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# ON HOCHSCHILD EXTENSIONS OF REDUCED AND CLEAN RINGS

**Hanxing Lin and Changchang Xi**
*School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing, China*

***We show that an arbitrary Hochschild extension of a reduced ring by a two-sided ideal is symmetric and reversible, and that any Hochschild extension of a clean ring by an arbitrary bimodule is clean. This generalizes a result of Kim and Lee, and provides many examples of clean rings.***

***Key Words:*** Hochschild extension; Reduced ring; Reversible ring; Symmetric ring.

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## 1. INTRODUCTION

Hochschild extensions of rings are often used to construct new rings with certain properties. In this note we shall use Hochschild extensions to obtain new symmetric, reversible, and clean rings. Note that reversible rings were studied by Cohn (1999) and many others. In Kim and Lee (2003) it was shown that the trivial extension of a reduced ring $R$ with identity by itself is reversible. We first show that an arbitrary Hochschild extension of a reduced ring by an arbitrary two-sided ideal is symmetric. Thus this result extends the above mentioned result in Kim and Lee (2003). Moreover, its conclusion is stronger than that in Kim and Lee (2003). In this way we can produce many examples of reversible and symmetric rings. Also, the proof in the note is completely different from the one in Kim and Lee (2003).

More precisely, our statement reads as follows.

**Theorem 1.1.** *Suppose $R$ is a reduced ring and $M$ a two-sided ideal in $R$. Then, for any Hochschild 2-cocycle $\alpha$, the Hochschild extension $H_\alpha(R, M)$ of $R$ by $M$ via $\alpha$ is both symmetric and reversible.*

Next, we prove the following result on clean rings. Note that clean rings are studied in Han and Nicholson (2001), Nicholson (1977), Sherman (1981) and many other articles.

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Address correspondence to Changchang Xi, School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China; Fax: +86-10-58802136; E-mail: xicc@bnu.edu.cn

**Theorem 1.2.** *Let $R$ be a clean ring with identity, and let $M$ be an $R$-$R$-bimodule. Then, for any Hochschild 2-cocycle $\alpha : R \times R \longrightarrow M$, the Hochschild extension $H_\alpha(R, M)$ of $R$ by $M$ via $\alpha$ is a clean ring.*

Note that it is easy to see that the trivial extension of $R$ by an $R$-$R$-bimodule $M$ is clean if and only if $R$ is clean.

## 2. PROOFS OF THE RESULTS

Throughout this article, $R$ will denote a ring not necessarily with identity. We start by recalling some basic definitions.

**Definition 2.1.** Let $M$ be an $R$-$R$-bimodule. A $\mathbb{Z}$-bilinear map $\alpha : R \times R \longrightarrow M$ is called a *Hochschild 2-cocycle* if for all $\lambda_1, \lambda_2, \lambda_3 \in R$ the following equation holds true:

$$\alpha(\lambda_1\lambda_2, \lambda_3) - \alpha(\lambda_1, \lambda_2\lambda_3) = \lambda_1\alpha(\lambda_2, \lambda_3) - \alpha(\lambda_1, \lambda_2)\lambda_3.$$

Given a Hochschild 2-cocycle $\alpha$, there is a ring $H_\alpha(R, M)$, called the *Hochschild extension of $R$ by $M$ via $\alpha$*, which is $R \oplus M$ as an abelian group, and the multiplication is defined by

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2 + \alpha(r_1, r_2))$$

for all $r_1, r_2 \in R$ and all $m_1, m_2 \in M$.

This is an associative ring. If $R$ has the identity 1, then $H_\alpha(R, M)$ has the identity $(1, -\alpha(1, 1))$ (see Cartan and Eilenberg, 1973, pp. 294–295). If $\alpha = 0$, the extension ring $H_0(R, M)$ is called the *trivial extension of $R$ by $M$* in the literature. Note that the Jacobson radical of $H_\alpha(R, M)$ is $(J(R), M)$, where $J(R)$ stands for the Jacobson radical of $R$.

We have the following standard fact.

**Lemma 2.2.** *Let $R$ be a ring, $e$ an element in $R$, and $I$ an ideal in $R$.*

*Suppose each element in $I$ is nilpotent. If $e^2 - e \in I$, then there is an idempotent element $f$ in $R$ such that $f - e \in I$. In this case we say that the idempotent $e$ modulo $I$ can be lifted.*

Recall that a ring $R$ with identity is said to be *clean* if each element $r$ in $R$ is a sum of an idempotent element $e$ and a unit $u$. The following fact concerning clean rings was pointed out in Nicholson (1977).

**Lemma 2.3.** *Let $R$ be a ring with identity. Then $R$ is clean if and only if $R/J(R)$ is clean and idempotents modulo $J(R)$ can be lifted.*

Given a ring $R$, we call $R$ a *reduced ring* if there is no nonzero nilpotent element in $R$. The ring $R$ is said to be *reversible* if $ab = 0$ implies that $ba = 0$ for each pair $a, b \in R$ (see Cohn, 1999). The ring $R$ is said to be *symmetric* if $abc = 0$ implies

$acb = 0$ for all $a, b, c \in R$. The word "symmetric" follows from the fact in Anderson and Camillo (1999) and Krempa and Niewieczeral (1977) that $R$ is symmetric if and only if $r_1 r_2 \ldots r_n = 0$ for a positive integer $n \geq 3$ implies $r_{\sigma(1)} r_{\sigma(2)} \ldots r_{\sigma(n)} = 0$ for any permutation $\sigma$ on $\{1, 2, \ldots, n\}$. This fact will be used in our proofs.

The following lemma reveals the relationship between these notions. For more information on these classes of rings one may refer to Anderson and Camillo (1999), Cohn (1999), Kim and Lee (2003), and Krempa and Niewieczeral (1977).

**Lemma 2.4.**

(1) *If $R$ is reduced, then $R$ is symmetric and reversible.*
(2) *If $R$ is symmetric and has an identity, then $R$ is reversible.*
(3) *If $R$ is reduced* (*or symmetric, or reversible*), *then every subring of $R$ is reduced* (*or symmetric, or reversible*).

*Proof.* (1) was shown in Anderson and Camillo (1999, Theorem I.3). (2) is easy. (3) follows from the definitions. □

Note that a symmetric ring may not be reversible in general. For example, the radical of the $3 \times 3$ upper triangular matrix algebra over a field is an easy counterexample. Conversely, a reversible ring with identity may not be a symmetric ring (see Anderson and Camillo, 1999 for a counterexample). However, as we have seen, if $R$ is symmetric with identity, then $R$ must be reversible.

*Proof of Theorem* 1.1. Pick any three elements $(r_i, m_i) \in H_\alpha(R, M)$ for $i = 1, 2, 3$, where $r_i \in R$ and $m_i \in M$. Suppose that $(r_1, m_1)(r_2, m_2)(r_3, m_3) = 0$. We have to show that $(r_1, m_1)(r_3, m_3)(r_2, m_2) = 0$. It follows from $(r_1, m_1)(r_2, m_2)(r_3, m_3) = 0$ that $r_1 r_3 r_2 = 0$ and

$$r_1 r_2 m_3 + r_1 m_2 r_3 + r_1 \alpha(r_2, r_3) + m_1 r_2 r_3 + \alpha(r_1, r_2 r_3) = 0. \tag{$*$}$$

Since $R$ is reduced, $R$ is symmetric by Lemma 2.4. Thus $r_{\sigma(1)} r_{\sigma(2)} r_{\sigma(3)} = 0$ for any permutation $\sigma$ of $\{1, 2, 3\}$. We put $s = r_1 r_3 m_2 + r_1 m_3 r_2 + r_1 \alpha(r_3, r_2) + m_1 r_3 r_2 + \alpha(r_1, r_3 r_2)$. It follows from $(*)$ that

$$\begin{aligned}
0 &= r_3 r_2 \big(r_1 r_2 m_3 + r_1 m_2 r_3 + r_1 \alpha(r_2, r_3) + m_1 r_2 r_3 + \alpha(r_1, r_2 r_3)\big) \\
&= r_3 r_2 r_1 r_2 m_3 + r_3 r_2 r_1 m_2 r_3 + r_3 r_2 r_1 \alpha(r_2, r_3) + r_3 r_2 m_1 r_2 r_3 + r_3 r_2 \alpha(r_1, r_2 r_3) \\
&= 0 + 0 + 0 + r_3 r_2 m_1 r_2 r_3 + r_3 r_2 \alpha(r_1, r_2 r_3) \\
&= r_3 r_2 m_1 r_2 r_3 + \alpha(r_3 r_2, r_1) r_2 r_3 + \alpha(r_3 r_2 r_1, r_2 r_3) - \alpha(r_3 r_2, r_1 r_2 r_3) \\
&= r_3 r_2 m_1 r_2 r_3 + \big(\alpha(r_3 r_2, r_1)\big) r_2 r_3 + 0 + 0 \\
&= (r_3 r_2 m_1 + \alpha(r_3 r_2, r_1)) r_2 r_3.
\end{aligned}$$

This implies that $(r_3 r_2 m_1 + \alpha(r_3 r_2, r_1)) r_3 r_2 = 0$ because $R$ is symmetric. Thus we have

$$\begin{aligned}
r_3 r_2 s &= r_3 r_2 (r_1 r_3 m_2 + r_1 m_3 r_2 + r_1 \alpha(r_3, r_2) + m_1 r_3 r_2 + \alpha(r_1, r_3 r_2)) \\
&= r_3 r_2 r_1 r_3 m_2 + r_3 r_2 r_1 m_3 r_2 + r_3 r_2 r_1 \alpha(r_3, r_2) + r_3 r_2 m_1 r_3 r_2 + r_3 r_2 \alpha(r_1, r_3 r_2)
\end{aligned}$$

$$
\begin{aligned}
&= r_3r_2m_1r_3r_2 + r_3r_2\alpha(r_1, r_3r_2)\\
&= r_3r_2m_1r_3r_2 + \alpha(r_3r_2, r_1)r_3r_2 + \alpha(r_3r_2r_1, r_3r_2) - \alpha(r_3r_2, r_1r_3r_2)\\
&= \big(r_3r_2m_1 + \alpha(r_3r_2, r_1)\big)r_3r_2\\
&= 0.
\end{aligned}
$$

Now we calculate $sr_3r_1$. It follows from $(*)$ that

$$
\begin{aligned}
0 &= r_3((r_1r_2m_3 + r_1m_2r_3 + r_1\alpha(r_2, r_3) + m_1r_2r_3 + \alpha(r_1, r_2r_3))r_1\\
&= r_3r_1r_2m_3r_1 + r_3r_1m_2r_3r_1 + r_3r_1\alpha(r_2, r_3)r_1 + r_3m_1r_2r_3r_1 + r_3\alpha(r_1, r_2r_3)r_1\\
&= r_3r_1m_2r_3r_1 + r_3r_1\alpha(r_2, r_3)r_1 + r_3\alpha(r_1, r_2r_3)r_1\\
&= r_3r_1m_2r_3r_1 + r_3r_1\alpha(r_2, r_3)r_1 + r_3\big(r_1\alpha(r_2r_3, r_1) + \alpha(r_1, r_2r_3r_1) - \alpha(r_1r_2r_3, r_1)\big)\\
&= r_3r_1\big(m_2r_3r_1 + \alpha(r_2, r_3)r_1 + \alpha(r_2r_3, r_1)\big).
\end{aligned}
$$

Since $R$ is symmetric, we have $r_1r_3(m_2r_3r_1 + \alpha(r_2, r_3)r_1 + \alpha(r_2r_3, r_1)) = 0$, and

$$
\begin{aligned}
sr_3r_1 &= r_1r_3m_2r_3r_1 + r_1m_3r_2r_3r_1 + r_1\alpha(r_3, r_2)r_3r_1 + m_1r_3r_2r_3r_1 + \alpha(r_1, r_3r_2)r_3r_1\\
&= r_1r_3m_2r_3r_1 + r_1(\alpha(r_3, r_2)r_3)r_1 + (\alpha(r_1, r_3r_2)r_3)r_1\\
&= r_1r_3m_2r_3r_1 + r_1r_3\alpha(r_2, r_3)r_1 + r_1\alpha(r_3, r_2r_3)r_1 - r_1\alpha(r_3r_2, r_3)r_1 + r_1\alpha(r_3r_2, r_3)r_1\\
&\quad + \alpha(r_1, r_3r_2r_3)r_1 - \alpha(r_1r_3r_2, r_3)r_1\\
&= r_1r_3m_2r_3r_1 + r_1r_3\alpha(r_2, r_3)r_1 + r_1r_3\alpha(r_2r_3, r_1) + r_1\alpha(r_3, r_2r_3r_1)\\
&\quad - r_1\alpha(r_3r_2r_3, r_1) + \alpha(r_1, r_3r_2r_3)r_1\\
&= r_1r_3\big(m_2r_3r_1 + \alpha(r_2, r_3)r_1 + \alpha(r_2r_3, r_1)\big)\\
&= 0.
\end{aligned}
$$

By a similar calculation, it follows from multiplying by $r_2r_1$ from right on the both sides of $(*)$ that $r_1r_2(m_3r_2r_1 + \alpha(r_3, r_2r_1)) = 0$. This implies that

$$
\begin{aligned}
r_2sr_1 &= r_2r_1r_3m_2r_1 + r_2r_1\alpha(r_3, r_2)r_1 + r_2m_1r_3r_2r_1 + r_2\alpha(r_1, r_3r_2)r_1\\
&= r_2r_1m_3r_2r_1 + r_2r_1\alpha(r_3, r_2r_1) + r_2r_1r_3\alpha(r_2, r_1) - r_2r_1\alpha(r_3r_2, r_1)\\
&\quad + r_2r_1\alpha(r_3r_2, r_1) + r_2\alpha(r_1, r_3r_2r_1) - r_2\alpha(r_1r_3r_2, r_1)\\
&= r_2r_1\big(m_3r_2r_1 + \alpha(r_3, r_2r_1)\big)\\
&= 0,
\end{aligned}
$$

because $r_1r_2(m_3r_2r_1 + \alpha(r_3, r_2r_1)) = 0$. Since $R$ is symmetric, it follows from $sr_3r_1 = r_3r_2s = r_2sr_1 = 0$ by permutations that $sr_1r_3 = sr_1m_3r_2 = sm_1r_3r_2 = 0$. Thus

$$
\begin{aligned}
s^2 &= s\big(r_1r_3m_2 + r_1m_3r_2 + r_1\alpha(r_3, r_2) + m_1r_3r_2 + \alpha(r_1, r_3r_2)\big)\\
&= sr_1\alpha(r_3, r_2) + s\alpha(r_1, r_3r_2)
\end{aligned}
$$

$$
\begin{aligned}
&= sr_1\alpha(r_3, r_2) + \alpha(s, r_1)(r_3r_2) + \alpha(sr_1, r_3r_2) - \alpha(s, r_1r_3r_2)\\
&= sr_1\alpha(r_3, r_2) + \alpha(s, r_1)(r_3r_2) + \alpha(sr_1, r_3r_2)\\
&= \alpha(sr_1, r_3)r_2 + \alpha(sr_1r_3, r_2) - \alpha(sr_1, r_3r_2) + \alpha(s, r_1)r_3r_2 + \alpha(sr_1, r_3r_2)\\
&= \alpha(sr_1, r_3)r_2 + \alpha(s, r_1)r_3r_2
\end{aligned}
$$

and

$$s^3 = \alpha(sr_1, r_3)r_2s + \alpha(s, r_1)r_3r_2s = \alpha(sr_1, r_3)r_2s = sr_1\alpha(r_3, r_2s).$$

This yields that $r_2s^3 = r_2sr_1\alpha(r_3, r_2s) = 0$ and

$$s^5 = \alpha(sr_1, r_3)r_2s^3 = 0.$$

Hence $s = 0$ since $R$ is reduced. Now we have

$$(r_1, m_1)(r_3, m_3)(r_2, m_2) = (r_1r_3r_2, s) = 0.$$

This shows that $H_\alpha(R, M)$ is symmetric, as desired.

Since we do not assume that $R$ has an identity, we need to prove that $H_\alpha(R, M)$ is reversible if $R$ is reduced. The proof is similar to the above proof. Let $(r_i, m_i) \in H_\alpha(R, M)$ for $i = 1, 2$, where $r_i \in R$ and $m_i \in M$. Suppose that $(r_1, m_1)(r_2, m_2) = 0$, we have to show that $(r_2, m_2)(r_1, m_1) = 0$. It follows from $(r_1, m_1)(r_2, m_2) = 0$ that $r_1r_2 = 0$ and

$$r_1m_2 + m_1r_2 + \alpha(r_1, r_2) = 0. \tag{$**$}$$

Since $R$ is reduced, $R$ is reversible by Lemma 2.4. Thus $r_2r_1 = 0$. It follows from $(**)$ that

$$
\begin{aligned}
0 &= (r_1m_2 + m_1r_2 + \alpha(r_1, r_2))r_1 = r_1m_2r_1 + m_1r_2r_1 + \alpha(r_1, r_2)r_1\\
&= r_1m_2r_1 + 0 + r_1\alpha(r_2, r_1) + \alpha(r_1, r_2r_1) - \alpha(r_1r_2, r_1)\\
&= r_1m_2r_1 + r_1\alpha(r_2, r_1) + \alpha(r_1, 0) - \alpha(0, r_1)\\
&= r_1(m_2r_1 + \alpha(r_2, r_1)),
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= r_2(r_1m_2 + m_1r_2 + \alpha(r_1, r_2)) = r_2r_1m_2 + r_2m_1r_2 + r_2\alpha(r_1, r_2)\\
&= 0 + r_2m_1r_2 + \alpha(r_2, r_1)r_2 + \alpha(r_2r_1, r_2) - \alpha(r_2, r_1r_2)\\
&= r_2m_1r_2 + \alpha(r_2, r_1)r_2 + \alpha(0, r_2) - \alpha(r_2, 0)\\
&= (r_2m_1 + \alpha(r_2, r_1))r_2.
\end{aligned}
$$

Now, let $m = r_2m_1 + m_2r_1 + \alpha(r_2, r_1)$. Then $m \in M$ and

$$r_1m = r_1(r_2m_1 + m_2r_1 + \alpha(r_2, r_1)) = r_1r_2m_1 + r_1(m_2r_1 + \alpha(r_2, r_1)) = 0.$$

Similarly, we have

$$mr_2 = (r_2m_1 + m_2r_1 + \alpha(r_2, r_1))r_2 = (r_2m_1 + \alpha(r_2, r_1))r_2 + m_2r_1r_2 = 0.$$

Thus we get

$$\begin{aligned}
m^2 &= (r_2m_1 + m_2r_1 + \alpha(r_2, r_1))m \\
&= r_2m_1m + m_2r_1m + \alpha(r_2, r_1)m \\
&= r_2m_1m + 0 + r_2\alpha(r_1, m) + \alpha(r_2, r_1m) - \alpha(r_2r_1, m) \\
&= r_2m_1m + r_2\alpha(r_1, m) \\
&= r_2(m_1m + \alpha(r_1, m))
\end{aligned}$$

and $m^3 = mm^2 = mr_2(m_1m + \alpha(r_1, m)) = 0$. Since $R$ is reduced, we must have $m = 0$. This implies that $(r_2, m_2)(r_1, m_1) = (r_2r_1, m) = (0, 0)$, as desired. This finishes the proof.

*Proof of Theorem* 1.2. We shall use Lemma 2.3 to prove Theorem 1.2.

Note that $(0, M)$ is a nilpotent ideal in $H_\alpha(R, M)$ and $H_\alpha(R, M)/(0, M) \simeq R$. It is clear that $H_\alpha(R, M)/(J(R), M) \simeq R/J(R)$ which is a clean ring. It remains to show that idempotents modulo the Jacobson radical of $H_\alpha(R, M)$ can be lifted. In fact, suppose $x$ is in $H_\alpha(R, M)$ and $x^2 - x \in (J(R), M)$. Let $\overline{R}$ be the ring $H_\alpha(R, M)/(0, M)$ and $\overline{J}$ its radical $(J(R), M)/(0, M)$. Since $x^2 - x \in (J(R), M)$ we see that the element $(x^2 - x) + (0, M)$ of $\overline{R}$ lies in $(J(R), 0) + (0, M) = \overline{J}$, the radical of $\overline{R}$, and that $x + (0, M)$ is an idempotent element in $\overline{R}/\overline{J}$. Since $\overline{R}$ is isomorphic to $R$, it follows from Lemma 2.3 that there is an idempotent element $f + (0, M) \in \overline{R}$ such that $(f - x) + (0, M) \in \overline{J}$, that is $f - x \in (J(R), M)$. By Lemma 2.2, there is an idempotent element $e \in H_\alpha(R, M)$ such that $e - f \in (0, M)$. Thus $e - x = (e - f) + (f - x) \in (0, M) + (J(R), M) = (J(R), M)$. This means that $x$ can be lifted modulo the radical of $H_\alpha(R, M)$. Thus $H_\alpha(R, M)$ is clean by Lemma 2.3. This finishes the proof.

Finally, let us point out the following fact which is a consequence of the statement that the trivial extension of $R$ by a bimodule $M$ is clean if and only if $R$ is clean, and was known already in Haghany and Varadarajan (1999, Proposition 6.3).

**Corollary 2.5.** *Let $R$ and $S$ be two rings with identity, and let $M$ be an $R$-$S$-bimodule. Then the matrix ring $\left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$ is clean if and only if both $R$ and $S$ are clean. In particular, $R$ is clean if and only if the $n \times n$ upper triangular matrix algebra over $R$ is clean for all $n \geq 2$.*

**Remark.** In this note we have considered the Hochschild extensions of reduced rings and clean rings. One may also consider the Hochschild extensions of reversible rings (or other type of rings) and ask if the corresponding Hochschild extensions are still reversible or semi-commutative (Lambek, 1971) (or of the same type). At moment we do not know the answer.

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