# Derived equivalences for Φ-Auslander-Yoneda algebras

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### Abstract

In this paper, we introduce Φ-Auslander-Yoneda algebras in a triangulated category with Φ a parameter set in $\mathbb{N}$, and present a method to construct new derived equivalences between these Φ-Auslander-Yoneda algebras (not necessarily Artin algebras), or their quotient algebras, from a given almost ν-stable derived equivalence. As consequences of our method, we have: (1) Suppose that $A$ and $B$ are representation-finite, self-injective Artin algebras with $_AX$ and $_BY$ additive generators for $A$ and $B$, respectively. If $A$ and $B$ are derived-equivalent, then the Φ-Auslander-Yoneda algebras of $X$ and $Y$ are derived-equivalent for every admissible set Φ. In particular, the Auslander algebras of $A$ and $B$ are both derived-equivalent and stably equivalent. (2) For a self-injective Artin algebra $A$ and an $A$-module $X$, the Φ-Auslander-Yoneda algebras of $A \oplus X$ and $A \oplus \Omega_A(X)$ are derived-equivalent for every admissible set Φ, where Ω is the Heller loop operator. Motivated by these derived equivalences between Φ-Auslander-Yoneda algebras, we consider constructions of derived equivalences for quotient algebras, and show, among other results, that a derived equivalence between two basic self-injective algebras may transfer to a derived equivalence between their quotient algebras obtained by factorizing out socles.

## 1 Introduction

Derived categories and derived equivalences were introduced by Grothendieck and Verdier in [13]. As is known, they have widely been used in many branches of mathematics and physics. One of the fundamental problems in the study of derived categories and derived equivalences is: how to construct derived equivalences ? On the one hand, Rickard's beautiful Morita theory for derived categories can be used to find all rings that are derived-equivalent to a given ring $A$ by determining all tilting complexes over $A$ (see [10] and [11]). On the other hand, a natural course of investigation on derived equivalences should be constructing new derived equivalences from given ones. In this direction, Rickard used tensor products and trivial extensions to produce new derived-equivalences in [10, 12], Barot and Lenzing employed one-point extensions to transfer certain a derived equivalence to a new one in [2]. Up to now, however, it seems that not much is known for constructing new derived equivalences based on given ones.

In this paper, we continue the consideration in this direction and provide, roughly speaking, two methods to construct new derived equivalences from given ones. One is to form Φ-Auslander-Yoneda algebras (see Section 3.1 for definition) of generators, or cogenerators over derived-equivalent algebras, and the other is to form quotient algebras of derived-equivalent algebras. We point out that our family of Φ-Auslander-Yoneda algebras include Auslander algebras, generalized Yoneda algebras

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and some of their quotients. Thus our method produces also derived equivalences between infinite-dimensional algebras.

To state our results, we first introduce a few terminologies.

Suppose that $F$ is a derived equivalence between two Artin algebras $A$ and $B$, with the quasi-inverse functor $G$. Further, suppose that

$$T^{\bullet}: \quad \cdots \longrightarrow 0 \longrightarrow T^{-n} \longrightarrow \cdots \longrightarrow T^{-1} \longrightarrow T^0 \longrightarrow 0 \longrightarrow \cdots$$

is a radical tilting complex over $A$ associated to $F$, and suppose that

$$\bar{T}^{\bullet}: \quad \cdots \longrightarrow 0 \longrightarrow \bar{T}^0 \longrightarrow \bar{T}^1 \longrightarrow \cdots \longrightarrow \bar{T}^n \longrightarrow 0 \longrightarrow \cdots$$

is a radical tilting complex over $B$ associated to $G$. The functor $F$ is called *almost $\nu$-stable* if $\mathrm{add}(\bigoplus_{i=-1}^{-n} T^i)$ $= \mathrm{add}(\bigoplus_{i=-1}^{-n} \nu_A T^i)$, and $\mathrm{add}(\bigoplus_{i=1}^{n} \bar{T}^i) = \mathrm{add}(\bigoplus_{i=1}^{n} \nu_B \bar{T}^i)$, where $\nu_A$ is the Nakayama functor for $A$. We have shown in [6] that an almost $\nu$-stable functor $F$ induces an equivalence functor $\bar{F}$ between the stable module categories $A$-$\underline{\mathrm{mod}}$ and $B$-$\underline{\mathrm{mod}}$. For further information on almost $\nu$-stable derived equivalences, we refer the reader to [6].

For a module $M$ over an algebra $A$, the *generalized Yoneda algebra* of $M$ is defined by $\mathrm{Ext}_A^*(M) := \bigoplus_{i \geq 0} \mathrm{Ext}_A^i(M,M)$. In case $M = A/\mathrm{rad}(A)$, the algebra $\mathrm{Ext}_A^*(M)$ is called the *Yoneda algebra* of $A$ in literature. We shall extend this notion to a more general situation, and introduce the $\Phi$-Auslander-Yoneda algebras with $\Phi$ a parameter set in $\mathbb{N}$ (for details see Subsection 3.1 below). We notice that a $\Phi$-Auslander-Yoneda algebra may not be an Artin algebra in general.

Our main result on $\Phi$-Auslander-Yoneda algebras of modules reads as follows:

**Theorem 1.1.** *Let $A$ and $B$ be two Artin algebras, and let $\bar{F} : A$-$\underline{\mathrm{mod}} \longrightarrow B$-$\underline{\mathrm{mod}}$ be the stable equivalence induced by an almost $\nu$-stable derived equivalence $F$ between $A$ and $B$. Suppose that $X$ is an $A$-module, we set $M := A \oplus X$ and $N := B \oplus \bar{F}(X)$. Let $\Phi$ be an admissible subset of $\mathbb{N}$, and define the $\Phi$-Auslander-Yoneda algebra of $M$ as $\mathrm{E}_A^{\Phi}(M) := \bigoplus_{i \in \Phi} \mathrm{Ext}_A^i(M,M)$. Then:*

*(1) The $\Phi$-Auslander-Yoneda algebras $\mathrm{E}_A^{\Phi}(M)$ and $\mathrm{E}_B^{\Phi}(N)$ are derived-equivalent.*

*(2) If $\Phi$ is finite, then there is an almost $\nu$-stable derived equivalence between $\mathrm{E}_A^{\Phi}(M)$ and $\mathrm{E}_B^{\Phi}(N)$. Thus $\mathrm{E}_A^{\Phi}(M)$ and $\mathrm{E}_B^{\Phi}(N)$ are also stably equivalent. In particular, there is an almost $\nu$-stable derived equivalence and a stable equivalence between $\mathrm{End}_A(M)$ and $\mathrm{End}_B(N)$.*

A dual version of Theorem 1.1 can be seen in Corollary 3.17 below.

Since Auslander algebra and generalized Yoneda algebra are two special cases of $\Phi$-Auslander-Yoneda algebras, Theorem 1.1 provides a large variety of derived equivalences between Auslander algebras, and between generalized Yoneda algebras, or their quotient algebras. Note that Theorem 1.1 (2) extends a result in [6, Proposition 6.1], where algebras were assumed to be finite-dimensional over a field, in order to employ two-sided tilting complexes in proofs, and where only endomorphism algebras were considered instead of general Auslander-Yoneda algebras. The existence of two-sided tilting complexes is guaranteed for Artin $R$-algebras that are projective as $R$-modules [11]. For general Artin algebras, however, we do not know the existence of two-sided tilting complexes. Hence, in this paper, we have to provide a completely different proof to the general result, Theorem 1.1.

As a direct consequence of Theorem 1.1, we have the following corollary concerning the Auslander algebras of self-injective algebras.

**Corollary 1.2.** *(1) For a self-injective Artin algebra $A$ and an $A$-module $Y$, the $\Phi$-Auslander-Yoneda algebras $\mathrm{E}_A^{\Phi}(A \oplus Y)$ and $\mathrm{E}_A^{\Phi}(A \oplus \Omega_A(Y))$ are derived-equivalent, where $\Omega$ is the Heller loop operator.*

*(2) Suppose that $A$ and $B$ are self-injective Artin algebras of finite representation type with $_AX$ and $_BY$ additive generators for $A$-mod and $B$-mod, respectively. If $A$ and $B$ are derived-equivalent, then*

*(i) the Auslander algebras of $A$ and $B$ are both derived and stably equivalent.*

*(ii) The generalized Yoneda algebras $\mathrm{Ext}_A^*(X)$ and $\mathrm{Ext}_B^*(Y)$ of $X$ and $Y$ are derived-equivalent.*

Notice that, in Corollary 1.2, the Auslander algebra of $A$ is a quotient algebra of the generalized Yoneda algebra $\text{Ext}_A^*(X)$ of the additive generator $X$. The next result shows another way to construct derived equivalences for quotient algebras.

**Theorem 1.3.** *Let $F : \mathscr{D}^{\text{b}}(A) \longrightarrow \mathscr{D}^{\text{b}}(B)$ be a derived equivalence between two self-injective basic Artin algebras $A$ and $B$. Suppose that $P$ is a direct summand of $_AA$, and $Q$ is a direct summand of $_BB$ such that $F(\text{soc}(P))$ is isomorphic to $\text{soc}(Q)$, where $\text{soc}(P)$ denotes the socle of the module $P$. Then the quotient algebras $A/\text{soc}(P)$ and $B/\text{soc}(Q)$ are derived-equivalent.*

The structure of this paper is organized as follows. In Section 2, we make some preparations for later proofs. In Section 3, we introduce the $\Phi$-Auslander-Yoneda algebras and prove Theorem 1.1 and its dual version, Corollary 3.17, which produces derived equivalences between the endomorphism algebras of cogenerators. Furthermore, we deduce a series of consequences of Theorem 1.1 for self-injective algebras, including Corollary 1.2. In Section 4, we provide several methods to construct derived equivalences for quotient algebras. First, we give a general criterion, and then apply it to self-injective algebras modulo socles, and to algebras modulo annihilators. In particular, we show Theorem 1.3, and point out a class of derived equivalences satisfying the conditions in Theorem 1.3.

## 2 Preliminaries

In this section, we shall recall basic definitions and facts on derived categories and derived equivalences, which are elementary elements in our proofs.

Throughout this paper, $R$ is a fixed commutative Artin ring. Given an $R$-algebra $A$, by an $A$-module we mean a unitary left $A$-module; the category of all finitely generated $A$-modules is denoted by $A$-mod, the full subcategory of $A$-mod consisting of projective (respectively, injective) modules is denoted by $A$-proj (respectively, $A$-inj). The stable module category $A$-$\underline{\text{mod}}$ of $A$ is, by definition, the quotient category of $A$-mod modulo the ideal generated by homomorphisms factorizing through projective modules. An equivalence between the stable module categories of two algebras is called a *stable equivalence*

An $R$-algebra $A$ is called an *Artin $R$-algebra* if $A$ is finitely generated as an $R$-module. For an Artin $R$-algebra $A$, we denote by $D$ the usual duality on $A$-mod, and by $\nu_A$ the Nakayama functor $D\text{Hom}_A(-, {}_AA) : A$-proj $\longrightarrow A$-inj. For an $A$-module $M$, we denote by $\Omega_A(M)$ the first syzygy of $M$, and call $\Omega_A$ the *Heller loop operator* of $A$. In this paper, we mainly concentrate us on Artin algebras and finitely generated modules.

Let $\mathcal{C}$ be an additive category.

For two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, we write $fg$ for their composition. But for two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ of categories, we write $GF$ for their composition instead of $FG$. For an object $X$ in $\mathcal{C}$, we denote by $\text{add}(X)$ the full subcategory of $\mathcal{C}$ consisting of all direct summands of finite direct sums of copies of $X$. An object $X$ in $\mathcal{C}$ is called an *additive generator* for $\mathcal{C}$ if $\text{add}(X) = \mathcal{C}$.

By a complex $X^\bullet$ over $\mathcal{C}$ we mean a sequence of morphisms $d_X^i$ between objects $X^i$ in $\mathcal{C}$: $\cdots \to X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} X^{i+2} \to \cdots$, such that $d_X^i d_X^{i+1} = 0$ for all $i \in \mathbb{Z}$, and write $X^\bullet = (X^i, d_X^i)$. For a complex $X^\bullet$, the *brutal truncation* $\sigma_{<i}X^\bullet$ of $X^\bullet$ is a subcomplex of $X^\bullet$ such that $(\sigma_{<i}X^\bullet)^k$ is $X^k$ for all $k < i$ and zero otherwise. Similarly, we define $\sigma_{\geqslant i}X^\bullet$. For a fixed $n \in \mathbb{Z}$, we denote by $X^\bullet[n]$ the complex obtained from $X^\bullet$ by shifting $n$ degrees, that is, $(X^\bullet[n])^0 = X^n$.

The category of all complexes over $\mathcal{C}$ with chain maps is denoted by $\mathscr{C}(\mathcal{C})$. The homotopy category of complexes over $\mathcal{C}$ is denoted by $\mathscr{K}(\mathcal{C})$. When $\mathcal{C}$ is an abelian category, the derived category of complexes over $\mathcal{C}$ is denoted by $\mathscr{D}(\mathcal{C})$. The full subcategory of $\mathscr{K}(\mathcal{C})$ and $\mathscr{D}(\mathcal{C})$ consisting of

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bounded complexes over $C$ is denoted by $\mathscr{K}^{\mathrm{b}}(C)$ and $\mathscr{D}^{\mathrm{b}}(C)$, respectively. As usual, for an algebra $A$, we simply write $\mathscr{C}(A)$ for $\mathscr{C}(A\text{-mod})$, $\mathscr{K}(A)$ for $\mathscr{K}(A\text{-mod})$ and $\mathscr{K}^{\mathrm{b}}(A)$ for $\mathscr{K}^{\mathrm{b}}(A\text{-mod})$. Similarly, we write $\mathscr{D}(A)$ and $\mathscr{D}^{\mathrm{b}}(A)$ for $\mathscr{D}(A\text{-mod})$ and $\mathscr{D}^{\mathrm{b}}(A\text{-mod})$, respectively.

It is well-known that, for an $R$-algebra $A$, the categories $\mathscr{K}(A)$ and $\mathscr{D}(A)$ are triangulated categories. For basic results on triangulated categories, we refer the reader to the excellent books [3] and [9].

Let $A$ be an Artin algebra. Recall that a homomorphism $f : X \to Y$ of $A$-modules is called a *radical map* if, for any module $Z$ and homomorphisms $h : Z \to X$ and $g : Y \to Z$, the composition $hfg$ is not an isomorphism. A complex over $A$-mod is called a *radical* complex if all of its differential maps are radical maps. Every complex over $A$-mod is isomorphic to a radical complex in the homotopy category $\mathscr{K}(A)$. If two radical complexes $X^{\bullet}$ and $Y^{\bullet}$ are isomorphic in $\mathscr{K}(A)$, then $X^{\bullet}$ and $Y^{\bullet}$ are isomorphic in $\mathscr{C}(A)$.

Two $R$-Artin algebras $A$ and $B$ are said to be *derived-equivalent* if their derived categories $\mathscr{D}^{\mathrm{b}}(A)$ and $\mathscr{D}^{\mathrm{b}}(B)$ are equivalent as triangulated categories. By a result of Rickard (see Lemma 2.2 below), two algebras $A$ and $B$ are derived-equivalent if and only if $B$ is isomorphic to the endomorphism algebra $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(T^{\bullet})$ of a tilting complex $T^{\bullet}$ over $A$. Recall that a complex $T^{\bullet}$ in $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ is called a *tilting complex* over $A$ if it satisfies

(1) $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A\text{-proj})}(T^{\bullet}, T^{\bullet}[n]) = 0$ for all $n \neq 0$, and

(2) $\mathrm{add}(T^{\bullet})$ generates $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ as a triangulated category.

It is known that, given a derived equivalence $F$ between $A$ and $B$, there is a unique (up to isomorphism in $\mathscr{K}^{\mathrm{b}}(A)$) tilting complex $T^{\bullet}$ over $A$ such that $F(T^{\bullet}) \simeq B$. This complex $T^{\bullet}$ is called a tilting complex *associated* to $F$. Recall that a complex $X^{\bullet}$ of $A$-modules is called *self-orthogonal* if $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(X^{\bullet}, X^{\bullet}[i]) = 0$ for every $i \neq 0$.

The following lemma, proved in [6, Lemma 2.2], will be used frequently in our proofs below.

**Lemma 2.1.** *Let $A$ be an arbitrary ring with identity, and let $A$-Mod be the category of all left (not necessarily finitely generated) $A$-modules. Suppose that $X^{\bullet}$ is a complex over $A$-Mod bounded above, and that $Y^{\bullet}$ is a complex over $A$-Mod bounded below. Let $m$ be an integer. If one of the following two conditions holds:*

*(1) $X^i$ is projective for all $i > m$ and $Y^j = 0$ for all $j < m$,*

*(2) $Y^j$ is injective for all $j < m$ and $X^i = 0$ for all $i > m$,*

*then the localization functor $\theta : \mathscr{K}(A\text{-Mod}) \longrightarrow \mathscr{D}(A\text{-Mod})$ induces an isomorphism $\theta_{X^{\bullet}, Y^{\bullet}} :$ $\mathrm{Hom}_{\mathscr{K}(A\text{-Mod})}(X^{\bullet}, Y^{\bullet}) \longrightarrow \mathrm{Hom}_{\mathscr{D}(A\text{-Mod})}(X^{\bullet}, Y^{\bullet})$.*

Thus, for the complexes $X^{\bullet}$ and $Y^{\bullet}$ given in Lemma 2.1, the computation of morphisms from $X^{\bullet}$ to $Y^{\bullet}$ in $\mathscr{D}(A\text{-Mod})$ is reduced to that in $\mathscr{K}(A\text{-Mod})$.

For later reference, we cite the following fundamental result on derived equivalences by Rickard (see [10, Theorem 6.4]) as a lemma.

**Lemma 2.2.** *[10] Let $\Lambda$ and $\Gamma$ be two rings. The following conditions are equivalent:*

*(a) $\mathscr{K}^{-}(\Lambda\text{-Proj})$ and $\mathscr{K}^{-}(\Gamma\text{-Proj})$ are equivalent as triangulated categories;*

*(b) $\mathscr{D}^{\mathrm{b}}(\Lambda\text{-Mod})$ and $\mathscr{D}^{\mathrm{b}}(\Gamma\text{-Mod})$ are equivalent as triangulated categories;*

*(c) $\mathscr{K}^{\mathrm{b}}(\Lambda\text{-Proj})$ and $\mathscr{K}^{\mathrm{b}}(\Gamma\text{-Proj})$ are equivalent as triangulated categories;*

*(d) $\mathscr{K}^{\mathrm{b}}(\Lambda\text{-proj})$ and $\mathscr{K}^{\mathrm{b}}(\Gamma\text{-proj})$ are equivalent as triangulated categories;*

*(e) $\Gamma$ is isomorphic to $\mathrm{End}(T)$, where $T$ is a tilting complex in $\mathscr{K}^{\mathrm{b}}(\Lambda\text{-proj})$.*

*Here $\Lambda$-Proj stands for the full subcategory of $\Lambda$-Mod consisting of all projective $\Lambda$-modules.*

Two rings $\Lambda$ and $\Gamma$ are called *derived-equivalent* if one of the above conditions (a)-(e) is satisfied. For Artin algebras, the two definitions of a derived equivalence coincide with each other.

# 3 Derived equivalences for $\Phi$-Auslander-Yoneda algebras

As is known, Auslander algebra is a key to characterizing representation-finite algebras, and Yoneda algebra plays a role in the study of the graded module theory of Koszul algebras. In this section, we shall first unify the two notions and introduce the so-called $\Phi$-Auslander-Yoneda algebra of an object in a triangulated category, where $\Phi$ is a parameter subset of $\mathbb{N}$, and then construct new derived equivalences between these $\Phi$-Auslander-Yoneda algebras from a given almost $\nu$-stable derived equivalence. In particular, Theorem 1.1 will be proved, and a series of its consequences will be deduced in this section.

## 3.1 Admissible sets and Auslander-Yoneda algebras

First, we introduce some special subsets of the set $\mathbb{N} := \{0, 1, 2, \cdots, \}$ of the natural numbers, and then define a class of algebras called Auslander-Yoneda algebras.

A subset $\Phi$ of $\mathbb{N}$ containing 0 is called an *admissible subset* of $\mathbb{N}$ if the following condition is satisfied:

*If $i$, $j$ and $k$ are in $\Phi$ such that $i + j + k \in \Phi$, then $i + j \in \Phi$ if and only if $j + k \in \Phi$.*

For instance, the sets $\{0, 3, 4\}$, $\{0, 1, 2, 3, 4\}$ are admissible subsets of $\mathbb{N}$. The following is a family of admissible subsets of $\mathbb{N}$.

Let $n$ be a positive integer, and let $m$ be a positive integer or positive infinity. We define

$$\Phi(n, m) := \{xn \mid x \in \mathbb{N}, 0 \le x < m + 1\}.$$

Then $\Phi(n, m)$ is an admissible subset in $\mathbb{N}$. Clearly, we have $\Phi(1, \infty) = \mathbb{N}, \Phi(1, 0) = \{0\}$, and $\Phi(1, m) = \{0, 1, 2, \cdots, m\}$.

Admissible subsets of $\mathbb{N}$ have the following simple properties.

**Proposition 3.1.** (1) *If $\Phi$ is an admissible subset of $\mathbb{N}$, then so is $m\Phi := \{mx \mid x \in \Phi\}$ for every $m \in \mathbb{N}$.*

(2) *If $\Phi_1$ and $\Phi_2$ are admissible subsets of $\mathbb{N}$, then so is $\Phi_1 \cap \Phi_2$. Moreover, the intersection of a family of admissible subsets of $\mathbb{N}$ is admissible.*

(3) *For a subset $\Phi \subseteq \mathbb{N}$ with $0 \in \Phi$, the set $\Phi^m := \{x^m \mid x \in \Phi\}$ is an admissible subset of $\mathbb{N}$ for every integer $m \ge 3$.*

*Proof.* The statements (1) and (2) follow easily from the definition of an admissible subset. Now we consider (3). We pick an integer $m \ge 3$. Let $i^m, j^m, k^m$ and $l^m$ be in $\Phi^m$ such that $i^m + j^m + k^m = l^m$. If $i^m + j^m \in \Phi^m$, then $i^m + j^m = t^m$ for some $t \in \Phi$. By Fermat's Last Theorem, one of the integers $i$ and $j$ is zero. If $j = 0$, then $j^m + k^m = k^m \in \Phi^m$. If $i = 0$, then $j^m + k^m = l^m \in \Phi^m$. Similarly, we can show that if $j^m + k^m \in \Phi^m$, then $i^m + j^m \in \Phi$. Hence the set $\Phi^m$ is an admissible subset of $\mathbb{N}$. $\square$

Note that $\Phi^2$ is not necessarily admissible in $\mathbb{N}$ even if $\Phi$ is an admissible subset of $\mathbb{N}$. For instance, if $\Phi = \{0, 3, 4, 5, 12, 13\}$, then $\Phi$ is admissible. Clearly, $3^2 + 4^2 + 12^2 = 13^2 \in \Phi^2$ and $3^2 + 4^2 = 5^2 \in \Phi^2$, but $4^2 + 12^2 \notin \Phi^2$, so $\Phi^2$ is not admissible.

Now, we use admissible subsets to define a class of associative algebras. Let us start with the following general situation.

Let $\Phi$ be a subset of $\mathbb{N}$. Given an $\mathbb{N}$-graded $R$-algebra $\Lambda = \bigoplus_{i \ge 0}^{\infty} \Lambda_i$, where $R$ is a commutative ring and each $\Lambda_i$ is an $R$-module with $\Lambda_i \Lambda_j \subseteq \Lambda_{i+j}$ for all $i, j \in \mathbb{N}$, we define an $R$-module $\Lambda(\Phi) := \bigoplus_{i \in \Phi} \Lambda_i$, and a multiplication in $\Lambda(\Phi)$: for $a_i \in \Lambda_i$ and $b_j \in \Lambda_j$ with $i, j \in \Phi$, we define $a_i \cdot b_j = a_i b_j$ if $i + j \in \Phi$, and zero otherwise. Then one can easily check that $\Lambda(\Phi)$ is an associative algebra if $\Phi$ is an admissible subset of $\mathbb{N}$.

This procedure can be applied to a triangulated category, in this special situation, the details which are needed in our proofs read as follows:

Let $\mathcal{T}$ be a triangulated $R$-category over a commutative Artin ring $R$, and let $\Phi$ be a subset in $\mathbb{N}$ containing 0. We denote by $E_{\mathcal{T}}^{\Phi}(-,-)$ the bifunctor

$$\bigoplus_{i \in \Phi} \mathrm{Hom}_{\mathcal{T}}(-,-[i]) : \mathcal{T} \times \mathcal{T} \longrightarrow R\text{-Mod},$$

$$(X,Y) \mapsto E_{\mathcal{T}}^{\Phi}(X,Y) := \bigoplus_{i \in \Phi} \mathrm{Hom}_{\mathcal{T}}(X,Y[i]).$$

Let $X, Y$ and $Z$ be objects in $\mathcal{T}$. For each $i \in \Phi$, let $\iota_i$ denote the canonical embedding of $\mathrm{Hom}_{\mathcal{T}}(X,Y[i])$ into $E_{\mathcal{T}}^{\Phi}(X,Y)$. For $i \notin \Phi$, we define $\iota_i$ to be the zero map from $\mathrm{Hom}_{\mathcal{T}}(X,Y[i])$ to $E_{\mathcal{T}}^{\Phi}(X,Y)$. An element in $E_{\mathcal{T}}^{\Phi}(X,Y)$ is of the form $(f_i)_{i \in \Phi}$, where $f_i$ is a morphism in $\mathrm{Hom}_{\mathcal{T}}(X,Y[i])$ for $i \in \Phi$. For simplicity, we shall just write $(f_i)$ for $(f_i)_{i \in \Phi}$, and each element $(f_i)$ in $E_{\mathcal{T}}^{\Phi}(X,Y)$ can be rewritten as $\sum_{i \in \Phi} \iota_i(f_i)$, where $\iota_i(f_i)$ denotes the image of $f_i$ under the map $\iota_i$.

Let $(f_i) \in \mathrm{E}_{\mathcal{T}}^{\Phi}(X,Y)$ and $(g_i) \in \mathrm{E}_{\mathcal{T}}^{\Phi}(Y,Z)$. We define a multiplication $(h_i) = (f_i)(g_i)$:

$$\mathrm{E}_{\mathcal{T}}^{\Phi}(X,Y) \times \mathrm{E}_{\mathcal{T}}^{\Phi}(Y,Z) \longrightarrow \mathrm{E}_{\mathcal{T}}^{\Phi}(X,Z)$$

$$\big((f_i),(g_i)\big) \mapsto (h_i),$$

where

$$h_i := \sum_{\substack{u,v \in \Phi \\ u+v=i}} f_u(g_v[u])$$

for each $i \in \Phi$. In particular, for $f \in \mathrm{Hom}_{\mathcal{T}}(X,Y[i])$ and $g \in \mathrm{Hom}_{\mathcal{T}}(X,Y[j])$ with $i,j \in \Phi$, we have

$$\iota_i(f)\,\iota_j(g) = \iota_{i+j}(f(g[i])).$$

Note that $\iota_{i+j} = 0$ if $i+j \notin \Phi$.

The next proposition explains further why we need to introduce admissible subsets.

**Proposition 3.2.** *Let $\mathcal{T}$ be a triangulated $R$-category with at least one non-zero object, and let $\Phi$ be a subset of $\mathbb{N}$ containing 0. Then $\mathrm{E}_{\mathcal{T}}^{\Phi}(V)$ together with the multiplication defined above is an associative $R$-algebra for every object $V \in \mathcal{T}$ if and only if $\Phi$ is an admissible subset of $\mathbb{N}$.*

*Proof.* If $\Phi$ is an admissible subset of $\mathbb{N}$, then it is straightforward to check that the multiplication on $\mathrm{E}_{\mathcal{T}}^{\Phi}(V)$ defined above is associative for all objects $V \in \mathcal{T}$. Now we assume that $\Phi$ is not an admissible subset, that is, there are integers $i,j,k \in \Phi$ satisfying: $i+j+k \in \Phi$, $i+j \in \Phi$, and $j+k \notin \Phi$. Let $X$ be a non-zero object in $\mathcal{T}$, and let $V := \bigoplus_{s=0}^{i+j+k} X[s]$. We consider the multiplication on $\mathrm{E}_{\mathcal{T}}^{\Phi}(V)$. By definition, the object $\bigoplus_{s=i}^{i+j+k} X[s]$ is a common direct summand of $V$ and $V[i]$. Let $f$ be the composition $V \xrightarrow{\pi} \bigoplus_{s=i}^{i+j+k} X[s] \xrightarrow{\lambda} V[i]$, where $\pi$ is the canonical projection and $\lambda$ is the canonical inclusion. Similarly, we define $g : V \longrightarrow \bigoplus_{s=j}^{i+j+k} X[s] \longrightarrow V[j]$ and $h : V \longrightarrow \bigoplus_{s=k}^{i+j+k} X[s] \longrightarrow V[k]$. Since $i+j \in \Phi$, we have $\big(\iota_i(f)\iota_j(g)\big)\iota_k(h) = \iota_{i+j}(f(g[i]))\iota_k(h) = \iota_{i+j+k}\big(f(g[i])(h[i+j])\big)$. One can check that $f(g[i])(h[i+j])$ is just the composition $V \longrightarrow X[i+j+k] \longrightarrow V[i+j+k]$, where the maps are canonical maps. Hence the map $\big(\iota_i(f)\iota_j(g)\big)\iota_k(h)$ is non-zero. Since $j+k \notin \Phi$, we have $\iota_j(g)\iota_k(h) = 0$, and consequently $\iota_i(f)\big(\iota_j(g)\iota_k(h)\big) = 0$. This shows that the multiplication of $\mathrm{E}_{\mathcal{T}}^{\Phi}(V)$ is not associative, and the proof is completed. $\square$

Note that $E_{\mathcal{T}}^{\mathbb{N}}(X)$ is an $\mathbb{N}$-graded associative $R$-algebra with $\mathrm{Hom}_{\mathcal{T}}(X,X[i])$ as $i$-th component. If we define $\Lambda := E_{\mathcal{T}}^{\mathbb{N}}(X)$, then $\Lambda(\Phi) = E_{\mathcal{T}}^{\Phi}(X)$.

From now on, we consider exclusively admissible subsets $\Phi$ of $\mathbb{N}$. Thus, for objects $X$ and $Y$ in $\mathcal{T}$, one has an $R$-algebra $E_{\mathcal{T}}^{\Phi}(X,X)$ (which may not be artinian), and a left $E_{\mathcal{T}}^{\Phi}(X,X)$-module $E_{\mathcal{T}}^{\Phi}(X,Y)$. For simplicity, we write $E_{\mathcal{T}}^{\Phi}(X)$ for $E_{\mathcal{T}}^{\Phi}(X,X)$.

In case $\Phi = \Phi(1,0)$, we see that $E_{\mathcal{T}}^{\Phi}(X)$ is the endomorphism algebra of the object $X$ in $\mathcal{T}$. In case $\Phi = \mathbb{N}$, we know that $E_{\mathcal{T}}^{\Phi}(X)$ is the generalized Yoneda algebra $\mathrm{Ext}_{\mathcal{T}}^*(X) = \bigoplus_{i \geq 0} \mathrm{Hom}_{\mathcal{T}}(X, X[i])$ of $X$. Particularly, let us take $\mathcal{T} = \mathscr{D}^b(A)$ with $A$ an Artin $R$-algebra. If $A$ is representation-finite and if $X$ is an additive generator for $A$-mod, then $E_{\mathcal{T}}^{\Phi(1,0)}(X)$ is the Auslander algebra of $A$; if we take $X = A/\mathrm{rad}(A)$, then $E_{\mathcal{T}}^{\Phi(1,\infty)}(X)$ is the usual Yoneda algebra of $A$. Thus the algebra $E_{\mathcal{T}}^{\Phi}(X)$ is a generalization of both Auslander algebra and Yoneda algebra. For this reason, the algebra $E_{\mathcal{T}}^{\Phi}(X)$ of $X$ in a triangulated category $\mathcal{T}$ is called the $\Phi$-*Auslander-Yoneda algebra* of $X$ in $\mathcal{T}$ in this paper.

If $\mathcal{T} = \mathscr{D}^b(A)$ with $A$ an Artin algebra, we simply write $E_A^{\Phi}(X)$ for $E_{\mathcal{T}}^{\Phi}(X)$, and $E_A^{\Phi}(X,Y)$ for $E_{\mathcal{T}}^{\Phi}(X,Y)$. If $\Phi$ is finite, or if the projective or injective dimension of $X$ is finite, then $E_A^{\Phi}(X)$ is an Artin $R$-algebra.

Note also that the algebra $E_{\mathcal{T}}^{\Phi(1,m)}(X)$ is a quotient algebra of $E_{\mathcal{T}}^{\mathbb{N}}(X)$, and the algebra $E_{\mathcal{T}}^{\Phi(n,m)}(X)$ is a subalgebra of $E_{\mathcal{T}}^{\Phi(1,nm)}(X)$. Nevertheless, if we take $\Phi = \{0,3,9\}$ and $X$ a simple module over the algebra $A := k[X]/(X^2)$ with $k$ a field, then $E_A^{\Phi}(X)$ is neither a subalgebra nor a quotient algebra of the generalized Yoneda algebra of $X$.

Let us remark that one may define the notion of an admissible subset of $\mathbb{Z}$ (or of a monoid $M$ with an identity $e$), and introduce $\Phi$-Auslander-Yoneda algebra of an object in an arbitrary $R$-category $\mathcal{C}$ with an additive self-equivalence functor (or a family of additive functors $\{F_g\}_{g \in M}$ from $\mathcal{C}$ to itself, such that $F_e = id_{\mathcal{C}}$ and $F_g F_h = F_{gh}$ for all $g, h \in M$). For our goals in this paper, we just formulate the admissible subsets for $\mathbb{N}$.

## 3.2 Almost ν-stable derived equivalences

We briefly recall some basic facts on almost ν-stable derived equivalences from [6], which are needed in proofs.

Let $A$ and $B$ be Artin algebras, and let $F : \mathscr{D}^b(A) \longrightarrow \mathscr{D}^b(B)$ be a derived equivalence between $A$ and $B$. Suppose that $Q^\bullet$ and $\bar{Q}^\bullet$ are the tilting complexes associated to $F$ and to a quasi-inverse $G$ of $F$, respectively. Now, we assume that $Q^i = 0$ for all $i > 0$, that is, the complex $Q^\bullet$ is of the form

$$0 \longrightarrow Q^{-n} \longrightarrow \cdots \longrightarrow Q^{-1} \longrightarrow Q^0 \longrightarrow 0.$$

In this case, the complex $\bar{Q}^\bullet$ may be chosen of the following form (see [6, Lemma 2.1], for example)

$$0 \longrightarrow \bar{Q}^0 \longrightarrow \bar{Q}^1 \longrightarrow \cdots \longrightarrow \bar{Q}^n \longrightarrow 0.$$

Set $Q := \bigoplus_{i=1}^n Q^{-i}$ and $\bar{Q} := \bigoplus_{i=1}^n \bar{Q}^i$. The functor $F$ is called an *almost ν-stable* derived equivalence provided $\mathrm{add}(_A Q) = \mathrm{add}(\nu_A Q)$ and $\mathrm{add}(_B \bar{Q}) = \mathrm{add}(\nu_B \bar{Q})$. A crucial property is that an almost ν-stable derived equivalence induces an equivalence between the stable module categories $A$-$\underline{\mathrm{mod}}$ and $B$-$\underline{\mathrm{mod}}$. Thus $A$ and $B$ share many common properties, for example, $A$ is representation-finite if and only if $B$ is representation-finite.

In the following lemma, we collect some basic facts on almost ν-stable derived equivalences, which will be used in our proofs.

**Lemma 3.3.** *Let $F : \mathscr{D}^b(A) \to \mathscr{D}^b(B)$ be an almost ν-stable derived equivalence between Artin algebras $A$ and $B$. Suppose that $Q^\bullet$ and $\bar{Q}^\bullet$ are the tilting complexes associated to $F$ and to its quasi-inverse $G$, respectively. Then:*

(1) *For each A-module X, the complex $F(X)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(B)$ to a radical complex $\bar{Q}_X^\bullet$ of the form*

$$0 \longrightarrow \bar{Q}_X^0 \longrightarrow \bar{Q}_X^1 \longrightarrow \cdots \longrightarrow \bar{Q}_X^n \longrightarrow 0,$$

*with $\bar{Q}_X^i \in \mathrm{add}({_B}\bar{Q})$ for all $i > 0$. Moreover, the complex $\bar{Q}_X^\bullet$ of this form is unique up to isomorphism in $\mathscr{C}^{\mathrm{b}}(B)$.*

(2) *For each B-module Y, the complex $G(Y)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(A)$ to a radical complex $Q_Y^\bullet$ of the form*

$$0 \longrightarrow Q_Y^{-n} \longrightarrow \cdots \longrightarrow Q_Y^{-1} \longrightarrow Q_Y^0 \longrightarrow 0,$$

*with $Q_Y^i \in \mathrm{add}({_A}Q)$ for all $i < 0$. Moreover, the complex $Q_Y^\bullet$ of this form is unique up to isomorphism in $\mathscr{C}^{\mathrm{b}}(B)$.*

(3) *There is a stable equivalence $\bar{F} : A\text{-}\underline{\mathrm{mod}} \longrightarrow B\text{-}\underline{\mathrm{mod}}$ with $\bar{F}(X) = \bar{Q}_X^0$ for each A-module X.*

(4) *There is a stable equivalence $\bar{G} : B\text{-}\underline{\mathrm{mod}} \longrightarrow A\text{-}\underline{\mathrm{mod}}$ with $\bar{G}(Y) = Q_Y^0$ for each B-module Y. Moreover, the functor $\bar{G}$ is a quasi-inverse of $\bar{F}$ defined in (3).*

(5) *For an A-module X, we denote by $\bar{Q}_X^+$ the complex $\sigma_{>0}\bar{Q}_X^\bullet$. Then $G(\bar{Q}_X^+)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(A)$ to a bounded complex $P_X^\bullet$ of projective-injective A-modules with $P_X^i = 0$ for all $i > 1$.*

*Proof.* The statement (1) follows from [6, Lemma 3.1]. The statement (2) is a direct consequence of the definition of an almost $\nu$-stable derived equivalence and [6, Lemma 3.2]. Note that the statements (3) and (4) follow from the proof of [6, Theorem 3.7], and the statement (5) is implied in the proof of [6, Proposition 3.6]. $\square$

For an Artin algebra $A$, let ${_A}E$ be the direct sum of all non-isomorphic indecomposable projective $A$-modules $P$ with the property: $\nu_A^i P$ is projective-injective for all $i \geq 0$. The $A$-module ${_A}E$ is called a *maximal $\nu$-stable A*-module.

## 3.3 Derived equivalences for Auslander-Yoneda algebras

Our main result in this section is the following theorem on derived equivalences between $\Phi$-Auslander-Yoneda algebras.

**Theorem 3.4.** *Let $F : \mathscr{D}^{\mathrm{b}}(A) \longrightarrow \mathscr{D}^{\mathrm{b}}(B)$ be an almost $\nu$-stable derived equivalence between two Artin algebras A and B, and let $\bar{F}$ be the stable equivalence defined in* Lemma 3.3 (3). *For an A-module X, we set $M := A \oplus X$ and $N := B \oplus \bar{F}(X)$. Suppose that $\Phi$ is an admissible subset in $\mathbb{N}$. Then we have the following:*

(1) *The algebras $\mathrm{E}_A^\Phi(M)$ and $\mathrm{E}_B^\Phi(N)$ are derived-equivalent.*

(2) *If $\Phi$ is finite, then there is an almost $\nu$-stable derived equivalence between $\mathrm{E}_A^\Phi(M)$ and $\mathrm{E}_B^\Phi(N)$. Thus $\mathrm{E}_A^\Phi(M)$ and $\mathrm{E}_B^\Phi(N)$ are also stably equivalent. In particular, there is an almost $\nu$-stable derived equivalence and a stable equivalence between $\mathrm{End}_A(M)$ and $\mathrm{End}_B(N)$.*

Thus, under the assumptions of Theorem 3.4, if $\Phi$ is finite, then the algebras $\mathrm{E}_A^\Phi(M)$ and $\mathrm{E}_B^\Phi(N)$ share many common invariants: for example, finiteness of finitistic and global dimensions, representation dimension, Hochschild cohomology, representation-finite type and so on.

The rest of this section is essentially devoted to the proof of Theorem 3.4. First of all, we need some preparations. Let us start with the following lemma that describes some basic properties of the algebra $\mathrm{E}_A^\Phi(V)$, where $V$ is an A-module and is considered as a complex concentrated on degree zero.

**Lemma 3.5.** *Let A be an Artin algebra, and let V be an A-module. Suppose that $V_1 \in \mathrm{add}({_A}V)$ and $V_2 \in A\text{-}\mathrm{mod}$. Then*

(1) *The $E_A^\Phi(V)$-module $E_A^\Phi(V,V_1)$ is projective and finitely generated, and there is an isomorphism*

$$\mu : E_A^\Phi(V_1,V_2) \longrightarrow \mathrm{Hom}_{E_A^\Phi(V)}(E_A^\Phi(V,V_1), E_A^\Phi(V,V_2)),$$

*which sends $(f_i) \in E_A^\Phi(V_1,V_2)$ to the morphism $\big((a_i) \mapsto (a_i)(f_i)\big)$ for $(a_i) \in E_A^\Phi(V,V_2)$. Moreover, if $V_3 \in \mathrm{add}(_AV)$ and $(g_i) \in E_A^\Phi(V_2,V_3)$, then $\mu((f_i)(g_i)) = \mu((f_i))\mu((g_i))$.*

(2) *The functor $E_A^\Phi(V,-) : \mathrm{add}(_AV) \longrightarrow E_A^\Phi(V)$-proj is faithful.*

(3) *If $V_1$ is projective or $V_2$ is injective, then the functor $E_A^\Phi(V,-)$ induces an isomorphism of $R$-modules:*

$$E_A^\Phi(V,-) : \mathrm{Hom}_A(V_1,V_2) \longrightarrow \mathrm{Hom}_{E_A^\Phi(V)}(E_A^\Phi(V,V_1), E_A^\Phi(V,V_2)).$$

(4) *If $\Phi$ is finite, and $P \in \mathrm{add}(_AV)$ is projective, then*

$$\nu_{E_A^\Phi(V)} E_A^\Phi(V,P) \simeq E_A^\Phi(V, \nu_A P).$$

*Proof.* (1) Since $E_A^\Phi(V,-)$ is an additive functor and since $V_1 \in \mathrm{add}(_AV)$, we know that $E_A^\Phi(V,V_1)$ is in $\mathrm{add}(E_A^\Phi(V))$, and consequently $E_A^\Phi(V,V_1)$ is a finitely generated projective $E_A^\Phi(V)$-module. Similarly, the $E_A^\Phi(V)$-module $E_A^\Phi(V,V_2)$ is also projective. To show that $\mu$ is an isomorphism, we can assume that $V_1$ is indecomposable by additivity. Let $\pi_1 : V \longrightarrow V_1$ be the canonical projection, and let $\lambda_1 : V_1 \longrightarrow V$ be the canonical injection. We define a map

$$\gamma : \mathrm{Hom}_{E_A^\Phi(V)}(E_A^\Phi(V,V_1), E_A^\Phi(V,V_2)) \longrightarrow E_A^\Phi(V_1,V_2)$$

by sending $\alpha \in \mathrm{Hom}_{E_A^\Phi(V)}(E_A^\Phi(V,V_1), E_A^\Phi(V,V_2))$ to $\iota_0(\lambda_1)\alpha\big(\iota_0(\pi_1)\big)$. By calculation, the morphism $(\gamma\mu)(\alpha) : E_A^\Phi(V,V_1) \longrightarrow E_A^\Phi(V,V_2)$ sends each $x \in E_A^\Phi(V,V_1)$ to $x\iota_0(\lambda_1)\alpha\big(\iota_0(\pi_1)\big) = \alpha\big(x\iota_0(\lambda_1)\iota_0(\pi_1)\big) = \alpha(x)$. This shows that $\gamma\mu = \mathrm{id}$. Similarly, one can check that $\mu\gamma = \mathrm{id}$. Hence $\mu$ is an isomorphism. The rest of (1) can be verified easily.

(2) Using definition, one can check that the map

$$E_A^\Phi(V,-) : \mathrm{Hom}_{\mathscr{D}^b(A)}(V_1,V_2) \longrightarrow \mathrm{Hom}_{E_A^\Phi(V)}(E_A^\Phi(V,V_1), E_A^\Phi(V,V_2))$$

is the composition of the embedding $\iota_0 : \mathrm{Hom}_A(V_1,V_2) \longrightarrow E_A^\Phi(V_1,V_2)$ with the isomorphism $\mu$ in (1). Hence $E_A^\Phi(V,-)$ is a faithful functor.

(3) If $V_1$ is projective or $V_2$ is injective, then the embedding

$$\iota_0 : \mathrm{Hom}_{\mathscr{D}^b(A)}(V_1,V_2) \longrightarrow E_A^\Phi(V_1,V_2)$$

is an isomorphism. Since $E_A^\Phi(V,-)$ is the composition of $\iota_0$ with the isomorphism $\mu$ in (1), the statement (3) follows.

(4) This follows directly from the following isomorphisms

$$
\begin{aligned}
\nu_{E_A^\Phi(V)} E_A^\Phi(V,P) &= D\mathrm{Hom}_{E_A^\Phi(V)}(E_A^\Phi(V,P), E_A^\Phi(V,V)) \\
&\simeq DE_A^\Phi(P,V) \quad \text{by (1)} \\
&= D\mathrm{Hom}_A(P,V) \\
&\simeq \mathrm{Hom}_A(V, \nu_A P) \\
&= E_A^\Phi(V, \nu_A P).
\end{aligned}
$$

Thus we have finished the proof. $\square$

From now on, we assume that $F : \mathscr{D}^b(A) \longrightarrow \mathscr{D}^b(B)$ is an almost $\nu$-stable derived equivalence with a quasi-inverse functor $G$, that $Q^\bullet$ and $\bar{Q}^\bullet$ are tilting complexes associated to $F$ and $G$, respectively, and that $\bar{F} : A\text{-}\underline{\mathrm{mod}} \longrightarrow B\text{-}\underline{\mathrm{mod}}$ is the stable equivalence defined by Lemma 3.3 (3). For an $A$-module $X$, we may assume that $F(X) = \bar{Q}_X^\bullet$ as in Lemma 3.3 (1), and define $_A M = A \oplus X$ and $_B N = B \oplus \bar{F}(X)$. By $\bar{T}^\bullet$ we denote the complex $\bar{Q}^\bullet \oplus \bar{Q}_X^\bullet$. Clearly, $\bar{T}^\bullet$ is in $\mathscr{K}^b(\mathrm{add}(_B N))$.

**Lemma 3.6.** *Keeping the notations above, we have the following:*

(1) $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\mathrm{add}(_BN))}(\bar{T}^\bullet, \bar{T}^\bullet[i]) = 0$ *for all* $i \neq 0$.

(2) $\mathrm{add}(\bar{T}^\bullet)$ *generates* $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(_BN))$ *as a triangulated category.*

*Proof.* Since $F(A) \simeq \bar{Q}^\bullet$, the complex $\bar{T}^\bullet$ is isomorphic to $F(M) = \bar{Q}_M^\bullet$. So, we consider $\bar{Q}_M^\bullet$ instead.

(1) Suppose $i < 0$. Then $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^\bullet, \bar{Q}_M^\bullet[i]) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(B)}(\bar{Q}_M^\bullet, \bar{Q}_M^\bullet[i])$ by Lemma 2.1. Since $F(M) = \bar{Q}_M^\bullet$, we have $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(B)}(\bar{Q}_M^\bullet, \bar{Q}_M^\bullet[i]) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(M, M[i]) = 0$. Hence $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^\bullet, \bar{Q}_M^\bullet[i]) = 0$ for all $i < 0$.

Let $\bar{Q}_M^+$ be the complex $\sigma_{>0}\bar{Q}_M^\bullet$. There is a distinguished triangle

$$(*) \qquad \bar{Q}_M^+ \xrightarrow{i_M} \bar{Q}_M^\bullet \xrightarrow{\pi_M} \bar{F}(M) \xrightarrow{\alpha_M} \bar{Q}_M^+[1]$$

in $\mathscr{K}^{\mathrm{b}}(B)$. Applying $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^\bullet, -)$ to $(*)$, we get an exact sequence

$$\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^\bullet, \bar{F}(M)[i-1]) \to \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^\bullet, \bar{Q}_M^+[i]) \to \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^\bullet, \bar{Q}_M^\bullet[i]) \to \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^\bullet, \bar{F}(M)[i])$$

for each integer $i$. Since $\bar{Q}_M^i = 0$ for all $i < 0$, we have $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^\bullet, \bar{F}(M)[i]) = 0$ for all $i > 0$. By Lemma 3.3 (5), $G(\bar{Q}_M^+)$ is isomorphic to a bounded complex $P_M^\bullet$ of projective-injective $A$-modules such that $P_M^i = 0$ for all $i > 1$. Thus, we have

$$
\begin{aligned}
\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^\bullet, \bar{Q}_M^+[i]) &\simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(B)}(\bar{Q}_M^\bullet, \bar{Q}_M^+[i]) \qquad (\text{since } \bar{Q}_M^+[i] \text{ is in } \mathscr{K}^{\mathrm{b}}(B\text{-inj})) \\
&\simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(G(\bar{Q}_M^\bullet), G(\bar{Q}_M^+)[i]) \\
&\simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(M, P_M^\bullet[i]) \\
&\simeq \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(M, P_M^\bullet[i]) \\
&= 0
\end{aligned}
$$

for all $i > 1$, and consequently $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^\bullet, \bar{Q}_M^\bullet[i]) = 0$ for all $i > 1$. To prove (1), it remains to show that $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^\bullet, \bar{Q}_M^\bullet[1]) = 0$. Using the above exact sequence, we only need to show that the induced map

$$\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^\bullet, \alpha_M) : \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^\bullet, \bar{F}(M)) \longrightarrow \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^\bullet, \bar{Q}_M^+[1])$$

is surjective. Note that $G(\bar{Q}_M^+)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(A)$ to a complex $P_M^\bullet$ of projective-injective modules such that $P_M^k = 0$ for all $k > 1$. Thus, we can form a commutative diagram

$$
\begin{array}{ccccccc}
P_M^\bullet & \xrightarrow{\phi_M} & M & \xrightarrow{\lambda} & \mathrm{con}(\phi_M) & \xrightarrow{p} & P_M^\bullet[1] \\
\downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} \\
G(\bar{Q}_M^+) & \xrightarrow{Gi_M} & GF(M) & \xrightarrow{G\pi_M} & G\bar{F}(M) & \xrightarrow{G\alpha_M} & G(\bar{Q}_M^+)[1]
\end{array}
$$

in $\mathscr{D}^{\mathrm{b}}(A)$, where the vertical maps are all isomorphisms, $\lambda$ and $p$ are the canonical morphisms, and where the morphism $\phi_M$ is chosen in $\mathscr{K}^{\mathrm{b}}(A)$ such that the first square is commutative. The distinguished triangle in the top row of the above diagram can be viewed as a distinguished triangle in $\mathscr{K}^{\mathrm{b}}(A)$. Applying $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(M, -)$ to this triangle, we can easily see that $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(M, p)$ is a surjective map since $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(M, M[1]) = 0$. By Lemma 2.1, the localization functor $\theta : \mathscr{K}^{\mathrm{b}}(A) \to \mathscr{D}^{\mathrm{b}}(A)$ induces two isomorphisms

$$\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(M, \mathrm{con}(\phi_M)) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(M, \mathrm{con}(\phi_M)) \text{ and } \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(M, P_M^\bullet[1]) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(M, P_M^\bullet[1]).$$

It follows that $\operatorname{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(M, p)$ is surjective. Since the vertical maps of the above diagram are all isomorphisms, the map $\operatorname{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(M, G\alpha_M)$ is surjective, or equivalently $\operatorname{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(G(\bar{Q}_M^{\bullet}), G\alpha_M)$ is surjective. Since $G$ is an equivalence, it follows that $\operatorname{Hom}_{\mathscr{D}^{\mathrm{b}}(B)}(\bar{Q}_M^{\bullet}, \alpha_M)$ is surjective. By Lemma 2.1 again, the localization functor $\theta : \mathscr{K}^{\mathrm{b}}(B) \to \mathscr{D}^{\mathrm{b}}(B)$ gives rise to isomorphisms

$$\operatorname{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^{\bullet}, \bar{F}(M)) \simeq \operatorname{Hom}_{\mathscr{D}^{\mathrm{b}}(B)}(\bar{Q}_M^{\bullet}, \bar{F}(M)) \text{ and } \operatorname{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^{\bullet}, \bar{Q}_M^{+}[1]) \simeq \operatorname{Hom}_{\mathscr{D}^{\mathrm{b}}(B)}(\bar{Q}_M^{\bullet}, \bar{Q}_M^{+}[1]).$$

Hence the map $\operatorname{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^{\bullet}, \alpha_M)$ is surjective, and consequently $\operatorname{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^{\bullet}, \bar{Q}_M^{\bullet}[1]) = 0$.

Altogether, we have shown that $\operatorname{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_M^{\bullet}, \bar{Q}_M^{\bullet}[i]) = 0$ for all $i \neq 0$. Since $\mathscr{K}^{\mathrm{b}}(B)$ is a full subcategory of $\mathscr{K}^{\mathrm{b}}(\operatorname{add}(_B N))$, we have $\operatorname{Hom}_{\mathscr{K}^{\mathrm{b}}(\operatorname{add}(_B N))}(\bar{Q}_M^{\bullet}, \bar{Q}_M^{\bullet}[i]) = 0$ for all $i \neq 0$. This proves (1).

(2) Since $\bar{Q}^{\bullet}$ is a tilting complex over $B$, $\operatorname{add}(\bar{Q}^{\bullet})$ generates $\mathscr{K}^{\mathrm{b}}(\operatorname{add}(_B B))$ as a triangulated category. By Lemma 3.3, $\bar{Q}_X^0 = \bar{F}(X)$ and all the terms of $\bar{Q}_X^{\bullet}$ other than $\bar{Q}_X^0$ are in $\operatorname{add}(_B B)$. Hence $\bar{F}(X)$ is in the triangulated subcategory generated by $\operatorname{add}(\bar{Q}^{\bullet} \oplus \bar{Q}_X^{\bullet})$, and consequently $\operatorname{add}(\bar{Q}^{\bullet} \oplus \bar{Q}_X^{\bullet})$ generates $\mathscr{K}^{\mathrm{b}}(\operatorname{add}(B \oplus \bar{F}(X)))$ as a triangulated category. Thus, the statement (2) follows. $\square$

The additive functor $\mathrm{E}_B^{\Phi}(N, -) : \operatorname{add}(_B N) \longrightarrow \mathrm{E}_B^{\Phi}(N)$-proj induces a triangle functor

$$\mathrm{E}_B^{\Phi\bullet}(N, -) : \mathscr{K}^{\mathrm{b}}(\operatorname{add}(_B N)) \longrightarrow \mathscr{K}^{\mathrm{b}}(\mathrm{E}_B^{\Phi}(N)\text{-proj}).$$

For each integer $i$, the $i$-th term of $\mathrm{E}_B^{\Phi\bullet}(N, \bar{T}^{\bullet})$ is $\mathrm{E}_B^{\Phi}(N, \bar{T}^i)$, and the differential map from $\mathrm{E}_B^{\Phi}(N, \bar{T}^i)$ to $\mathrm{E}_B^{\Phi}(N, \bar{T}^{i+1})$ is $\mathrm{E}_B^{\Phi}(N, d)$, where $d : \bar{T}^i \longrightarrow \bar{T}^{i+1}$ is the differential map of $\bar{T}^{\bullet}$.

**Lemma 3.7.** *The complex $\mathrm{E}_B^{\Phi\bullet}(N, \bar{T}^{\bullet})$ is a tilting complex over $\mathrm{E}_B^{\Phi}(N)$.*

*Proof.* Let $i \neq 0$, and let $f^{\bullet}$ be a morphism in $\operatorname{Hom}_{\mathscr{K}^{\mathrm{b}}(\mathrm{E}_B^{\Phi}(N)\text{-proj})}(\mathrm{E}_B^{\Phi\bullet}(N, \bar{T}^{\bullet}), \mathrm{E}_B^{\Phi\bullet}(N, \bar{T}^{\bullet})[i])$. Then we have a commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathrm{E}_B^{\Phi}(N, \bar{T}^0) & \xrightarrow{\mathrm{E}_B^{\Phi}(N, d)} \mathrm{E}_B^{\Phi}(N, \bar{T}^1) & \xrightarrow{\mathrm{E}_B^{\Phi}(N, d)} \cdots \\
& & \downarrow f^0 & \downarrow f^1 & \\
\mathrm{E}_B^{\Phi}(N, \bar{T}^{i-1}) & \xrightarrow{\mathrm{E}_B^{\Phi}(N, d)} & \mathrm{E}_B^{\Phi}(N, \bar{T}^i) & \xrightarrow{\mathrm{E}_B^{\Phi}(N, d)} \mathrm{E}_B^{\Phi}(N, \bar{T}^{i+1}) & \xrightarrow{\mathrm{E}_B^{\Phi}(N, d)} \cdots .
\end{array}$$

Note that the term $\mathrm{E}_B^{\Phi}(N, \bar{T}^i)$ is zero if $i < 0$. Since all the terms of $\bar{T}^{\bullet}$ other than $\bar{T}^0$ are projective-injective, and since $i \neq 0$, we see from Lemma 3.5 (3) that $f^k = \mathrm{E}_B^{\Phi}(N, g^k)$ for some $g^k : \bar{T}^k \longrightarrow \bar{T}^{k+i}$ for all integers $k$. It follows from the above commutative diagram that, for each integer $k$, we have

$$\mathrm{E}_B^{\Phi}(N, d)\mathrm{E}_B^{\Phi}(N, g^{k+1}) - \mathrm{E}_B^{\Phi}(N, g^k)\mathrm{E}_B^{\Phi}(N, d) = 0,$$

or equivalently $\mathrm{E}_B^{\Phi}(N, dg^{k+1} - g^k d) = 0$. Since $\mathrm{E}_B^{\Phi}(N, -) : \operatorname{add}(_B N) \longrightarrow \mathrm{E}_B^{\Phi}(N)$-proj is a faithful functor by Lemma 3.5 (2), we have $dg^{k+1} - g^k d = 0$ for all integers $k$, and consequently $g^{\bullet} := (g^k)$ is in $\operatorname{Hom}_{\mathscr{K}^{\mathrm{b}}(\operatorname{add}(_B N))}(\bar{T}^{\bullet}, \bar{T}^{\bullet}[i])$ and $f^{\bullet} = \mathrm{E}_B^{\Phi\bullet}(N, g^{\bullet})$. By Lemma 3.6 (1), the map $g^{\bullet}$ is null-homotopic, and consequently $f^{\bullet} = \mathrm{E}_B^{\Phi\bullet}(N, g^{\bullet})$ is null-homotopic. Thus, we have proved that

$$\operatorname{Hom}_{\mathscr{K}^{\mathrm{b}}(\mathrm{E}_B^{\Phi}(N)\text{-proj})}(\mathrm{E}_B^{\Phi\bullet}(N, \bar{T}^{\bullet}), \mathrm{E}_B^{\Phi\bullet}(N, \bar{T}^{\bullet})[i]) = 0$$

for all non-zero integers $i$.

By definition, the triangle functor $\mathrm{E}_B^{\Phi\bullet}(N, -) : \mathscr{K}^{\mathrm{b}}(\operatorname{add}(_B N)) \longrightarrow \mathscr{K}^{\mathrm{b}}(\mathrm{E}_B^{\Phi}(N)\text{-proj})$ sends $N$ to $\mathrm{E}_B^{\Phi}(N)$. The full triangulated subcategory of $\mathscr{K}^{\mathrm{b}}(\operatorname{add}(_B N))$ generated by $\operatorname{add}(\bar{T}^{\bullet})$ contains $N$ by Lemma 3.6 (2), and so $\mathrm{E}_B^{\Phi}(N)$ is in the full triangulated subcategory of $\mathscr{K}^{\mathrm{b}}(\mathrm{E}_B^{\Phi}(N)\text{-proj})$ generated by $\operatorname{add}(\mathrm{E}_B^{\Phi\bullet}(N, \bar{T}^{\bullet}))$. Hence $\operatorname{add}(\mathrm{E}_B^{\Phi\bullet}(N, \bar{T}^{\bullet}))$ generates $\mathscr{K}^{\mathrm{b}}(\mathrm{E}_B^{\Phi}(N)\text{-proj})$ as a triangulated category. This finishes the proof. $\square$

In the following, we shall prove that the endomorphism algebra of the complex $\mathrm{E}_B^{\Phi\bullet}(N, \bar{T}^{\bullet})$ is isomorphic to $\mathrm{E}_A^{\Phi}(M)$. For this purpose, we first prove the following lemma.

**Lemma 3.8.** *Keeping the notations above, for each A-module V, we have:*
  (1) *For each positive integer k, there is an isomorphism*

$$\theta_k : \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(V,V[k]) \longrightarrow \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(B)}(\bar{F}(V),\bar{F}(V)[k]).$$

*Here we denote the image of g under $\theta_k$ by $\theta_k(g)$.*
  (2) *For each pair of positive integers k and l, the $\theta_k$ and $\theta_l$ in* (1) *satisfy*

$$\theta_k(g)(\theta_l(h)[k]) = \theta_{k+l}(g(h[k]))$$

*for all $g \in \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(V,V[k])$ and $h \in \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(V,V[l])$.*

*Proof.* By Lemma 3.3, we may assume that $F(V)$ is the complex $\bar{Q}_V^\bullet$ defined in Lemma 3.3 (1), and therefore $\bar{F}(V) = \bar{Q}_V^0$. As before, the complex $\sigma_{>0}\bar{Q}_V^\bullet$ is denoted by $\bar{Q}_V^+$. Thus, we have a distinguished triangle in $\mathscr{D}^{\mathrm{b}}(B)$:

$$\bar{Q}_V^+ \xrightarrow{\; i_V \;} F(V) \xrightarrow{\; \pi_V \;} \bar{F}(V) \xrightarrow{\; \alpha_V \;} \bar{Q}_V^+[1].$$

(1) For a morphism $f : V \longrightarrow V[k]$, we can form the following commutative diagram in $\mathscr{D}^{\mathrm{b}}(B)$

$$
\begin{array}{ccccccc}
\bar{Q}_V^+ & \xrightarrow{\; i_V \;} & F(V) & \xrightarrow{\; \pi_V \;} & \bar{F}(V) & \xrightarrow{\; \alpha_V \;} & \bar{Q}_V^+[1] \\
\downarrow{\scriptstyle a_f} & & \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle b_f} & & \downarrow{\scriptstyle a_f[1]} \\
\bar{Q}_V^+[k] & \xrightarrow{\; i_V[k] \;} & F(V)[k] & \xrightarrow{\; \pi_V[k] \;} & \bar{F}(V)[k] & \xrightarrow{\; \alpha_V[k] \;} & \bar{Q}_V^+[k+1].
\end{array}
$$

The map $b_f$ exists because the composition $i_V F(f)(\pi_V[k])$ belongs to $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(B)}(\bar{Q}_V^+,\bar{F}(V)[k]) = 0$. If there is another map $b_f' : \bar{F}(V) \longrightarrow \bar{F}(V)[k]$ such that $\pi_V b_f' = F(f)(\pi_V[k])$, then $\pi_V(b_f - b_f') = 0$, and $b_f - b_f'$ factorizes through $\bar{Q}_V^+$. But $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(B)}(\bar{Q}_V^+[1],\bar{F}(V)[k]) \simeq \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{Q}_V^+[1],\bar{F}(V)[k]) = 0$. Hence $b_f = b_f'$, that is, the map $b_f$ is uniquely determined by the above commutative diagram. Thus, we can define a morphism

$$\theta_k : \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(V,V[k]) \longrightarrow \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(B)}(\bar{F}(V),\bar{F}(V)[k])$$

by sending $f$ to $b_f$. We claim that this $\theta_k$ is an isomorphism.

In fact, it is surjective: For each map $b : \bar{F}(V) \longrightarrow \bar{F}(V)[k]$, the composition $\pi_V b(\alpha_V[k])$ belongs to $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(B)}(F(V),\bar{Q}_V^+[k+1]) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(GF(V),G(\bar{Q}_V^+)[k+1])$. By Lemma 3.3 (5), the complex $G(\bar{Q}_V^+)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(A)$ to a bounded complex $P_V^\bullet$ of projective-injective A-modules such that $P_V^i = 0$ for all $i > 1$. Hence $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(GF(V),G(\bar{Q}_V^+)[k+1]) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(V,P_V^\bullet[k+1]) = 0$, and the map $\pi_V b(\alpha_V[k])$ is zero. It follows that there is a morphism $u : F(V) \longrightarrow F(V)[k]$ such that $u(\pi_V[k]) = \pi_V b$. Since $F$ is an equivalence, we have $u = F(f)$ for some $f : V \longrightarrow V[k]$, and consequently $b = \theta_k(f)$. This shows that $\theta_k$ is a surjective map.

Now we show that $\theta_k$ is injective: Assume that $\theta_k(f) = b_f = 0$. Then the composition $F(f)(\pi_V[k]) = 0$, and consequently $F(f)$ factorizes through $\bar{Q}_V^+[k]$. It follows that $GF(f)$ factorizes through $G(\bar{Q}_V^+)[k] \simeq P_V^\bullet[k]$, or equivalently, the map $f : V \longrightarrow V[k]$ factorizes through the bounded complex $P_V^\bullet$ of projective-injective A-modules, say $f = gh$ for some $g : V \longrightarrow P_V^\bullet$ and $h : P_V^\bullet \longrightarrow V[k]$. Since $k > 0$, and since both $g$ and $h$ can be chosen to be chain maps, we see immediately that $f = gh = 0$. This shows that the map $\theta_k$ is injective, and therefore $\theta_k$ is an isomorphism.

(2) By the above discussion, we have

$$
\begin{aligned}
\pi_V \theta_k(g)(\theta_l(g)[k]) &= (F(g)(\pi_V[k]))(\theta_l(h)[k]) \\
&= F(g)((\pi_V \theta_l(h))[k]) \\
&= F(g)((F(h)(\pi_V[l]))[k]) \\
&= (F(g)F(h[k]))(\pi_V[k+l]) \\
&= F(g(h[k]))(\pi_V[k+l]).
\end{aligned}
$$

By the definition of $\theta_{k+l}$, we have $\theta_{k+l}(g(h[k])) = \theta_k(g)(\theta_l(g)[k])$. $\square$

**Remark:** Let $f$ be in $\mathrm{Hom}_{\mathscr{D}^b(A)}(V,V)$, and let $g$ be in $\mathrm{Hom}_{\mathscr{D}^b(A)}(V,V[k])$ for some $k > 0$. If $t_f : \bar{F}(V) \longrightarrow \bar{F}(V)$ is a morphism such that $\pi_V t_f = F(f)\pi_V$, then, by a proof similar to Lemma 3.8(3), we have

$$t_f \theta_k(g) = \theta_k(fg) \qquad \text{and} \qquad \theta_k(g)(t_f[k]) = \theta_k(g(f[k])).$$

For instance, by Lemma 2.1, we can assume that the map $F(f) : \bar{Q}_V^\bullet \longrightarrow \bar{Q}_V^\bullet$ is induced by a chain map $p^\bullet$, that is, $F(f) = p^\bullet$ in $\mathscr{D}^b(B)$. Since the map $\pi_V$ is the canonical map from $\bar{Q}_V^\bullet$ to $\bar{Q}_V^0$, we see that the map $p^0 : \bar{F}(V) \longrightarrow \bar{F}(V)$ satisfies the condition $\pi_V p^0 = F(f)\pi_V$. Therefore, by the above discussion, we have

$$p^0 \theta_k(g) = \theta_k(fg) \text{ and } \theta_k(g)(p^0[k]) = \theta_k(g(f[k])).$$

**Proposition 3.9.** $\mathrm{End}_{\mathscr{K}^b(\mathrm{E}_B^\Phi(N)\text{-proj})}(\mathrm{E}_B^{\Phi\bullet}(N,\bar{T}^\bullet))$ *is isomorphic to* $\mathrm{E}_A^\Phi(M)$.

*Proof.* Let $(f_i)$ be in $\mathrm{E}_A^\Phi(M)$. By our assumption, we have $\bar{T}^\bullet = F(M)$. By Lemma 2.1, the morphism $F(f_0) : \bar{T}^\bullet \longrightarrow \bar{T}^\bullet$ is equal in $\mathscr{D}^b(B)$ to a chain map. For simplicity, we shall assume that $F(f_0)$ is a chain map. Recall that $\bar{F}(M) = \bar{T}^0$ by the definition of $\bar{F}$ (see Lemma 3.3 (3) ).

Now we set $\Phi^+ := \Phi \backslash \{0\}$. For each $k \in \Phi^+$, by Lemma 3.8, we have a map $\theta_k(f_k) : \bar{F}(M) \longrightarrow \bar{F}(M)[k]$. This gives rise to a morphism

$$\mu\big(\iota_k(\theta_k(f_k))\big) : \mathrm{E}_B^\Phi(N,\bar{T}^0) \longrightarrow \mathrm{E}_B^\Phi(N,\bar{T}^0),$$

where $\mu$ is the isomorphism defined in Lemma 3.5 (1) and $\iota_k$ is the embedding from $\mathrm{Hom}_{\mathscr{D}^b(B)}(\bar{T}^0,\bar{T}^0[k])$ to $\mathrm{E}_B^\Phi(\bar{T}^0,\bar{T}^0)$. We claim that the composition of $\mu\big(\iota_k(\theta_k(f_k))\big)$ with the differential $\mathrm{E}_B^\Phi(N,d) : \mathrm{E}_B^\Phi(N,\bar{T}^0) \longrightarrow \mathrm{E}_B^\Phi(N,\bar{T}^1)$ is zero.

Indeed, by the proof of Lemma 3.5 (2), we have $\mathrm{E}_B^\Phi(N,d) = \mu(\iota_0(d))$. Thus,

$$
\begin{aligned}
\mu\big(\iota_k(\theta_k(f_k))\big)\mathrm{E}_B^\Phi(N,d) &= \mu\big(\iota_k(\theta_k(f_k))\big)\mu(\iota_0(d)) && (\text{ by the proof of Lemma 3.5 (2) }) \\
&= \mu\big(\iota_k(\theta_k(f_k))\iota_0(d)\big) && (\text{ by Lemma 3.5(1) }) \\
&= \mu\big(\iota_k(\theta_k(f_k)d[k])\big) = 0 && (\text{ since } \theta_k(f_k)d[k] : \bar{T}^0 \longrightarrow \bar{T}^1[k] \text{ must be zero }).
\end{aligned}
$$

Thus, the map $\mu\big(\iota_k(\theta_k(f_k))\big)$ gives rise to an endomorphism of $\mathrm{E}_B^{\Phi\bullet}(N,\bar{T}^\bullet)$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{E}_B^\Phi(N,\bar{T}^0) & \xrightarrow{\mathrm{E}_B^\Phi(N,d)} & \mathrm{E}_B^\Phi(N,\bar{T}^1) & \longrightarrow & \cdots & \longrightarrow & \mathrm{E}_B^\Phi(N,\bar{T}^n) & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle\mu(\iota_k(\theta_k(f_k)))} & & \Big\downarrow{\scriptstyle 0} & & & & \Big\downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathrm{E}_B^\Phi(N,\bar{T}^0) & \xrightarrow{\mathrm{E}_B^\Phi(N,d)} & \mathrm{E}_B^\Phi(N,\bar{T}^1) & \longrightarrow & \cdots & \longrightarrow & \mathrm{E}_B^\Phi(N,\bar{T}^n) & \longrightarrow & 0.
\end{array}
$$

We denote this endomorphism by $\tilde{\theta}_k(f_k)$. Now, we define a map

$$\eta : \mathrm{E}_A^\Phi(M) \longrightarrow \mathrm{End}_{\mathscr{K}^b(\mathrm{E}_B^\Phi(N)\text{-proj})}(\mathrm{E}_B^{\Phi\bullet}(N,\bar{T}^\bullet))$$

by sending $(f_i)$ to

$$\mathrm{E}_B^{\Phi\bullet}(N,F(f_0)) + \sum_{k \in \Phi^+} \tilde{\theta}_k(f_k).$$

We claim that $\eta$ is an algebra homomorphism. This will be shown with help of the next lemma.

**Lemma 3.10.** *Let $(f_i)$ and $(g_i)$ be in $\mathrm{E}_A^{\Phi}(M)$, and let $k, l$ be in $\Phi^+$. Then the following hold:*

(1) $\tilde{\theta}_k(f_k)\tilde{\theta}_l(g_l) = \begin{cases} \tilde{\theta}_{k+l}(f_k(g_l[k])), & k+l \in \Phi; \\ 0, & k+l \notin \Phi. \end{cases}$

(2) $\mathrm{E}_B^{\Phi\bullet}(N, F(f_0))\tilde{\theta}_k(g_k) = \tilde{\theta}_k(f_0 g_k)$.

(3) $\tilde{\theta}_k(f_k)\mathrm{E}_B^{\Phi\bullet}(N, F(g_0)) = \tilde{\theta}_k(f_k(g_0[k]))$.

*Proof.* (1). By Lemma 3.8 (2), we have

$$\iota_k\big(\theta_k(f_k)\big)\iota_l\big(\theta_l(f_l)\big) = \iota_{k+l}\big(\theta_{k+1}(f_k(g_l[k]))\big).$$

If $k + l \in \Phi$, then it follows that $\tilde{\theta}_k(f_k)\tilde{\theta}_l(f_l) = \tilde{\theta}_{k+l}(f_k(g_l[k]))$ by applying $\mu$. If $k + l \notin \Phi$, then $\iota_{k+l} = 0$, and consequently $\iota_k\big(\theta_k(f_k)\big)\iota_l\big(\theta_l(f_l)\big) = 0$. Therefore $\tilde{\theta}_k(f_k)\tilde{\theta}_l(f_l) = 0$ for $k + l \notin \Phi$.

(2) and (3). By definition, the map $\mathrm{E}_B^{\Phi\bullet}(N, F(f_0))^0 : \mathrm{E}_B^{\Phi}(N, \bar{T}^0) \longrightarrow \mathrm{E}_B^{\Phi}(N, \bar{T}^0)$ is $\mathrm{E}_B^{\Phi}(N, F(f_0)^0) = \mu(\iota_0(F(f_0)^0))$, where $F(f_0)^0 : \bar{T}^0 \longrightarrow \bar{T}^0$ is induced by the chain map $F(f_0)$ from $\bar{T}^\bullet$ to $\bar{T}^\bullet$. By the remark just before Lemma 3.9, we have

$$\iota_0(F(f_0)^0)\iota_k\big(\theta_k(g_k)\big) = \iota_k\big(\theta_k(f_0 g_k)\big) \text{ and } \iota_k\big(\theta_k(f_k)\big)\iota_0(F(g_0)^0) = \iota_k\big(\theta_k(f_k(g_0[k]))\big).$$

Applying $\mu$ to these equalities, one can easily see that

$$\mathrm{E}_B^{\Phi\bullet}(N, F(f_0))\tilde{\theta}_k(g_k) = \tilde{\theta}_k(f_0 g_k) \text{ and } \tilde{\theta}_k(f_k)\mathrm{E}_B^{\Phi\bullet}(N, F(g_0)) = \tilde{\theta}_k(f_k(g_0[k])).$$

These are precisely the (2) and (3). $\square$

Now, we continue the proof of Lemma 3.9: With Lemma 3.10 in hand, it is straightforward to check that $\eta$ is an algebra homomorphism. In the following we first show that $\eta$ is injective.

Pick an $(f_i)$ in $\mathrm{E}_A^{\Phi}(M)$, let $p^\bullet := \eta((f_i))$. Then we have

$$p^0 = \mathrm{E}_B^{\Phi}(N, F(f_0)^0) + \sum_{k \in \Phi^+} \mu\big(\iota_k(\theta_k(f_k))\big)$$

and $p^i = \mathrm{E}_B^{\Phi}(N, F(f_0)^i)$ for all $i > 0$. If $p^\bullet = 0$, then there is map $h^i : \mathrm{E}_B^{\Phi}(N, \bar{T}^i) \longrightarrow \mathrm{E}_B^{\Phi}(N, \bar{T}^{i-1})$ for $i > 0$ such that $p^0 = \mathrm{E}_B^{\Phi}(N, d)h^1$ and $p^i = \mathrm{E}_B^{\Phi}(N, d)h^{i+1} + h^i \mathrm{E}_B^{\Phi}(N, d)$ for all $i > 0$. Since $\bar{T}^i$ is projective-injective for all $i > 0$, it follows from Lemma 3.5 (3) that, for each $i > 0$, we have $h^i = \mathrm{E}_B^{\Phi}(N, u^i)$ for some $u^i : \bar{T}^i \longrightarrow \bar{T}^{i-1}$. Hence

$$\mathrm{E}_B^{\Phi}(N, F(f_0)^0) + \sum_{k \in \Phi^+} \mu\big(\iota_k(\theta_k(f_k))\big) = \mathrm{E}_B^{\Phi}(N, d)\mathrm{E}_B^{\Phi}(N, u^1) = \mathrm{E}_B^{\Phi}(N, du^1).$$

This yields that

$$\mu\big(\iota_0(F(f_0)^0 - du^1)\big) = \mathrm{E}_B^{\Phi}(N, F(f_0)^0 - du^1) = \sum_{k \in \Phi^+} \mu\big(\iota_k(\theta_k(f_k))\big).$$

Since $\mu$ is an isomorphism, and since $\mathrm{E}_B^{\Phi}(N, \bar{T}^0) = \bigoplus_{k \in \Phi} \mathrm{Hom}_{\mathscr{D}^b(B)}(N, \bar{T}^0[k])$ is a direct sum, we get $F(f_0)^0 = du^1$ and $\theta_k(f_k) = 0$ for all $k \in \Phi^+$. Since $\theta_k$ is an isomorphism by Lemma 3.8, we have $f_k = 0$ for all $k \in \Phi^+$. Now for each $i > 0$, we have

$$\mathrm{E}_B^{\Phi}(N, F(f_0)^i) = p^i = \mathrm{E}_B^{\Phi}(N, d)\mathrm{E}_B^{\Phi}(N, u^{i+1}) + \mathrm{E}_B^{\Phi}(N, u^i)\mathrm{E}_B^{\Phi}(N, d).$$

Hence $\mathrm{E}_B^{\Phi}(N, F(f_0)^i - du^{i+1} - u^i d) = 0$. By Lemma 3.5 (2), the functor $\mathrm{E}_B^{\Phi}(N, -)$ is faithful. Therefore, we get $F(f_0)^i = du^{i+1} + u^i d$ for $i > 0$. Note that we have shown that $F(f_0)^0 = du^1$. Hence

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the morphism $F(f_0)$ is null-homotopic, that is, $F(f_0) = 0$, and therefore $f_0 = 0$. Altogether, we get $(f_i) = 0$. This shows that $\eta$ is injective.

Finally, we show that $\eta$ is surjective. For $p^\bullet$ in $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(\mathrm{E}_B^\Phi(N)\text{-proj})}(\mathrm{E}_B^{\Phi\bullet}(N, \bar{T}^\bullet))$, we can assume that $p^i = \mathrm{E}_B^\Phi(N, t_i)$ with $t_i : \bar{T}^i \longrightarrow \bar{T}^i$ for $i > 0$ since $\bar{T}^i$ is projective-injective for $i > 0$. By Lemma 3.5 (1), we may assume further that $p^0 = \mu\left(\sum_{k \in \Phi} \iota_k(s_k)\right)$ with $s_k : \bar{T}^0 \longrightarrow \bar{T}^0[k]$ for $k \in \Phi$. By the proof of Lemma 3.5 (3), we have $\mu\left(\iota_0(s_0)\right) = \mathrm{E}_B^\Phi(N, s_0)$. Thus, $p^0 = \mathrm{E}_B^\Phi(N, s_0) + \sum_{k \in \Phi^+} \mu\left(\iota_k(s_k)\right)$. It follows from $\mathrm{E}_B^\Phi(N, d)p^1 = p^0 \mathrm{E}_B^\Phi(N, d)$ that

$$
\begin{aligned}
\mathrm{E}_B^\Phi(N, dt_1) &= \mathrm{E}_B^\Phi(N, s_0 d) + \sum_{k \in \Phi^+} \mu\left(\iota_k(s_k)\right)\mu\left(\iota_0(d)\right) \\
&= \mathrm{E}_B^\Phi(N, s_0 d) + \sum_{k \in \Phi^+} \mu\left(\iota_k(s_k(d[k]))\right) \\
&= \mathrm{E}_B^\Phi(N, s_0 d) \quad (\text{because } s_k(d[k]) : \bar{T}^0 \longrightarrow \bar{T}^1[k] \text{ must be zero for } k > 0).
\end{aligned}
$$

Hence $dt_1 = s_0 d$ since $\mathrm{E}_B^\Phi(N, -)$ is a faithful on $\mathrm{add}(N)$. For each $i > 0$, by the fact $\mathrm{E}_B^\Phi(N, d)p^{i+1} = p^i \mathrm{E}_B^\Phi(N, d)$, we get $dt_{i+1} = t_i d$. This gives rise to a morphism $\alpha^\bullet$ in $\mathrm{End}_{\mathscr{K}^{\mathrm{b}}(B)}(\bar{T}^\bullet)$ by defining $\alpha^0 := s_0$ and $\alpha^i := t_i$ for all $i > 0$. By Lemma 2.1 and the fact that $F$ is an equivalence, we conclude that $\alpha^\bullet = F(f_0)$ for some $f_0 \in \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(M, M)$. The map $p^\bullet - \mathrm{E}_B^{\Phi\bullet}(N, \alpha^\bullet)$ is a chain map $\beta^\bullet$ from $\mathrm{E}_B^{\Phi\bullet}(N, \bar{T}^\bullet)$ to itself with $\beta^0 = \sum_{k \in \Phi^+} \mu\left(\iota_k(s_k)\right)$ and $\beta^k = 0$ for all $k > 0$. By Lemma 3.8, we can write $s_k = \theta_k(f_k)$ with $f_k : M \longrightarrow M[k]$ for all $k \in \Phi^+$. Thus $\beta^0 = \sum_{k \in \Phi^+} \mu\left(\iota_k(s_k)\right) = \sum_{k \in \Phi^+} \mu\left(\iota_k(\theta_k(f_k))\right)$, and $p^\bullet - \mathrm{E}_B^{\Phi\bullet}(N, \alpha^\bullet) = \sum_{k \in \Phi^+} \tilde{\theta}_k(f_k)$. Consequently, we get

$$
p^\bullet = \mathrm{E}_B^{\Phi\bullet}(N, \alpha^\bullet) + \sum_{k \in \Phi^+} \tilde{\theta}_k(f_k) = \mathrm{E}_B^{\Phi\bullet}(N, F(f_0)) + \sum_{k \in \Phi^+} \tilde{\theta}_k(f_k) = \eta\left((f_i)\right)
$$

for $(f_i) \in \mathrm{E}_A^\Phi(M)$. Hence $\eta$ is surjective. This finishes the proof of Lemma 3.9. $\square$

**Lemma 3.11.** *Let $F : \mathscr{D}^{\mathrm{b}}(\Lambda) \longrightarrow \mathscr{D}^{\mathrm{b}}(\Gamma)$ be a derived equivalence between Artin $R$-algebras $\Lambda$ and $\Gamma$, and let $P^\bullet$ be a tilting complex associated to $F$. Suppose that the following two conditions are satisfied.*

*(1) All the terms of $P^\bullet$ in negative degrees are zero, and all the terms of $P^\bullet$ in positive degrees are in $\mathrm{add}(_\Lambda W)$ for some projective $\Lambda$-module $_\Lambda W$ with $\mathrm{add}(\nu_\Lambda W) = \mathrm{add}(_\Lambda W)$.*

*(2) For the module $_\Lambda W$ in (1), the complex $F(_\Lambda W)$ is isomorphic to a complex in $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(_\Gamma V))$ for some projective $\Gamma$-module $_\Gamma V$ with $\mathrm{add}(\nu_\Gamma V) = \mathrm{add}(_\Gamma V)$.*

*Then the quasi-inverse of $F$ is an almost $\nu$-stable derived equivalence.*

*Proof.* Let $G$ be a quasi-inverse of $F$. By the definition of almost $\nu$-stable equivalences, we need to consider the tilting complex associated to $G$. This is equivalent to considering $F(\Lambda)$.

Since $P^\bullet$ is a tilting complex over $\Lambda$, it is well-known that $_\Lambda \Lambda$ is in $\mathrm{add}(\bigoplus_{i \in \mathbb{Z}} P^i)$ which is contained in $\mathrm{add}(P^0 \oplus W)$ by the assumption (1). Hence $F(_\Lambda \Lambda)$ is in $\mathrm{add}(F(P^0 \oplus W))$. Let $P^+$ be the complex $\sigma_{>0} P^\bullet$. There is a distinguished triangle

$$
P^+ \longrightarrow P^\bullet \longrightarrow P^0 \longrightarrow P^+[1]
$$

in $\mathscr{D}^{\mathrm{b}}(\Lambda)$. Applying $F$, we get a distinguished triangle

$$
F(P^+) \longrightarrow F(P^\bullet) \longrightarrow F(P^0) \longrightarrow F(P^+)[1]
$$

in $\mathscr{D}^{\mathrm{b}}(\Gamma)$. By definition, there is an isomorphism $F(P^\bullet) \simeq \Gamma$ in $\mathscr{D}^{\mathrm{b}}(\Gamma)$. By the assumption (1), we have $P^+ \in \mathscr{K}^{\mathrm{b}}(\mathrm{add}(_\Lambda W))$, and consequently $F(P^+)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(\Gamma)$ to a complex $R^\bullet$ in $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(_\Gamma V))$ by Assumption (2). Thus, the complex $F(P^0)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(\Gamma)$ to the mapping

cone of a chain map from $R^\bullet$ to $_\Gamma\Gamma$. This implies that $F(P^0)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(\Gamma)$ to a complex $S^\bullet$ in $\mathscr{K}^{\mathrm{b}}(\Gamma\text{-proj})$ such that $S^i \in \mathrm{add}(_\Gamma V)$ for all $i \neq 0$. By the assumption (2) again, the complex $F(_\Lambda W)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(\Gamma)$ to a complex in $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(_\Gamma V))$. Hence $F(P^0) \oplus F(_\Lambda W)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(\Gamma)$ to a complex $U^\bullet$ in $\mathscr{K}^{\mathrm{b}}(\Gamma\text{-proj})$ such that $U^i \in \mathrm{add}(_\Gamma V)$ for all $i \neq 0$. Note that $F(\Lambda) \in \mathrm{add}(F(P^0) \oplus F(_\Lambda W))$. Therefore, the complex $F(\Lambda)$ is isomorphic in $\mathscr{D}^{\mathrm{b}}(\Gamma)$ to a complex $\bar{P}^\bullet$ in $\mathscr{K}^{\mathrm{b}}(\Gamma\text{-proj})$ such that $\bar{P}^i \in \mathrm{add}(_\Gamma V)$ for all $i \neq 0$. Since $P^i = 0$ for all $i < 0$, we see from [6, Lemma 2.1] that $\bar{P}^\bullet$ has zero homology in all positive degrees. Hence we can assume that $\bar{P}^i = 0$ for all $i > 0$.

Thus, the complex $\bar{P}^\bullet \simeq F(\Lambda)$ is a tilting complex associated to $G$ and satisfies that $\bar{P}^i = 0$ for all $i > 0$ and $\bar{P}^i \in \mathrm{add}(_\Gamma V)$ for all $i < 0$. The complex $P^\bullet$ is a tilting complex associated to $F$ and satisfies that $P^i = 0$ for all $i < 0$ and $P^i \in \mathrm{add}(_\Lambda W)$ for all $i > 0$. Since $\mathrm{add}(\nu_\Lambda W) = \mathrm{add}(_\Lambda W)$, and since $\mathrm{add}(\nu_\Gamma V) = \mathrm{add}(_\Gamma V)$, it follows from [6, Proposition 3.8 (3)] that the functor $G$ is an almost $\nu$-stable derived equivalence. $\square$

Now we prove our main result, Theorem 3.4, in this section.

**Proof of Theorem 3.4**. The statement (1) follows from Lemma 3.7, Proposition 3.9 and Lemma 2.2. It remains to prove statement (2). Now we suppose that $\Phi$ is finite. Then $\mathrm{E}_A^\Phi(M)$ and $\mathrm{E}_B^\Phi(N)$ are Artin $R$-algebras.

Let $_A E$ be a maximal $\nu$-stable $A$-module, and let $_B \bar{E}$ be a maximal $\nu$-stable $B$-module. Then $_A E$ can be viewed as a direct summand of $_A M$. Let $\bar{Q}_E^\bullet$ be $F(_A E)$ defined in Lemma 3.3 (1). Then $\bar{Q}_E^\bullet$ is a direct summand of $\bar{Q}_M^\bullet = \bar{Q}^\bullet \oplus \bar{Q}_X^\bullet$. Note that $\bar{Q}_M^\bullet$ is just the complex $\bar{T}^\bullet$ considered in Proposition 3.9. Now we consider the isomorphism $\eta$ in the proof of Proposition 3.9. Let $e$ be the idempotent in $\mathrm{End}_A(M)$ corresponding to the direct summand $_A E$. Then $\iota_0(e)$ is the idempotent in $\mathrm{E}_A^\Phi(M)$ corresponding to the direct summand $\mathrm{E}_A^\Phi(M,E)$ of $\mathrm{E}_A^\Phi(M)$. By definition, $\eta(\iota_0(e))$ is $\mathrm{E}_B^{\Phi\bullet}(N, F(e))$, which is the idempotent in $\mathrm{End}_{\mathrm{E}_B^\Phi(N)\text{-proj}}(\bar{T}^\bullet)$ corresponding to $\mathrm{E}_B^{\Phi\bullet}(N, \bar{Q}_E^\bullet)$. Hence the derived equivalence $\hat{F} : \mathscr{D}^{\mathrm{b}}(\mathrm{E}_A^\Phi(M)) \longrightarrow \mathscr{D}^{\mathrm{b}}(\mathrm{E}_B^\Phi(N))$ induces by the isomorphism $\eta$ in the proof of Proposition 3.9 sends $\mathrm{E}_A^\Phi(M,E)$ to $\mathrm{E}_B^{\Phi\bullet}(N, \bar{Q}_E^\bullet)$. By [6, Lemma 3.9], the functor $F$ induces an equivalence between the triangulated categories $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(_A E))$ and $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(_B \bar{E}))$. Hence $\bar{Q}_E^\bullet = F(_A E)$ is in $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(_B \bar{E}))$, the complex $\mathrm{E}_B^{\Phi\bullet}(N, \bar{Q}_E^\bullet)$ belongs to $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(\mathrm{E}_B^\Phi(N, \bar{E})))$ and consequently $\hat{F}$ induces a full, faithful triangle functor

$$\hat{F} : \mathscr{K}^{\mathrm{b}}(\mathrm{add}(\mathrm{E}_A^\Phi(M,E))) \longrightarrow \mathscr{K}^{\mathrm{b}}(\mathrm{add}(\mathrm{E}_B^\Phi(N,\bar{E}))).$$

Since $\mathrm{add}(_A E)$ clearly generates $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(_A E))$ as a triangulated category, we see immediately that $\mathrm{add}(\bar{Q}_E^\bullet)$ generates $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(_B \bar{E}))$ as a triangulated category. This implies that $\mathrm{add}(\mathrm{E}_B^{\Phi\bullet}(N, \bar{Q}_E^\bullet))$ generates $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(\mathrm{E}_B^\Phi(N,\bar{E})))$ as a triangulated category. This shows that

$$\hat{F} : \mathscr{K}^{\mathrm{b}}(\mathrm{add}(\mathrm{E}_A^\Phi(M,E))) \longrightarrow \mathscr{K}^{\mathrm{b}}(\mathrm{add}(\mathrm{E}_B^\Phi(N,\bar{E})))$$

is dense, and therefore an equivalence. Let $\hat{G}$ be a quasi-inverse of the derived equivalence $\hat{F}$. Then the functor $\hat{G}$ also induces an equivalence between the triangulated categories $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(\mathrm{E}_B^\Phi(N,\bar{E})))$ and $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(\mathrm{E}_A^\Phi(M,E)))$. This implies that the complex $\hat{G}(\mathrm{E}_B^\Phi(N,\bar{E}))$ is isomorphic to a complex in $\mathscr{K}^{\mathrm{b}}(\mathrm{add}(\mathrm{E}_A^\Phi(M,E)))$.

Now we use Lemma 3.11 to complete the proof. In fact, the complex $\mathrm{E}_B^{\Phi\bullet}(N,\bar{T}^\bullet)$ is a tilting complex associated to the derived equivalence $\hat{G} : \mathscr{D}^{\mathrm{b}}(\mathrm{E}_B^\Phi(N)) \longrightarrow \mathscr{D}^{\mathrm{b}}(\mathrm{E}_A^\Phi(M))$. By definition, the $B$-module $\bar{Q}$ is in $\mathrm{add}(_B \bar{E})$. Thus, the term $\mathrm{E}_B^\Phi(N,\bar{T}^i)$ of $\mathrm{E}_B^{\Phi\bullet}(N,\bar{T}^\bullet)$ in degree $i$ is in $\mathrm{add}(\mathrm{E}_B^\Phi(N,\bar{E}))$ for all $i > 0$, and it follows from Lemma 3.5 (4) that

$$\mathrm{add}(\nu_{\mathrm{E}_B^\Phi(N)}\mathrm{E}_B^\Phi(N,\bar{E})) = \mathrm{add}(_{\mathrm{E}_B^\Phi(N)}\mathrm{E}_B^\Phi(N,\nu_B \bar{E})) = \mathrm{add}(_{\mathrm{E}_B^\Phi(N)}\mathrm{E}_B^\Phi(N,\bar{E})).$$

Similarly, we have $\mathrm{add}(\nu_{\mathrm{E}_A^\Phi(M)}\mathrm{E}_A^\Phi(M,E)) = \mathrm{add}(_{\mathrm{E}_A^\Phi(M)}\mathrm{E}_A^\Phi(M,E))$. Hence, by Lemma 3.11, the functor $\hat{F}$ is an almost $\nu$-stable derived equivalence.

The statements on stable equivalence in Theorem 3.4 follow from [6, Theorem 1.1]. This finishes the proof. □

Note that the proof of Theorem 3.4 (2) shows also that if both $E_A^\Phi(M)$ and $E_B^\Phi(N)$ are Artin $R$-algebras, then the conclusion of Theorem 3.4 (2) is valid.

Let us remark that, in case of finite-dimensional algebras over a field, the special case for $\Phi = \Phi(1,0)$ in Theorem 3.4 about stable equivalence was proved in [6, Proposition 6.1] by using two-sided tilting complexes, and the conclusion there guarantees a stable equivalence of Morita type. But the proof there in [6] does not work here any more, since we do not have two-sided tilting complexes in general for Artin algebras.

As a consequence of Theorem 3.4, we have the following corollary.

**Corollary 3.12.** *Let $F : \mathscr{D}^b(A) \longrightarrow \mathscr{D}^b(B)$ be a derived equivalence between self-injective Artin algebras $A$ and $B$, and let $\phi$ be the stable equivalence induced by $F$. Then, for each $A$-module $X$ and each admissible subset $\Phi$ of $\mathbb{N}$, the $\Phi$-Auslander-Yoneda algebras $E_A^\Phi(A \oplus X)$ and $E_B^\Phi(B \oplus \phi(X))$ are derived-equivalent. Particularly, the generalized Yoneda algebras $\mathrm{Ext}_A^*(A \oplus X)$ and $\mathrm{Ext}_B^*(B \oplus \phi(X))$ are derived-equivalent. Moreover, if $\Phi$ is finite, then $E_A^\Phi(A \oplus X)$ and $E_B^\Phi(B \oplus \phi(X))$ are stably equivalent.*

*Proof.* There is an integer $i$ such that $F[i]$ is an almost $\nu$-stable derived equivalence. Let $\phi_1$ be the stable equivalence induced by $F[i]$. Then $\phi(X) \simeq \phi_1 \Omega^i(X)$ in $B$-<u>mod</u> for every $A$-module $X$, where $\Omega^i$ is the $i$-th syzygy operator of $A$. By the definition of an almost $\nu$-stable derived equivalence, either $[i]$ or $[-i]$ is almost $\nu$-stable. Hence $E^\Phi(A \oplus X)$ and $E_A^\Phi(A \oplus \Omega^i(X))$ are derived-equivalent by Theorem 3.4. Thus, by Theorem 3.4 again, the algebras $E_A^\Phi(A \oplus \Omega^i(X))$ and $E_B^\Phi(B \oplus \phi_1 \Omega^i(X))$ are derived-equivalent. The stable equivalence follows from [6, Theorem 1.1]. Thus the proof is completed. □

As a direct consequence of Corollary 3.12, we have the following corollary concerning Auslander algebras.

**Corollary 3.13.** *Suppose that $A$ and $B$ are self-injective Artin algebras of finite representation type. If $A$ and $B$ are derived-equivalent, then the Auslander algebras of $A$ and $B$ are both derived and stably equivalent.*

Let us remark that the notion of a stable equivalence of Morita type for finite-dimensional algebras can be formulated for Artin $R$-algebras. But, in this case, we do not know if a stable equivalence of Morita type between Artin algebras induces a stable equivalence since we do not know whether a projective $A$-$A$-bimodule is projective as a one-sided module when the ground ring is a commutative Artin ring. So, Theorem 3.4 (2), Corollary 3.12 (1) and Corollary 3.13 ensure a stable equivalence between the endomorphism algebras of generators over Artin algebras, while the main result in [6, Section 6] ensures a stable equivalence of Morita type between the endomorphism algebras of generators over finite-dimensional algebras.

Note that if $A$ and $B$ are not self-injective, then Corollary 3.13 may fail. For a counterexample, we just check the following two algebras $A$ and $B$, where $A$ is given by the path algebra of the quiver $\circ \to \circ \to \circ$, and $B$ is given by $\circ \xrightarrow{\alpha} \circ \xrightarrow{\beta} \circ$ with the relation $\alpha\beta = 0$. Clearly, $B$ is the endomorphism algebra of a tilting $A$-module. Note that the Auslander algebras of $A$ and $B$ have different numbers of non-isomorphic simple modules, and therefore are never derived-equivalent since derived equivalences preserve the number of non-isomorphic simple modules [7]. Notice that, though $A$ and $B$ are derived-equivalent, there is no almost $\nu$-stable derived equivalence between $A$ and $B$ since $A$ and $B$ are not stably equivalent. This example shows also that Theorem 3.4 may fail if we drop the almost $\nu$-stable condition.

The following question relevant to Corollary 3.13 might be of interest.

**Question.** Let $A$ and $B$ be self-injective Artin algebras of finite representation type with ${}_A X$ and ${}_B Y$ additive generators for $A$-mod and $B$-mod, respectively. Suppose that there is a natural number $i$ such that the algebras $\mathrm{E}_A^{\Phi(1,i)}(X)$ and $\mathrm{E}_B^{\Phi(1,i)}(Y)$ are derived-equivalent. Are $A$ and $B$ derived-equivalent ?

We remark that Asashiba in [1] gave a complete classification of representation-finite self-injective algebras up to derived equivalence.

For a self-injective Artin $R$-algebra $A$, we know that the shift functor $[-1]: \mathscr{D}^{\mathrm{b}}(A) \longrightarrow \mathscr{D}^{\mathrm{b}}(A)$ is an almost $\nu$-stable derived equivalence, and this functor induces a stable functor $\bar{F}: A\text{-}\underline{\mathrm{mod}} \longrightarrow A\text{-}\underline{\mathrm{mod}}$, which is isomorphic to $\Omega_A(-)$, the Heller loop operator. Thus we have the following corollary to Theorem 3.4, which extends [5, Corollary 3.7] in some sense.

**Corollary 3.14.** *Let $A$ be a self-injective Artin algebra. Then, for any admissible subset $\Phi$ of $\mathbb{N}$ and for any $A$-module $X$, we have a derived equivalence between $\mathrm{E}_A^\Phi(A \oplus X)$ and $\mathrm{E}_A^\Phi(A \oplus \Omega_A(X))$. Moreover, if $\Phi$ is finite, then there is an almost $\nu$-stable derived equivalence between $\mathrm{E}_A^\Phi(A \oplus X)$ and $\mathrm{E}_A^\Phi(A \oplus \Omega_A(X))$. Thus they are stably equivalent.*

Let us mention the following consequence of Corollary 3.14.

**Corollary 3.15.** *Let $A$ be a self-injective Artin algebra, and let $J$ be the Jacobson radical of $A$ with the nilpotency index $n$. Then:*

*(1) For any $1 \leq j \leq n-1$ and for any admissible subset $\Phi$ of $\mathbb{N}$, the $\Phi$-Auslander-Yoneda algebras $\mathrm{E}_A^\Phi(A \oplus \bigoplus_{i=1}^{j} A/J^i)$ and $\mathrm{E}_A^\Phi(A \oplus \bigoplus_{i=1}^{j} J^i)$ are derived-equivalent.*

*(2) The global dimension of $\mathrm{End}_A({}_A A \oplus J \oplus J^2 \oplus \cdots \oplus J^{n-1})$ is at most $n$.*

*(3) The global dimension of $\mathrm{End}_A({}_A A \oplus \bigoplus_{i=1}^{n-1} A/\mathrm{soc}^i({}_A A))$ is at most $n$.*

*(4) The global dimension of $\mathrm{End}_A({}_A A \oplus \mathrm{soc}({}_A A) \oplus \cdots \oplus \mathrm{soc}^{n-1}({}_A A))$ is at most $n$.*

*Proof.* Since the syzygy of $\bigoplus_{i=1}^{j} A/J^i$ is $\bigoplus_{i=1}^{j} J^i$ up to a projective summand, we have (1) immediately from Corollary 3.14. The statement (2) follows from [6, Corollary 4.3] together with a result of Auslander, which says that, for any Artin algebra $A$, the global dimension of $\mathrm{End}_A(A \oplus \bigoplus_{i=1}^{n-1} A/J^i)$ is at most $n$.

Since ${}_A A$ is injective, we know that $\mathrm{add}({}_A A) = \mathrm{add}(D(A_A))$. It follows from $D(A_A/J_A^i) \simeq \mathrm{soc}^i(D(A_A))$ that

$$\mathrm{End}_{A^{op}}\big(A_A \oplus \bigoplus_{i=1}^{n-1} A/J_A^i\big) \simeq \Big(\mathrm{End}_A\big(D(A_A \oplus \bigoplus_{i=1}^{n-1} A/J_A^i)\big)\Big)^{op} \simeq \Big(\mathrm{End}_A\big(D(A_A) \oplus \bigoplus_{i=1}^{n-1} \mathrm{soc}^i(D(A_A))\big)\Big)^{op}.$$

The latter is Morita equivalent to $\big(\mathrm{End}_A({}_A A \oplus \bigoplus_{i\geq 1}^{n-1} \mathrm{soc}^i({}_A A))\big)^{op}$. This shows (4). The statement (3) follows from (4), Corollary 3.14 and [6, Corollary 4.3]. $\square$

Finally, we state a dual version of Theorem 3.4, which will produce a derived equivalence between the endomorphism algebras of cogenerators. First, we point out the following facts.

**Lemma 3.16.** *Let $F: \mathscr{D}^{\mathrm{b}}(A) \longrightarrow \mathscr{D}^{\mathrm{b}}(B)$ be an almost $\nu$-stable derived equivalence with a quasi-inverse functor $G$. Suppose $D$ is the usual duality. Then we have the following.*

*(1) The functor $DGD: \mathscr{D}^{\mathrm{b}}(B^{op}) \longrightarrow \mathscr{D}^{\mathrm{b}}(A^{op})$ is an almost $\nu$-stable derived equivalence with a quasi-inverse functor $DFD$.*

*(2) Let $\bar{F}: A\text{-}\underline{\mathrm{mod}} \longrightarrow B\text{-}\underline{\mathrm{mod}}$ and $\overline{DFD}: A^{op}\text{-}\underline{\mathrm{mod}} \longrightarrow B^{op}\text{-}\underline{\mathrm{mod}}$ be the stable equivalence defined in Lemma 3.3 (3) and (4), respectively. Then, for each $A$-module $X$, there is an isomorphism $D\bar{F}(X) \simeq \overline{DFD}(D(X))$ in $B^{op}\text{-}\mathrm{mod}$.*

*Proof.* (1) Suppose that $Q^\bullet$ and $\bar{Q}^\bullet$ are tilting complexes associated to $F$ and $G$, respectively. We assume that $Q^\bullet$ and $\bar{Q}^\bullet$ are radical complexes. There is an isomorphism

$$DGD(\mathrm{Hom}_B^\bullet(\bar{Q}^\bullet, {}_BB)) \simeq DG(\nu_B\bar{Q}^\bullet) \simeq D\nu_B G(\bar{Q}^\bullet) \simeq D\nu_A({}_AA) \simeq \mathrm{Hom}_A({}_AA, {}_AA) \simeq A_A$$

Similarly, we have $DFD(\mathrm{Hom}_A^\bullet(Q^\bullet, {}_AA)) \simeq B_B$. Consequently, the complexes $P^\bullet := \mathrm{Hom}_B^\bullet(\bar{Q}^\bullet, {}_BB)$ and $\bar{P}^\bullet := \mathrm{Hom}_A^\bullet(Q^\bullet, {}_AA)$ are tilting complexes associated to $DGD$ and $DFD$, respectively. Since ${}_B\bar{Q} = \bigoplus_{i=1}^n \bar{Q}^i$, we have $\mathrm{Hom}_B(\bar{Q}, {}_BB) = \bigoplus_{i=1}^n P^{-i}$. Moreover,

$$\nu_{B^{\mathrm{op}}}(\mathrm{Hom}_B(\bar{Q}, {}_BB)) \simeq D({}_B\bar{Q}) \simeq \mathrm{Hom}_B(\nu_B^-(\bar{Q}), {}_BB) \in \mathrm{add}(\mathrm{Hom}_B(\bar{Q}, {}_BB))$$

since $\nu_B^- \bar{Q}$ is in $\mathrm{add}({}_B\bar{Q})$. Hence $\mathrm{add}(\nu_{B^{\mathrm{op}}}(\mathrm{Hom}_B(\bar{Q}, {}_BB))) = \mathrm{add}(\mathrm{Hom}_B(\bar{Q}, {}_BB))$. Similarly, we have $\mathrm{Hom}_A(Q, {}_AA) = \bigoplus_{i=1}^n \bar{P}^i$ and $\mathrm{add}(\nu_{A^{\mathrm{op}}}(\mathrm{Hom}_A(Q, {}_AA))) = \mathrm{add}(\mathrm{Hom}_A(Q, {}_AA))$, and consequently $DGD$ is an almost $\nu$-stable derived equivalence. Clearly, the functors $DGD$ and $DFD$ are mutually quasi-inverse functors. This proves (1).

(2) For each $A$-module $X$, we have $DFD(D(X)) = DF(X)$. By Lemma 3.3 (2), the complex $DFD(D(X))$ is isomorphic to a complex $P_{D(X)}^\bullet$ of the form

$$0 \longrightarrow P_{D(X)}^{-n} \longrightarrow \cdots \longrightarrow P_{D(X)}^0 \longrightarrow 0$$

with $P_{D(X)}^i \in \mathrm{add}(\mathrm{Hom}_B(\bar{Q}, {}_BB))$ for all $i < 0$ and $\overline{DFD}(D(X)) = P_{D(X)}^0$. Consequently, the complex $F(X)$ is isomorphic to $D(P_{D(X)}^\bullet)$ of the form

$$0 \longrightarrow D(P_{D(X)}^0) \longrightarrow \cdots \longrightarrow D(P_{D(X)}^{-n}) \longrightarrow 0$$

with $D(P_{D(X)}^0)$ being in degree zero and $D(P_{D(X)}^i) \in \mathrm{add}(\nu_B\bar{Q}) = \mathrm{add}({}_B\bar{Q})$ for all $i > 0$. By Lemma 3.3 (1) and (3), we have $\bar{F}(X) \simeq D(P_{D(X)}^0) = D\overline{DFD}(D(X))$ in $B$-mod. This finishes the proof. $\square$

Clearly, for an Artin algebra $A$ and an $A$-module $V$, the algebra $\mathrm{E}_\Lambda^\Phi(V)$ is isomorphic to the opposite algebra of $\mathrm{E}_{A^{\mathrm{op}}}^\Phi(D(V))$ for every admissible subset $\Phi$ of $\mathbb{N}$.

**Corollary 3.17.** *Let $F : \mathscr{D}^{\mathrm{b}}(A) \longrightarrow \mathscr{D}^{\mathrm{b}}(B)$ be an almost $\nu$-stable derived equivalence between two Artin algebras $A$ and $B$, and let $\bar{F}$ be the stable equivalence defined in* Lemma 3.3. *For each $A$-module $X$, set $M = D(A_A) \oplus X$ and $N = D(B_B) \oplus \bar{F}(X)$. Suppose that $\Phi$ is an admissible subset of $\mathbb{N}$. Then*
(1) *The $\Phi$-Auslander-Yoneda algebras $\mathrm{E}_A^\Phi(M)$ and $\mathrm{E}_B^\Phi(N)$ are derived-equivalent.*
(2) *If $\Phi$ is finite, then there is an almost $\nu$-stale derived equivalence between $\mathrm{E}_A^\Phi(M)$ and $\mathrm{E}_B^\Phi(N)$.*

*Proof.* We consider the $A^{\mathrm{op}}$-module $DM = A_A \oplus D(X)$ and the $B^{\mathrm{op}}$-module $DN = B_B \oplus D\bar{F}(X)$. By Lemma 3.16, we see that $D\bar{F}(X) \simeq \overline{DFD}(D(X))$. Let $G$ be a quasi-inverse of $F$. Then the functor $DGD$ is an almost $\nu$-stable derived equivalence by Lemma 3.16 (1), and $\overline{DFD}$ is a quasi-inverse of $\overline{DGD}$. Thus, by Theorem 3.4 and by Lemma 3.16 (1), the corollary follows. $\square$

# 4   Derived equivalences for quotient algebras

In the previous section, we have seen that there are many derived equivalences between quotient algebras of $\Phi$-Auslander-Yoneda algebras that are derived-equivalent (see Theorem 3.4 and Subsection 3.1). This phenomenon gives rise to a general question: How to construct a new derived equivalence for quotient algebras from the given one between two given algebras ? In this section, we shall consider this question and provide methods to transfer a derived equivalence between two given algebras to a derived equivalence between their quotient algebras. In particular, we shall prove Theorem 1.3

## 4.1 Derived equivalences for algebras modulo ideals

Let us start with the following general setting.

Suppose that $A$ is an Artin $R$-algebra over a commutative Artin ring $R$, and suppose that $I$ is an ideal in $A$. We denote by $\overline{A}$ the quotient algebra $A/I$ of $A$ by the ideal $I$. The category $\overline{A}$-mod can be regarded as a full subcategory of $A$-mod. Also, there is a canonical functor from $A$-mod to $\overline{A}$-mod which sends each $X \in A$-mod to $\overline{X} := X/IX$. This functor induces a functor $^- : \mathscr{C}(A) \longrightarrow \mathscr{C}(\overline{A})$, which is defined as follows: for a complex $X^\bullet = (X^i)_{i \in \mathbb{Z}}$ of $A$-modules, let $IX^\bullet$ be the sub-complex of $X^\bullet$ in which the $i$-th term is the submodule $IX^i$ of $X^i$; we define $\overline{X}^\bullet$ to be the quotient complex of $X^\bullet$ modulo $IX^\bullet$. The action of $^-$ on a chain map can be defined canonically. Thus $^-$ is a well-defined functor. For each complex $X^\bullet$ of $A$-modules, we have the following canonical exact sequence of complexes:

$$0 \longrightarrow IX^\bullet \xrightarrow{i^\bullet} X^\bullet \xrightarrow{\pi^\bullet} \overline{X}^\bullet \longrightarrow 0.$$

For a complex $Y^\bullet$ of $\overline{A}$-modules, this sequence induces another exact sequence of $R$-modules:

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{C}(A)}(\overline{X}^\bullet, Y^\bullet) \xrightarrow{\pi^*} \mathrm{Hom}_{\mathscr{C}(A)}(X^\bullet, Y^\bullet) \xrightarrow{i^*} \mathrm{Hom}_{\mathscr{C}(A)}(IX^\bullet, Y^\bullet).$$

Since $Y^\bullet$ is a complex of $\overline{A}$-modules, the map $i^*$ must be zero, and consequently $\pi^*$ is an isomorphism. Now we show that $\pi^*$ actually induces an isomorphism between $\mathrm{Hom}_{\mathscr{K}(A)}(\overline{X}^\bullet, Y^\bullet)$ and $\mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, Y^\bullet)$.

**Lemma 4.1.** *Suppose that $A$ is an Artin algebra and $I$ is an ideal in $A$. Let $\overline{A}$ be the quotient algebra of $A$ modulo $I$. If $X^\bullet$ is a complex of $A$-modules and $Y^\bullet$ is a complex of $\overline{A}$-modules, then we have a natural isomorphism of $R$-modules*

$$\pi^* : \mathrm{Hom}_{\mathscr{K}(A)}(\overline{X}^\bullet, Y^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, Y^\bullet).$$

*Proof.* Note that we have already an isomorphism

$$\pi^* : \mathrm{Hom}_{\mathscr{C}(A)}(\overline{X}^\bullet, Y^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{C}(A)}(X^\bullet, Y^\bullet).$$

Clearly, $\pi^*$ sends null-homotopic maps to null-homotopic maps. This means that $\pi^*$ induces an epimorphism

$$\pi^* : \mathrm{Hom}_{\mathscr{K}(A)}(\overline{X}^\bullet, Y^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, Y^\bullet).$$

Now let $f^\bullet : \overline{X}^\bullet \to Y^\bullet$ be a chain map such that $\pi^*(f^\bullet) = \pi^\bullet f^\bullet$ is null-homotopic. Then there is a homomorphism $h^i : X^i \to Y^{i-1}$ for each integer $i$ such that $\pi^i f^i = h^i d_Y^{i-1} + d_X^i h^{i+1}$. Note that $h^i$ factorizes through $\pi^i$, that is, $h^i = \pi^i g^i$ for some $g^i : \overline{X}^i \to Y^{i-1}$. Hence we have

$$
\begin{aligned}
\pi^i f^i \quad &= h^i d_Y^{i-1} + d_X^i h^{i+1} \\[2mm]
&= \pi^i g^i d_Y^{i-1} + d_X^i \pi^{i+1} g^{i+1} \\[2mm]
&= \pi^i g^i d_Y^{i-1} + \pi^i d_{\overline{X}}^i g^{i+1} \\[2mm]
&= \pi^i (g^i d_Y^{i-1} + d_{\overline{X}}^i g^{i+1}).
\end{aligned}
$$

It follows that $f^i = g^i d_Y^{i-1} + d_{\overline{X}}^i g^{i+1}$ since $\pi^i$ is surjective for each $i$. Therefore, the map $f^\bullet$ is null-homotopic. Thus $\pi^*$ is injective. $\square$

For any complexes $X^\bullet$ and $X'^\bullet$ over $A$-mod, we have a natural map

$$\eta : \mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, X'^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{K}(A)}(\overline{X}^\bullet, \overline{X'}^\bullet),$$

which is the composition of $\pi_*^\bullet : \mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, X'^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, \overline{X'}^\bullet)$ with the map $(\pi^*)^{-1} : \mathrm{Hom}_{\mathscr{K}(A)}(X^\bullet, \overline{X'}^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{K}(A)}(\overline{X}^\bullet, \overline{X'}^\bullet)$ defined in Lemma 4.1. In particular, if $X^\bullet = X'^\bullet$, then we get an algebra homomorphism

$$\eta : \mathrm{End}_{\mathscr{K}(A)}(X^\bullet) \longrightarrow \mathrm{End}_{\mathscr{K}(A)}(\overline{X}^\bullet).$$

Now, let $T^\bullet$ be a tilting complex over $A$, and let $B = \mathrm{End}_{\mathscr{K}(A)}(T^\bullet)$. Further, suppose that $I$ is an ideal in $A$. By the above discussion, there is an algebra homomorphism

$$\eta : \mathrm{End}_{\mathscr{K}(A)}(T^\bullet) \longrightarrow \mathrm{End}_{\mathscr{K}(A)}(\overline{T}^\bullet).$$

Let $J_I$ be the kernel of $\eta$, which is an ideal of $B$. Since $(\pi^*)^{-1}$ is an isomorphism, we see that $J_I$ is the kernel of the map $\pi_*^\bullet : \mathrm{End}_{\mathscr{K}(A)}(T^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet, \overline{T}^\bullet)$. In fact, $J_I$ is also the set of all endomorphisms of $T^\bullet$ which factorize through the embedding $IT^\bullet \longrightarrow T^\bullet$. We denote quotient algebra $B/J_I$ by $\overline{B}$.

In the following, we study when the complex $\overline{T}^\bullet$ is a tilting complex over the quotient algebra $\overline{A}$ and induces a derived equivalence between $\overline{A}$ and $\overline{B}$. The following result supplies an answer to this question.

**Theorem 4.2.** *Let $A$ be an Artin algebra, and let $T^\bullet$ be a tilting complex over $A$ with the endomorphism algebra $B = \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet)$. Suppose that $I$ is an ideal in $A$, and $\overline{A} := A/I$. Let $\overline{B}$ be the quotient algebra of $B$ modulo $J_I$. Then $\overline{T}^\bullet$ is a tilting complex over $\overline{A}$ and induces a derived equivalence between $\overline{A}$ and $\overline{B}$ if and only if $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, IT^\bullet[i]) = 0$ for all $i \neq 0$ and $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(\overline{T}^\bullet, \overline{T}^\bullet[-1]) = 0$.*

*Proof.* First, we assume $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, IT^\bullet[i]) = 0$ for all $i \neq 0$ and $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(\overline{T}^\bullet, \overline{T}^\bullet[-1]) = 0$. Applying the functor $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^\bullet, -)$ to the distinguished triangle

$$IT^\bullet \xrightarrow{i^\bullet} T^\bullet \xrightarrow{\pi^\bullet} \overline{T}^\bullet \longrightarrow IT^\bullet[1],$$

for each integer $i$, we get an exact sequence

$$\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^\bullet, T^\bullet[i]) \longrightarrow \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^\bullet, \overline{T}^\bullet[i]) \longrightarrow \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^\bullet, IT^\bullet[i+1]),$$

which is isomorphic to the exact sequence

$$(*) \qquad \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, T^\bullet[i]) \longrightarrow \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, \overline{T}^\bullet[i]) \longrightarrow \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, IT^\bullet[i+1]).$$

Since the first and third terms of $(*)$ are zero for $i \neq 0, -1$, the middle term $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, \overline{T}^\bullet[i])$ must be zero for $i \neq 0, -1$. Thus, taking our assumption into account, we have

$$\begin{aligned} \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(\overline{A}\text{-proj})}(\overline{T}^\bullet, \overline{T}^\bullet[i]) \ &\simeq \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(\overline{T}^\bullet, \overline{T}^\bullet[i]) \\ &\simeq \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, \overline{T}^\bullet[i]) \\ &= 0 \end{aligned}$$

for all $i \neq 0$. Thus $\overline{T}^\bullet$ is self-orthogonal in $\mathscr{D}^{\mathrm{b}}(\overline{A})$.

Note that the functor

$$(A/I) \otimes_A^{\mathbf{L}} - : \mathscr{K}^{\mathrm{b}}(A\text{-proj}) \longrightarrow \mathscr{K}^{\mathrm{b}}(\overline{A}\text{-proj})$$

sends $T^\bullet$ to $\overline{T}^\bullet$. Let $C$ be the full triangulated subcategory of $\mathscr{K}^{\mathrm{b}}(\overline{A}\text{-proj})$ generated by $\mathrm{add}(\overline{T}^\bullet)$, and let $\mathcal{D}$ be a full triangulated subcategory of $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$ consisting of those $X^\bullet$ for which $(A/I)\otimes_A^{\mathbf{L}} X^\bullet$ belongs to $C$. Then $\mathcal{D}$ contains $\mathrm{add}(T^\bullet)$. Therefore $\mathcal{D} = \mathscr{K}^{\mathrm{b}}(A\text{-proj})$, and consequently $\mathrm{add}(\overline{A})$ is contained in $C$. Thus $C = \mathscr{K}^{\mathrm{b}}(\overline{A}\text{-proj})$, and $\overline{T}^\bullet$ is a tilting complex over the quotient algebra $\overline{A}$. Since $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, IT^\bullet[1]) = 0$, by the exact sequence $(*)$, we have a surjective map $\pi_*^\bullet :$ $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, T^\bullet) \to \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, \overline{T}^\bullet)$. Therefore, the algebra homomorphism $\eta : \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet) \to \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(\overline{T}^\bullet)$ is an epimorphism. Hence

$$\overline{B} = \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet)/\mathrm{Ker}(\eta) \simeq \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(\overline{T}^\bullet) \simeq \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(\overline{A})}(\overline{T}^\bullet).$$

Consequently, the tilting complex $\overline{T}^\bullet$ induces a derived equivalence between $\overline{A}$ and $\overline{B}$.

Conversely, we assume that $\overline{T}^\bullet$ is a tilting complex over $\overline{A}$ and induces a derived equivalence between $\overline{A}$ and $\overline{B}$. Then $\mathrm{Hom}_{\mathscr{K}(\overline{A})}(\overline{T}^\bullet, \overline{T}^\bullet[i]) = 0$ for all $i \neq 0$. Note that, for each integer $i$, we have an exact sequence

$$(**) \qquad \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, \overline{T}^\bullet[i-1]) \longrightarrow \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, IT^\bullet[i]) \longrightarrow \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, T^\bullet[i]).$$

Since $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, \overline{T}^\bullet[i-1]) \simeq \mathrm{Hom}_{\mathscr{K}(\overline{A})}(\overline{T}^\bullet, \overline{T}^\bullet[i-1])$ and since $T^\bullet$ is self-orthogonal, the first and third terms of $(**)$ are zero for $i \neq 0, 1$. It follows that $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, IT^\bullet[i]) = 0$ for all $i \neq 0, 1$. We claim that $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, IT^\bullet[1]) = 0$. Indeed, we consider the following exact sequence

$$\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, IT^\bullet) \xrightarrow{\;i_*^\bullet\;} \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, T^\bullet) \xrightarrow{\;\pi_*^\bullet\;} \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, \overline{T}^\bullet) \longrightarrow$$

$$\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, IT^\bullet[1]) \longrightarrow \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, T^\bullet[1]) = 0.$$

Since the kernel of $\pi_*^\bullet$ is $J_I$, the image of $\pi_*^\bullet$ is isomorphic to $\overline{B}$ as $R$-modules. But we already have $\overline{B} \simeq \mathrm{End}_{\mathscr{K}^{\mathrm{b}}(A)}(\overline{T}^\bullet)$, which is isomorphic to $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, \overline{T}^\bullet)$ as an $R$-module. Hence the map $\pi_*^\bullet$ is surjective, and $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, IT^\bullet[1]) = 0$. Clearly, $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(\overline{T}^\bullet, \overline{T}^\bullet[-1]) = 0$. Altogether, we have shown that $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^\bullet, IT^\bullet[i]) = 0$ for all $i \neq 0$ and $\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(\overline{T}^\bullet, \overline{T}^\bullet[-1]) = 0$. This completes the proof of Theorem 4.2. $\square$

## 4.2 Derived equivalences for self-injective algebras modulo socles

In the following, we shall use Theorem 4.2 to prove our second main result in this paper. Let us first prove the following lemma.

**Lemma 4.3.** *Let A be a self-injective basic algebra, and let P be a direct summand of $_AA$.*

*(1) If J is an ideal of A such that $_AJ \simeq {}_A\mathrm{soc}(P)$, then $J = \mathrm{soc}(P)$.*

*(2) If $T^\bullet$ is a radical tilting complex over A such that the endomorphism algebra of $T^\bullet$ is self-injective and basic, then $T^i \simeq \nu_A T^i$ for all integers i.*

*Proof.* (1) Let $e$ be the sum of the idempotents corresponding to the simple direct summands of $\mathrm{soc}(P)$. By assumption, we have $J \subseteq \mathrm{soc}(A)$ and $eJ = J$. Hence $J = eJ \subseteq e(\mathrm{soc}(A)) = \mathrm{soc}(P)$, and consequently $J = \mathrm{soc}(P)$.

(2) Let $B$ be the endomorphism algebra of $T^\bullet$. Then there is a derived equivalence $F : \mathscr{D}^{\mathrm{b}}(A) \longrightarrow \mathscr{D}^{\mathrm{b}}(B)$ such that $F(T^\bullet) \simeq B$. Since $B$ is a self-injective basic algebra, and since $F$ commutes with the Nakayama functor $\nu$, we have $F(\nu_A T^\bullet) \simeq \nu_B F(T^\bullet) \simeq \nu_B B \simeq B \simeq F(T^\bullet)$. Consequently, we have $T^\bullet \simeq \nu_A T^\bullet$ in $\mathscr{D}^{\mathrm{b}}(A)$. Since $A$ is self-injective, we see that $\nu_A T^\bullet$ is also a complex in $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$. Hence $\nu_A T^\bullet \simeq T^\bullet$ in $\mathscr{K}^{\mathrm{b}}(A\text{-proj})$, and consequently $T^\bullet \simeq \nu_A T^\bullet$ in $\mathscr{C}^b(A)$ since both $T^\bullet$ and $\nu_A T^\bullet$ are radical complexes. Thus, the statement (2) follows. $\square$

**Theorem 4.4.** *Suppose that A and B are basic self-injective Artin algebras, and that $F : \mathscr{D}^{\mathrm{b}}(A) \longrightarrow \mathscr{D}^{\mathrm{b}}(B)$ is a derived equivalence. Let P be a direct summand of ${}_A A$, and let P' be a direct summand of ${}_B B$ such that $F(\mathrm{soc}(P))$ is isomorphic to $\mathrm{soc}(P')$. Then the quotient algebras $A/\mathrm{soc}(P)$ and $B/\mathrm{soc}(P')$ are derived-equivalent.*

*Proof.* Since A and B are basic self-injective algebras, $\mathrm{soc}(P)$ and $\mathrm{soc}(P')$ are ideals in A and B, respectively. In the following, we shall verify that the conditions of Theorem 4.2 are satisfied by the ideal $\mathrm{soc}(P)$ in A and the tilting complex $T^{\bullet}$ associated to F.

Since $F(\mathrm{soc}(P))$ is isomorphic to $\mathrm{soc}(P')$, we can assume that $P = \bigoplus_{i=1}^{s} P_i$ and $P' = \bigoplus_{i=1}^{s} P_i'$, where P and P' are indecomposable such that $F(\mathrm{soc}(P_i))$ is isomorphic to $\mathrm{soc}(P_i')$ for all $i = 1, \cdots, s$. Let $D_i$ be the endomorphism ring of $\mathrm{soc}(P_i)$, which is a division ring. Since $F(\mathrm{soc}(P_i)) \simeq \mathrm{soc}(P_i')$, we see that $D_i$ is isomorphic to $\mathrm{End}_B(\mathrm{soc}(P_i'))$. Note that a radical map $f : M_1 \to M_2$ between two projective modules $M_1$ and $M_2$ has image contained in $\mathrm{rad}(M_2)$. Since all the differential maps of $T^{\bullet}$ are radical maps, the image of $d_T^k$ is contained in $\mathrm{rad}(T^{k+1})$ for all integers k. It follows that

$$
\begin{aligned}
\mathrm{Hom}_A(T^n, \mathrm{soc}(P_i)) &\simeq \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^{\bullet}[n], \mathrm{soc}(P_i)) \\
&\simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^{\bullet}[n], \mathrm{soc}(P_i)) \\
&\simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(B)}(B[n], \mathrm{soc}(P_i')) \\
&= 0
\end{aligned}
$$

for all $n \neq 0$. Hence, for each integer $n \neq 0$, the module $\nu_A^{-1} P_i$ is not a direct summand of $T^n$. Since $T^n \simeq \nu_A T^n$ (Lemma 4.3(2)), we infer that $P_i$ is not a direct summand of $T^n$ for all $n \neq 0$. Recall that $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^{\bullet}, \mathrm{soc}(P_i)) \simeq \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(B)}(B, \mathrm{soc}(P_i')) \simeq \mathrm{soc}(P_i')$ as $D_i^{\mathrm{op}}$-modules. Since B is basic, we see that $\mathrm{soc}(P_i')$ is one-dimensional over $D_i^{\mathrm{op}}$. Hence $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^{\bullet}, \mathrm{soc}(P_i))$ is one-dimensional over $D_i^{\mathrm{op}}$. It follows that $\nu_A^{-1} P_i$ is a direct summand of $T^0$ with multiplicity 1. Since $\nu_A T^0 \simeq T^0$, we see that $P_i$ is a direct summand of $T^0$ with multiplicity 1. Note that $\mathrm{soc}(P_i)X = 0$ for any A-module X if $P_i$ is not a direct summand of X. Hence $\mathrm{soc}(P_i)T^{\bullet}$ is isomorphic to the stalk complex $\mathrm{soc}(P_i)P_i = \mathrm{soc}(P_i)$. Therefore

$$
\mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^{\bullet}, \mathrm{soc}(P)T^{\bullet}[n]) = \mathrm{Hom}_{\mathscr{K}^{\mathrm{b}}(A)}(T^{\bullet}, \oplus_{i=1}^{s} \mathrm{soc}(P_i)[n]) = 0
$$

for all $n \neq 0$.

Let $\overline{T}^{\bullet}$ be the quotient complex $T^{\bullet}/(\mathrm{soc}(P)T^{\bullet})$. There is a canonical triangle in $\mathscr{D}^{\mathrm{b}}(A)$:

$$
\mathrm{soc}(P)T^{\bullet} \xrightarrow{\lambda} T^{\bullet} \longrightarrow \overline{T}^{\bullet} \longrightarrow (\mathrm{soc}(P)T^{\bullet})[1].
$$

Applying $\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^{\bullet}, -)$ to this triangle, we have an exact sequence of B-modules:

$$
0 \longrightarrow \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^{\bullet}, \overline{T}^{\bullet}[-1]) \longrightarrow \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^{\bullet}, \mathrm{soc}(P)T^{\bullet}) \xrightarrow{\lambda_*} \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^{\bullet}, T^{\bullet}).
$$

We claim that $\lambda_*$ is a monomorphism. Since $\mathrm{soc}(P)T^{\bullet}$ is isomorphic to $\bigoplus_{i=1}^{s} \mathrm{soc}(P_i)T^{\bullet}$, the map $\lambda$ can be written as $(\lambda_1, \cdots, \lambda_s)^{tr}$, where $\lambda_i : \mathrm{soc}(P_i)T^{\bullet} \to T^{\bullet}$ is the canonical map, and where *tr* stands for the transpose of a matrix. Now we consider the following commutative diagram of B-modules:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^{\bullet}, \mathrm{soc}(P_i)T^{\bullet}) & \xrightarrow{(\lambda_i)_*} & \mathrm{Hom}_{\mathscr{D}^{\mathrm{b}}(A)}(T^{\bullet}, T^{\bullet}) \\
\Big\downarrow{\simeq} & & \Big\downarrow{\simeq} \\
\mathrm{Hom}_B(B, F(\mathrm{soc}(P_i)T^{\bullet})) & \xrightarrow{F(\lambda_i)_*} & \mathrm{Hom}_B(B, B).
\end{array}
$$

Since $\lambda_i \neq 0$, we see that $F(\lambda_i)$ is nonzero. Moreover, $F(\mathrm{soc}(P_i)T^{\bullet}) \simeq F(\mathrm{soc}(P_i)) \simeq \mathrm{soc}(P_i')$. This implies that $F(\mathrm{soc}(P_i))$ is a simple B-module for all i. Hence $F(\lambda_i)_*$ must be injective. To show

that $\lambda_*$ is injective, it suffices to show that $F(\lambda)_*$ is injective. This is equivalent to proving that $(F(\lambda_1)_*, \cdots, F(\lambda_s)_*)^{tr}$ is injective. For this, we use induction on $s$. If $s = 1$, the foregoing discussion shows that this is true. Now we assume $s > 1$. Then the kernel $K$ of $(F(\lambda_1)_*, \cdots, F(\lambda_s)_*)^{tr}$ is the pull-back of $(F(\lambda_1)_*, \cdots, F(\lambda_{s-1})_*)^{tr}$ and $F(\lambda_s)_*$ both of which are monomorphisms by induction hypothesis. Thus $K$ is isomorphic to a submodule of both $\mathrm{Hom}_{\mathscr{D}^b(A)}(T^\bullet, \bigoplus_{i=1}^{s-1} \mathrm{soc}(P))$ and $\mathrm{Hom}_{\mathscr{D}^b(A)}(T^\bullet, \mathrm{soc}(P_s))$. However, the $B$-modules $\mathrm{Hom}_{\mathscr{D}^b(A)}(T^\bullet, \mathrm{soc}(P_i)) \simeq \mathrm{soc}(P_i'), i = 1, \cdots, s$, are pairwise non-isomorphic simple $B$-modules since $B$ is basic. This implies that $K = 0$. Hence $\lambda_*$ is injective, and therefore $\mathrm{Hom}_{\mathscr{D}^b(A)}(T^\bullet, \overline{T}^\bullet[-1]) = 0$. Since

$$\mathrm{Hom}_{\mathscr{K}^b(A)}(\overline{T}^\bullet, \overline{T}^\bullet[-1]) \simeq \mathrm{Hom}_{\mathscr{K}^b(A)}(T^\bullet, \overline{T}^\bullet[-1]) \simeq \mathrm{Hom}_{\mathscr{D}^b(A)}(T^\bullet, \overline{T}^\bullet[-1]),$$

it follows that $\mathrm{Hom}_{\mathscr{K}^b(A)}(\overline{T}^\bullet, \overline{T}^\bullet[-1]) = 0$. Hence the complex $T^\bullet$ and the ideal $\mathrm{soc}(P)$ satisfy all conditions in Theorem 4.2. Thus $A/\mathrm{soc}(P)$ and $B/J$ are derived-equivalent, where $J$ is the ideal of $B$ consisting of maps $b$ factorizing through the canonical map $\mathrm{soc}(P)T^\bullet \longrightarrow T^\bullet$. Moreover, $J$ is isomorphic to $\mathrm{Hom}_{\mathscr{K}^b(A)}(T^\bullet, \mathrm{soc}(P))$ as $B$-modules, and the latter is isomorphic to $\mathrm{soc}(P')$. By Lemma 4.3 (1), we have $J = \mathrm{soc}(P')$, and the theorem is proved. $\square$

We give a criterion to judge when a derived equivalence satisfies the condition in Theorem 4.4.

**Proposition 4.5.** *Let $T^\bullet = (T^i, d^i)$ be a tilting complex associated to a derived equivalence $F$ between self-injective basic Artin algebras $A$ and $B$, and let $P$ be an indecomposable projective $A$-module. Suppose we have the following:*
*(1) $P \notin \mathrm{add}(\nu_A T^i)$ for all $i \neq 0$;*
*(2) the multiplicity of $P$ as a direct summand of $\nu_A T^0$ is one.*
*Let $T_P^\bullet$ be the indecomposable direct summand of $T^\bullet$ such that $P$ is a direct summand of $\nu_A(T_P^0)$, and let $\bar{P}$ be the projective $B$-module $\nu_B(\mathrm{Hom}_{\mathscr{K}^b(A\text{-proj})}(T^\bullet, T_P^\bullet))$. Then $F(\mathrm{soc}(_AP)) \simeq \mathrm{soc}(_B\bar{P})$.*

*Proof.* We know that the Nakayama functor sends $P$ to the injective envelope of $\mathrm{top}(_AP)$. From (1) it follows that $\mathrm{Hom}_A(T^i, \mathrm{soc}(_AP)) = 0$ for all $i \neq 0$. Consequently, $\mathrm{Hom}_{\mathscr{D}^b(A)}(T^\bullet, \mathrm{soc}(_AP)[i]) = 0$ for all $i \neq 0$. This means that $F(\mathrm{soc}(_AP))$ is isomorphic in $\mathscr{D}^b(B)$ to a $B$-module $X$ that is indecomposable. Now we have the following isomorphisms:

$$\mathrm{Hom}_B(B, X) \simeq \mathrm{Hom}_{\mathscr{D}^b(A)}(T^\bullet, \mathrm{soc}(_AP)) \simeq \mathrm{Hom}_{\mathscr{D}^b(A)}(T_P^\bullet, \mathrm{soc}(_AP)) \simeq \mathrm{Hom}_B(\nu_B^- \bar{P}, X).$$

Hence $\mathrm{soc}(_B\bar{P})$ is the only simple $B$-module which occurs as a composition factor of $X$. If $X$ were not simple, then we would get a nonzero homomorphism $X \to \mathrm{top}(X) \to \mathrm{soc}(X) \to X$, which is not an isomorphism. This is a contradiction since $\mathrm{End}_B(X) \simeq \mathrm{End}_{\mathscr{D}^b(B)}(F(\mathrm{soc}(_AP)) \simeq \mathrm{End}_A(\mathrm{soc}(_AP))$ is a division ring. Hence $X$ is simple and isomorphic to $\mathrm{soc}(_B\bar{P})$. This finishes the proof. $\square$

## 4.3 Derived equivalences for algebras modulo annihilators

Now, we turn to another construction for derived-equivalent quotient algebras by using idempotent elements, which can be regarded as another consequence of Theorem 4.2.

**Lemma 4.6.** *Let $e$ be an idempotent of an Artin algebra $A$. Then there is a unique left ideal $I$ of $A$, which is maximal with respect to the property $eI = 0$. Moreover, the $I$ is an ideal of $A$. If, in addition, $\mathrm{add}(Ae) = \mathrm{add}(D(eA))$, then $Ie = 0$.*

*Proof.* Note that such a left ideal $I$ in $A$ exists, and any left ideal $L$ in $A$ with $eL = 0$ is contained in $I$. Clearly, $I$ is a left ideal in $A$. We have to show that $I$ is a right ideal in $A$. Let $x \in A$ and $a \in I$.

Since the right multiplying with $x$ is a homomorphism $\varphi$ from ${}_AA$ to ${}_AA$, we see that the image $\varphi(I)$ of $I$ under $\varphi$ is a left ideal in $A$. Since $eI = 0$, we have $\varphi(I) \subseteq I$, and $ax \in I$.

Suppose $\mathrm{add}(Ae) = \mathrm{add}(D(eA))$. It follows from

$$0 = eI = \mathrm{Hom}_A(Ae, I) \simeq \mathrm{Hom}_A(I, D(eA))$$

that $\mathrm{Hom}_A(I, Ae) = 0$. Clearly, the map $\psi : I \to Ae$ giving by $x \mapsto xe$ is a homomorphism from $I$ to $Ae$. Thus $\psi = 0$ and $Ie = 0$. $\square$

Let $A$ be an Artin algebra and $e$ an idempotent of $A$ such that $\mathrm{add}(Ae) = \mathrm{add}(D(eA))$. By a result in [4], there is a tilting complex $T^\bullet$ associated to $e$, which is defined in the following way: suppose $\varphi$ is a minimal right $\mathrm{add}(Ae)$-approximation of $A$. Then we form the following complex:

$$T_f^\bullet : \quad 0 \longrightarrow Q_1 \overset{\varphi}{\longrightarrow} A \longrightarrow 0$$

with $A$ in degree zero. Let $T_e^\bullet := (Ae)[1]$. The tilting complex $T^\bullet$ associated with $e$ is defined to be the direct sum of $T_e^\bullet$ and $T_f^\bullet$. Let $\lambda_e : T_e^\bullet \to T^\bullet$ be the canonical inclusion and $p_e : T^\bullet \to T_e^\bullet$ the canonical projection. Then $\tilde{e} := p_e \lambda_e$ is an idempotent in $B := \mathrm{End}_{\mathscr{K}(A)}(T^\bullet)$, which corresponds to the summand $T_e^\bullet$ of $T^\bullet$. Thus, there is a derived equivalence $F : \mathscr{D}^{\mathrm{b}}(A) \longrightarrow \mathscr{D}^{\mathrm{b}}(B)$, which sends $T_e^\bullet$ to $B\tilde{e}$, and $T_f^\bullet$ to $B(1 - \tilde{e})$. Let $\nabla(e)$ and $\nabla(\tilde{e})$ be the ideal $I$ of $A$ and $B$ defined by $e$ and $\tilde{e}$ in Lemma 4.6, respectively. With these notations in mind, we have the following proposition.

**Proposition 4.7.** *Let $A$ be an Artin algebra and $e$ an idempotent element in $A$ such that $\mathrm{add}(D(eA)) = \mathrm{add}(Ae)$. Suppose that $T^\bullet = T_e^\bullet \oplus T_f^\bullet$ is the tilting complex defined by the idempotent $e$ and $B = \mathrm{End}_{\mathscr{K}(A)}(T^\bullet)$. Let $\tilde{e}$ be the idempotent element in $\mathrm{End}_{\mathscr{K}(A)}(T^\bullet)$ corresponding to $T_e^\bullet$. Then $A/\nabla(e)$ is derived-equivalent to $B/\nabla(\tilde{e})$.*

*Proof.* Let $F : \mathscr{D}^{\mathrm{b}}(A) \longrightarrow \mathscr{D}^{\mathrm{b}}(B)$ be the derived equivalence given by the tilting complex $T^\bullet$. Then $F(T_e^\bullet) = F(Ae)[1]) \simeq B\tilde{e}$.

The complex $\nabla(e)T^\bullet$ is isomorphic to $\nabla(e)$ because, by Lemma 4.6, we have $\nabla(e)Ae = 0$ and $\nabla(e)T^\bullet = \nabla(e)$, which is a complex with the only non-zero term $\nabla(e)$ in degree zero. It is easy to see that $\mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet, \nabla(e)[i]) = 0$ for all $i \neq 0$. Let $\overline{T}^\bullet$ be the quotient complex $T^\bullet/\nabla(e)T^\bullet$. Then $\overline{T}^\bullet$ is of the following form:

$$0 \longrightarrow Ae \oplus Q_1 \overset{\left[\begin{smallmatrix} 0 \\ \overline{\varphi} \end{smallmatrix}\right]}{\longrightarrow} \overline{A} \longrightarrow 0 \ ,$$

where $\overline{A} = A/\nabla(e)$, and where $\overline{\varphi}$ is the composition of $\varphi$ with the canonical surjection from $A$ to $\overline{A}$. Since $\mathrm{Hom}_{\mathscr{K}(A)}(T_f^\bullet, T_e^\bullet[-1]) = 0$, we get $\mathrm{Hom}_A(\mathrm{Coker}(\varphi), Ae) = 0$. Moreover, since $\mathrm{Coker}(\overline{\varphi})$ is a quotient module of $\mathrm{Coker}(\varphi)$, we have $\mathrm{Hom}_A(\mathrm{Coker}(\overline{\varphi}), Ae) = 0$. Thus

$$\mathrm{Hom}_{\mathscr{K}(A)}(\overline{T}^\bullet, \overline{T}^\bullet[-1]) = 0.$$

By Theorem 4.2, $\overline{T}^\bullet$ is a tilting complex over $A/\nabla(e)$, and $A/\nabla(e)$ is derived-equivalent to $B/J$, where $J = \{\alpha^\bullet \in \mathrm{End}_{\mathscr{K}(A)}(T^\bullet) \mid \alpha^\bullet \pi^\bullet = 0\}$, and where the map $\pi^\bullet$ is the canonical map from $T^\bullet$ to $\overline{T}^\bullet$. Note that $\nabla(e)T_e^\bullet = 0$. This allows us to rewrite $\pi^\bullet$ as

$$T_e^\bullet \oplus T_f^\bullet \overset{\left[\begin{smallmatrix} 1 & 0 \\ 0 & \pi_f^\bullet \end{smallmatrix}\right]}{\longrightarrow} T_e^\bullet \oplus T_f^\bullet/(\nabla(e)T_f^\bullet).$$

For any $\alpha^\bullet \in J$, we can write $\alpha^\bullet$ as

$$T_e^\bullet \oplus T_f^\bullet \overset{\left[\begin{smallmatrix} \alpha_{11}^\bullet & \alpha_{12}^\bullet \\ \alpha_{21}^\bullet & \alpha_{22}^\bullet \end{smallmatrix}\right]}{\longrightarrow} T_e^\bullet \oplus T_f^\bullet.$$

Since $\alpha^\bullet \pi^\bullet = 0$, we have $\alpha^\bullet_{11} = 0 = \alpha^\bullet_{21}$ and $\alpha^\bullet_{12}\pi^\bullet_f = 0 = \alpha^\bullet_{22}\pi^\bullet_f$. Hence $\alpha^\bullet_{12} : T^\bullet_e \to T^\bullet_f$ factorizes through $\nabla(e)T^\bullet_f = \nabla(e)$. But $\mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet_e, \nabla(e)) = 0$. This implies that $\alpha^\bullet_{12} = 0$. Consequently, $J$ consists of maps $\alpha^\bullet$ of the form

$$\begin{bmatrix} 0 & 0 \\ 0 & \alpha^\bullet_{22} \end{bmatrix}$$

with $\alpha^\bullet_{22}\pi^\bullet_f = 0$. Therefore $\tilde{e}J = 0$ and $J \subseteq \nabla(\tilde{e})$. By the proof of Theorem 4.2, we know that the quotient $B$-module $B/J$ is isomorphic to $\mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet, \overline{T}^\bullet)$. Note that we have a distinguished triangle

$$A/\nabla(e) \longrightarrow \overline{T}^\bullet \longrightarrow (Ae \oplus Q_1)[1] \longrightarrow (A/\nabla(e))[1]$$

in $\mathscr{K}(A)$. Applying the functor $\mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet, -)$ to this triangle, we get an exact sequence

$$\mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet, A/\nabla(e)) \longrightarrow \mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet, \overline{T}^\bullet) \longrightarrow \mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet, (Ae \oplus Q_1)[1]).$$

By the maximality of $\nabla(e)$, the quotient $A/\nabla(e)$ has no submodule $X$ with $eX = 0$. Since $\varphi$ is a right $\mathrm{add}(Ae)$-approximation of $A$, we have $e(\mathrm{Coker}(\varphi)) = 0$. It follows that $\mathrm{Hom}_A(\mathrm{Coker}(\varphi), A/\nabla(e)) = 0$. Hence we have $\mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet, A/\nabla(e)) = 0$. Consequently, $\mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet, \overline{T}^\bullet)$ can be embedded in $\mathrm{Hom}_{\mathscr{K}(A)}(T^\bullet, (Ae \oplus Q_1)[1])$, which is in $\mathrm{add}(B\tilde{e}) = \mathrm{add}(D(\tilde{e}B))$. This means that $J$ is the maximal submodule of $B$ with $\tilde{e}J = 0$. Hence $J = \nabla(\tilde{e})$, and this finishes the proof. $\square$

We point out that there is another type of construction by passing derived equivalences between two given algebras to that between their quotient algebras, namely, forming endomorphism algebras first, and then passing to stable endomorphism algebras. For details of this construction, we refer the reader to [5, Corollary 1.2, Corollary 1.3].

Now, we end this paper by two simple examples to illustrate our results.

**Example 1.** Let $k$ be a field, and let $A$ be a $k$-algebra given by the quiver
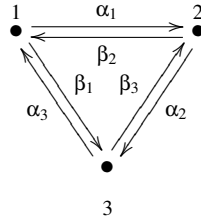


with relations $\alpha_i\beta_{i+1} - \beta_i\alpha_{i+2} = \alpha_i\alpha_{i+1} = \beta_i\beta_{i-1} = 0$, where the subscripts are considered modulo 3. This algebra is isomorphic to the group algebra of the alternative group $A_4$ if $k$ has characteristic 2. Let $e_2$ be the idempotent corresponding to the vertex 2, and let $T^\bullet$ be the tilting complex $T^\bullet$ associated with $e_2$. Then the endomorphism algebra $B$ of $T^\bullet$ is given by the quiver
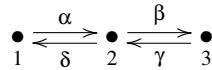


with relations $\alpha\delta = \gamma\beta = \delta\alpha\beta\gamma - \beta\gamma\delta\alpha = 0$. Note that $B$ is isomorphic to the principal block of the group algebra of $A_5$ if $k$ has characteristic 2. It is easy to see that the idempotent $\tilde{e}_2$ is the idempotent correspond to the vertex 2 in the quiver of $B$. Thus, by Proposition 4.7, the algebras $A/\nabla(e_2)$ and $B/\nabla(\tilde{e}_2)$ are derived-equivalent. A calculation shows that $A/\nabla(e_2) = A/\langle\alpha_2\beta_3\rangle$ and $B/\nabla(\tilde{e}_2) = B/\langle\beta\gamma\delta\alpha\rangle$. Note that the quotient algebras $A/\langle\alpha_2\beta_3\rangle$ and $B/\langle\beta\gamma\delta\alpha\rangle$ are stably equivalent of Morita type by a result in [8]. Thus $A/\langle\alpha_2\beta_3\rangle$ and $B/\langle\beta\gamma\delta\alpha\rangle$ are not only derived-equivalent, but also stably equivalent of Morita type.

**Example 2.** Let $m \geq 3$ be an integer, and let $A = k[t]/(t^m)$, the quotient algebra of the polynomial algebra $k[t]$ over a field $k$ in one variable $t$ modulo the ideal generated by $t^m$. Let $X$ be the simple $A$-module $k$. Then $\mathrm{E}_A^{\mathbb{N}}(A \oplus X)$ and $\mathrm{E}_A^{\mathbb{N}}(A \oplus \Omega_A(X))$ are infinite-dimensional $k$-algebras which can be described by quivers with relations:

$$\mathrm{E}_A^{\mathbb{N}}(A \oplus X) \qquad\qquad \mathrm{E}_A^{\mathbb{N}}(A \oplus \Omega_A(X))$$



$$\alpha^{m-1} - \beta\gamma = \alpha\beta = \gamma\alpha = \gamma\beta = 0; \qquad\qquad xz_i = z_i y = 0, i = 1, 2;$$
$$\delta_i \gamma = \beta\delta_i = 0, i = 1, 2; \qquad\qquad z_1^2 = z_1 z_2 - z_2 z_1 = 0;$$
$$\delta_1^2 = \delta_1\delta_2 - \delta_2\delta_1 = 0. \qquad\qquad (yx)^{m-1} = 0.$$

By Theorem 3.4, or Corollary 3.14, the two algebras $\mathrm{E}_A^{\mathbb{N}}(A \oplus X)$ and $\mathrm{E}_A^{\mathbb{N}}(A \oplus \Omega_A(X))$ are derived-equivalent.

Let $n \geq 1$ be a natural number. Then the finite-dimensional $k$-algebra $\mathrm{E}_A^{\Phi(1,n)}(A \oplus X)$ is the quotient of $\mathrm{E}_A^{\mathbb{N}}(A \oplus X)$ by the ideal generated by $\delta_2^{[\frac{n}{2}]+1}$ for $n$ an odd number, and by $\delta_1\delta_2^{n/2}$ and $\delta_2^{n/2+1}$ for $n$ an even number, where $[\frac{n}{2}]$ is is the largest integer less than or equal to $n/2$, and the finite-dimensional algebra $\mathrm{E}_A^{\Phi(1,n)}(A \oplus \Omega_A(X))$ is the quotient of $\mathrm{E}_A^{\mathbb{N}}(A \oplus \Omega_A(X))$ by the ideal generated by $z_2^{[\frac{n}{2}]+1}$ for $n$ an odd number, and by $z_1 z_2^{n/2}$ and $z_2^{n/2+1}$ for $n$ an even number. By Corollary 3.14, we know that $\mathrm{E}_A^{\Phi(1,n)}(A \oplus X)$ and $\mathrm{E}_A^{\Phi(1,n)}(A \oplus \Omega_A(X))$ are both derived and stably equivalent.

# References

[1] H. ASASHIBA, The derived equivalence classification of representation-finite selfinjective algebras. *J. Algebra* **214** (1999) 182-221.

[2] M. BAROT and H. LENZING, One-point extensions and derived equivalences. *J. Algebra* **264** (2003) 1-5.

[3] D. HAPPEL, *Triangulated categories in the representation theory of finite dimensional algebras.* Cambridge Univ. Press, Cambridge, 1988.

[4] M. HOSHINO and Y. KATO, Tilting complexes defined by idempotents. *Comm. Algebra* **30** (2002) 83-100.

[5] W. HU and C. C. XI, Almost $\mathcal{D}$-split sequences and derived equivalences. Preprint, available at : http://math.bnu.edu.cn/~ccxi/Papers/Articles/xihu-2.pdf, 2007.

[6] W. HU and C. C. XI, Derived equivalences and stable equivalence of Morita type, I. *Nagoya Math. J.* (in press). Preprint is available at : http://math.bnu.edu.cn/ ccxi/Papers/Articles/xihu-3.pdf, 2008.

[7] B. KELLER, Invariance and localization for cyclic homology of DG-algebras. *J. Pure Appl. Alg.* **123** (1998) 223-273.

[8] Y. M. LIU and C. C. XI, Constructions of stable equivalences of Morita type for finite dimensional algebras, I. *Trans. Amer. Math. Soc.* **358** (2006), no.6, 2537-2560.

[9] A. NEEMAN, *Triangulated categories.* Annals of Mathematics Studies 148, Princeton University Press, Princeton and Oxford, 2001.

[10] J. RICKARD, Morita theory for derived categories. *J. London Math. Soc.* **39** (1989) 436-456.

[11] J. RICKARD, Derived equivalences as derived functors. *J. London Math. Soc.* **43** (1991) 37-48.

[12] J. RICKARD, Derived categories and stable equivalences. *J. Pure Appl. Algebra* **64** (1989) 303-317.

[13] J. L. VERDIER, Catégories dérivées, etat O, *Lecture Notes in Mathematics* **569**, Springer-Verlag, Berlin, 1977, 262-311.

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