrm{flat.dim}\left((R/S)_S\right)\right\}.$$

However, this can be concluded from the following exact sequence of right R'-modules (also right S-modules):

$$0 \longrightarrow R' \longrightarrow \mathrm{Hom}_S(R, R/S) \longrightarrow \mathrm{Hom}_S(S, R/S) \longrightarrow 0,$$

which is obtained by applying $\mathrm{Hom}_S(-, R/S)$ to the exact sequence $0 \to S \to R \to R/S \to 0$. Now, the statement (1) follows from Theorem 1.2(1). $\qquad \square$

Let $\lambda : R \to S$ and $\mu : R \to T$ be ring homomorphisms, and let M be an S-T-bimodule with $m \in M$. Recall that an exact context (λ, μ, M, m) is called an *exact pair* if $M = S \otimes_R T$ and $m = 1 \otimes 1$. In this case, we simply say that (λ, μ) is an exact pair. By [6, Corollary 4.4], if the map λ in the exact context is a ring epimorphism, then the pair (λ, μ) is exact. Moreover, by [6, Remark 5.2], for an exact pair (λ, μ), we have $T \boxtimes_R S \simeq S \sqcup_R T$, the coproduct of the R-rings of S and T.

COROLLARY 3.19. *Let* $\lambda : R \to S$ *be a ring epimorphism and* M *an* S-S-*bimodule such that* $\mathrm{Tor}_i^R(M, S) = 0$ *for all* $i \geqslant 1$. *If* $_RS \in \mathscr{P}^{<\infty}(R)$, *then*

(a) $\mathrm{fin.dim}(S \ltimes M) \leqslant \mathrm{fin.dim}(S) + \mathrm{fin.dim}(R \ltimes M) + 1$;
(b) $\mathrm{fin.dim}(S) \leqslant \mathrm{fin.dim}(R) + \mathrm{fin.dim}(S \ltimes M)$.

Proof. We define $T := R \ltimes M$, $\mu : R \to T$ to be the inclusion from R into T, and $\widetilde{\lambda} : R \ltimes M \to S \ltimes M$ to be the canonical map induced from λ. By Lemma 2.2, the ring $S \ltimes M$, together with the inclusion $\rho : S \to S \ltimes M$ and $\widetilde{\lambda} : T \to S \ltimes M$, is the coproduct of S and T over R.

Now, we show that (λ, μ) is an exact pair. Actually, the split exact sequence $0 \to R \xrightarrow{\mu} T \to M \to 0$ of R-R-bimodules implies $_RT_R \simeq R \oplus M$ as R-R-bimodules. Since λ is a ring epimorphism and M is an S-S-bimodule, the map

$$S \otimes_R T \longrightarrow S \ltimes M, \ s \otimes (r, m) \mapsto (sr, sm)$$

for $s \in S$ and $m \in M$, is an isomorphism of S-T-bimodules. Under this isomorphism, we can identify the map $\mu' = id_S \otimes \mu : S \to S \otimes T$ with the inclusion $\rho : S \to S \ltimes M$, and the map $\lambda' = \lambda \otimes id_T : T \to S \otimes_R T$ with $\widetilde{\lambda}$. Note that $0 \to S \xrightarrow{\rho} S \ltimes M \to M \to 0$ is also a split exact sequence of S-S-bimodules. It follows that $\mathrm{Cok}(\mu) \simeq \mathrm{Cok}(\rho) \simeq M$ as R-R-bimodules, and therefore the following sequence of R-R-bimodules:

$$0 \longrightarrow R \xrightarrow{(\lambda, \mu)} S \oplus T \xrightarrow{\binom{\rho}{-\widetilde{\lambda}}} S \ltimes M \longrightarrow 0$$

is exact. This means that the pair (λ, μ) is exact.

Consequently, $T \boxtimes_R S \simeq S \sqcup_R T \simeq S \ltimes M$ as rings. Note that $\mathrm{Tor}_i^R(T, S) \simeq \mathrm{Tor}_i^R(R \oplus M, S) \simeq \mathrm{Tor}_i^R(M, S) = 0$ for all $i \geqslant 1$. Thus Corollary 3.19(a) follows immediately from Theorem 1.2(2)(a).

Now we turn to the proof of Corollary 3.19(b).

Note that, if Theorem 1.2(2)(b) is applied to the exact pair (λ, μ), then $\mathrm{fin.dim}(S) \leqslant \mathrm{fin.dim}(R) + \mathrm{fin.dim}(S \ltimes M) + \max\{1, \mathrm{proj.dim}(_RS)\} + 3$. So, to obtain the better upper bound given in Corollary 3.19(b), we need the following statement:

($*$) Let $f : \Lambda \to \Gamma$ and $g : \Gamma \to \Lambda$ be ring homomorphisms such that $fg = \mathrm{Id}_\Lambda$. If $\mathrm{fin.dim}(_\Gamma\Gamma \otimes_\Lambda^{\mathbb{L}} -) = s < +\infty$, then $\mathrm{fin.dim}(\Lambda) \leqslant \mathrm{fin.dim}(\Gamma) - s$.

To show ($*$), we set $F := {}_\Gamma\Gamma \otimes_\Lambda^{\mathbb{L}} - : \mathscr{D}(\Lambda) \to \mathscr{D}(\Gamma)$. If $\mathrm{fin.dim}(F) = -\infty$, then ($*$) is automatically true. So, we suppose that $\mathrm{fin.dim}(F) = s$ is an integer and $\Lambda \neq 0$. Since $F(\Lambda) \simeq \Gamma \neq 0$, we have $s \leqslant 0$. Let $X \in \mathscr{P}^{<\infty}(\Lambda)$. Then there exists a finite projective resolution $0 \to P_n \to \cdots \to P_1 \to P_0 \to X \to 0$ of $_\Lambda X$ with all P_i in Λ-proj. Now we define $Y := \Omega_\Lambda^{-s}(X)$, the $(-s)$th syzygy module of $_\Lambda X$ (with respect to this resolution). Thus $Y \in \mathscr{P}^{<\infty}(\Lambda)$. It

follows from $\mathrm{fin.dim}(F) = s$ that $\mathrm{Tor}_j^\Lambda(\Gamma, Y) = \mathrm{Tor}_{j-s}^\Lambda(\Gamma, X) \simeq H^{s-j}(F(X)) = 0$ for all $j > 0$. Hence $\Gamma \otimes_\Lambda Y \in \mathscr{P}^{<\infty}(\Gamma)$ and $\Gamma \otimes_\Lambda \Omega_\Lambda^i(Y) = \Omega_\Gamma^i(\Gamma \otimes_\Lambda Y) \oplus Q_i$ for all $i \geqslant 0$, where all Q_i are finitely generated projective Γ-modules. Further, we may suppose $\mathrm{fin.dim}(\Gamma) = t < \infty$. Then $\mathrm{proj.dim}(_\Gamma\Gamma \otimes_\Lambda Y) \leqslant t$, and therefore $\Gamma \otimes_\Lambda \Omega_\Lambda^t(Y) = \Omega_\Gamma^t(\Gamma \otimes_\Lambda Y) \oplus Q_t \in \Gamma$-proj. Due to $fg = \mathrm{Id}_\Lambda$, we have $\Omega_\Lambda^t(Y) \simeq \Lambda \otimes_\Gamma (\Gamma \otimes_\Lambda \Omega_\Lambda^t(Y)) \in \Lambda$-proj. Consequently,

$$\mathrm{proj.dim}(_\Lambda X) \leqslant \mathrm{proj.dim}(_\Lambda Y) - s \leqslant \mathrm{proj.dim}(_\Lambda \Omega_\Lambda^t(Y)) + t - s \leqslant t - s.$$

Thus $\mathrm{fin.dim}(\Lambda) \leqslant \mathrm{fin.dim}(\Gamma) - s$. This finishes the proof of $(*)$.

Now, we take $\Lambda := S$ and $\Gamma := S \ltimes M$. Let $f : S \to S \ltimes M$ and $g : S \ltimes M \to S$ be the canonical injection and surjection, respectively. Then $fg = \mathrm{Id}_S$. We assume $S \neq 0$. Then $_\Gamma\Gamma \otimes_\Lambda^\mathbb{L} \Lambda = \Gamma \neq 0$ and $\mathrm{fin.dim}(_\Gamma\Gamma \otimes_\Lambda^\mathbb{L} -) \leqslant 0$. Suppose $\mathrm{fin.dim}(R) = m < \infty$. Due to $(*)$, in order to show Corollary 3.19(b), we only need to prove $\mathrm{fin.dim}(_\Gamma\Gamma \otimes_S^\mathbb{L} -) \geqslant -m$. This is equivalent to $\mathrm{Tor}_n^S(\Gamma, X) \simeq \mathrm{Tor}_n^S(M, X) = 0$ for all $X \in \mathscr{P}^{<\infty}(S)$ and all $n > m$.

To check the latter, we first prove $\mathrm{Tor}_j^S(M, N) \simeq \mathrm{Tor}_j^R(M, N)$ for any S-module N and for all $j \geqslant 1$. Indeed, let $P^\bullet$ be a deleted projective resolution of the R^{op}-module M. Since $\mathrm{Tor}_i^R(M, S) = 0$ for all $i \geqslant 1$, $P^\bullet \otimes_R S$ is a deleted projective resolution of the S^{op}-module $M \otimes_R S$. Note that $M \otimes_R S \simeq M$ as S^{op}-modules since $\lambda : R \to S$ is a ring epimorphism and M is an S^{op}-module. It follows that $P^\bullet \otimes_R S$ is a deleted projective resolution of the S^{op}-module M. Since $(P^\bullet \otimes_R S) \otimes_S N \simeq P^\bullet \otimes_R N$ as complexes, $\mathrm{Tor}_j^S(M, N) \simeq \mathrm{Tor}_j^R(M, N)$ for all $j \geqslant 1$.

Let $_SX \in \mathscr{P}^{<\infty}(S)$. Since $\mathrm{proj.dim}(_RS) < \infty$, the Change of Rings Theorem implies $\mathrm{proj.dim}(_RX) \leqslant \mathrm{proj.dim}(_SX) + \mathrm{proj.dim}(_RS) < \infty$. Hence $\mathrm{proj.dim}(_RX) \leqslant m = \mathrm{fin.dim}(R)$ and $\mathrm{Tor}_n^S(M, X) \simeq \mathrm{Tor}_n^R(M, X) = 0$ if $n > m$. This means $\mathrm{fin.dim}(_\Gamma\Gamma \otimes_S^\mathbb{L} -) \geqslant -m$. Now, by the result $(*)$, $\mathrm{fin.dim}(S) \leqslant \mathrm{fin.dim}(\Gamma) + m = \mathrm{fin.dim}(S \ltimes M) + \mathrm{fin.dim}(R)$. This completes the proof of Corollary 3.19(b). $\qquad\square$

We remark that the statement $(*)$ also implies that for the trivial extension of R by an R-R-bimodule M, we always have $\mathrm{fin.dim}(R) \leqslant \mathrm{fin.dim}(R \ltimes M) + \mathrm{flat.dim}(M_R)$.

As an application of Theorem 1.2, we have the following result for pullback rings.

COROLLARY 3.20. *Let R be a ring, and let I_1 and I_2 be ideals of R such that $I_1 \cap I_2 = 0$. Then*

(1) $\mathrm{fin.dim}(R) \leqslant \mathrm{fin.dim}(R/I_1) + \mathrm{fin.dim}(R/I_2) + \max\{1, \mathrm{flat.dim}((R/I_2)_R)\} + 1$;
(2) *suppose $\mathrm{Tor}_i^R(I_2, I_1) = 0$ for all $i \geqslant 0$. If $_RR/I_1 \in \mathscr{P}^{<\infty}(R)$, then*
   (a) $\mathrm{fin.dim}(R/(I_1 + I_2)) \leqslant \mathrm{fin.dim}(R/I_1) + \mathrm{fin.dim}(R/I_2) + 1$;
   (b) $\mathrm{fin.dim}(R/I_1) \leqslant \mathrm{fin.dim}(R) + \mathrm{fin.dim}(R/(I_1 + I_2)) + \max\{1, \mathrm{proj.dim}(_R(R/I_1))\} + 3$.

Proof. Let $\lambda : R \to S := R/I_1$ and $\mu : R \to T := R/I_2$ be the canonical surjective ring homomorphisms. Since $I_1 \cap I_2 = 0$, $(\lambda, \mu, R/(I_1 + I_2), 1)$ is an exact context, where 1 is the identity of the ring $R/(I_1 + I_2)$. Even more, since R is a pullback of the surjective maps $R \to R/I_i$ over $R/(I_1 + I_2)$, the pair (λ, μ) is exact (for example, see [6, Section 3]). So $T \boxtimes_R S \simeq S \sqcup_R T$ as rings. Note that $S \sqcup_R T = (R/I_1) \sqcup_R (R/I_2) = R/(I_1 + I_2)$ by Lemma 2.3(2). Thus $T \boxtimes_R S \simeq R/(I_1 + I_2)$ as rings.
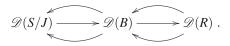
Now, we apply Theorem 1.2 to show Corollary 3.20. Clearly, it remains to check that if $\mathrm{Tor}_i^R(I_2, I_1) = 0$ for $i \geqslant 0$, then $\mathrm{Tor}_i^R(R/I_2, R/I_1) = 0$ for all $i > 0$. In fact, for $i > 2$, we have $\mathrm{Tor}_i^R(R/I_2, R/I_1) \simeq \mathrm{Tor}_{i-2}^R(I_2, I_1) = 0$ by assumption. Note that $\mathrm{Tor}_1^R(R/I_2, R/I_1) \simeq (I_2 \cap I_1)/(I_2 I_1) = 0$ and $\mathrm{Tor}_2^R(R/I_2, R/I_1) \simeq \mathrm{Tor}_1^R(I_2, R/I_1) = \mathrm{Ker}(f)$ where $f : I_2 \otimes_R I_1 \to I_2 I_1$ is the multiplication map. Since $I_2 \otimes_R I_1 = 0$ by assumption, $\mathrm{Tor}_2^R(R/I_2, R/I_1) \simeq \mathrm{Ker}(f) = 0$. Thus $\mathrm{Tor}_i^R(R/I_2, R/I_1) = 0$ for all $i > 0$. $\qquad\square$

Finally, we apply Theorem 1.2 to homological ring epimorphisms. First of all, we present a method to construct new homological ring epimorphisms from given ones.

LEMMA 3.21.   *Let $\lambda : R \to S$ be a homological ring epimorphism. Suppose that I is an ideal of R such that the image J' of I under λ is a left ideal in S and that the restriction of λ to I is injective. Let J be the ideal of S generated by J'. Then the following are equivalent.*

(1) *The homomorphism $\widetilde{\lambda} : R/I \to S/J$ induced from λ is homological.*
(2) $\mathrm{Tor}_i^{R/I}(J/J', S/J) = 0$ *for all $i \geqslant 1$.*
(3) *The multiplication map $I \otimes_R S \to J$ is an isomorphism and $\mathrm{Tor}_j^R(I, S) = 0$ for all $j \geqslant 1$.*
(4) $\mathrm{Tor}_j^R(R/I, S) = 0$ *for all $j \geqslant 1$.*

Let $B := \left(\begin{smallmatrix} S & S/J' \\ 0 & R/I \end{smallmatrix} \right)$. If one of the above statements holds true, then there exists a recollement of derived module categories:

$$\mathscr{D}(S/J) \longrightarrow \mathscr{D}(B) \longrightarrow \mathscr{D}(R) \ .$$

Proof.   We take $T := R/I$ and $\mu : R \to T$ to be the canonical surjective homomorphism of rings. Note that J' is a left ideal of S. Thus $S \otimes_R T = S \otimes_R (R/I) \simeq S/(S \cdot I) = S/J'$. On the one hand, the pair (λ, μ) is exact if and only if $\lambda|_I : I \to J'$ is an isomorphism. On the other hand, by Lemma 2.3(2), $S \sqcup_R T = S \sqcup (R/I) = S/J$ with $J = J'S$, and the ring homomorphism $\phi : T \to S \sqcup_R T$ in [6, Proposition 5.7] can be chosen as the canonical map $\widetilde{\lambda} : R/I \to S/J$ induced from λ. Thus (1) and (4) are equivalent by [6, Proposition 5.7]. Moreover, the recollement follows from [6, Theorem 1.1].

In the following, we shall show that (3) and (4) are equivalent.

Applying the tensor functor $- \otimes_R S$ to the exact sequence $0 \to I \to R \to R/I \to 0$, we obtain

$$\mathrm{Tor}_1^R(R/I, S) \simeq \mathrm{Ker}(\delta) \ \text{ and } \ \mathrm{Tor}_{j+1}^R(R/I, S) \simeq \mathrm{Tor}_j^R(I, S) \ \text{ for all } j \geqslant 1,$$

where $\delta : I \otimes_R S \to J$ is the multiplication map defined by $x \otimes s \mapsto (x)\lambda s$ for $x \in I$ and $s \in S$. This implies that (4) is equivalent to (3).

Now we show that (1) and (2) are equivalent.

According to Lemma 2.3(1) and the fact that λ is a ring epimorphism, $\widetilde{\lambda}$ is a ring epimorphism. By assumption, J' is a left ideal of S, and therefore $S \otimes_R (R/I) \simeq S/(S \cdot I) = S/J'$. Thanks to a general result in the proof of [6, Lemma 5.6], $\mathrm{Tor}_i^{R/I}(S/J', W) \simeq \mathrm{Tor}_i^{R/I}(S \otimes_R (R/I), W) = 0$ for all $i \geqslant 1$ and all S/J-modules W. It then follows that $\mathrm{Tor}_i^{R/I}(S/J', S/J) = 0$ for all $i \geqslant 1$. Consider the short exact sequence of right R/I-modules:

$$0 \longrightarrow J/J' \longrightarrow S/J' \longrightarrow S/J \longrightarrow 0.$$

If we apply the functor $- \otimes_{R/I} (S/J)$ to this sequence, then $\mathrm{Tor}_i^{R/I}(J/J', S/J) \simeq \mathrm{Tor}_{i+1}^{R/I}(S/J, S/J)$ for all $i \geqslant 1$ and the connecting homomorphism $\mathrm{Tor}_1^{R/I}(S/J, S/J) \to (J/J') \otimes_{R/I} (S/J)$ is injective.

If $\mathrm{Tor}_1^{R/I}(S/J, S/J) = 0$, then $\mathrm{Tor}_j^{R/I}(S/J, S/J) = 0$ for all $j \geqslant 1$ if and only if $\mathrm{Tor}_i^{R/I}(J/J', S/J) = 0$ for all $i \geqslant 1$. This implies that (1) and (2) are equivalent. So it is enough to demonstrate that $\mathrm{Tor}_1^{R/I}(S/J, S/J) = 0$ always holds under the assumptions of Lemma 3.21. However, this is true if we can show $(J/J') \otimes_{R/I} (S/J) = 0$.

In fact, if $C \to D$ is a ring epimorphism, then, for any D-module X and any right D-module Y, we have $D \otimes_C X \simeq X$ as D-modules, and $Y \otimes_C D \simeq Y$ as right D-modules. This fact, together with properties of ring epimorphisms, implies the following isomorphisms:

$$(J/J') \otimes_{R/I} (S/J) \simeq (J/J') \otimes_R (S/J) \simeq (J/J') \otimes_R (S \otimes_R (S/J)) \simeq ((J/J') \otimes_R S) \otimes_R (S/J).$$

Since $SJ' = J'$ and $JJ' \subseteq J'$, we deduce $((J/J') \otimes_R S)J' = 0$. This means that $(J/J') \otimes_R S$ is a right S/J-module. Remark that the composite of the two ring epimorphisms $R \to S$ and $S \to S/J$ is again a ring epimorphism. It follows that $((J/J') \otimes_R S) \otimes_R (S/J) \simeq (J/J') \otimes_R S$ as right S/J-modules.

In the following, we shall show $(J/J') \otimes_R S = 0$. Actually, applying the functor $- \otimes_R S$ to the exact sequence

$$0 \longrightarrow J' \overset{\alpha}{\longrightarrow} J \longrightarrow J/J' \longrightarrow 0$$

of right R-modules, we get another exact sequence

$$J' \otimes_R S \overset{\alpha \otimes_R S}{\longrightarrow} J \otimes_R S \longrightarrow (J/J') \otimes_R S \longrightarrow 0$$

of right S-modules. Since J is a right S-module and $\lambda : R \to S$ is a ring epimorphism, the multiplication map $\psi : J \otimes_R S \to J$, given by $x \otimes s \mapsto xs$ for $x \in J$ and $s \in S$, is an isomorphism. Note that the map $(\alpha \otimes_R S)\psi : J' \otimes_R S \to J$ is surjective. This yields that $\alpha \otimes_R S$ is surjective and $(J/J') \otimes_R S = 0$. Hence $\mathrm{Tor}_1^{R/I}(S/J, S/J) = 0$. This finishes the proof. $\square$

A special case of Lemma 3.21 appears in trivial extensions. Let $\lambda : R \to S$ be a homomorphism of rings and M be an S-S-bimodule. Then λ is homological if and only if $\widetilde{\lambda} : R \ltimes M \to S \ltimes M$ is homological. The necessity of this condition follows from [**6**, Theorem 1.1] and the proof of Corollary 3.19. The sufficiency can be seen from Lemma 3.21.

Applying Theorem 1.2 to the exact pair (λ, μ) in the proof of Lemma 3.21, we obtain the following estimations on finitistic dimensions, which can be applied to a class of examples of Milnor squares in the following corollary.

COROLLARY 3.22. *Let $\lambda : R \to S$ be a homological ring epimorphism. Suppose that I is an ideal of R such that the image J' of I under λ is a left ideal in S and that the restriction of λ to I is injective. Let J be the ideal of S generated by J'. Suppose that one of the conditions* (1)–(4) *in Lemma* 3.21 *holds. Then*

(1) $\mathrm{fin.dim}(R) \leqslant \mathrm{fin.dim}(S) + \mathrm{fin.dim}(R/I) + \max\{1, \mathrm{flat.dim}((R/I)_R)\} + 1$;
(2) *if* $_RS \in \mathscr{P}^{<\infty}(R)$, *then*
    (a) $\mathrm{fin.dim}(S) \leqslant \mathrm{fin.dim}(R)$ *and* $\mathrm{fin.dim}(S/J) \leqslant \mathrm{fin.dim}(R/I)$;
    (b) $\mathrm{fin.dim}(B) \leqslant \mathrm{fin.dim}(R) + \mathrm{fin.dim}(S/J) + \max\{1, \mathrm{proj.dim}(_RS)\} + 3$,   *where* $B := \left(\begin{smallmatrix} S & S/J' \\ 0 & R/I \end{smallmatrix}\right)$.

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